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Critical Configurations

A classification of all critical configurations for any number of projective views

Thesis submitted for the degree of Philosophiae Doctor

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To my children, Henrik and Sofie

Abstract

Structure from motion is the process of recovering cameras and a 3D scene from a set of images. Generally, all information can be recovered if enough images and image points are provided. There are, however, certain cases where such a recovery is impossible; these are called *critical configurations*. This thesis classifies all critical configurations for any number of projective cameras. We show that they form well-known algebraic varieties, such as quadratic surfaces and curves of degree at most four. We also improve upon earlier results both by finding previously unknown critical configurations and by showing that some configurations previously believed to be critical are in fact not.

Sammendrag

Structure form motion er prosessen hvor man finner tilbake informasjon om et 3D-objekt eller -landskap fra et sett av bilder. Vanligvis kan man finne tilbake all informasjon dersom man har tilstrekkelig mange bilder. Det finnes, imidlertid, tilfeller hvor dette ikke lar seg gjøre; disse kalles *kritiske konfigurasjoner*. Denne avhandlingen klassifiserer alle kritiske konfigurasjoner for ethvert antall projektive kamera. Disse kritiske konfigurasjonene viser seg å forme velkjente algebraiske varieteter, slik som kvadratiske flater og kurver av grad høyst fire. Avhandlingen forbedrer også tidligere resultater både ved å finne tidligere ukjente kritiske konfigurasjoner, og ved å vise at noen konfigurasjoner man tidligere trodde var kritiske, ikke er kritiske likevel.

Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of *Philosophiae Doctor* at the University of Oslo. The research presented here was conducted at the University of Oslo, under the supervision of professor Kristian Ranestad, and cosupervised by docent Kathlén Kohn at KTH Royal Institute of Technology. This work was supported by the Norwegian National Security Authority.

The thesis is a collection of four papers, presented in chronological order of writing. Papers I, II and IV classify the critical configurations for 2, 3, and 4+ views respectively. Paper III gives conditions for compatibility of fundamental matrices, which are necessary for the classification of 4+ views. The papers are preceded by an introductory chapter that provides background information and motivation for the work. Paper III is a joint work with Felix Rydell. I am the sole author of the remaining papers.

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First and foremost, I would like to express my deepest appreciation to my two supervisors, Kristian Ranestad and Kathlén Kohn. To Kristian for lending his time, experience, and intuition whenever the need arose, and to Kathlén for introducing me to the field of computer vision and the people working in it. Both have been invaluable, and without your guidance, enthusiasm, and belief in my work, I would not have gotten this far.

I am also thankful to my co-author Felix for joining me in solving the compatibility problem, a key result in my thesis that ensures that my Erdős number is four.

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Finally, I am eternally grateful to my beloved wife for taking care of first one, then two, small children while I have been free to travel and work on my PhD. Thank you for always encouraging me, and for helping me finish this degree.

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Martin Bråtelund
Oslo, December 2023

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Paper I

Bråtelund, M. "Critical configurations for two projective views, a new approach". In: *Journal of Symbolic Computation*. Vol. 120, (2024), DOI: 10.1016/j.jsc.2023.102226.

Paper II

Bråtelund, M. "Critical configurations for three projective views". In: *Mathematics Scandinavica*. Vol. 129, (2023), pp. 401–444. DOI: 10.7146/math.scand.a-139788.

Paper III

Bråtelund, M. and Rydell, F. "Compatibility of Fundamental Matrices for Complete Viewing Graphs". In: *Proceedings of the IEEE/CVF International Conference on Computer Vision (ICCV)*. (2023), pp. 3328–3336.

Paper IV

Bråtelund, M. "A Classification of Critical Configurations for any Number of Projective Views". *Submitted for publication.*

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Chapter 1

Introduction

1.1 Background

This thesis is on the topic of algebraic vision and multi-view geometry. As such, it falls somewhere between the more theoretical field of algebraic geometry and the more applied field of computer vision while not fully belonging to either. More specifically, it deals with the broader problem of *structure from motion*, and in particular with the problem of *critical configurations*.

Structure from motion is among the oldest and most well-studied problems in computer vision. Given a set of 2-dimensional images, the task is to reconstruct a scene in 3-space and find the camera positions in this scene. Early works on this problem long predate the first computers, with some going back to Taylor and Lagrange in the 16th century [Stu11]. Over time, many techniques were developed for solving these problems for varying camera models and under different assumptions on the space and image points [Har97; HZ04; May93; MF92]. In general, one can uniquely recover all information about the original scene if one has enough images and enough points in each image. There are, however, some configurations of cameras and points whose 3D information can not be uniquely recovered from the images. These are called *critical configurations*.

The first mention of critical configurations appears in Krames's paper from 1941 [Kra41], where he proves that for two cameras and any number of points

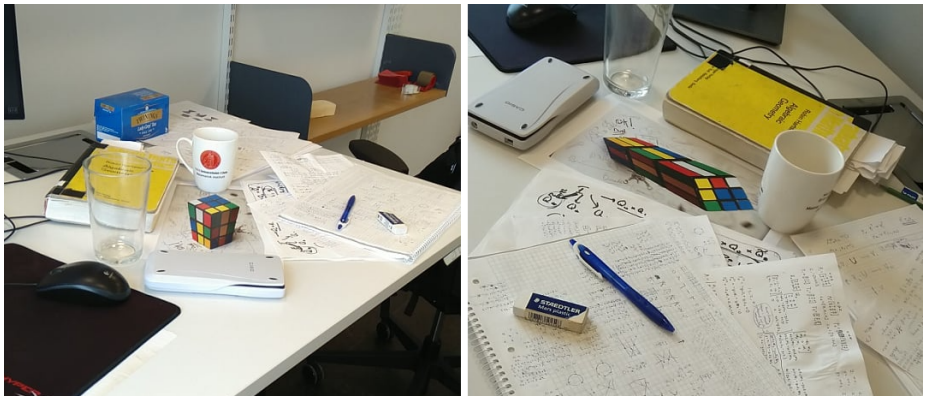

Figure 1.1: From these two images of the same scene, one can recover the 3D properties of the objects shown. For instance, one can tell that the Rubik's Cube in the center of each image is, in fact, no cube at all. This is impossible if one only has the first image.

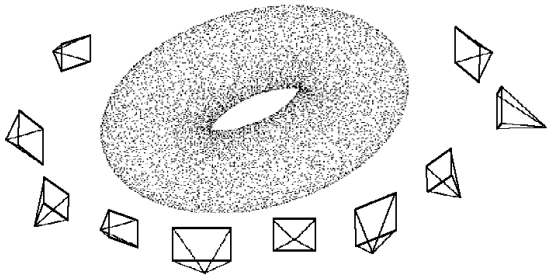

Figure 1.2: Cameras viewing a torus. In the model we use, the intrinsic parameters of the camera are unknown, and the 3D scene consists only of points[1].

all lying on a smooth quadratic surface, the two images will have three possible 3D reconstructions. Over time, much work has been done to understand critical configurations for various settings [Ber+20; BT07; Buc88; Buc92; Duf+19; Duf+20; Har00; HK02; HK07; KBP08; KHA01; LAO17]. The setting in the papers used in this thesis is one where the 3D scene consists of points in $\mathbb{P}^3$ viewed by *uncalibrated projective pinhole* cameras, meaning intrinsic parameters of the camera, such as focal length, skew coefficient, and principal point are unknown (see Figure 1.2). However, the new techniques developed in the papers can be extended to allow classification in other settings.

While interesting from a purely theoretical viewpoint, critical configurations are also relevant in practical applications. In real-life reconstruction problems, critical configurations rarely show up when enough data is available (noise alone should be enough to avoid the critical configurations). However, it has been shown that as the configurations approach the critical ones, reconstruction algorithms can become less stable [BTB07; HK07; LF94]. Moreover, when doing reconstruction with little data available, the chances of running into critical configurations greatly increase.

1.2 Summary of Papers

Paper I classifies all critical configurations for one and two views. For one view of n points, a configuration is critical if the camera center and the points span a space of dimension less than n. For two views, a configuration is critical only if all the points and both camera centers lie on a quadric surface containing real lines. A key contribution of this paper is the introduction of new techniques for the classification of critical configurations. These are the techniques used for the classification in Papers II and IV. We also hope these new techniques can be used in other, more complicated scenarios

[1]The model allows for an infinite number of points, so the points may (and often will) form a curve or a surface

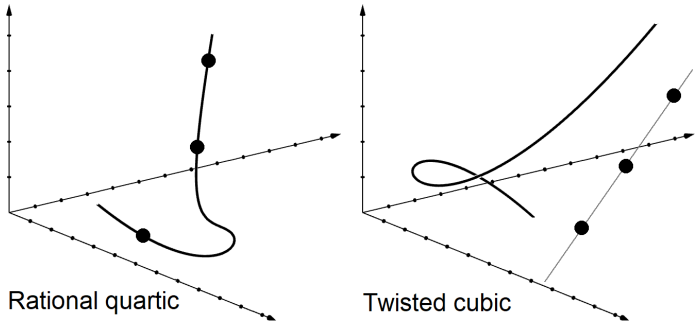

Figure 1.3: Two examples of critical configurations for three views (cameras pictured as black dots). While these 3D configurations are quite different, the images they produce will be the same, namely that of a plane cubic curve.

(e.g., cameras observing lines [Bre+22; Buc92]) in future work. Paper 1 uses these techniques to confirm the main result of [HK07] for two cameras, and to provide proofs for many assertions given without proof in [HK07]. Moreover, it gives an explicit description of the relation between a critical configuration and its conjugate(s), that is, the other configurations that produce the same images.

Paper II uses the techniques from Paper I to classify the critical configurations for three views. In this case, the critical configurations consist of points lying on the intersection of three quadratic surfaces, each passing through two of the three camera centers. This intersection will usually consist of algebraic curves of degree at most four (see example in Figure 1.3), although critical configurations where the points form surfaces or finite point sets also exist. This paper, like Paper I, confirms many results from [HK07] using new proofs. It also provides new results on conditions for criticality and multi-view ideals. Lastly, it presents one previously unknown critical configuration consisting of points lying on a twisted cubic passing through exactly two of the three camera centers.

Paper III solves the compatibility problem of fundamental matrices. Given two cameras, their images are related by some incidence relation[2] telling us which points in the first image correspond to which points in the second. This relation can be described by a 3×3 matrix of rank 2 called the *fundamental matrix*. For any 3×3 matrix of rank 2, one can find two cameras for which it is the fundamental matrix. For three such matrices, however, certain conditions need to be satisfied for there to exist three cameras having them as fundamental matrices. These are called the *compatibility conditions*.

[2] Often called the epipolar constraint

For three cameras, these conditions appear in [HZ04] and are used extensively in Paper II. This is because the critical configurations for three views need to lie on the intersection of three surfaces that are described by three *compatible* fundamental matrices. Likewise, for n cameras, they need to lie on the intersection of surfaces described by a set of $\binom{n}{2}$ compatible fundamental matrices, one for each pair of cameras.

While the three-view case was well known. For four or more cameras, the answer was less clear, with multiple papers claiming that the conditions for three cameras were sufficient for any number of cameras. Paper III provides a counterexample to that claim, and more importantly, it solves the compatibility problem for any number of cameras by giving explicit polynomial conditions for when a set of $\binom{n}{2}$ fundamental matrices comes from a set of n cameras. It covers the general case, but also all cases where the cameras are in a degenerate position. It also improves upon the work of Kasten et al. [Kas+19] by showing that the eigenvalue condition in their main theorem can be dropped. Lastly, the compatibility conditions are needed to make any sort of classification of critical configurations for four or more views. This paper is a joint work with Felix Rydell.

Paper IV finishes the classification of critical configurations. Using the results from Papers I and II, along with the compatibility conditions from Paper III, it classifies all critical configurations for four or more cameras. It also includes most results from the first two papers, sometimes with new proofs, to give a complete classification of any number of views in a single paper. This paper proves that apart from the six- and seven-point configurations, every critical configuration for three views remains critical when more cameras are added. Additionally, the six-point configuration will remain critical if the new cameras lie on the unique quadric defined by the six points and the first three cameras. The paper also presents one new critical configuration for three views; a degenerate of the elliptic quartic curve, previously believed to be non-critical.

Papers

Paper I

Critical configurations for two projective views, a new approach

Martin Bråtelund

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The published version contains an error in the table on page 17, where the bottom two entries are swapped. This is corrected in this version, along with minor typographical corrections.

Abstract

This article develops new techniques to classify critical configurations for 3D scene reconstruction from images taken by unknown cameras. Generally, all information can be uniquely recovered if enough images and image points are provided, but there are certain cases where unique recovery is impossible; these are called *critical configurations*. In this paper, we use an algebraic approach to study the critical configurations for two projective cameras. We show that all critical configurations lie on quadric surfaces, and classify exactly which quadrics constitute a critical configuration. The paper also describes the relation between the different reconstructions when unique reconstruction is impossible.

I.1 Introduction

In computer vision, one of the main problems is that of *structure from motion*, where given a set of 2-dimensional images the task is to reconstruct a scene in 3-space and find the camera positions in this scene. Over time, many techniques have been developed for solving these problems for varying camera models and under different assumptions on the space and image points [HZ04; May93]. In general, with enough images and enough points in each image, one can uniquely recover all information about the original scene. However, there are also some configurations of points and images where a unique recovery is never possible. These are called *critical configurations*.

Much work has been done to understand critical configurations for various settings [Ber+20; BT07; Buc88; Buc92; Har00; HK02; HK07; KHA01], with results dating as far back as 1941 [Kra41]. While interesting from

a purely theoretical viewpoint, critical configurations also play a part in practical applications. Even though critical configurations are rare in real-life reconstruction problems when enough data is available (due to noise), it has been shown that as the configurations approach the critical ones, reconstruction algorithms tend to become less and less stable [BTB07; HK07; LF94].

Our new techniques confirm the main result of [HK07] for two cameras, and provide proofs for many unproven assertions in [HK07] (see Section 4). Moreover, they yield explicit maps between different critical configurations that have the same images (see Section 5). We also give a complete description of all critical configurations in the case of a single camera (see Section 3). Our companion paper [Brå23] uses the techniques developed in this article to correct the classification of [HK07] for three cameras. We plan to use these new techniques to classify critical configurations for any number of views, as well as using them in other, more complicated scenarios (e.g., cameras observing lines [Bre+22; Buc92], or rolling-shutter cameras [ASP16]) in future work.

The main result of this paper is the classification of the critical configurations for two views in Theorem I.4.11.

I.2 Background

We refer the reader to [HZ04] for the basics on computer vision and multi-view geometry.

Let $\mathbb{C}$ denote the complex numbers, and let $\mathbb{P}^n$ denote the projective space over the vector space $\mathbb{C}^{n+1}$. Projection from a point $p \in \mathbb{P}^3$ is a linear map

$$P\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2.$$

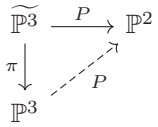
We refer to such a projection and its projection center p as a *camera* and its *camera center* (following established terminology, we use the words *camera* and *view* interchangeably). Following this theme, we refer to points in $\mathbb{P}^3$ as *space points* and points in $\mathbb{P}^2$ as *image points*. Similarly, $\mathbb{P}^2$ will be referred to as an *image*.

Once a basis is chosen in $\mathbb{P}^3$ and $\mathbb{P}^2$, a camera can be represented by a 3×4 matrix of full rank called the *camera matrix*. The camera center is then given as the kernel of the matrix. For the most part, we make no distinction between a camera and its camera matrix, referring to both simply as cameras. We use the *real projective pinhole camera model*, meaning that we require a camera matrix to be of full rank and to have only real entries.

Remark I.2.1. Throughout the paper, whenever we talk about cameras it is to be understood that a choice of basis has been made, both on the images and 3-space.

Since the map P is not defined at the camera center p, it is not a morphism. This problem can be mended by taking the blow-up. Let $\widetilde{\mathbb{P}^3}$ be the blow-up of

$\mathbb{P}^3$ in the camera center of P. We then get the following diagram:



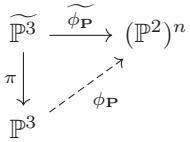
where π denotes the blow-down of $\widetilde{\mathbb{P}^3}$. This gives a morphism from $\widetilde{\mathbb{P}^3}$ to $\mathbb{P}^2$. For ease of notation, we retain the symbol P and the names *camera* and *camera center*, although one should note that in $\widetilde{\mathbb{P}^3}$ the camera center is no longer a point, but an exceptional divisor.

Definition I.2.2. Given an n-tuple of cameras $\mathbf{P} = (P_1, \dots, P_n)$, with camera centers $p_1, \dots, p_n$, let $\widetilde{\mathbb{P}^3}$ denote the blow-up of $\mathbb{P}^3$ in the n camera centers. We define the *joint camera map* to be the map

$$\begin{aligned}\widetilde{\phi_{\mathbf{P}}} = P_1 \times \cdots \times P_n \colon \widetilde{\mathbb{P}^3} &\to (\mathbb{P}^2)^n, \\ x &\mapsto (P_1(x), \dots, P_n(x)).\end{aligned}$$

Remark I.2.3. Throughout the paper, we assume that all camera centers are distinct.

This again gives a commutative diagram



The reason we use the blow-up $\widetilde{\mathbb{P}^3}$ rather $\mathbb{P}^3$ is to turn the cameras (and hence the joint camera map) into morphisms rather than rational maps. This ensures that the image of the joint camera map is Zariski closed, turning it into a projective variety.

Definition I.2.4. We denote the image of the joint camera map $\widetilde{\phi_{\mathbf{P}}}$ as the *multi-view variety* of $P_1, \dots, P_n$. The set of all homogeneous polynomials vanishing on $\mathrm{Im}(\widetilde{\phi_{\mathbf{P}}})$ is an ideal that we denote as the *multi-view ideal*.

Notation. Most works on computer vision do not use the blow-up, defining cameras/the joint camera map as rational maps rather than morphisms. Hence, they tend to define the multi-view variety as the *closure* of the image rather than as the image itself. While our definition of the multi-view variety seems different, it is equivalent to that used in other works, like [APT21].

Notation. While the multi-view variety is always irreducible, we use the term *variety* to also include reducible algebraic sets.

Definition I.2.5. Given a set of points $S \subset (\mathbb{P}^2)^n$, a *reconstruction* of S is an n-tuple of cameras $\mathbf{P} = (P_1, \dots, P_n)$ and a set of points $X \subset \widetilde{\mathbb{P}^3}$ such that $S = \widetilde{\phi_{\mathbf{P}}}(X)$ where $\widetilde{\phi_{\mathbf{P}}}$ is the joint camera map of the cameras $P_1, \dots, P_n$.

Definition I.2.6. Given a configuration of cameras and points $(P_1, \dots, P_n, X)$, we refer to $\widetilde{\phi_{\mathbf{P}}}(X) \subset (\mathbb{P}^2)^n$ as the *images* of $(P_1, \dots, P_n, X)$.

Given a set of image points $S \subset (\mathbb{P}^2)^n$ as well as a reconstruction $(P_1, \dots, P_n, X)$ of S, note that any scaling, rotation, translation, or more generally, any real projective transformation of $(P_1, \dots, P_n, X)$ does not change the images, giving rise to a large family of reconstructions of S. However, we are not interested in differentiating between these reconstructions.

Definition I.2.7. Given a set of points $S \subset (\mathbb{P}^2)^n$, let $(\mathbf{P}, X)$ and $(\mathbf{Q}, Y)$ be two reconstructions of S, let $\overline{X}$ and $\overline{Y}$ denote the blow-downs of X and Y respectively and let P_i' and Q_i' be the matrix representation of P_i and Q_i respectively. The two reconstructions of S are considered *equivalent* if there exists an element $A \in \mathrm{PGL}(4)$, such that

$$\begin{aligned} A(\overline{X}) &= \overline{Y}, \\ P_i' A^{-1} &= Q_i', \quad \forall i. \end{aligned}$$

From now on, whenever we talk about a configuration of cameras and points, it is to be understood as unique up to such an isomorphism/action of PGL(4), and two configurations are considered different only if they are not isomorphic/do not lie in the same orbit under this group action. As such, we consider a reconstruction to be unique if it is unique up to action by PGL(4).

Definition I.2.8. Given a configuration of cameras and points $(P_1, \dots, P_n, X)$, a *conjugate configuration* is a configuration $(Q_1, \dots, Q_n, Y)$, nonequivalent to the first, such that $\widetilde{\phi_{\mathbf{P}}}(X) = \widetilde{\phi_{\mathbf{Q}}}(Y)$. Pairs of points $(x, y) \in X \times Y$ are called *conjugate points* if $\widetilde{\phi_{\mathbf{P}}}(x) = \widetilde{\phi_{\mathbf{Q}}}(y)$.

Definition I.2.9. A configuration of cameras $(P_1, \dots, P_n)$ and points $X \subset \widetilde{\mathbb{P}^3}$ is said to be a *critical configuration* if it has at least one conjugate configuration. A critical configuration $(P_1, \dots, P_n, X)$ is said to be *maximal* if there exists no critical configuration $(P_1, \dots, P_n, X')$ such that $X \subsetneq X'$.

Hence, a configuration is critical if and only if the images it produces do not have a unique reconstruction.

Remark I.2.10. Various definitions of critical configurations exist. For instance, [Kra41] considers the cone with two cameras on the same generator to be critical, while it fails to be critical by our definition. We use a definition similar to the one in [HK07], except we are working in $\widetilde{\mathbb{P}^3}$. If one considers the blow-down, our definition matches the results in [HK07].

Definition I.2.11. Let $\mathbf{P}$ and $\mathbf{Q}$ be two n-tuples of cameras, let $\widetilde{\mathbb{P}^3}_P$ and $\widetilde{\mathbb{P}^3}_Q$ denote the blow-up of $\mathbb{P}^3$ in the camera centers of $\mathbf{P}$ and $\mathbf{Q}$ respectively. Projecting the fiber product

$$\widetilde{\mathbb{P}^3}_P \times_{\mathbb{P}^2} \widetilde{\mathbb{P}^3}_Q = \left\{ (x, y) \in \widetilde{\mathbb{P}^3}_P \times \widetilde{\mathbb{P}^3}_Q \mid \widetilde{\phi_{\mathbf{P}}}(x) = \widetilde{\phi_{\mathbf{Q}}}(y) \right\}$$

to the first coordinate gives a variety, X. We call X the *set of critical points of P with respect to Q*.

This definition is motivated by the following fact:

Proposition I.2.12. *Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be two (different) n-tuples of cameras, and let X and Y be their respective sets of critical points. Then $(\boldsymbol{P}, X)$ is a critical configuration, with $(\boldsymbol{Q}, Y)$ as its conjugate. Furthermore, $(\boldsymbol{P}, X)$ is maximal with respect to $\boldsymbol{Q}$ in the sense that if there exists a critical configuration $(\boldsymbol{P}, X')$ with $X \subsetneq X'$ then its conjugate consists of cameras different from $\boldsymbol{Q}$*

Proof. It follows from Definition I.2.11 that for each point $x \in X$, we have a conjugate point $y \in Y$. Hence the two configurations have the same images, so they are both critical configurations, conjugate to one another.

The (partial) maximality follows from the fact that if we add a point x_0 to X that does not lie in the set of critical points, there is (by Definition I.2.11) no point $y_0 \in \widetilde{\mathbb{P}^3}_Q$ such that $\widetilde{\phi_{\mathbf{P}}}(x_0) = \widetilde{\phi_{\mathbf{Q}}}(y_0)$. Hence $(\mathbf{P}, X)$ will no longer be critical. ■

For two different pairs of cameras, the sets of critical points turn out to be the quadric surfaces $\widetilde{S_P}, \widetilde{S_Q}$ described in Section I.4.

The goal of this paper is to classify all maximal critical configurations for three cameras. The reason we focus primarily on the maximal ones is that every critical configuration is contained in a maximal one and (when working with more than one camera) the converse is true as well, any subconfiguration of a critical configuration is itself critical.

We conclude this section with a final, useful property of critical configurations, namely that the only property of the cameras we need to consider when exploring critical configurations is the position of their camera centers (i.e. change of coordinates in the images does not affect criticality).

Proposition I.2.13 ([HK07, Proposition 3.7])**.** *Let $(P_1, \dots, P_n)$ be n cameras with centers $p_1, \dots, p_n$, and let $(P_1, \dots, P_n, X)$ be a critical configuration.*
If $(P'_1, \dots, P'_n)$ is a set of cameras sharing the same camera centers, the configuration $(P'_1, \dots, P'_n, X)$ is critical as well.

Proof. Since P_i and P'_i share the same camera center and the camera center determines the map uniquely up to a choice of coordinates, there exists some $H_i \in \mathrm{PGL}(3)$ such that $P'_i = H_i P_i$. Let $(Q_1, \dots, Q_n, Y)$ be a conjugate to $(P_1, \dots, P_n, X)$. Then $(H_1 Q_1, \dots, H_n Q_n, Y)$ is a conjugate to $(H_1 P_1, \dots, H_n P_n, X) = (P'_1, \dots, P'_n, X)$, so this configuration is critical as well. ■

I.3 The one-view case

Reconstruction of a 3D-scene from the image of one projective camera is generally considered impossible, so most papers start with the two-view case. Still, for

the sake of completeness, we give a summary of the critical configurations for one camera.

Let P be a camera, and let p be its camera center, we then have the joint camera map:

$$\widetilde{\phi_P}\colon \widetilde{\mathbb{P}^3} \to \mathbb{P}^2$$

For any point $x \in \mathbb{P}^2$, the fiber over x is a line through p, so no point can be uniquely recovered. From this, one might assume that every configuration with one camera is critical. However, this is only the case if our configuration consists of sufficiently many points.

Theorem I.3.1. *A configuration of one point and one camera is never critical. A configuration of one camera and $n > 1$ points is critical if and only if the camera center along with the n points span a space of dimension less than n.*

Proof. For the first part, note that up to a projective transformation, there exists only one configuration of one point and one camera. In other words, any configuration of one point and one camera can be taken to any other such configuration by simply changing coordinates. By Definition I.2.7 this makes them equivalent, which means that only one reconstruction exists.

The same turns out to be the case if the configuration is such that the camera center along with the $n > 1$ points span a space of dimension n. In $\mathbb{P}^3$, one can never span a space of dimension greater than 3, so this implies that $n \leq 3$. Furthermore, if the n points along with the camera center span a space of dimension n, then the points and camera center lie in general position. However, for $n \leq 4$ points (fewer than 3 points + one camera center) there exists only one configuration (up to action with PGL(4)) where all points are in general position. This means (by Definition I.2.7) that all reconstructions are equivalent.

Now it only remains to show that a configuration is critical if the camera center along with the $n > 1$ points span a space of dimension less than n. Indeed, if the points along with the camera center span a space of dimension less than n, the image points span a space of dimension $m < n - 1$. Then there are at least two nonequivalent reconstructions; one reconstruction where the points span a space of dimension m not containing the camera center and one where they span a space of dimension $m + 1$ which contains the camera center. ■

I.4 The two-view case

I.4.1 The multi-view ideal

We start the study of the case of two cameras P_1 and P_2 by understanding their multi-view variety. We assume, here and throughout the rest of the paper, that all cameras have distinct centers. The two cameras define the joint camera map:

$$\widetilde{\phi_P}\colon \widetilde{\mathbb{P}^3} \to \mathbb{P}^2 \times \mathbb{P}^2.$$

Proposition I.4.1. *For two cameras (P_1, P_2), the joint camera map $\widetilde{\phi_P}$ takes the line spanned by the two camera centers to a point, and is an embedding everywhere else.*

Proof. For $x = (x_1, x_2) \in \mathbb{P}^2 \times \mathbb{P}^2$, the preimage of x is given by

$$\widetilde{\phi_P}^{-1}(x) = l_1 \cap l_2,$$

where l_i is the line $P_i^{-1}(x_1)$. The line l_i passes through the camera center p_i, so the intersection of the two lines is a single point unless they are both equal to the line spanned by the camera centers. ■

This means that the multi-view variety $\mathrm{Im}(\widetilde{\phi_P})$ is an irreducible singular 3-fold in $\mathbb{P}^2 \times \mathbb{P}^2$. It is described by a single bilinear polynomial F_P, which we call *the fundamental form* of P_1, P_2.

The fundamental form is well-studied in the literature and is often represented by a 3×3 matrix of rank 2 called the *fundamental matrix*. See [HZ04, section 9.2] for a geometric construction of the fundamental matrix. We use the construction in [APT21] where the fundamental form is given as the determinant of a 6×6 matrix:

I.4.2 The multi-view variety

Proposition I.4.2 (The fundamental form)**.** *[HZ04, Sections 9.2 and 17.1] For two cameras P_1, P_2, the multi-view variety $\mathrm{Im}(\widetilde{\phi_P}) \subset \mathbb{P}^2 \times \mathbb{P}^2$ is the vanishing locus of a single, bilinear, rank 2 form F_P, called the fundamental form (or fundamental matrix).*

$$F_P(\boldsymbol{x}, \boldsymbol{y}) = \det \begin{bmatrix} P_i & \boldsymbol{x} & \boldsymbol{0} \\ P_j & \boldsymbol{0} & \boldsymbol{y} \end{bmatrix},$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the variables in the first and second image respectively.

Proof. By Proposition I.4.1, the multi-view variety for two cameras is an irreducible, 3-fold. It follows that the multi-view ideal is generated by a single polynomial. Let $(\mathbf{x}, \mathbf{y})$ be a generic point in the multi-view variety, then there exists a point $\mathbf{X} \in \mathbb{P}^3$ such that $P_1(\mathbf{X}) = \lambda_1 \mathbf{x}$ and $P_2(\mathbf{X}) = \lambda_2 \mathbf{y}$, then

$$\begin{bmatrix} P_i & \mathbf{x} & \mathbf{0} \\ P_j & \mathbf{0} & \mathbf{y} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ -\lambda_1 \\ -\lambda_2 \end{bmatrix} = 0.$$

Since the matrix has a non-zero kernel, the determinant F_P has to vanish on $(\mathbf{x}, \mathbf{y})$. Now we need only show that the determinant is irreducible to prove that $F_P(\mathbf{x}, \mathbf{y})$ generates the multi-view ideal. Irreducibility follows from the fact that the polynomial is of rank 2 (a reducible polynomial is always of rank 1) which in turn follows from the fact that F_P satisfies

$$F_P(e^2_{P_1}, -) = F_P(-, e^1_{P_2}) = 0, \tag{I.1}$$

where

$$e^j_{P_i} = P_i(p_j). \tag{I.2}$$

■

Remark I.4.3. Recall that the entries in the camera matrix are real, this means that the fundamental form always has real coefficients.

Definition I.4.4. The *epipoles* $e^j_{P_i}$ are the image points we get by mapping the j-th camera center to the i-th image

$$e^j_{P_i} = P_i(p_j).$$

The fundamental form satisfies

$$F_P(e^2_{P_1}, -) = F_P(-, e^1_{P_2}) = 0,$$

in other words, it vanishes in either epipole. This means that the fundamental form is of rank 2 (also follows from $\mathrm{Im}(\widetilde{\phi_P})$ being singular), so for each pair of cameras, we get a bilinear form of rank 2. The following result states that the converse is also true, i.e. that any bilinear form of rank 2 is the fundamental form for some pair of cameras.

Theorem I.4.5. *There is a* $1:1$ *correspondence between bilinear forms of rank two, and pairs of cameras* P_1, P_2 *(up to action by* $\mathrm{PGL}(4)$*)*

Proof. By Proposition I.4.2, the fundamental form of two cameras is a real bilinear form of rank 2. The converse follows from Theorem 9.13. in [HZ04]. ■

With these results, we can move on to classifying all the critical configurations for two views. We start with a special type of critical configuration:

I.4.3 Trivial critical configurations

Definition I.4.6. A configuration (P_1, P_2, X) is said to be a *non-trivial critical configuration* if it has a conjugate configuration (Q_1, Q_2, Y) satisfying

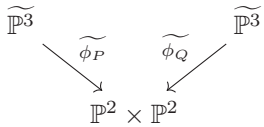
$$F_P \neq F_Q.$$

Critical configurations not satisfying this property exist, they are called trivial. If (P_1, P_2, X) is a trivial critical configuration, then all its conjugates (Q_1, Q_2, Y) have the same fundamental form as the cameras P_1, P_2. By Theorem I.4.5 this means that, after a change of coordinates, $Q_1 = P_1$ and $Q_2 = P_2$. Since the cameras are the same, Proposition I.4.1 tells us that the sets X and Y are equal, with the exception of any point lying on the line spanned by the two camera centers. It is a well-known fact that no number of cameras can differentiate between points lying on a line containing all the camera centers, hence the name "trivial".

The focus of this paper is the non-trivial critical configurations. A classification of the trivial critical configurations for any number of views can be found in [HK07] Section 4, or in [HZ04] Chapter 22.

I.4.4 Critical configurations for two views

Let us consider a non-trivial critical configuration (P_1, P_2, X). Since it is critical, there exists a conjugate configuration (Q_1, Q_2, Y) giving the same images in $\mathbb{P}^2 \times \mathbb{P}^2$. The two sets of cameras define two joint-camera maps $\widetilde{\phi_P}$ and $\widetilde{\phi_Q}$.



Now, since the configuration is critical, we have that $\widetilde{\phi_P}(X) = \widetilde{\phi_Q}(Y)$. As such, the two sets X and Y both map (with their respective maps) into the intersection of the two multi-view varieties $\mathrm{Im}(\widetilde{\phi_P}) \cap \mathrm{Im}(\widetilde{\phi_Q})$. Taking the preimage of this intersection under $\widetilde{\phi_P}$, we get a variety S in $\widetilde{\mathbb{P}^3}$ which needs to contain all the points in X. Moreover, if (P_1, P_2, X) is maximal, then $X = S$. As such, classifying all non-trivial maximal critical configurations can be done by classifying all possible intersections between two multi-view varieties, and then examining what these intersections pull back to in $\widetilde{\mathbb{P}^3}$. This is made even simpler by the fact that the pullback of $\mathrm{Im}(\widetilde{\phi_P}) \cap \mathrm{Im}(\widetilde{\phi_Q})$ is just the variety we get by pulling back the fundamental form F_Q (the fundamental form F_P pulls back to the zero polynomial).

The pullback of a bilinear form describes the strict (or proper) transform of a quadric surface[1]. It follows that X and Y lie on the strict transform of two quadric surfaces (quadrics) which we denote by S_P and S_Q respectively. Let $\widetilde{S_P}$ and $\widetilde{S_Q}$ denote their strict transforms. These quadrics are given by the following equations:

$$\begin{aligned} S_P(x) &= F_Q(P_1(x), P_2(x)), \\ S_Q(x) &= F_P(Q_1(x), Q_2(x)). \end{aligned} \tag{I.3}$$

The two quadrics have the following properties:

Lemma I.4.7 (Lemma 5.10 in [HK07])**.**

I.4.7.1. *The quadric S_P contains the camera centers p_1, p_2.*

I.4.7.2. *The quadric S_P is ruled (contains real lines).*

Proof. The first item follows from the nature of the pullback, whereas the second is because we are working with forms of rank 2. Detailed proof is given in [HK07]. ■

There are only four quadrics containing lines (up to choice of coordinates), these are illustrated in Figure I.1.

The discussion so far can be summarized as follows:

[1] This is a surface over the complex numbers. The real points on this surface will generally also form a surface, but we will later see that there is one exception, namely when the surface is the union of two complex conjugate planes, and only their line of intersection is real

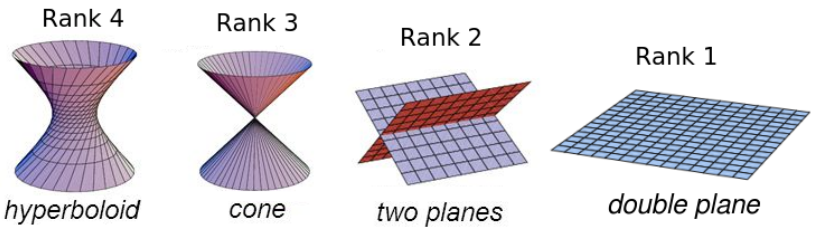


Figure I.1: All types of real ruled quadrics.

Theorem I.4.8 (Lemma 5.10 in [HK07])**.** *Let (P_1, P_2, X) be a non-trivial critical configuration. Then there exists a ruled quadric S_P passing through the camera centers p_1, p_2, whose strict transform contains the points X.*

Theorem I.4.8 tells us that all non-trivial critical configurations have their points and camera centers lying on a ruled quadric. The converse, however, is not always true. For instance, we will soon see that cameras and camera centers all lying on a cone is not critical if the cone contains the line spanned by the two camera centers (see Figure I.2). Let us instead give a partial converse:

Lemma I.4.9. *Let (P_1, P_2, X) be a configuration of cameras and points such that X is contained in the strict transform $\widetilde{S_P}$ of a ruled quadric S_P that passes through both the camera centers. Then for each real bilinear form F_Q of rank 2 such that:*

$$S_P(x) = F_Q(P_1(x), P_2(x)),$$

there exists a conjugate configuration to (P_1, P_2, X).

Proof. Assume such a bilinear form F_Q exists. By Theorem I.4.5 there exists a pair of cameras (Q_1, Q_2) such that F_Q is their fundamental form. Since X lies on $\widetilde{S_P}$, we have

$$\widetilde{\phi_P}(X) \subset \mathrm{Im}(\widetilde{\phi_Q}).$$

Then for every point $x \in X$ we can find a point y such that

$$\widetilde{\phi_P}(x) = \widetilde{\phi_Q}(y).$$

Let Y be the set of these points y. Then (Q_1, Q_2, Y) is a conjugate to (P_1, P_2, X). This can be repeated for each bilinear form of rank 2, giving unique conjugate configurations. ∎

The problem of determining which configurations are critical is now reduced to finding out which quadrics are the pullbacks of real bilinear forms of rank 2.

Let $\mathbf{F}$ denote the space of bilinear forms on $\mathbb{P}^2 \times \mathbb{P}^2$. Since all such forms can be represented by a 3×3 matrix (up to scaling) we have that $\mathbf{F}$ is isomorphic to $\mathbb{P}^8$. The fundamental form F_P of the pair (P_1, P_2) is an element in $\mathbf{F}$.

Lemma I.4.10. *There is a* $1:1$ *correspondence between the set of real quadrics in* $\mathbb{P}^3$ *passing through* p_1, p_2*, and the set of real lines in* $\mathbf{F}$ *passing through* F_P*.*

Proof. Let $L \subset \mathbf{F}$ be a line through F_P, and let $F_0 \neq F_P$ be some point on L. Every point $F \in L$ can be written as $\alpha F_0 + \beta F_P$ for some $[\alpha : \beta] \in \mathbb{P}^1$. But then we have for $F \neq F_P$:

$$\begin{aligned} F(P_1(x), P_2(x)) &= \alpha F_0(P_1(x), P_2(x)) + \beta F_P(P_1(x), P_2(x)), \\ &= \alpha F_0(P_1(x), P_2(x)) + \beta \cdot 0, \\ &= F_0(P_1(x), P_2(x)). \end{aligned}$$

Hence, the equation $F(P_1(x), P_2(x))$ describes the same quadric for all points $F \neq F_P$ on L.

Next, a quadric S passing through p_1 and p_2 is fixed by a set of 7 points $\{x_1, \ldots, x_7\}$ on S in generic position. Demanding that a bilinear form pulls back to a quadric passing through a specific point is one linear constraint in $\mathbf{F}$. So with seven generic points, there is exactly one line L through F_P such that the forms on this line pull back to S. ■

Using this $1:1$ correspondence and Lemma I.4.9, the problem has been reduced to determining which quadrics correspond to lines in $\mathbf{F}$ containing at least one viable form of rank 2. Let $\mathbf{F}_2$ denote the Zariski closure of the rank 2 locus. Since $\mathbf{F}_2$ is a hypersurface of degree 3, a generic line L will contain two forms of rank 2 in addition to F_P. There are also other possibilities, listed in the tables below (the underlying computations for these tables can be found in the appendix.

We start with the cases where the line L corresponding to S_P has a finite number of intersections with the rank 2 locus $\mathbf{F}_2$.

Intersection points	S_P
All three intersection points are distinct real points	A smooth quadric, cameras not on a line
Two intersection points F_P and F_Q, $L \cap \mathbf{F}_2$ has multiplicity 2 at F_Q	A cone, two cameras not on a line, neither camera on a vertex
Two intersection points F_P and F_Q, $L \cap \mathbf{F}_2$ has multiplicity 2 at F_P	A smooth quadric, cameras lie on a line

These are the cases where we have at least one real rank 2 form $F \in L$ different from F_P. There are, however, some cases where there are no viable forms:

Intersection points	S_P
The two other intersection points are complex conjugates	A smooth non-ruled quadric
The two other intersection points are of rank 1	Union of two planes, cameras in different planes
The two intersection points are equal to F_P	A cone, both cameras on a line, neither camera at the vertex

In the case where L is contained in $\mathbf{F}_2$, all the forms on L are of rank 2 (with the possible exception of at most 2 that can be of rank 1). As such, rather than looking at where the intersections are, we look at the epipoles of the forms in L:

Epipoles	S_P
All forms have different epipoles	Two planes, cameras lying in same plane
All forms share the same right epipole, the left epipoles trace a line[2]	Two planes, one camera on the intersection of the planes
All forms share the same right epipole, the left epipoles trace a conic[2]	Cone, one camera at the vertex
All forms share the same right and left epipole	Two (possibly complex) planes, both cameras lying on the intersection of the planes
All forms share the same right and left epipole AND the two rank one forms on L coincide	Double plane (as a set it is equal to a plane, but every point has multiplicity 2)

With this, we have a classification of all maximal critical configurations for two views:

Theorem I.4.11. *A configuration (P_1, P_2, X) is non-trivially critical if and only if there exists a real quadric S_P containing the camera centers p_1, p_2 and whose strict transform contains X, and S_P is one of the quadrics in Table I.1. (illustrated in Figure I.2)*

Recall that by Definition I.2.7, we required two conjugate configurations to not be projectively equivalent. Yet by Theorem I.4.11, most critical configurations have conjugates that are of the same type. Now, while there is indeed some $A \in \mathrm{PGL}(4)$ taking any smooth quadric S_P to any other smooth quadric S_Q, we will soon see that the map taking a point in S_P to its conjugate on S_Q certainly does not lie in $\mathrm{PGL}(4)$. In the final section, we give a description of the map taking a point to its conjugate, to make it clear that it is not a projective transformation.

[2]The statement also holds if we swap "left" and "right".

Quadric S_P	**Conjugate quadric**	**Conjugates**
Smooth quadric, cameras not on a line	Same	2
Smooth quadric, cameras on a line	Cone, cameras not on a line	1
Cone, cameras not on a line	Smooth quadric, cameras on a line	1
Cone, one camera at vertex, other one not	Same	∞
Two planes, cameras in the same plane	Same	∞
Two planes, one camera on the singular line	Same	∞
Two planes, cameras on the singular line	Same	∞
A double plane, cameras in the plane	Same	∞

Table I.1: A list of all possible non-trivial critical configurations and their conjugates, as well as the number of conjugates for each configuration.

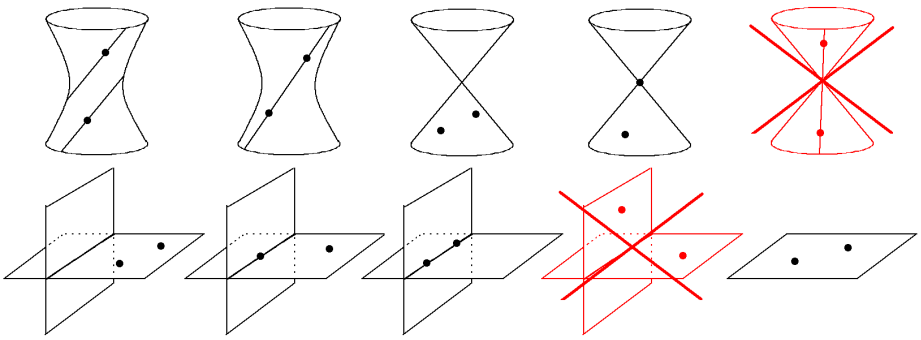


Figure I.2: Illustration of the blow-downs of the non-trivial critical configurations for two views. The two marked with crosses are not critical.

I.5 Maps between quadrics

I.5.1 Epipolar lines

Before we can describe the map taking a point to its conjugate, we need to point out a certain pair of lines on S_P. Given two pairs of cameras (P_1, P_2) and (Q_1, Q_2), let $\widetilde{S_P}$ be the pullback of F_Q using $\widetilde{\phi_P}$, and define

$$\widetilde{g^2_{P_1}} = P_1^{-1}(e^2_{Q_1}) = P_1^{-1}(Q_1(q_2)), \tag{I.4}$$

$$\widetilde{g^1_{P_2}} = P_2^{-1}(e^1_{Q_2}) = P_2^{-1}(Q_2(q_1)). \tag{I.5}$$

The blowdown of $\widetilde{g^2_{P_1}}$ and $\widetilde{g^1_{P_2}}$ are two lines on S_P, we denote them by $g^j_{P_i}$. Since they are the pullback of the epipoles from the other set of cameras, so we call them *epipolar lines*.

Epipolar lines are key in understanding the relation between points on $\widetilde{S_P}$ and points on its conjugate $\widetilde{S_Q}$. They also play an important role in the study of critical configurations for more than 2 cameras, so let us give a brief analysis of these lines.

Lemma I.5.1 (Lemma 5.10, Definition 5.11 in [HK07])**.**

1. *The line $g^j_{P_i}$ lies on S_P and passes through p_i.*
2. *Any point lying on both $g^j_{P_i}$ and $g^i_{P_j}$ is a singular point on S_P.*
3. *Any point in the singular locus of S_P that lies on one of the lines also lies on the other.*
4. *If S_P is the union of two planes, $g^j_{P_i}$ and $g^i_{P_j}$ lie in the same plane.*

The first two properties are taken from Lemma 5.10 in [HK07], the last two are neither stated nor proven in the paper. Nevertheless, the authors seem to have been aware of all four properties.

Proof.

1-2. See proof of Lemma 5.10 in [HK07].

3. For ease of reading, we use matrix notation. As such, S_P, F_Q, and P_i are represented by matrices of dimensions 4×4, 3×3 and 3×4 respectively. In particular, S_P is represented by the *symmetric* matrix $P_1^T F_Q P_2 + P_2^T F_Q^T P_1$.

   If $x_0 \in g^2_{P_1}$ lies in the singular locus of S_P, then we have

$$\begin{aligned} S_P x_0 &= (P_1^T F_Q P_2 + P_2^T F_Q^T P_1) x_0, \\ &= P_1^T F_Q P_2 x_0 + P_2^T F_Q^T e^2_{Q_1}), \\ &= P_1^T F_Q P_2 x_0. \end{aligned}$$

   However, since x_0 lies in the singular locus of S_P, this expression is equal to zero. Since both the camera matrices are of full rank, the only way we can get zero is if $x_0 = p_2$ or if $P_2 x_0 = e^1_{Q_2}$. In either case, it follows that x_0 lies on $g^1_{P_2}$ as well.

4. Assume there exist cameras P_1, P_2, Q_1, Q_2 such that S_P is the union of two planes and the epipolar lines $g^j_{P_i}$ lie in different planes. When the quadric S_P consists of two planes, one of the planes, which we denote by Π, will (by Theorem I.4.11) contain both camera centers. As such, the only way that the epipolar lines can lie in different planes is if one of the camera centers, say p_2, lies on the intersection of the two planes and the other

does not (if both lie on the intersection, then by 3., the epipolar lines must both be equal to the intersection of the two planes).

Recall that the quadric S_P and its conjugate S_Q (also two planes) are both pullbacks of the surface $\mathrm{Im}(\widetilde{\phi_P}) \cap \mathrm{Im}(\widetilde{\phi_Q}) \subset \mathbb{P}^2 \times \mathbb{P}^2$. The map $\widetilde{\phi_P}$ takes the plane Π to the product of two lines in $\mathbb{P}^2 \times \mathbb{P}^2$. The line in the first image passes through the epipole $e^2_{Q_1}$, whereas the line in the second image does not pass through the epipole $e^1_{Q_2}$ (this is because Π contains one of the epipolar lines but not the other). The problem is now that neither of the planes on S_Q can map to the product of these two lines, since any such plane would have to be both

a) a plane passing through q_2 but not through q_1 (because in the second image, the line does not pass through $e^1_{Q_2}$) and

b) a plane passing through both q_1 and q_2 (because the plane maps to a line in both images),

which gives us a contradiction. It follows that there are no P_1, P_2, Q_1, Q_2 such that S_P is the union of two planes and the epipolar lines $g^j_{P_i}$ lie in different planes, so whenever S_P is the union of two planes, the epipolar lines lie in the same plane. ■

Definition I.5.2. Let S_P be a quadric surface and let p_1, p_2 be two distinct points on S_P. A pair of lines $g^2_{P_1}, g^1_{P_2}$ is called *permissible* if $g^2_{P_1}, g^1_{P_2}$ satisfy the four conditions in Lemma I.5.1.

Proposition I.5.3. *Let P_1, P_2 be two cameras, and let S_P be a quadric passing through their camera centers. The configuration $(P_1, P_2, \widetilde{S_P})$ is critical if and only if S_P contains a permissible pair of lines. Furthermore, if p_1, p_2 do not both lie in the singular locus of S_P, there is a 1:1 correspondence between permissible pairs of lines and configurations conjugate to $(P_1, P_2, \widetilde{S_P})$.*

Proof. The first part can be proven by comparing the quadrics in Table I.1 to the set of quadrics containing the required lines, and noting that they are the same. We leave this to the reader.

The second part is immediate for the three cases where there is a finite number of conjugates since the number of conjugates is equal to the number of pairs of epipolar lines and each conjugate comes with its unique choice of lines. For the remaining quadrics S_P (cone and two planes), there exists a pencil of fundamental forms, where each $F_Q \neq F_P$ satisfies

$$S_P(x) = F_Q(P_1(x), P_2(x)).$$

By the tables on pages 17 and 18, we get that each form in the pencil yields a different pair of lines as long as the camera centers do not both lie in the singular locus of S_P. For $F_Q \neq F_P$, the lines are permissible, while for F_P, the lines coincide (not permissible). Note also that on each of these quadrics, the set of permissible lines forms a set whose Zariski closure is also a pencil

(the pair where the two lines coincide lie in the closure, but the pair itself is not permissible). We now have a map from the pencil of fundamental forms to the pencil of permissible lines. Since the map is not constant, it needs to be surjective, and since no two fundamental forms in the pencil share the same left and right kernel, it is also injective. Hence there is a 1:1 correspondence between permissible pairs of lines and configurations conjugate to (P_1, P_2, S_P). ■

I.5.2 Maps between quadrics

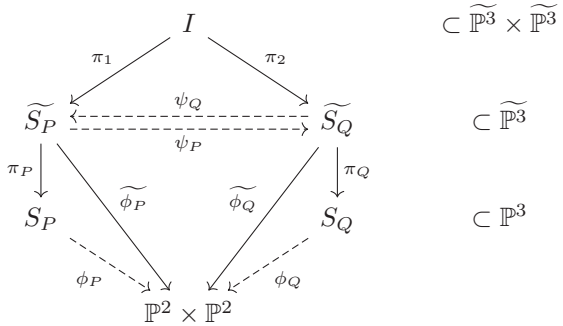
Let us now have a closer look at the map ψ taking a point to its conjugate. Given two pairs of cameras P_1, P_2 and Q_1, Q_2 with camera centers p_1, p_2 and q_1, q_2 respectively, let the quadrics S_P and S_Q and the epipolar lines g be defined as before (Equations (I.3), (I.4), and (I.5)). Let

$$\pi_P \colon \widetilde{\mathbb{P}^3_P} \to \mathbb{P}^3$$

be the blow-up of $\mathbb{P}^3$ in the two camera centers p_1, p_2 and let $\widetilde{S_P}$ denote the strict transform of S_P, and similarly for Q. Define the incidence variety:

$$I = \left\{ (x, y) \in \widetilde{S_P} \times \widetilde{S_Q} \mid \widetilde{\phi_P}(x) = \widetilde{\phi_Q}(y) \right\},$$

where ϕ is the joint-camera map. If we can understand I, we will know the exact relation between points on one quadric and the other. We have the following the commutative diagram:



We study I by studying the fibers of the projection map π_1. For any point $x \in \widetilde{S_P}$, we have

$$\pi_1^{-1}(x) = l_1 \cap l_2,$$

where l_i is the line $Q_i^{-1}(P_i(x))$. We will rarely refer to this formula explicitly, but it is the foundation for the analysis of the fibers.

Lemma I.5.4.

I.5.4.1. *If $\widetilde{S_P}$ is smooth, the map π_1 is an isomorphism.*

I.5.4.2. *The map ψ_P taking a point $x \in \widetilde{S_P}$ to its conjugate is a birational map, defined everywhere except the intersection $\widetilde{g^2_{P_1}} \cap \widetilde{g^1_{P_2}}$.*

I.5.4.3. *The quadric S_P is singular if and only if the quadric S_Q contains the line spanned by the camera centers q_1, q_2.*

I.5.4.4. *For any point $x \in \widetilde{g^2_{P_1}}$ or $\widetilde{g^1_{P_2}}$ (but not both), the conjugate $\psi_P(x)$ is a point on the exceptional divisor we get when blowing up q_2 or q_1 respectively. In particular, on the blow-down $S_P \subset \mathbb{P}^3$, the conjugate to any point $x \in g^2_{P_1}$ or $x \in g^1_{P_2}$ (apart from the camera centers themselves) is the camera center q_2 or q_1 respectively.*

Proof.

1. As long as x does not lie in $\widetilde{g^2_{P_1}} \cap \widetilde{g^1_{P_2}}$, the two lines l_i do not coincide, so the fiber $\pi_1^{-1}(x)$ consists of a single point. By Lemma I.5.1, any point lying on this intersection is a singular point on S_P, so if S_P is smooth, then $\widetilde{g^2_{P_1}}$ and $\widetilde{g^1_{P_2}}$ do not intersect. Then every fiber is a singleton, meaning that π_1 is injective. Furthermore, for each point $x \in \widetilde{S_P}$ there is at least one point $y \in \widetilde{S_Q}$ such that $\widetilde{\phi_P}(x) = \widetilde{\phi_Q}(y)$, so π_1 is surjective as well.

2. As mentioned above, $\pi_1^{-1}(x)$ is a singleton if x does not lie in $\widetilde{g^2_{P_1}} \cap \widetilde{g^1_{P_2}}$. If it does, on the other hand, the fiber is a line $L \in I$ such that $\pi_2(L)$ is the strict transform of the line spanned by the camera centers q_1, q_2, so in this case the conjugate is not unique. Since the same is true for the map taking a point $y \in \widetilde{S_Q}$ to its conjugate, the map is birational.

3. If S_P is singular, the epipolar line $g^2_{P_1}$ passes through some singular point $x \in \widetilde{S_P}$ (as do all other lines, see Figure I.1). By Lemma I.5.1, the other epipolar line $g^1_{P_2}$ passes through the same point. Since the two epipolar lines intersect, the quadric $\widetilde{S_Q}$ must contain the line spanned by the camera centers.

   Conversely, if S_Q contains the line spanned by the camera centers, the conjugate to any point on the strict transform of this line is the intersection $\widetilde{g^2_{P_1}} \cap \widetilde{g^1_{P_2}}$ on $\widetilde{S_P}$. By Lemma I.5.1, the blow-down of these points are always singular points on S_P.

4. Let x be a point lying on the epipolar line $\widetilde{g^2_{P_1}}$, but not on $\widetilde{g^1_{P_2}}$. We have $P_1(x) = e^2_{Q_1}$ and $P_2(x) \neq e^1_{Q_2}$. The conjugate to x is then

$$\psi(x) = l_1 \cap l_2 = Q_1^{-1}(P_1(x)) \cap Q_2^{-1}(P_2)(x).$$

   The line l_1 is here the line spanned by the camera centers q_1, q_2 , while l_2 is a different line, passing through q_2. Their intersection must then be the camera center q_2. The proof for the other epipolar line is the same. ■

Remark I.5.5. When the two pairs cameras P_1, P_2 and Q_1, Q_2 are known, one can explicitly compute the map ψ_P as a rational function. For some representatives of a general point $X \in S_P$ and its conjugate $\psi_P(X) \in S_Q$, we have

$$\underbrace{\begin{pmatrix} Q_1 & P_1X & 0 \\ Q_2 & 0 & P_2X \end{pmatrix}}_{A} \cdot \begin{pmatrix} \psi_Q(X) \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} P_1 & Q_1\psi_Q(X) & 0 \\ P_2 & 0 & Q_2\psi_Q(X) \end{pmatrix} \cdot \begin{pmatrix} X \\ 1 \\ 1 \end{pmatrix} = 0.$$

Using the adjugate of A, we can get an expression for ψ_Q as a rational function in the homogeneous coordinates of X.

Let us now give two results on how ϕ_P acts on the curves on S_P:

Definition I.5.6. Let S_P be a smooth quadric or cone and let C_P be a curve on S_P. We say that C_P is of type (a, b, c_1, c_2), where c_i is the multiplicity of C_P in the camera center p_i and where a and b is:

- If S_P is smooth, a is the number of times C_P intersects a generic line in the same family as the epipolar lines $g^j_{P_i}$, and b is the number of times it intersects the lines in the other family, (meaning (a, b) is the bidegree of C_P).
- If S_P is a cone, a is the number of times C_P intersects each line outside of the vertex, and b the number of times it intersects each line.

Proposition I.5.7 ([HK07, Lemma 8.32])**.** *Let S_P be a smooth quadric or a cone, and let $C_P \subset S_P$ be a curve of type (a, b, c_1, c_2), such that C_P does not contain either of the epipolar lines. Then the conjugate curve $C_Q \subset S_Q$ is of type $(a, a + b - c_1 - c_2, a - c_2, a - c_1)$.*

An illustration of how ψ_P acts on the lines on a quadric/cone can be found in Figures I.3 and I.4.

Definition I.5.8. Let S_P be two planes with two camera centers not both lying on the intersection and let C_P be a curve on S_P. We say that C_P is of type (a, b, c_0, c_1, c_2), where a is the degree of the curve in the plane with the epipolar lines, b the degree of the curve in the other plane, c_0 is the multiplicity of C_P in the intersection of the epipolar lines $g^j_{P_i}$ and c_1, c_2 is the multiplicity in the camera centers p_1, p_2 respectively.

Proposition I.5.9. *Let S_P be two planes with two camera centers not both lying on the intersection, and let $C_P \subset S_P$ be a curve of type (a, b, c_0, c_1, c_2) such that C_P does not contain either of the epipolar lines or the line spanned by the camera centers. Then the conjugate curve $C_Q \subset S_Q$ is of type $(2a - c_0 - c_1 - c_2, b, a - c_1 - c_2, a - c_0 - c_2, a - c_0 - c_1)$.*

The goal of the remainder of this section will be to give a rigorous proof of Lemma 8.32 in [HK07] and to generalize this result to also hold for singular quadrics (that is, to prove Propositions I.5.7 and I.5.9). While the method described in Remark I.5.5 gives us an explicit expression for ψ_P, using this to

prove Propositions I.5.7 and I.5.9 is somewhat difficult. We instead switch our approach to a more classical one, using intersection theory. The map ψ_P taking a point on $\widetilde{S_P}$ to its conjugate can be described by describing the pullback of the hyperplane sections of $\widetilde{S_Q}$. For such hyperplane sections H, the pullbacks $\psi_P^{-1}(H)$ will be curves on $\widetilde{S_P}$. Finding the class of $\psi_P^{-1}(H)$ in the Chow ring will give us the information we need about ψ_P. We refer the reader to [EH16] (chapter 2 in particular) for the basics on intersection theory used in the rest of the section.

By Lemma I.5.4, the map ψ is defined everywhere except the intersection of the two epipolar lines, so if we assume that they do not intersect, we get the diagram below:

$$\begin{array}{ccccc} \widetilde{\mathbb{P}^3_P} & \supset & \widetilde{S_P} & \underset{\psi_P}{\overset{\psi_Q}{\leftrightarrows}} & \widetilde{S_Q} \subset \widetilde{\mathbb{P}^3_Q} \\ & & \pi_P \downarrow & & \downarrow \pi_Q \\ \mathbb{P}^3 & \supset & S_P & \dashleftarrow\!\dashrightarrow & S_Q \subset \mathbb{P}^3 \end{array}$$

If either pair of epipolar lines DO intersect, however (this happens whenever the quadrics are not smooth), then the map ψ is not a morphism, but rather a rational map as it is not defined on the intersection. This can be mended by first blowing up the intersection of the epipolar lines, let

$$\pi_{g_P} \colon \widetilde{\mathbb{P}^3} \to \mathbb{P}^3$$

denote this blow-up, and let π_P, like before, be the blow-up in the two camera centers p_i. Let $\overline{S_P}$ be the strict transform of S_P after the first blow-up, and let $\widetilde{S_P}$ be the strict transform of $\overline{S_P}$ after the second. And similarly for Q. We then get the following diagram:

$$\begin{array}{ccccc} \widetilde{\mathbb{P}^3_P} & \supset & \widetilde{S_P} & \underset{\psi_P}{\overset{\psi_Q}{\leftrightarrows}} & \widetilde{S_Q} \subset \widetilde{\mathbb{P}^3_Q} \\ & & \pi_P \downarrow & & \downarrow \pi_Q \\ \widetilde{\mathbb{P}^3} & \supset & \overline{S_P} & \dashleftarrow\!\dashrightarrow & \overline{S_Q} \subset \widetilde{\mathbb{P}^3} \\ & & \pi_{g_P} \downarrow & & \downarrow \pi_{g_Q} \\ \mathbb{P}^3 & \supset & S_P & \dashleftarrow\!\dashrightarrow & S_Q \subset \mathbb{P}^3 \end{array}$$

We can now cover the different cases one by one, but first, let us give a result which will be helpful to determine the intersection multiplicities of curves on $\widetilde{S_P}$:

Proposition I.5.10 ([EH16, Proposition 2.19])**.** *Let S be a smooth projective surface and $\pi \colon \widetilde{S} \to S$ the blow-up of S at a point p; let $E \in A^1(S)$ be the class of the exceptional divisor.*

I.5.10.1. *$A(\widetilde{S}) = A(S) \oplus \mathbb{Z}E$ as abelian groups.*

I.5.10.2. $\pi^*\alpha \cdot \pi^*\beta = \pi^*(\alpha\beta)$ *for any* $\alpha, \beta \in A^1(S)$.

I.5.10.3. $E \cdot \pi^*\alpha = 0$ *for any* $\alpha \in A^1(S)$.

I.5.10.4. $E^2 = -1$.

Smooth quadric

We start by describing the map ψ_P in the case where the quadric S_P is smooth, i.e. a hyperboloid of one sheet. Let L_2 be the class of the total transform of a line in the same family as $g^j_{P_i}$, and let L_1 be the class of the total transform of a line in the other family. Let E_i be the class of the exceptional divisor we get when blowing up p_i.

On S_P, two lines from the same family do not intersect, whereas two lines in different families intersect once. By Proposition I.5.10 (2) this is also the case on $\widetilde{S_P}$, by (3) L_i will not intersect E_i, and by (4), $E_i^2 = -1$. We get the following intersection multiplicity table:

$\cdot$	L_1	L_2	E_1	E_2
L_1	0	1	0	0
L_2	1	0	0	0
E_1	0	0	-1	0
E_2	0	0	0	-1

Next, let H_Q be the pullback of a hyperplane section from the conjugate configuration $\widetilde{S_Q}$. We want to find the class of H_Q on $\widetilde{S_P}$.

First, let us consider what the map ψ_P does to lines in each family. Let l be a generic curve in L_2, in each image, $P_i(l)$ is a line, furthermore, since l does not intersect $\widetilde{g^j_{P_i}}$, $P_i(l)$ is a line not passing through the epipole $e^j_{P_i} \in \mathbb{P}^2$. It follows that $\psi(l)$ is a curve appearing as the intersection of $\widetilde{S_Q}$ and two planes (the preimages $Q_i^{-1}(P_i(l))$), each passing through exactly one camera center. In other words: $\psi(l)$ is a line, which means it intersects a generic hyperplane once. For a generic curve l' in L_1, $P_i(l')$ is a line passing through the epipole, hence $\psi(l')$ is a curve appearing as the intersection of $\widetilde{S_Q}$ and two planes, both passing through the camera centers, in other words, a conic curve. This means $\psi(l')$ will intersect a generic hyperplane twice.

Furthermore, by Lemma I.5.4 (4), the lines $g^j_{P_i}$, belonging to the class $L_2 - E_i$, map to camera centers on S_Q. These will not intersect a generic hyperplane. Finally, H_Q has self-intersection 2. This gives us the following equations:

$$H_Q L_2 = 1, \quad H_Q L_1 = 2, \quad H_Q(L_2 - E_i) = 0, \quad H_Q^2 = 2.$$

Let $H_Q = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 E_1 + \alpha_4 E_2$. Using the intersection multiplicity table, we solve equations above for α_i and get $H_Q = L_1 + 2L_2 - E_1 - E_2$. In

other words, the hyperplane sections on $\widetilde{S_Q}$ pull back to curves of bidegree $(1,2)$ passing through both camera centers.

Cone

Next, let S_P be a cone. Since the two epipolar lines intersect in the vertex, we first blow up $\mathbb{P}^3$ in the vertex of the cone and then in the camera centers (see diagram above). Let L be the class of the total transform (with the second blow-up) of the strict transform (with the first blow-up) of a line on S_P, let E_0 be the class of the exceptional divisor which is the blow-up of the vertex and let E_i be the class of the exceptional divisor we get when blowing up p_i.

Since the cone is singular, we can not apply Proposition I.5.10 to the first blow-up, we need a separate argument. Blowing up the vertex of the cone, and taking the strict transform of the lines from S_P, we get curves L which no longer intersect one another, so $L^2 = 0$, they do, however, intersect the exceptional divisor E_0, so $LE_0 = 1$. Moreover, a hyperplane section is of class $H = 2L + E_0$, blowing up the cone, the resulting surface still has degree 2, so $H^2 = 2$, it follows that $E_0^2 = -2$. When we next blow up the two camera centers we are blowing up a smooth surface, so Proposition I.5.10 applies, we get the following intersection multiplicity table:

$\cdot$	L	E_0	E_1	E_2
L	0	1	0	0
E_0	1	-2	0	0
E_1	0	0	-1	0
E_2	0	0	0	-1

Again, let H_Q be the pullback of a hyperplane from the conjugate configuration $\widetilde{S_Q}$. By arguments similar to the smooth quadric case we get the following equations:

$$H_Q L = 1, \quad H_Q(L - E_i) = 0, \quad H_Q^2 = 2.$$

Furthermore, if neither camera center on S_P is on the vertex, the conjugate quadric S_Q is a smooth quadric with both cameras lying on the same line, the preimage of this line (under ψ_P) is E_0, hence $H_Q E_0 = 1$. On the other hand, if one of the cameras, say p_1, lies on the vertex, S_Q is a cone and the strict transform $E_0 - E_1$ collapses to the vertex on S_Q, meaning $H_Q(E_0 - E_1) = 0$, so $H_Q E_0 = 1$ in this case also.

It follows that $H_Q = 3L + E_0 - E_1 - E_2$ in both cases.

Two planes, at most one camera center on the intersection

Let S_P be the union of two planes with at most one camera center lying on the intersection. In this case, S_Q is reducible as well and ψ_P takes the plane with

the camera centers p_i to the plane with the camera centers q_i and the plane with no camera centers to the plane with no camera centers. As such we can consider the planes separately. The map ψ_P restricted to the plane with no camera centers is an isomorphism, so we need only consider how ψ acts on the plane with the camera centers.

Since the two epipolar lines $g^j_{P_i}$ intersect in a point, we blow up this point, and then the camera centers to get a morphism ψ_P. Let L be the class of the total transform of a line in the plane containing the camera centers, let E_i be the class of the exceptional divisor we get when blowing up p_i, and let E_0 be the class of the exceptional divisor we get when blowing up the intersection of the epipolar lines. The plane is smooth, so by Proposition I.5.10 we get the following intersection multiplicity table:

$\cdot$	L	E_0	E_1	E_2
L	1	0	0	0
E_0	0	-1	0	0
E_1	0	0	-1	0
E_2	0	0	0	-1

Again, let H_Q be the pullback of a hyperplane from the conjugate configuration $\widetilde{S_Q}$. The strict transform of the line spanned by the two camera centers on S_P is mapped to the intersection of the epipolar lines on S_Q, similarly, the two epipolar lines map to camera centers. There are three corresponding lines on S_Q whose strict transform maps to points on S_P. Lastly, since we are working with a single plane, H_Q has self-intersection 1 this time. This gives us the equations:

$$H_Q(L - E_1 - E_2) = 0, \qquad H_Q(L - E_0 - E_1) = 0, \qquad H_Q(L - E_0 - E_2) = 0,$$
$$H_Q E_i = 1, \qquad H_Q^2 = 1.$$

It follows that $H_Q = 2L - E_0 - E_1 - E_2$.

One or two planes, both cameras in singular locus

Finally, there is the case where S_P (and hence S_Q) is either a double plane or two planes with both cameras lying on their intersection. In this case, the map ψ_P is not defined on the line spanned by the two camera centers p_1, p_2 as any point on this line is conjugate to any point on the line spanned by q_1, q_2. Outside this line, ψ_P acts simply as a linear transformation on each of the two (or one) planes. Note however that in the case of two planes the two linear transformations need not coincide on the intersection, for instance, two intersecting lines, one from each plane, might be taken to two disjoint lines.

Remark I.5.11. In the case where S_P is a double plane, ψ_P acts as a linear transformation on S_P outside the line spanned by the camera centers. As such,

it is in many ways similar to the trivial critical configurations, although it is, by Definition I.4.6, non-trivial.

With all these results in place, we can now prove Propositions I.5.7 and I.5.9. Recall that Definition I.5.6 defines the type of a curve to be (a, b, c_1, c_2) where c_1 and c_2 is the multiplicity in each camera center, and where a and b is the number of times it intersects the epipolar lines and the other lines respectively (or in the case of a cone, the number of times it intersects a generic line outside the vertex, and the number of times it does so in total respectively). In other words, a curve of type (a, b, c_1, c_2) belongs to the class $aL_1 + bL_2 - c_1E_1 - c_2E_2$ on a smooth quadric, whereas on the cone, it belongs to the class $(a+b)L+aE_0-c_1E_1-c_2E_2$.

Proof of Proposition I.5.7. Let $C_P \subset S_P$ be a curve of type (a, b, c_1, c_2) and let $C_Q \subset S_Q$ be of type (a', b', c_1', c_2'). By Lemma I.5.4 (4), the pullback of the camera center $q_2 \in S_Q$ is the epipolar line $g^2_{P_1}$, which belongs to the class $L_2 - E_1$. C_P intersects this line $a - c_1$ times. It follows that $c_2' = a - c_1$, similarly, $c_1' = a - c_2$.

Assume now that S_P is smooth. A hyperplane section on S_Q is of type $(1, 1, 0, 0)$, we have shown above that the pullback of such a section is of type $(1, 2, 1, 1)$ on S_P. Certain hyperplane sections of S_Q are reducible, reducing into two components of type $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$, with each component moving in a pencil. Their pullback should do the same. The only way for a $(1, 2, 1, 1)$-curve to reduce into two such components is as $(0, 1, 0, 0)$ and $(1, 1, 1, 1)$, since bidegreee $(0, 2)$ would mean it is reducible, and the bidegree $(0, 1)$-component passing through a camera center would mean it can not move. This means that the pullback of the $(1, 0, 0, 0)$ component is of type $(1, 1, 1, 1)$ (since this is the component intersecting the epipolar lines, it should correspond to the component through the camera centers on the other side), similarly, the pullback of the $(0, 1, 0, 0)$-component is of type $(0, 1, 0, 0)$. It follows that $a' = a$ and $b' = a + b - c_1 - c_2$, meaning C_Q is of type $(a, a + b - c_1 - c_2, a - c_2, a - c_1)$.

The proof in the case where S_P is a cone follows a similar pattern: in this case, a hyperplane section $(1, 1, 0, 0)$ on S_Q pulls back to a curve in the class $3L + E_0 - E_1 - E_2$, that is, a curve of type $(1, 2, 1, 1)$. Regardless of whether S_Q is a cone or smooth quadric, there are (like in the previous case) hyperplane sections of S_Q which reduce to two irreducible components, each free to move in a pencil. Similarly, their pullback must be reducible into two such components. The only way a curve in $3L + E_0 - E_1 - E_2$ reduces in such a way is as L and $2L + E_0 - E_1 - E_2$ since $3L$ and $2L$ would both be further reducible, and the curve in class L passing through a camera center would prevent it from moving. L is of type $(0, 1, 0, 0)$ and $2L + E_0 - E_1 - E_2$ is of type $(1, 1, 1, 1)$. Following the same argument as in the smooth quadric case, C_Q is of type $(a, a + b - c_1 - c_2, a - c_2, a - c_1)$ in this case also. ∎

Proof of Proposition I.5.9. Let $C_P \subset S_P$ be a curve of type (a, b, c_0, c_1, c_2) and let $C_Q \subset S_Q$ be of type $(a', b', c_0', c_1', c_2')$. By Lemma I.5.4 (4) the pullback of the camera center $q_2 \in S_Q$ is the epipolar line $g^2_{P_1}$, which belongs to the class $L - E_0 - E_1$. C_P intersects this line $a - c_0 - c_1$ times. It follows that

$c_2' = a - c_0 - c_1$, similarly, we get $c_1' = a - c_0 - c_2$. Moreover, by Lemma I.5.4 (3) the pullback of the intersection of the epipolar lines, is the lines spanned by the two camera centers (class $L - E_1 - E_2$), it follows that $c_0' = a - c_1 - c_2$.

A generic line in the plane containing the camera centers on S_Q pulls back to a curve in the class $2L - E_0 - E_1 - E_2$ on S_P, C_P intersects this $2a - c_0 - c_1 - c_2$ times, it follows that $a' = 2a - c_0 - c_1 - c_2$. Lastly, ϕ_P is an isomorhpism on the final plane, so $b' = b$. This means that C_Q is of type $(2a - c_0 - c_1 - c_2, b, a - c_1 - c_2, a - c_0 - c_2, a - c_0 - c_1)$. ■

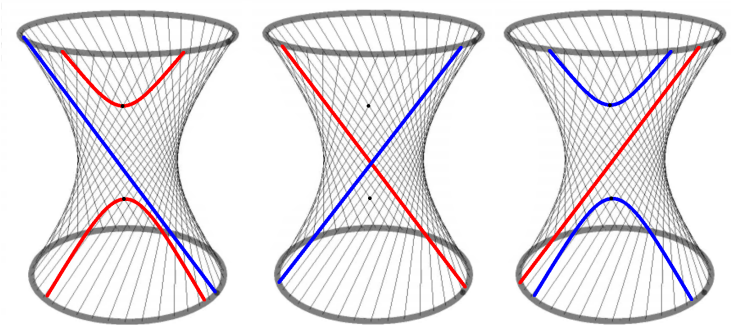


Figure I.3: Illustration of the map taking a point to its conjugate. The smooth quadric has two conjugates (original in the middle), the left is the one we get if we take the lines in the "A" family to be the epipolar lines, whereas the one on the right is the one we get if we choose "B". In both cases the lines in the family with the epipolar lines are preserved, whereas the lines in the other family are mapped to conic curves.

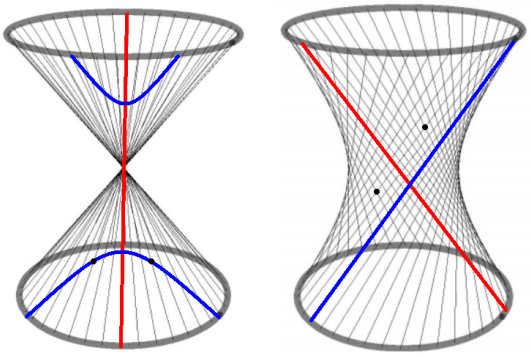


Figure I.4: Illustration of the map taking a point to its conjugate. The line spanned by the camera centers (right) maps to the vertex on the cone. The other lines in this family map to conics. Lines in the other family are preserved.

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Appendix I.A

Proof behind tables in Section I.4.4

Let P_1, P_2 be two cameras with camera centers p_1, p_2, and fundamental form F_P. Let $\mathbf{F}$ denote the space of bilinear forms on $\mathbb{P}^2 \times \mathbb{P}^2$. By Lemma I.4.10, there is an isomorphism between the set of quadrics passing through p_1, p_2, and lines $L \subset \mathbf{F}$ that pass through F_P. The set of bilinear forms that are not of full rank is denoted by $\mathbf{F}_2$. There are several different ways a line l through F_P can intersect $\mathbf{F}_2$, namely:

When we have a finite number of intersections:

- Three distinct, real intersection points (one of which is F_P).
- Three distinct intersection points, two of which are complex conjugates.
- Two distinct intersection points of rank 2, F_P has multiplicity 2.
- Two distinct intersection points of rank 2, the one that is not F_P has multiplicity 2.
- Only one intersection point: F_P with multiplicity 3.
- Two distinct intersection points, F_P and one which has rank 1. The latter will have multiplicity 2.

Then there are the ones where L lies in $\mathbf{F}_2$; here we look at the kernels, and at the number of rank 1 forms on L:

- Two distinct, real, rank 1 forms (implies all bilinear forms share the same left- and right kernel).
- Two distinct, complex, rank 1 forms (implies all bilinear forms share the same left- and right kernel).
- One form of rank 1, which has multiplicity 2 (implies all bilinear forms share the same left- and right kernel)

- One form of rank 1, which has multiplicity 1 (implies all bilinear forms share the same left- OR right kernel, but never both)
- No forms of rank 1, all bilinear forms share the same left- or right kernel
- No forms of rank 1, all bilinear forms have distinct kernels.

This gives us a total of 12 different configurations to check. For each of these, we want to know what kind of quadric/camera configuration it corresponds to in $\mathbb{P}^3$.

Let $\mathrm{PGL}(3,\mathbb{R})$ be the projective general linear group of degree 3 (the group of real, invertible 3×3 matrices up to scale). This group acts on $\mathbf{F}$ with an action that can be represented by matrix multiplication. Let $\mathrm{PGL}(3)_{F_P}$ be the subgroup of $\mathrm{PGL}(3)$ that fixes F_P. This gives us a group that acts on the $\mathbb{P}^7$ of lines passing through F_P. This group will never take a line with one of the 12 configurations above to a line with a different configuration. For instance, a real invertible matrix can not take three distinct real points, to anything other than three distinct real points. In particular, this means that the 12 configurations above all lie in distinct orbits under this group action.

Recall the isomorphism between the set of quadrics through p_1, p_2 and the set of lines through F_P. It gives us a similar group action on the set of quadrics passing through the camera centers p_1, p_2, namely the subgroup of $\mathrm{PGL}(4)$ that fixes the two camera centers. Here too, the camera/quadric configurations fall into 12 different orbits (not listed). Now we only need to check one representative from each orbit to find the exact correspondences.

By Proposition I.2.13, the only property of the cameras that matters when considering critical configurations is their center. Furthermore, one pair of distinct points in $\mathbb{P}^3$ is no different from any other pair. This means that when we make computations for the two view case, we are free to pick any two cameras P_1, P_2 with distinct centers. We use

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The fundamental matrix of these two cameras is

$$F_P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now we can go ahead and check the 12 possible configurations listed earlier. This will be done by picking a fundamental matrix F_0 such that the line l spanned by F_P and F_0 intersects $\mathbf{F}_2$ the way we want and then checking what quadric it corresponds to. The full results are given in Table I.2.

Table I.2: Possible intersections between l and $\mathbf{F}_2$, and their corresponding quadrics.

Matrix F_0	**Intersections between l and $\mathbf{F}_2$**	**Configuration in $\mathbb{P}^3$**
$\begin{bmatrix}0&0&0\\1&0&0\\0&0&1\end{bmatrix}$	Three distinct, real intersection points, (one of which is F_P)	Smooth ruled quadric, camera centers not on same line
$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	Three distinct intersection points, two of which are complex conjugates.	Smooth, non-ruled quadric
$\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$	Two distinct intersection points, the one that is not F_P has multiplicity 2	A cone, cameras on different lines, no camera at vertex
$\begin{bmatrix}0&0&0\\0&0&1\\1&0&0\end{bmatrix}$	Two distinct intersection points, F_P has multiplicity 2	Smooth quadric, camera centers lie on the same line
$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$	Only one intersection point: F_P with multiplicity 3	Cone, both cameras lie on the same line, neither lies at the vertex
$\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$	Two distinct intersection points, F_P and one which has rank 1. The latter has multiplicity 2	Union of two planes, camera centers in different planes
$\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}$	l lies in $\mathbf{F}_2$, and contains two distinct, real, rank 1 forms[3]	Union of two planes, camera centers lie on the intersection of the planes
$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$	l lies in $\mathbf{F}_2$, and contains two distinct, complex, rank 1 forms[3]	Union of two complex conjugate planes, both cameras on intersection.
$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$	l lies in $\mathbf{F}_2$, and contains one form of rank 1, which has multiplicity 2[3]	A single plane with multiplicity 2
$\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	l lies in $\mathbf{F}_2$, and contains one form of rank 1, which has multiplicity 1[4]	Union of two planes, one camera on the intersection
$\begin{bmatrix}1&0&1\\1&1&0\\0&0&0\end{bmatrix}$	l lies in $\mathbf{F}_2$, and contains no forms of rank 1, all bilinear forms share the same left or right kernel	Cone, one camera at the vertex
$\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}$	l lies in $\mathbf{F}_2$, and contains no forms of rank 1, all bilinear forms have distinct kernels.	Union of two planes, cameras in same plane, neither at the intersection

[3] This implies all bilinear forms share the same left and right kernel
[4] This implies all bilinear forms share the same left or right kernel

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Paper II

Critical configurations for three projective views

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This is the published version of the paper, which contains the following error (corrected in Paper IV): On page 72, in the paragraph before Lemma II.6.16, the paper claims that a configuration of points lying on a twisted cubic passing through exactly one of the three camera centers is not critical. While this is generally true, the configuration will be critical if the line spanned by the two camera centers not lying on the twisted cubic is a secant to the twisted cubic. In fact; any configuration of three cameras and points all lying on the union of a twisted cubic and a secant line is critical, this is a degeneration of the elliptic/singular quartic curve in Proposition 6.12 and can be proven to be critical using a similar proof. If one includes this configuration, the classification in the paper is complete.

Abstract

The problem of structure from motion is concerned with recovering the 3-dimensional structure of an object from a set of 2-dimensional images taken by unknown cameras. Generally, all information can be uniquely recovered if enough images and point correspondences are provided, yet there are certain cases where unique recovery is impossible; these are called *critical configurations*. We use an algebraic approach to study the critical configurations for three projective cameras. We show that all critical configurations lie on the intersection of quadric surfaces, and classify exactly which intersections constitute a critical configuration.

II.1 Introduction

In computer vision, one of the main problems is that of *structure from motion*, where given a set of 2-dimensional images the task is to reconstruct a scene in 3-space and find the camera positions in this scene. Over time, many techniques have been developed for solving these problems for varying camera models and under different assumptions on the space and image points [HZ04; May93]. In

general, with enough images and enough points in each image, one can uniquely recover the original scene. However, there are also some 3D-configurations of points and cameras where a unique recovery from the images is never possible. These are called *critical configurations*.

Much work has been done to understand critical configurations for various settings [Ber+20; BT07; Buc88; Buc92; Har00; HK02; HK07; KHA01], with results dating as far back as 1941 [Kra41]. While interesting from a purely theoretical viewpoint, critical configurations also play a part in practical applications. Even though critical configurations are rare in real-life reconstruction problems when enough data is available (due to noise), it has been shown that as the configurations approach the critical ones, reconstruction algorithms tend to become less and less stable [BTB07; HK07; LF94].

We will study the critical configurations for three projective cameras using a new approach. The idea is as follows (full details in Section II.3): A set of n cameras $\mathbf{P}$ defines a rational map $\phi_{\mathbf{P}} \colon \mathbb{P}^3 \dashrightarrow (\mathbb{P}^2)^n$, which extends to a morphism under blow-up of the camera centers:

$$\begin{array}{ccc} \mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3) & \xrightarrow{\widetilde{\phi_{\mathbf{P}}}} & (\mathbb{P}^2)^n \\ {\scriptstyle \pi_{\mathbf{P}}}\downarrow & \nearrow_{\phi_{\mathbf{P}}} & \\ \mathbb{P}^3 & & \end{array}$$

A set of n cameras $\mathbf{P}$ and a set of points $X \subset \mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3)$ is referred to as a *configuration*. A configuration is critical if there exists another set of cameras $\mathbf{Q}$ and points Y such that $\widetilde{\phi_{\mathbf{P}}}(X) = \widetilde{\phi_{\mathbf{Q}}}(Y)$. The map $\widetilde{\phi_{\mathbf{P}}}$ takes $\mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3)$ to a variety in $(\mathbb{P}^2)^n$ called the *multi-view variety*. Since X and Y both map into the intersection of the multi-view varieties of $\widetilde{\phi_{\mathbf{P}}}$ and $\widetilde{\phi_{\mathbf{Q}}}$, one can classify critical configurations by classifying all possible intersections of multi-view varieties.

We use this approach to recover many of the results in [HK07]. Figure II.1 shows the blow-down of some of the critical configurations for three views, that is, it shows the position of the camera centers (large dots), along with the blow-down $\pi_{\mathbf{P}}(X) \subset \mathbb{P}^3$ of the points (in most cases, a curve). The critical configurations not shown in Figure II.1 are subconfigurations of the ones shown. The main result of this paper is as follows:

Theorem II.1.1. *A configuration of three projective cameras $\boldsymbol{P}$ and 8 or more points $X \subset \mathrm{Bl}_{\boldsymbol{P}}(\mathbb{P}^3)$ form a critical configuration if and only if the points X lie on the strict transform of one of the varieties described in Figure II.1, or if they form a subset of such a variety. A configuration of three projective cameras and 6 or fewer points is always a critical configuration.*

The case of 7 points and three projective cameras is covered in Proposition II.6.19.

The result in Theorem II.1.1 is similar to what was found in [HK07], although one critical configuration (twisted cubic through two camera centers) is new. Still, even though many of the results are the same, the techniques for obtaining

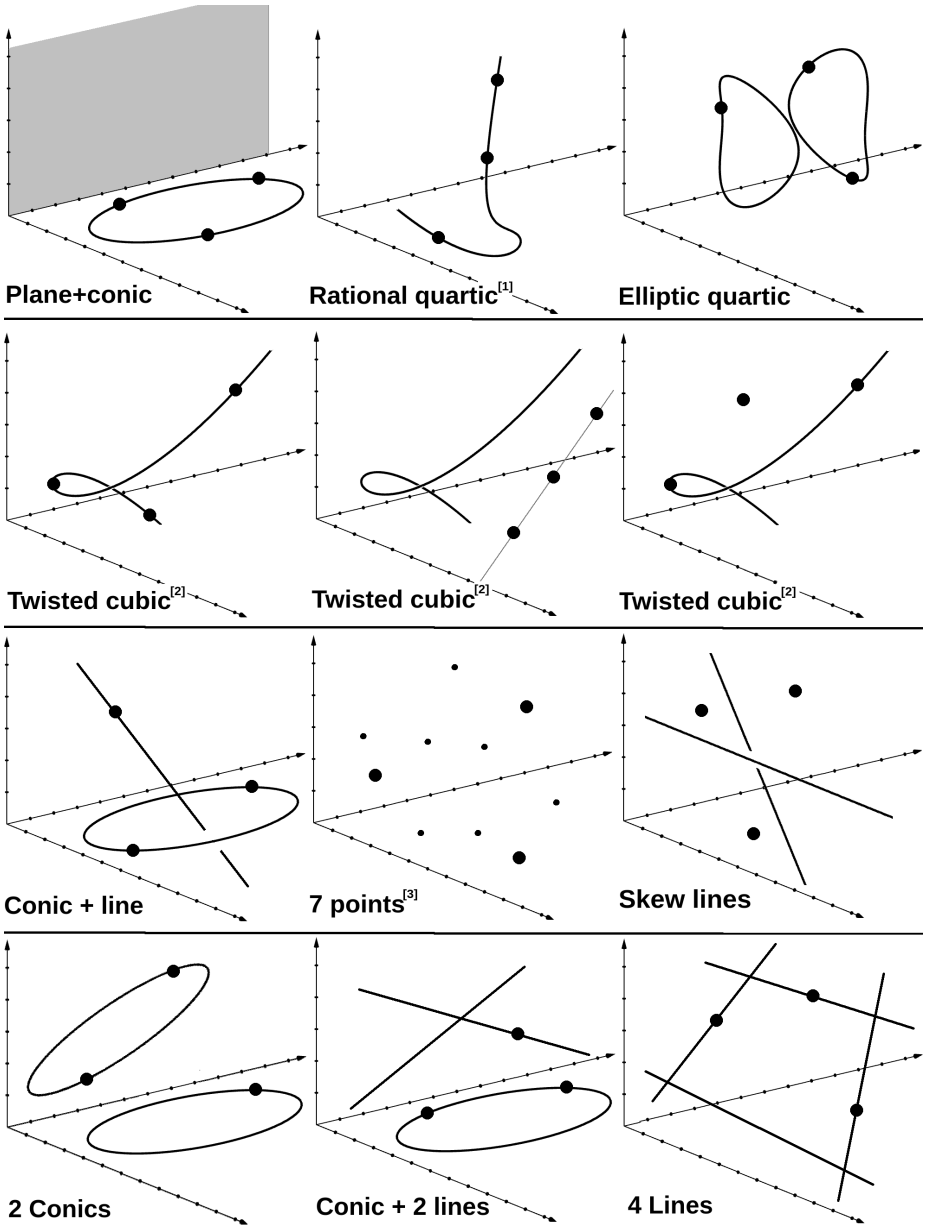


Figure II.1: The blow-down of some of the critical configurations for three views.
[1] The rational quartic may have a node or cusp.
[2] The twisted cubic may degenerate to the union of a conic and a line, or to three lines.
[3] The 7-point configuration must satisfy the conditions in Proposition II.6.19.

them are new. We use the techniques developed in [Brå23] and classify the critical configurations by considering the possible intersections of multi-view varieties, rather than relying on computations and ad hoc examples. We also hope that these new techniques will allow classification of critical configurations in more complicated scenarios in future work.

In Section II.2, we introduce the main concepts, such as cameras and critical configurations, as well as some general results on critical configurations. Section II.3 describes the approach used to find critical configurations. Section II.4 gives a summary of the key results from the two-view case necessary for the three-view classification. Sections II.5 and II.6 make up the main part of this paper. Here we show that critical configurations for three views need to lie on the intersection of three ruled quadrics and describe exactly which such intersections constitute a critical configuration.

II.2 Background

As this paper deals with a very similar topic as [Brå23], this section is more or less the same as Section 2 in [Brå23]. We refer the reader to [HZ04] for the basics on computer vision and multi-view geometry.

Let $\mathbb{C}$ denote the complex numbers, and let $\mathbb{P}^n$ denote the projective space over the vector space $\mathbb{C}^{n+1}$. Projection from a point $p \in \mathbb{P}^3$ is a linear map

$$P' \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2.$$

We refer to such a projection and its projection center p as a *camera* and its *camera center* in $\mathbb{P}^3$ (following established terminology, we use the words *camera* and *view* interchangeably). Following this theme, we refer to points in $\mathbb{P}^3$ as *space points* and points in $\mathbb{P}^2$ as *image points*. Similarly, $\mathbb{P}^2$ will be referred to as an *image*.

Once coordinates are chosen in $\mathbb{P}^3$ and $\mathbb{P}^2$, a camera can be represented by a 3×4 matrix of full rank called the *camera matrix*. The camera center is then given as the kernel of the matrix. For the most part, we make no distinction between a camera and its camera matrix, referring to both simply as cameras. We use the *real projective pinhole camera model*, meaning that we require a camera matrix to be of full rank and to have only real entries.

Remark II.2.1. Throughout the paper, whenever we talk about cameras it is to be understood that a choice of coordinates has been made, both on the images and 3-space.

Since the map P' is not defined at the camera center p, it is not a morphism. This problem can be mended by taking the blow-up. Let $\mathrm{Bl}_P(\mathbb{P}^3)$ be the blow-up of $\mathbb{P}^3$ in the camera center of P'. We then get the following diagram:

$$\begin{array}{ccc} \mathrm{Bl}_P(\mathbb{P}^3) & \xrightarrow{P} & \mathbb{P}^2 \\ {\scriptstyle \pi_P}\downarrow & \nearrow_{P'} & \\ \mathbb{P}^3 & & \end{array}$$

where π_P denotes the blow-down of $\mathrm{Bl}_P(\mathbb{P}^3)$. This gives a morphism from $\mathrm{Bl}_P(\mathbb{P}^3)$ to $\mathbb{P}^2$. We denote the cameras by P' when talking about rational maps and P when talking about morphisms (see diagram above). For ease of notation, we retain the names *camera* and *camera center*, although one should note that in $\mathrm{Bl}_P(\mathbb{P}^3)$ the camera center is no longer a point, but an exceptional divisor.

Definition II.2.2. A set of cameras P_i is said to have *collinear camera centers* if the camera centers of P_i' lie on a line.

Remark II.2.3. Throughout the paper, when dealing with multiple cameras, we assume that all camera centers are distinct.

Definition II.2.4. Given an n-tuple of cameras $\mathbf{P} = (P_1, \dots, P_n)$, with camera centers $p_1, \dots, p_n$, let $\mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3)$ denote the blow-up of $\mathbb{P}^3$ in the n camera centers and let π_i denote the blow-down

$$\pi_i \colon \mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3) \to \mathrm{Bl}_{P_i}(\mathbb{P}^3).$$

We define the *joint camera map* to be the map

$$\widetilde{\phi_{\mathbf{P}}} = P_1 \times \cdots \times P_n \colon \mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3) \to (\mathbb{P}^2)^n, \quad x \mapsto (P_1(\pi_1(x)), \dots, P_n(\pi_n(x))).$$

This again gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3) & \xrightarrow{\widetilde{\phi_{\mathbf{P}}}} & (\mathbb{P}^2)^n \\ {\scriptstyle \pi_{\mathbf{P}}}\downarrow & \nearrow_{\phi_{\mathbf{P}}} & \\ \mathbb{P}^3 & & \end{array}$$

Remark II.2.5. The reason we use the blow-up $\mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3)$ rather than $\mathbb{P}^3$ is to turn the cameras (and hence the joint camera map) into morphisms rather than rational maps. This ensures that the image of the joint camera map is Zariski closed, turning it into a projective variety. This in turn makes the maximal critical configurations (Definition II.2.12) Zariski closed, simplifying the classification, otherwise, the maximal configurations would be constructible sets (see Remark II.4.9).

Definition II.2.6. We denote the image of the joint camera map $\widetilde{\phi_{\mathbf{P}}}$ as the *multi-view variety* of $P_1, \dots, P_n$. The set of all multi-homogeneous polynomials vanishing on $\mathrm{Im}(\widetilde{\phi_{\mathbf{P}}})$ is an ideal that we denote as the *multi-view ideal*.

Notation. Most works on computer vision do not use the blow-up, defining cameras/the joint camera map as rational maps rather than morphisms. Hence, they tend to define the multi-view variety as the *closure* of the image rather than as the image itself. While our definition of the multi-view variety seems different, it is equivalent to that used in other works, like [APT21].

Notation. While the multi-view variety is always irreducible, we use the term *variety* to also include reducible algebraic sets.

Theorem II.2.7 ([APT21, Theorem 3.7])**.** *The ideal of the multi-view variety is generated only by bilinear and trilinear forms. In particular, it is generated by the determinant of*

$$\begin{bmatrix} P_i' & \boldsymbol{x} & \boldsymbol{0} \\ P_j' & \boldsymbol{0} & \boldsymbol{y} \end{bmatrix},$$

for each pair of cameras, and the 7×7 *minors of*

$$\begin{bmatrix} P_i' & \boldsymbol{x} & \boldsymbol{0} & \boldsymbol{0} \\ P_j' & \boldsymbol{0} & \boldsymbol{y} & \boldsymbol{0} \\ P_k' & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{z} \end{bmatrix},$$

for each triple. Here $\boldsymbol{x}$*,* $\boldsymbol{y}$ *and* $\boldsymbol{z}$ *are the* 3×1 *vectors with variables in the* i*-th,* j*-th and* k*-th image respectively.*

Definition II.2.8. Given a set of points $S \subset (\mathbb{P}^2)^n$, a *reconstruction* of S is a configuration of cameras and points $(P_1, \dots, P_n, X)$ such that $S = \widetilde{\phi_{\mathbf{P}}}(X)$ where $\widetilde{\phi_{\mathbf{P}}}$ is the joint camera map of the cameras $P_1, \dots, P_n$.

Definition II.2.9. Given a configuration of cameras and points $(P_1, \dots, P_n, X)$, we refer to $\widetilde{\phi_{\mathbf{P}}}(X) \subset (\mathbb{P}^2)^n$ as the *images* of $(P_1, \dots, P_n, X)$.

Note that every configuration of cameras and points is a reconstruction of its images, so every configuration (of cameras and points) is a reconstruction and vice versa.

Given a set of image points $S \subset (\mathbb{P}^2)^n$ as well as a reconstruction $(P_1, \dots, P_n, X)$ of S, note that any scaling, rotation, translation, or more generally, any real projective transformation of $(P_1, \dots, P_n, X)$ does not change the images, giving rise to a large family of reconstructions of S. However, we are not interested in differentiating between these reconstructions.

Definition II.2.10. Given a set of points $S \subset (\mathbb{P}^2)^n$, let $(P_1, \dots, P_n, X)$ and $(Q_1, \dots, Q_n, Y)$ be two reconstructions of S. Let $\overline{X} = \pi_{\mathbf{P}}(X)$ and $\overline{Y} = \pi_{\mathbf{Q}(Y)}$ and let P_i' and Q_i' be the matrix representation of P_i and Q_i respectively. The two reconstructions of S are considered *equivalent* if there exists an element $A \in \mathrm{PGL}(4)$, such that

$$\begin{aligned} &A(\overline{X}) = \overline{Y}, \\ &P_i' A^{-1} = Q_i', \quad \forall i. \end{aligned}$$

From now on, whenever we talk about a configuration of cameras and points, it is to be understood as unique up to the action above, and two configurations will be considered different only if they are nonequivalent. As such, we consider a reconstruction to be unique if it is unique up to the action above.

Definition II.2.11. Two configurations of cameras and points $(P_1, \dots, P_n, X)$, and $(Q_1, \dots, Q_n, Y)$, are called *conjugate configurations* if they are nonequivalent reconstructions of the same set. That is, if they satisfy $\widetilde{\phi_{\mathbf{P}}}(X) = \widetilde{\phi_{\mathbf{Q}}}(Y)$, but

not the conditions in Definition II.2.10. Pairs of points $(x, y) \in X \times Y$ are called *conjugate points* if $\widetilde{\phi_{\mathbf{P}}}(x) = \widetilde{\phi_{\mathbf{Q}}}(y)$.

Definition II.2.12. A configuration of cameras and points $(P_1, \dots, P_n, X)$ is said to be a *critical configuration* if it has at least one conjugate configuration. A critical configuration $(P_1, \dots, P_n, X)$ is said to be *maximal* if there exists no critical configuration $(P_1, \dots, P_n, X')$ such that $X \subsetneq X'$.

Hence, a configuration is critical if and only if the images it produces do not have a unique reconstruction.

Remark II.2.13. Various definitions of critical configurations exist. For instance, [Kra41] considers the cone with two cameras on the same generator to be critical, while it fails to be critical by our definition. We use a definition similar to the one in [HK07], except we are working in $\mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3)$. If one considers the blow-down, our definition matches the results in [HK07].

Definition II.2.14. Let $\mathbf{P}$ and $\mathbf{Q}$ be two n-tuples of cameras, let $\mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3)$ and $\mathrm{Bl}_{\mathbf{Q}}(\mathbb{P}^3)$ denote the blow-up of $\mathbb{P}^3$ in the camera centers of $\mathbf{P}$ and $\mathbf{Q}$ respectively. Define

$$I = \left\{ (x, y) \in \mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3) \times \mathrm{Bl}_{\mathbf{Q}}(\mathbb{P}^3) \mid \widetilde{\phi_{\mathbf{P}}}(x) = \widetilde{\phi_{\mathbf{Q}}}(y) \right\}.$$

The projection of I to each coordinate gives us two varieties, X and Y, which we denote X as the *set of critical points of P with respect to Q*, and similarly for Y.

This definition is motivated by the following fact:

Proposition II.2.15. *Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be two n-tuples of cameras such that there is no $A \in \mathrm{PGL}(4)$ satisfying $P_i' A^{-1} = Q_i'$ for all i. Let X be the critical points of P with respect to Q and conversely for Y. Then $(\boldsymbol{P}, X)$ is a critical configuration, with $(\boldsymbol{Q}, Y)$ as its conjugate. Furthermore, $(\boldsymbol{P}, X)$ is maximal with respect to $\boldsymbol{Q}$ in the sense that if there exists a critical configuration $(\boldsymbol{P}, X')$ with $X \subsetneq X'$, then its conjugate consists of cameras different from $\boldsymbol{Q}$.*

Proof. (2.12 in [Brå23]): It follows from Definition II.2.14 that for each point $x \in X$, we have a conjugate point $y \in Y$. Hence the two configurations have the same images. Inequivalence follows from the fact that the cameras lie in different orbits under $\mathrm{PGL}(4)$, so the second point in Definition II.2.10 can not be satisfied. Hence they are both critical configurations, conjugate to one another.

The (partial) maximality follows from the fact that if we add a point x_0 to X that does not lie in the set of critical points, there is (by Definition II.2.14) no point $y_0 \in \mathrm{Bl}_{\mathbf{Q}}(\mathbb{P}^3)$ such that $\widetilde{\phi_{\mathbf{P}}}(x_0) = \widetilde{\phi_{\mathbf{Q}}}(y_0)$. ∎

The goal of this paper is to classify all maximal critical configurations for three cameras. The reason we focus primarily on the maximal ones is that every critical configuration is contained in a maximal one and (when working with more than one camera) the converse is true as well, any subconfiguration of a critical configuration is itself critical.

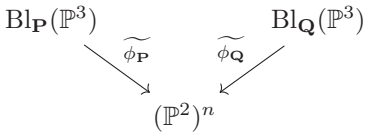
We conclude this section with a final, useful property of critical configurations, namely that the only property of the cameras we need to consider when exploring critical configurations is the position of their camera centers (i.e. change of coordinates in the images does not affect criticality).

Proposition II.2.16 ([HK07, Proposition 3.7])**.** *Let $(P_1, \dots, P_n)$ be n cameras with centers $p_1, \dots, p_n$, and let $(P_1, \dots, P_n, X)$ be a critical configuration. If $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ is a set of cameras sharing the same camera centers, the configuration $(\mathcal{P}_1, \dots, \mathcal{P}_n, X)$ is critical as well.*

Proof. Since P_i and $\mathcal{P}_i$ share the same camera center and the camera center determines the map uniquely up to a choice of coordinates, there exists some $H_i \in \mathrm{PGL}(3)$ such that $\mathcal{P}_i = H_i P_i$. Let $(Q_1, \dots, Q_n, Y)$ be a conjugate to $(P_1, \dots, P_n, X)$. Then $(H_1 Q_1, \dots, H_n Q_n, Y)$ is a conjugate to $(H_1 P_1, \dots, H_n P_n, X) = (\mathcal{P}_1, \dots, \mathcal{P}_n, X)$, so this configuration is critical as well. ■

II.3 Approach

Consider a critical configuration $(P_1, \dots, P_n, X)$. Since it is critical, there exists a conjugate configuration $(Q_1, \dots, Q_n, Y)$ giving the same images in $(\mathbb{P}^2)^n$. The two sets of cameras define two joint-camera maps $\widetilde{\phi_{\mathbf{P}}}$ and $\widetilde{\phi_{\mathbf{Q}}}$.



The image of X lies in the multi-view variety $\mathrm{Im}(\widetilde{\phi_{\mathbf{P}}}) \subseteq (\mathbb{P}^2)^n$ and the image of Y lies in $\mathrm{Im}(\widetilde{\phi_{\mathbf{Q}}})$. As such, the two sets of points X and Y lie in such a way that they both map (with their respective maps) into the intersection of the two multi-view varieties $\mathrm{Im}(\widetilde{\phi_{\mathbf{P}}}) \cap \mathrm{Im}(\widetilde{\phi_{\mathbf{Q}}})$.

Remark II.3.1. This is one of the points at which the choice of coordinates in Remark II.2.1 is needed. Without it, there is no way to identify the $(\mathbb{P}^2)^n$ containing $\mathrm{Im}(\widetilde{\phi_{\mathbf{P}}})$ with that containing $\mathrm{Im}(\widetilde{\phi_{\mathbf{Q}}})$.

Lemma II.3.2. *Let $(P_1, \dots, P_n)$ be a tuple of $n \geq 2$ cameras. If the camera centers do not all lie on a line, the joint camera map $\widetilde{\phi_{\mathbf{P}}}$ is an embedding. If the camera centers all lie on a line, the joint camera map contracts this line and is an embedding everywhere else.*

Proof. Let $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{P}^2)^n$ be a point in the image of $\widetilde{\phi_{\mathbf{P}}}$. The preimage of $\mathbf{x}$ is the intersection $\bigcap_{i=1}^{n} l_{P_i}$ where $l_{P_i} = P_i^{-1}(x_i)$, that is, the strict transform of a line passing through the camera center p_i. As such, they intersect in a single point unless they are all equal to the strict transform of a line passing through

all camera centers. Hence the joint camera map is bijective outside of a potential line spanned by the camera centers.

To show that this bijective map is indeed an embedding, we need to show that no tangent collapses. For contradiction, assume there exists a point $x \in \mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3)$ and a tangent line L (different from the strict transform of the line containing all camera centers, if such a line exists) which collapses under $\widetilde{\phi_{\mathbf{P}}}$. First, if L is not contained in an exceptional divisor, then L is the strict transform of a line L' in $\mathbb{P}^3$. There is at least one camera center, say p_1, that does not lie on L'. Let π_i be the projection onto the i-th factor, the map

$$f\colon \mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3) \xrightarrow{\widetilde{\phi_{\mathbf{P}}}} (\mathbb{P}^2)^n \xrightarrow{\pi_1} \mathbb{P}^2$$

restricted to L is a smooth embedding, of L into $\mathbb{P}^2$. Since it factors through $(\mathbb{P}^2)^n$, the map $\widetilde{\phi_{\mathbf{P}}}$ is also a smooth embedding of this line, which violates our assumption that L is collapsed by $\widetilde{\phi_{\mathbf{P}}}$.

On the other hand, if L is contained in an exceptional divisor, say the one we get when blowing up p_1, the map f above is again a smooth embedding when restricted to L, by the same argument as above, $\widetilde{\phi_{\mathbf{P}}}$ does not collapse L in this case either. It follows that the map $\widetilde{\phi_{\mathbf{P}}}$ does not collapse any tangents, and hence is an embedding. ■

This shows that the two multi-view varieties $\mathrm{Im}(\widetilde{\phi_{\mathbf{P}}})$ and $\mathrm{Im}(\widetilde{\phi_{\mathbf{Q}}})$ are irreducible 3-folds, so their intersection is either a surface, a curve, or a set of points. Taking the preimage under $\widetilde{\phi_{\mathbf{P}}}$, $\mathrm{Im}(\widetilde{\phi_{\mathbf{P}}}) \cap \mathrm{Im}(\widetilde{\phi_{\mathbf{Q}}})$ pulls back to a surface, curve, or point set in $\mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3)$, which is exactly the set of critical points of P. As such, *classifying all maximal critical configurations can be done by classifying all possible intersections between two multi-view varieties.* The configuration of points in $\mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3)$ can be found by taking the pullback of $\mathrm{Im}(\widetilde{\phi_{\mathbf{P}}}) \cap \mathrm{Im}(\widetilde{\phi_{\mathbf{Q}}})$ under $\widetilde{\phi_{\mathbf{P}}}$.

Proposition II.3.3. *Let $(P_1, \dots, P_n, X)$ be a maximal critical configuration. Then X is the intersection of surfaces $\widetilde{S^{ij}}$ (one for each pair of cameras) and surfaces $\widetilde{S^{ijk}}$ (one or more for each triple of cameras). The surfaces satisfy the following properties:*

- *The blow-down S^{ij} of $\widetilde{S^{ij}}$ is a quadric surface containing the camera centers p_i and p_j. The blow-down S^{ijk} of $\widetilde{S^{ijk}}$ is a cubic surface containing the camera centers p_i, p_j, and p_k.*
- *$\widetilde{S^{ij}}$ is the surface one gets when taking the strict transform of S^{ij} when blowing up p_i, p_j and the total transform under the remaining blow-ups. $\widetilde{S^{ijk}}$ is the surface one gets when taking the strict transform of S^{ijk} when blowing up p_i, p_j, p_k and the total transform under the remaining blow-ups.*

Proof. By Theorem II.2.7, the ideal of the multi-view variety is generated only by bilinear and trilinear polynomials. Under the rational map $\phi_{\mathbf{P}}$, a polynomial in

$(\mathbb{P}^2)^n$ of multidegree $(d_1, \dots, d_n)$ pulls back to a polynomial of degree $d_1 + \cdots + d_n$ with multiplicity d_i in the camera center p_i. This gives us the surfaces S^{ij} and S^{ijk} in $\mathbb{P}^3$, whose intersection is the blow-down of X.

The variety X itself, however, is the pullback of the same polynomials under the morphism $\widetilde{\phi_{\mathbf{P}}}$. Under this morphism, a polynomial of multidegree $(d_1, \dots, d_n)$ pulls back to a divisor in the class $(d_1 + \cdots + d_n)H - d_1E_1 - \cdots - d_nE_n$, where H is the class of a hyperplane and E_i is the class of the exceptional divisor we get when blowing up p_i. In particular, a bilinear form with $d_i, d_j = 1$ will pull back to a surface in the class $2H - E_i - E_j$, that is, the strict transform of a quadric surface under the blow-up of p_i, p_j and the total transform under the remaining blow-ups. ■

Remark II.3.4. Note that if two sets of cameras satisfy point 2 in Definition II.2.10 their multi-view varieties are equal. Hence the multi-view varieties being different (which is the case we study throughout the paper) is enough to ensure that the two configurations are inequivalent.

II.4 Key results from the two-view case

While the main focus of this paper is the critical configurations for three views, there are some key results from the two-view case which can help simplify many of the three-view arguments. This section gives a summary of the main results on critical configurations for two views. A full analysis of the two-view case can be found in [Brå23] or [HK07].

II.4.1 The multi-view variety

Proposition II.4.1 (The fundamental form)**.** *[HZ04, Sections 9.2 and 17.1] For two cameras P_1, P_2 with distinct centers, the multi-view variety $\operatorname{Im}(\widetilde{\phi_{\boldsymbol{P}}}) \subset \mathbb{P}^2 \times \mathbb{P}^2$ is the vanishing locus of a single, bilinear, rank 2 form F_P, called the fundamental form (or fundamental matrix).*

$$F_P(\boldsymbol{x}, \boldsymbol{y}) = \det \begin{bmatrix} P_1' & \boldsymbol{x} & \boldsymbol{0} \\ P_2' & \boldsymbol{0} & \boldsymbol{y} \end{bmatrix},$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the variables in the first and second image respectively.

In the literature, this bilinear form is usually represented by a 3×3 matrix called the *fundamental matrix*.

Proof. By Lemma II.3.2, the multi-view variety for two cameras is an irreducible 3-fold in $\mathbb{P}^2 \times \mathbb{P}^2$. It follows that the multi-view ideal in the multi-graded ring of $\mathbb{P}^2 \times \mathbb{P}^2$ can be generated by a single polynomial. Let $(\mathbf{x}, \mathbf{y})$ be a generic point in the multi-view variety, then there exists a point $\mathbf{X} \in \mathbb{P}^3$ such that $P_1(\mathbf{X}) = \lambda_1 \mathbf{x}$

and $P_2(\mathbf{X}) = \lambda_2 \mathbf{y}$, then

$$\underbrace{\begin{bmatrix} P_1' & \mathbf{x} & \mathbf{0} \\ P_2' & \mathbf{0} & \mathbf{y} \end{bmatrix}}_{A} \begin{bmatrix} \mathbf{X} \\ -\lambda_1 \\ -\lambda_2 \end{bmatrix} = 0.$$

Since A has a non-zero kernel, the determinant F_P has to vanish on $(\mathbf{x}, \mathbf{y})$. Now we need only show that the determinant is irreducible to prove that $F_P(\mathbf{x}, \mathbf{y})$ generates the multi-view ideal. To do this, we show that F_P is of rank 2, this is sufficient since a reducible polynomial is always of rank 1.

Let us find the left kernel of F_P, that is, all values of $\mathbf{x}$ such that $F_P(\mathbf{x}, -) = 0$. The fifth column of A has zero as its bottom three entries, so the only way to get this as a linear combination of the bottom three entries of the other 5 columns is as $P_2' \cdot p_2 + 0 \cdot \mathbf{y}$ (there are no other options since $\mathbf{y}$ is a set of variables, and P_2' is of full rank. It follows that the left kernel of F_P consists only of the point $P_1' \cdot p_2$, so F_P is indeed of rank 2. ■

Definition II.4.2. The special points mentioned at the end of the proof of Proposition II.4.1:

$$e^j_{P_i} = P_i(p_j), \tag{II.1}$$

are called *epipoles*. For the fundamental form of P_i, P_j, they satisfy

$$F_P(e^j_{P_i}, -) = F_P(-, e^i_{P_j}) = 0. \tag{II.2}$$

By Proposition II.4.1, the multi-view ideal for two views is generated by a rank 2 bilinear form. The converse follows from Theorem 9.13. in [HZ04].

Theorem II.4.3. *There is a* $1:1$ *correspondence between real bilinear forms of rank two, and multi-view ideals for two views.*

II.4.2 Critical quadrics

Since the multi-view ideal is generated by a bilinear form and since such a form pulls back to the strict transform of a quadric surface passing through the two camera centers in $\mathbb{P}^3$, critical configurations for two views need to lie on the strict transform of quadrics (recall Proposition II.3.3). Furthermore, since the bilinear forms vanish in the two epipoles, the quadrics in question need to contain a certain pair of lines (the pullback of each epipole):

Definition II.4.4. Let p_1, p_2 be two camera centers, and let S_P be a quadric surface containing both camera centers. A pair of real (not necessarily distinct) lines $g^2_{P_1}, g^1_{P_2}$ are called *permissible* if they satisfy the following properties.

II.4.4.1. The line $g^j_{P_i}$ lies on S_P and passes through p_i.

II.4.4.2. Any point lying on both $g^j_{P_i}$ and $g^i_{P_j}$ is a singular point on S_P.

II.4.4.3. Any point in the singular locus of S_P which lies on one of the lines also lies on the other.

II.4.4.4. If S_P is the union of two planes, $g^j_{P_i}$ and $g^i_{P_j}$ lie in the same plane.

This definition is similar to the definition of permissible lines in [HK07, Definition 5.11], but differs in that it allows for permissible lines in the case where the quadric S_P is a double plane. On the double plane, every point is singular, so two lines are permissible only if they are both equal to the line spanned by the camera centers. Figure II.2 shows all quadrics containing at least one pair of permissible lines.

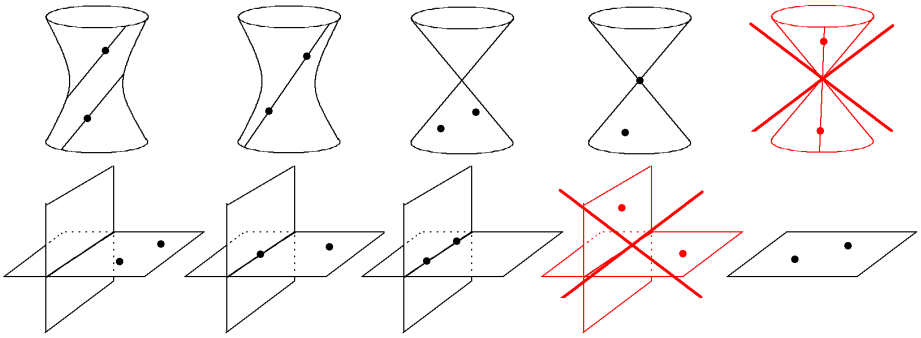


Figure II.2: Illustration showing all critical quadrics with two marked camera centers, i.e. those containing a pair of permissible lines. The ones marked in red are not critical and do not contain such a pair of lines.

Theorem II.4.5 ([Brå23, Theorem 4.11])**.** *Let (P_1, P_2, X) be a configuration of cameras and points. The configuration is critical if and only if there exists a real quadric $S_P \in \mathbb{P}^3$ containing both camera centers p_i and a pair of permissible lines, whose strict transform in $\mathrm{Bl}_{\boldsymbol{P}}(\mathbb{P}^3)$ contains all the points X.*

[Brå23, Theorem 4.11] is formulated slightly differently, requiring the quadrics to be among those in Table II.1 rather than to be ones containing two permissible lines. It can easily be checked that these two conditions are equivalent.

Remark II.4.6. The theorem requires the quadric to contain a pair of permissible lines for the configuration to be critical. These two lines are the pre-images of the two epipoles $e^2_{Q_1}, e^1_{Q_2}$ of F_Q (where F_Q denotes the fundamental form of the two cameras in the conjugate configuration).

Proposition II.4.7 ([Brå23, Proposition 5.3])**.** *Let S_P be a quadric, and let P_1, P_2 be two cameras with camera centers p_1, p_2, not both lying in the singular locus of S_P. Let $\widetilde{S_P}$ denote the strict transform of S_P in $\mathrm{Bl}_{\boldsymbol{P}}(\mathbb{P}^3)$. There is a 1:1 correspondence between pairs of permissible lines on S_P, and configurations conjugate to $(P_1, P_2, \widetilde{S_P})$.*

Quadric S_P	**Conjugate quadric**	**Conjugates**
Smooth quadric, cameras not on a line	Same	2
Smooth quadric, cameras on a line	Cone, cameras not on a line	1
Cone, cameras not on a line	Smooth quadric, cameras on a line	1
Cone, one camera at vertex, other one not	Same	∞
Two planes, cameras in the same plane	Same	∞
Two planes, one camera on the singular line	Same	∞
Two planes, cameras on the singular line	Same	∞
A double plane, cameras in the plane	Same	∞

Table II.1: A list of all possible critical configurations and their conjugates, as well as the number of conjugates for each configuration. Taken from [Brå23, Table 1]

Since we will be working with these quadrics extensively throughout the rest of the paper, it makes sense to distinguish the quadrics that give rise to a critical configuration from the ones that do not:

Definition II.4.8. Given two cameras P_1, P_2, a *critical quadric* is a quadric S_P along with two permissible lines $g^2_{P_1}$ and $g^1_{P_2}$ such that $(P_1, P_2, \widetilde{S_P})$ is a critical configuration.

Remark II.4.9. In Section II.2, we defined critical configurations to lie in $\mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3)$ rather than $\mathbb{P}^3$ to ensure that the maximal critical configurations would be closed varieties. Had we defined them to lie in $\mathbb{P}^3$, the critical configurations for two views would not be quadrics as above, but rather the quadrics minus the permissible lines.

II.4.3 Relations between quadrics

By Theorem II.4.5, any critical configuration (P_1, P_2, X) needs to lie on the strict transform of a quadric S_P. Similarly, the conjugate configuration (Q_1, Q_2, Y) needs to lie on the strict transform of a quadric (denoted S_Q). Each point $x \in S_P$ has a conjugate point $y \in S_Q$ such that $\widetilde{\phi_{\mathbf{P}}}(x) = \widetilde{\phi_{\mathbf{Q}}}(y)$. We give some relations between these conjugate pairs.

Proposition II.4.10 ([HK07, Table 1] or [Brå23, Table 1])**.** *When the quadric S_P is a hyperboloid with two camera centers on the same line, the conjugate S_Q is a cone with no camera center at the vertex, and vice versa. In all other cases, the quadrics S_P and S_Q are of the same type (see Figure II.2 for types).*

Theorem II.4.5 and Proposition II.4.7 provide a complete classification of all critical configurations for two views as well as the number of conjugates of each one. These results, together with Proposition II.4.10, are summarized in Table II.1

By Proposition II.4.7, there exists one conjugate configuration for each pair of permissible lines. The choice of lines impacts the relation between points on $\widetilde{S_P}$ and its conjugate $\widetilde{S_Q}$. For the next three results, assume a choice of permissible lines $g^2_{P_1}, g^1_{P_2}$ on S_P has been made, and let $\widetilde{g^2_{P_1}}$ and $\widetilde{g^1_{P_2}}$ denote their strict transforms.

Proposition II.4.11 ([Brå23, Lemma 5.4])**.** *The map taking a point $x \in \widetilde{S_P}$ to its conjugate is a birational map, defined everywhere except the intersection $\widetilde{g^2_{P_1}} \cap \widetilde{g^1_{P_2}}$.*

Proposition II.4.12 ([Brå23, Section 5.2])**.** *Points on the strict transform of the line $g^j_{P_i}$ on $\widetilde{S_P}$ map to the exceptional divisor (camera center on the blow-down) E_{q_j} on $\widetilde{S_Q}$. Points lying on the line spanned by $\widetilde{p_1}, \widetilde{p_2}$ on $\widetilde{S_P}$ map to the intersection of the two permissible lines $g^2_{Q_1}, g^1_{Q_2}$ on $\widetilde{S_Q}$ (a point which by Definition II.4.4 is singular).*

This gives us 2 (or 3, if the line spanned by p_1, p_2 is contained in S_P) curves on $\widetilde{S_P}$ which collapse to points on $\widetilde{S_Q}$. All other lines/curves are mapped to lines/curves.

Definition II.4.13. Let S_P be an irreducible critical quadric and let C_P be a curve on S_P. We say that C_P is of type (a, b, c_1, c_2), where c_i is the multiplicity of C_P in the camera center p_i and where a and b are:

- If S_P is smooth, a is the number of times C_P intersects a generic line in the same family as the permissible lines $g^j_{P_i}$, and b is the number of times it intersects the lines in the other family. (meaning (a, b) is the bidegree of C_P).
- If S_P is a cone, a is the number of times C_P intersects each line outside of the vertex, and b the number of times it intersects each line.

Proposition II.4.14 ([Brå23, Proposition 5.6])**.** *Let S_P be a smooth quadric or a cone, and let $C_P \subset \widetilde{S_P}$ be the strict transform of a curve of type (a, b, c_1, c_2) not containing either of the permissible lines. Then the conjugate curve $C_Q \subset \widetilde{S_Q}$ is the strict transform of a curve of type $(a, a + b - c_1 - c_2, a - c_2, a - c_1)$.*

Definition II.4.15. Let S_P be two planes with two camera centers not both lying on the intersection and let C_P be a curve on S_P. We say that C_P is of type (a, b, c_0, c_1, c_2), where a is the degree of the curve in the plane with the epipolar lines, b the degree of the curve in the other plane, c_0 is the multiplicity of C_P in the intersection of the epipolar lines $g^j_{P_i}$ and c_1, c_2 is the multiplicity in the camera centers p_1, p_2 respectively.

Proposition II.4.16 ([Brå23, Proposition 5.8])**.** *Let S_P be two planes with two camera centers not both lying on the intersection and let $C_P \subset S_P$ be a curve of type (a, b, c_0, c_1, c_2) such that C_P does not contain either of the epipolar lines or the lines spanned by the camera centers. Then the conjugate curve $C_Q \subset S_Q$ is of type $(2a - c_0 - c_1 - c_2, b, a - c_1 - c_2, a - c_0 - c_2, a - c_0 - c_1)$.*

II.5 Preliminaries for the three-view case

We start with the observation that any critical configuration for three views is critical for each pair of views.

Theorem II.5.1. *Let $(P_1, \dots, P_n, X)$ be a critical configuration, and let $\mathcal{P} \subset \{P_1, \dots, P_n\}$ be a nonempty subset. We get the following commutative diagram:*

$$\begin{array}{ccc} \mathrm{Bl}_{\boldsymbol{P}}(\mathbb{P}^3) & & \\ \downarrow{\scriptstyle g_P} & \searrow^{\pi} & \\ \mathrm{Bl}_{\mathcal{P}}(\mathbb{P}^3) & \xrightarrow{f} & \mathbb{P}^3 \end{array}$$

The configuration $(\mathcal{P}, g_P(X))$ is either a maximal critical configuration or a subconfiguration of a maximal critical configuration.

Proof. Since $(P_1, \dots, P_n, X)$ is a critical configuration, there exists a conjugate configuration $(Q_1, \dots, Q_n, Y)$. Each camera P_i has a conjugate camera Q_i. Let $\mathcal{Q} \subset \{Q_1, \dots, Q_n\}$ be the set of cameras conjugate to $\mathcal{P}$. Then the cameras $\mathcal{P}$ along with the points

$$\widetilde{\phi_{\mathcal{P}}}^{-1}(\mathrm{Im}(\widetilde{\phi_{\mathcal{P}}}) \cap \mathrm{Im}(\widetilde{\phi_{\mathcal{Q}}}))$$

is a critical configuration with

$$(\mathcal{Q}, \widetilde{\phi_{\mathcal{Q}}}^{-1}(\mathrm{Im}(\widetilde{\phi_{\mathcal{P}}}) \cap \mathrm{Im}(\widetilde{\phi_{\mathcal{Q}}})))$$

as a conjugate. Finally, since $(\mathcal{P}, g_P(X))$ and $(\mathcal{Q}, g_Q(Y))$ (where g_Q is defined similarly to g_P) produce the same image in $(\mathbb{P}^2)^m$, $g_P(X)$ is either equal to or a subset of $\mathcal{P}^{-1}(\mathrm{Im}(\widetilde{\phi_{\mathcal{P}}}) \cap \mathrm{Im}(\widetilde{\phi_{\mathcal{Q}}}))$. ∎

Definition II.5.2. Let $(P_1, \dots, P_n, X)$ be a configuration of cameras and points, and let $\left\{S_P^{ij}\right\}$ be a set of $\binom{n}{2}$ quadrics. Let $\widetilde{S_P^{ij}}$ denote the surface we get in $\mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3)$ when taking the strict transform of S_P^{ij} under the blow-up of the camera centers p_i, p_j, and the total transform under the remaining blow-ups. The configuration $(P_1, \dots, P_n, X)$ is said to *lie on the intersection of the quadrics* $\left\{S_P^{ij}\right\}$ if each quadric S_P^{ij} contains the camera centers p_i, p_j, and $\widetilde{S_P^{ij}}$ contains X.

Corollary II.5.3. *For $n \geq 2$ views, the critical configurations always lie on the intersection of $\binom{n}{2}$ critical quadrics.*

Proof. This follows from Theorem II.5.1 along with the fact that for two views, the critical configurations lie on critical quadrics. It also follows directly from Proposition II.3.3. ■

In particular, for the three-view case, the critical configurations lie on the intersection of three critical quadrics. The task of classifying the critical configurations for three views, however, is not as simple as just classifying the possible intersections of the strict transforms of three quadrics. Even if a configuration is critical for each pair of views, it need not be critical for all three views. There are two issues, namely the *compatibility of the quadrics*, and the *ambiguous points*. Before we tackle these issues, let us state a result on the generators of the multi-view ideal for three views.

Lemma II.5.4. *For the three cameras, the multi-view ideal contains a three-dimensional family of bilinear- and a ten-dimensional family of trilinear forms.*

Proof. The first part is simply an application of Theorem II.2.7 to the case of three cameras. To show that the ideal contains a ten-dimensional family of trilinear forms, note that there is a 27-dimensional family of trilinear forms on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, which pulls back to a 17-dimensional family of cubics passing through the three camera centers (20 cubics, less 3 for the constraint of passing through the camera centers). Hence a 10-dimensional family of trilinear forms pull back to the zero polynomial and thus lie in the multi-view ideal. ■

Lemma II.5.5. *Let P_1, P_2, P_3 be three cameras, and let $F_P^{12}, F_P^{13}, F_P^{23}$ be their fundamental forms.*

II.5.5.1. *The camera centers are collinear if and only if there are exactly two linear syzygies between the fundamental forms.*

II.5.5.2. *The camera centers are not collinear if and only if there are no linear syzygies between the fundamental forms*

Proof.

1. Let P_1, P_2, P_3 be three cameras whose cameras all lie on a line l, and let Π_1 and Π_2 be two distinct planes passing through l. These planes are mapped to lines in each image, let $f^j_{P_i}$ be the polynomial describing the line $P_i(\Pi_j)$, and let F be the matrix

$$F = \begin{bmatrix} f^1_{P_1} & f^1_{P_2} & f^1_{P_3} \\ f^2_{P_1} & f^2_{P_2} & f^2_{P_3} \end{bmatrix}.$$

   Under the joint camera map of the first two cameras ϕ_{P_1,P_2}, the minor $f^1_{P_1} f^2_{P_2} - f^2_{P_1} f^1_{P_2}$ pulls back to the zero polynomial, the only polynomial with this property is the fundamental form F_P^{12}. Hence the three minors of F are exactly the fundamental forms F_P^{ij}. Since the fundamental forms can be written as the minors of a 2×3 matrix, there are two linear syzygies between them.

For the converse, let there be two linear syzygies between the three fundamental forms. This means they can be written as the 2×2 minors of a 2×3 matrix

$$F' = \begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix}.$$

We then have that $F_P^{12} = \det \begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ and $F_P^{13} = \det \begin{pmatrix} a & c \\ a' & c' \end{pmatrix}$. Setting the first column in the first matrix to zero gives us the epipole $e_{P_1}^2$ and doing so for the second matrix gives us the epipole $e_{P_1}^3$, but since these columns are equal, then so are the epipoles, hence the camera centers must lie on a line.

2. Assume there exists at least one linear syzygy between the F_P^{ij}, and let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be the coordinates in the first, second, and third image respectively. Then there exist three linear polynomials f_i such that

$$f_3(\mathbf{z})F_P^{12}(\mathbf{x},\mathbf{y}) + f_2(\mathbf{y})F_P^{13}(\mathbf{x},\mathbf{z}) + f_1(\mathbf{z})F_P^{23}(\mathbf{y},\mathbf{z}) = 0$$

This holds for all $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, in particular, if $\mathbf{z} = e_{P_3}^2$, we get

$$f_3(e_{P_3}^2)F_P^{12}(\mathbf{x},\mathbf{y}) = -f_2(\mathbf{y})F_P^{13}(\mathbf{x},e_{P_3}^2).$$

If $F_P^{13}(\mathbf{x},e_{P_3}^2)$ was nonzero, this would imply that F_P^{12} is reducible, and hence of rank 1, meaning it is not a fundamental form. Hence $F_P^{13}(\mathbf{x},e_{P_3}^2)$ must be zero. But this means that $e_{P_3}^1 = e_{P_3}^2$, which means the camera centers are collinear. The converse, that the F_P^{ij} have at least one linear syzygy if the camera centers are collinear follows from item 1. ■

Proposition II.5.6. *If the three camera centers p_1, p_2, p_3 are not collinear, the multi-view ideal is generated by the fundamental forms along with one trilinear form. If the camera centers lie on a line, the ideal is generated by the fundamental forms along with three trilinear forms.*

Proof. By Lemma II.5.4 the multi-view ideal contains three fundamental forms. Each of these can be used to generate a 3-dimensional family of trilinear forms by multiplying it with the missing variable. By Lemma II.5.5, if the camera centers are not collinear, there are no linear syzygies between the fundamental forms, so they generate a 9-dimensional family of trilinear forms in the ideal. In this case, we need only add one trilinear form to the set of generators.

On the other hand, if the camera centers are collinear, we get two linear syzygies, so they generate only a 7-dimensional family of trilinear forms. In this case, we need to add three trilinear forms to get the necessary generators. ■

II.5.1 Compatible triples of quadrics

For three views, the multi-view ideal is generated by three fundamental forms, namely the ones we get from each pair of cameras (as well 1 or 3 trilinear forms, which we consider later). In the two-view case, any fundamental form (bilinear form of rank 2) comes from some pair of cameras (Theorem II.4.3). In the three-view case, however, three arbitrary fundamental forms will generally not come from some triple of cameras. This motivates the following definition:

Definition II.5.7. Let $\left\{F_Q^{ij} \mid 1 \leq i \leq j \leq n\right\}$ be a set of $\binom{n}{2}$ fundamental forms. The forms are said to be *compatible* if there exist n cameras $Q_1, ..., Q_n$ such that F_Q^{ij} is the fundamental form of Q_i, Q_j.

Definition II.5.8. Given n cameras $P_1, ..., P_n$ we say a set of quadrics $\left\{S_P^{ij}\right\}$ is *compatible* (with respect to the cameras P_i), if they are the pullback (with $\phi_{\mathbf{P}}$) of a compatible set of fundamental forms.

For a configuration (P_1, P_2, P_3, X) to be critical, it is not enough that it lies on the intersection of three critical quadrics, the three quadrics also need to be compatible. Only then can we find three conjugate cameras Q_1, Q_2, Q_3 which in turn can be used to construct a conjugate configuration. A detailed study of compatible fundamental forms can be found in [BR23; HZ04]. Recall (from (II.2)) that the *epipoles* $e^j_{Q_i}$ are the unique points satisfying

$$F_Q^{12}(e^2_{Q_1}, -) = F_Q^{12}(-, e^1_{Q_2}) = 0.$$

Theorem II.5.9 ([BR23, Proposition 5.8])**.** *Let F_Q^{12}, F_Q^{13}, F_Q^{23} be a triple of rank 2 bilinear forms, and let $e^j_{Q_i}$ be their epipoles. Then the triple is compatible and comes from a collinear triple of cameras if and only if*

$$e^2_{Q_1} = e^3_{Q_1}, \quad e^1_{Q_2} = e^3_{Q_2}, \quad e^1_{Q_3} = e^2_{Q_3}, \tag{II.3}$$

and

$$(F_Q^{12})^T [e^2_{Q_1}]_\times F_Q^{13} = F_Q^{23}, \tag{II.4}$$

where the product is taken to be matrix multiplication of the matrix representation of the fundamental forms, and where $[a]_\times$ denotes the unique 3×3 matrix such that $[a]_\times b = a \times b$ for all b.

Theorem II.5.10 ([HZ04, Section 15.4])**.** *Let F_Q^{12}, F_Q^{13}, F_Q^{23} be a triple of rank 2 bilinear forms, and let $e^j_{Q_i}$ be their epipoles. Then the triple is compatible and comes from a non-collinear triple of cameras if and only if*

$$e^2_{Q_1} \neq e^3_{Q_1}, \quad e^1_{Q_2} \neq e^3_{Q_2}, \quad e^1_{Q_3} \neq e^2_{Q_3}, \tag{II.5}$$

and

$$F_Q^{12}(e^3_{Q_1}, e^3_{Q_2}) = F_Q^{13}(e^2_{Q_1}, e^2_{Q_3}) = F_Q^{23}(e^1_{Q_2}, e^1_{Q_3}) = 0. \tag{II.6}$$

Since we are working with quadrics in $\mathbb{P}^3$, we want to translate these conditions on the fundamental forms, to conditions on the quadrics. The remainder of this subsection is used to give two results stating these conditions. We start with the collinear case, that of Theorem II.5.9 (clarification: this is NOT the case where the camera centers p_i are collinear, but the case where the camera centers q_i of the conjugate configuration are collinear).

Proposition II.5.11 (Collinear case). *Let P_1, P_2, P_3 be a triple of cameras, and let S_P^{12}, S_P^{13}, S_P^{23} be a triple of critical quadrics. Then the three quadrics come from a triple of fundamental forms that satisfy the conditions in Theorem II.5.9, if and only if there exist three lines L_1, L_2, L_3 satisfying:*

II.5.11.1. *$p_i \in L_i$.*

II.5.11.2. *$L_i \subseteq S_P^{ij} \cap S_P^{ik}$.*

II.5.11.3. *L_i and L_j form a permissible pair on S_P^{ij}.*

and any point $x \in S_P^{12} \cap S_P^{13}$ not lying on L_1 also lies on S_P^{23}.

Proof. ($\Rightarrow$): Let S_P^{12}, S_P^{13}, S_P^{23} be a compatible (in the collinear sense) triple of critical quadrics. This means they are the pullback of a triple of fundamental forms satisfying the conditions in Theorem II.5.9. By Theorem II.4.5, each quadric contains a pair of permissible lines, the pullback of the epipoles. By Equation (II.3) the two epipoles coincide in each image, so the two lines passing through p_i have to coincide as well. This gives us 3 lines satisfying the three conditions above.

Next, let x be any point in $S_P^{12} \cap S_P^{13}$ not lying on L_1, and denote $x_i = P_i(x)$. Since x lies on S_P^{1i}, $F_Q^{1i}x_i$ is the unique line in the first image spanned by x_1 and the epipole $e_1^2 = e_1^3$. Now, since $F_Q^{12}x_2$ and $F_Q^{13}x_3$ define the same line, we must have

$$\begin{aligned} 0 &= x_2^T (F_Q^{12})^T [e_{Q_1}^2]_\times F_Q^{13} x_3, \\ &= x_2^T F_Q^{23} x_3, \\ &= x^T P_2'^T F_Q^{23} P_3' x, \\ &= x^T S_P^{23} x. \end{aligned}$$

It follows that x lies on S_P^{23} as well

($\Leftarrow$): Let S_P^{ij} be three quadrics satisfying the conditions above. Since L_i and L_j form a permissible pair on S_P^{ij}, there exists (by Proposition II.4.7) bilinear forms F_Q^{ij} of rank 2 such that S_P^{ij} is the pullback (under $\phi_{\mathbf{P}}$) of F_Q^{ij}, and the lines L_i, L_j are the pullback of the epipoles of F_Q^{ij} (see Remark II.4.6). Since the line L_i is the pullback of both the epipole $e_{Q_i}^j$ and the epipole $e_{Q_i}^k$, the two epipoles are equal. Hence, the epipoles satisfy Equation (II.3).

Let $S_P^{23'}$ denote the quadric defined by the polynomial

$$x^T P_2'^T F_Q^{12T} [e_{Q_1}^2]_\times F_Q^{13} P_1' x.$$

For the next part, we want to show that the $S_P^{23'} = S_P^{23}$. If so, the three quadrics S_P^{ij} are the pullback of the three fundamental forms F_Q^{12}, F_Q^{13} and $F_Q^{12T}[e_{Q_1}^2]_\times F_Q^{13}$, since this triple of fundamental forms is compatible, so is the triple of quadrics. By assumption, the set $(S_P^{12} \cap S_P^{13}) - L_1$ lies on all three quadrics. If the quadrics do not contain a common surface, this set is a (possibly reducible) cubic curve C. The curve C lies on both S_P^{23} and $S_P^{23'}$. Moreover, since $F_Q^{12T}[e_{Q_1}^2]_\times F_Q^{13}$ has the same left and right kernels as F_Q^{23}, $S_P^{23'}$ must also contain the two lines L_2, L_3. The union of a cubic curve with two lines uniquely determines a quadric, so we have $S_P^{23} = S_P^{23'}$. A similar argument can be made if the quadrics contain a common surface, and in particular if they are equal. ■

Among other things, this shows that as long as the cameras are not in the position shown in Figure II.3, three copies of the same quadric form a compatible triple. For the remainder of the section, we assume that compatible triples of quadrics come from triples of bilinear forms that satisfy the non-collinear conditions (Equations (II.5) and (II.6)), rather than Equations (II.3) and (II.4).

Theorem II.5.12 (**General case**). *Let P_1, P_2, P_3 be a triple of cameras, and let S_P^{12}, S_P^{13}, S_P^{23} be a triple of critical quadrics. Then the three quadrics come from a triple of fundamental forms that satisfy the conditions in Theorem II.5.10, if and only if there exist six lines $g_{P_i}^j$ such that:*

II.5.12.1. *$g_{P_i}^j$ and $g_{P_j}^i$ form a permissible pair on S_P^{ij} (see Definition II.4.4).*

II.5.12.2. *The intersection of the plane spanned by $g_{P_i}^j$ and $g_{P_i}^k$ with the plane spanned by $g_{P_j}^i$ and $g_{P_j}^k$ lies on S_P^{ij}.*

Proof. ($\Rightarrow$): If the triple of quadrics is compatible, they are the pullback (under $\phi_{\mathbf{P}}$) of a compatible triple of rank 2 bilinear forms. The pullback of a rank 2 bilinear form will always be a quadric with two permissible lines, one through each camera center. Hence, the first part is satisfied.

Among the six permissible lines, there are 2 lines on each quadric and 2 through each camera center. By (II.5), for each camera center, the two lines passing through it are distinct. Hence the two lines through each camera center p_i span a plane, which we denote by Π_i.

Let y be any point lying in the intersection of Π_i and Π_j. Recall Remark II.4.6 which tells us that $P_i'(g_{P_i}^j) = e_{Q_i}^j$. We then have

$$\begin{aligned} P_i'(y) &= \alpha e_{Q_i}^j + \beta e_{Q_i}^k, \\ P_j'(y) &= \gamma e_{Q_j}^i + \delta e_{Q_j}^k. \end{aligned}$$

for some $[\alpha : \beta], [\gamma : \delta] \in \mathbb{P}^1$. Hence

$$\begin{aligned} S_P^{ij}(y) &= F_Q^{ij}(P_i'(y), P_j'(y)) \\ &= F_Q^{ij}(\alpha e_{Q_i}^j + \beta e_{Q_i}^k, \gamma e_{Q_j}^i + \delta e_{Q_j}^k) \end{aligned}$$

$$
\begin{aligned}
&= F_Q^{ij}(\beta e_{Q_i}^k, \delta e_{Q_j}^k) \\
&= 0,
\end{aligned}
$$

(final equality follows from (II.6)) which proves that y lies on S_P^{ij}. Hence S_P^{ij} contains the whole intersection of Π_i, Π_j.

($\Leftarrow$): For the converse, assume that the quadrics contain 6 lines satisfying the two conditions. Since 1. is satisfied, there exists a F_Q^{ij} such that S_P^{ij} is the pullback of F_Q^{ij}, and the lines $g_{P_i}^j$ and $g_{P_j}^i$ are the pullbacks of the epipoles of F_Q^{ij} (by Remark II.4.6 and Proposition II.4.7). Denote the plane spanned by $g_{P_i}^j$ and $g_{P_i}^k$ by Π_i.

Let x and y be the two points where $\Pi_i \cap \Pi_j$ intersects $g_{P_i}^k$ and $g_{P_j}^k$ respectively (in the case where $\Pi_i \cap \Pi_j$ is a plane, let x and y be any two points lying on $g_{P_i}^k$ and $g_{P_j}^k$ respectively). If x and y are equal, we get

$$
\begin{aligned}
0 &= S_P^{ij}(x) \\
&= F_Q^{ij}(P_i'(x), P_j'(y) \\
&= F_Q^{ij}(e_{Q_i}^k, e_{Q_j}^k),
\end{aligned}
$$

proving that F_Q^{ij} satisfies (II.6). On the other hand, if they are different, they span a line lying in $\Pi_i \cap \Pi_j$, which in turn lies in S_P^{ij}. In other words:

$$
\alpha x + \beta y \in S_P^{ij}
$$

for all $[\alpha : \beta] \in \mathbb{P}^1$. Now

$$
\begin{aligned}
P_i'(x) &= e_{Q_i}^k & \qquad P_j'(x) &= \gamma' e_{Q_j}^i + \delta' e_{Q_j}^k \\
P_i'(y) &= \gamma e_{Q_i}^j + \delta e_{Q_i}^k & \qquad P_j'(y) &= e_{Q_j}^k
\end{aligned}
$$

for some $[\gamma : \delta], [\gamma' : \delta'] \in \mathbb{P}^1$, so

$$
\begin{aligned}
0 &= S_P^{ij}(\alpha x + \beta y) \\
&= F_Q^{ij}(P_i'(\alpha x + \beta y), P_j'(\alpha x + \beta y)) \\
&= F_Q^{ij}(\alpha e_{Q_i}^k + \beta(\gamma e_{Q_i}^j + \delta e_{Q_i}^k), \alpha(\gamma' e_{Q_j}^i + \delta' e_{Q_j}^k) + \beta e_{Q_j}^k) \\
&= F_Q^{ij}((\alpha + \beta\delta) e_{Q_i}^k, (\alpha\delta' + \beta) e_{Q_j}^k) \\
&= (\alpha + \beta\delta)(\alpha\delta' + \beta) F_Q^{ij}(e_{Q_i}^k, e_{Q_j}^k).
\end{aligned}
$$

This holds for all $[\alpha : \beta] \in \mathbb{P}^1$. In particular, we can take $[\alpha : \beta]$ to be such that $(\alpha + \beta\delta)(\alpha\delta' + \beta) \neq 0$, proving that F_Q^{ij} satisfies (II.6). Hence, S_P^{12}, S_P^{13}, S_P^{23} forms a compatible triple of quadrics in this case also. ■

II.5.2 Ambiguous points

Having dealt with the first issue, the fact that not all triples of quadrics provide three conjugate cameras Q_1, Q_2, Q_3, we move on to the second issue, namely that

even if a configuration (P_1, P_2, P_3, X) lies on the intersection of three compatible, critical quadrics, it might not be a critical configuration.

By Proposition II.3.3 and Proposition II.5.6, the set of critical points is the intersection of the strict transform of three quadrics and one or three cubic surfaces. Hence, there might be some points in the intersection $\widetilde{S_P^{12}} \cap \widetilde{S_P^{13}} \cap \widetilde{S_P^{23}}$ which are not critical.

Proposition II.5.13. *Given two triples of cameras* (P_1, P_2, P_3) *and* (Q_1, Q_2, Q_3)*, and a point* $x \in \widetilde{S_P^{12}} \cap \widetilde{S_P^{13}} \cap \widetilde{S_P^{23}}$*. The three lines*

$$l'_{Q_i} = \overline{Q_i'^{-1}(P_i(x))} \subset \mathbb{P}^3$$

either have a point in common or all lie in the same plane.

Proof. The surface $\widetilde{S_P^{ij}}$ is exactly the collection of points x for which the two lines l'_{Q_i} and l'_{Q_j} intersect one another, so if $x \in \widetilde{S_P^{12}} \cap \widetilde{S_P^{13}} \cap \widetilde{S_P^{23}}$ each pair of lines l'_{Q_i} will intersect. This leaves two options, the l'_{Q_i} are either three lines lying in the same plane, or they have at least one point in common. ■

Definition II.5.14. Given two triples of cameras, the intersection $\widetilde{S_P^{12}} \cap \widetilde{S_P^{13}} \cap \widetilde{S_P^{23}}$ is the union of two subvarieties: the critical points, for which l'_{Q_i} pass through a common point, and the *ambiguous points*, for which the lines l_{Q_i} are coplanar.

Remark II.5.15. The two subvarieties need not be disjoint, nor do either of them need to be a proper subvariety. For instance, when the camera centers p_i are collinear, the ambiguous points form all of $\widetilde{S_P^{12}} \cap \widetilde{S_P^{13}} \cap \widetilde{S_P^{23}}$ and contain the critical points as a subvariety.

Remark II.5.16. The two subvarieties of $\widetilde{S_P^{12}} \cap \widetilde{S_P^{13}} \cap \widetilde{S_P^{23}}$ come from two subvarieties of $V(F_Q^{12}, F_Q^{13}, F_Q^{23})$. The critical points come from the multi-view variety, whereas the ambiguous points come from a subvariety that we also refer to as *ambiguous*, making no distinction between $\mathrm{Bl}_{\mathbf{P}}(\mathbb{P}^3)$ and $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.

The issue of these ambiguous points is well-known, and descriptions can be found in [HZ04, Section 15.3], and, for a more reconstruction-oriented angle, in [HK07, Section 6]. In this section, we recall a key result by Hartley and Kahl [HK07, Theorem 6.17], although approached from a more algebraic setting; we also provide a few additional results on the ambiguous points in the case where the cameras are collinear.

II.5.2.1 The non-collinear case

Let (Q_1, Q_2, Q_3) be three cameras whose camera centers are not collinear. Consider the ideal generated by the three fundamental forms:

$$I = (F_Q^{12}, F_Q^{13}, F_Q^{23}).$$

This is not the multi-view ideal, since it lacks the final trilinear form (Proposition II.5.6), so $V(I)$ contains at least some points not lying in the multi-view variety.

Lemma II.5.17. *When the three cameras are non-collinear, the multi-view variety is isomorphic to* $\mathrm{Bl}_{\boldsymbol{P}}(\mathbb{P}^3)$*, while the ambiguous subvariety is isomorphic to* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$*. The latter is the product of the three lines we get when projecting the plane spanned by the camera centers* q_1, q_2, q_3 *onto each image.*

Proof. The first part is a restatement of Lemma II.3.2. As for the ambiguous subvariety, it corresponds to the case where the lines l'_{Q_i} are coplanar. Since l'_{Q_i} passes through q_i, the only plane containing these lines is the plane spanned by the camera centers q_i. The family of triples of lines lying in that plane describes exactly the $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ above. ■

Adding the final trilinear form to the ideal removes the ambiguous subvariety and leaves only the multi-view variety.

So how does this affect the critical configurations? Let (P_1, P_2, P_3) be a triple of cameras, different from (Q_1, Q_2, Q_3). The pullback (using $\widetilde{\phi_{\mathbf{P}}}$) of the multi-view ideal $\mathrm{Im}(\widetilde{\phi_{\mathbf{Q}}})$ is the set of critical points. The pullback of $V(I)$ is the intersection of the strict transforms of three quadrics. This intersection is the union of the critical points and the ambiguous points.

Lemma II.5.18. *Let* P_1*,* P_2*,* P_3*, and* Q_1*,* Q_2*,* Q_3 *be two triples of cameras, such that the camera centers* q_i *are not collinear. Then the ambiguous points on the* P*-side are the intersection of the strict transforms of three planes, one through each camera center, where the plane through the* i*-th camera center is:*

$$\begin{aligned}\Pi_i &= \mathrm{span}(g^j_{P_i}, g^k_{P_i}) \\ &= \mathrm{span}(P_i^{-1}(e^j_{Q_i}), P_i^{-1}(e^k_{Q_i})) \\ &= \mathrm{span}(P_i^{-1}(Q_i(q_j)), P_i^{-1}(Q_i(q_k))\end{aligned}$$

Proof. By Lemma II.5.17, the ambiguous subvariety in $V(I)$ is a $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, the product of the three lines we get by projecting the plane spanned by q_1, q_2, q_3 onto each image. In the i-th image, this line is the line spanned by the two epipoles $e^j_{Q_i}$ and $e^k_{Q_i}$, whose pre-image (under P_i) is the strict transform of the plane spanned by the two lines $g^j_{P_i}$ and $g^k_{P_i}$ (recall Remark II.4.6). Hence, the ambiguous subvariety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ pulls back to the intersection of the strict transforms of three planes in $\widetilde{\mathbb{P}^3}$. ■

Lemma II.5.19. *Let* P_1*,* P_2*,* P_3*, and* Q_1*,* Q_2*,* Q_3 *be two triples of cameras, such that the camera centers* q_i *are not collinear. Then the ambiguous points on the* P*-side make up one of the following varieties:*

- *The strict transform of the plane spanned by the three camera centers* p_i*.*
- *The strict transform of a line.*

- *A single point.*

In all cases, the variety they form is contained in $\widetilde{S_P^{12}} \cap \widetilde{S_P^{13}} \cap \widetilde{S_P^{23}}$.

Proof. By Lemma II.5.18, the ambiguous points are the intersection of the strict transform of three planes, one through each camera center, so they form either a point, a line, or a plane. In the event of a plane, it must be the one spanned by the camera centers. Furthermore, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a component of $V(I)$, so its pullback is contained in the pullback of $V(I)$, which is the intersection $\widetilde{S_P^{12}} \cap \widetilde{S_P^{13}} \cap \widetilde{S_P^{23}}$. ■

Theorem II.5.20 ([HK07, Theorem 6.17])**.** *Let* P_1, P_2, P_3 *be a triple of cameras, let* S_P^{12}, S_P^{13}, S_P^{23} *be a compatible (in the non-collinear sense) triple of quadrics, and let* X *be the intersection of their strict transforms. The compatible triple of quadrics comes with six permissible lines, spanning three planes. Let* x *be the intersection of the strict transforms of these three planes. Then the cameras* P_i *along with the points in* $\overline{X - x}$ *constitute a critical configuration.*

Proof. Since the three quadrics S_P^{ij} are compatible, there exist three non-collinear cameras Q_1, Q_2, Q_3 giving rise to these quadrics. Now by Lemma II.5.19, this means that all of X, with the possible exception of a point or the strict transform of a line or plane, lies in the set of critical points. This point/line/plane is exactly the intersection of the three planes spanned by the six lines. Hence the set $X - x$ lies in the set of critical points. By continuity, this is also the case for $\overline{X - x}$, meaning that the cameras P_i along with the points in $\overline{X - x}$ constitute a critical configuration. ■

Remark II.5.21. In the case where x is the strict transform a plane, it might happen that the configuration $(P_1, P_2, P_3, \overline{X - x})$ is not a maximal critical configuration, as the intersection of the critical and ambiguous points need do not lie in the closure of $X - x$. For instance, in the case where the three quadrics are all equal to the union of two planes, x is the strict transform of the plane containing the camera centers. $(P_1, P_2, P_3, \overline{X - x})$ is not a maximal critical configuration in this case, as there is a conic curve in x which also lies in the set of critical points (see Proposition II.6.6).

Theorem II.5.20 tells us all we need to classify the (non-collinear) critical configurations for three views. We need only check in what ways three compatible quadrics can intersect, and then remove the strict transform of a point/line/plane if needed. In particular, if the intersection $S_P^{12} \cap S_P^{13} \cap S_P^{23}$ does not contain a plane, a line, or any isolated points, the whole intersection is part of a critical configuration.

II.5.2.2 The collinear case

We now consider the other case, where (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) are two triples of cameras, such that the camera centers q_1, q_2, q_3 lie on a line. In this

case,

$$V(I) = V(F_Q^{12}, F_Q^{13}, F_Q^{23}) \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$$

is still the union of the same two subvarieties: the multi-view variety, and the ambiguous one. In this case, however, the latter is not a $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ but rather all of $V(I)$, containing the multi-view variety as a divisor. Indeed, whenever the lines l'_{Q_i} intersect in a point, the three lines are coplanar, since the camera centers q_i lie on a line.

Since the set of critical points is contained in the ambiguous points, it is not enough to simply remove a point, line, or plane. It follows that Theorem II.5.20 does not hold in the case where the cameras on the other side are collinear. Unlike the non-collinear case, the permissible lines on the quadrics S_P^{ij} are not enough to determine the critical points. We do, however, have the following result, which holds for both the collinear and non-collinear cases:

Proposition II.5.22. *The intersection of the multi-view variety with the ambiguous points forms a codimension 1 subvariety of the latter.*

Proof. In the family of triples of coplanar lines, requiring the third line to pass through the intersection of the first two is (generally) a linear constraint. Hence the triples of lines all having at least one point in common form a codimension 1 subvariety of this family. ■

Hence, the ambiguous points in $S_P^{12} \cap S_P^{13} \cap S_P^{23}$ contain a subvariety that also lies in the set of critical points. This subvariety has codimension at most 1 (exactly 1 if we assume that the ambiguous points are not contained in the set of critical points).

II.6 Critical configurations for three views

The goal of this section is to provide proof of Theorem II.1.1. By Corollary II.5.3, and the discussion in Section II.5.1, any critical configuration for three views lies on the intersection of three compatible quadrics. Furthermore, any such intersection is critical, with the possible exception of some number of ambiguous points (usually a point, line, or plane). As such, our approach is to consider all possible intersections of three quadrics, and for each case check whether it exists as the intersection of a compatible triple and if so, remove any potential ambiguous points. This will give us all critical configurations for three views. We will start with the cases where the quadrics intersect in a surface (Section II.6.1), then a curve (Section II.6.2), and finally in a finite number of points (Section II.6.3).

Remark II.6.1. While the intersection of quadrics may contain non-reduced components, the definition of critical configuration is a set-theoretical one. This means that the multiplicity of components does not affect criticality. As such, we consider all components to be reduced.

Remark II.6.2. While the critical configurations for two views tend to have a finite number of conjugates, this is generally not the case for three views. In fact, the only critical configurations for three views that have a finite number of conjugates are the ones consisting of 7 or fewer points. The rest all have an infinite number of conjugates (although the dimension of the family of conjugates may vary). The proofs will usually revolve around constructing a single conjugate, but the choices made in the construction should make it clear that many conjugates exist when different choices are made.

II.6.1 Quadrics intersecting in a surface

In the two-view case, the points in a maximal critical configuration always form a surface. We begin by checking whether any such configurations remain critical in the three-view case. In particular, we want to examine whether there exist any critical configurations where the points form a surface. There are two ways three quadrics can intersect in a surface, either by all being equal or by all being reducible and sharing a common plane.

II.6.1.1 The smooth quadric/cone

We start with the case where the quadrics S_P^{ij} are all the same smooth quadric or cone (denoted simply S_P). In this case, it is impossible to find 6 lines satisfying the conditions in Theorem II.5.12, so one cannot find a non-collinear conjugate triple of cameras Q_1, Q_2, Q_3 such that $\widetilde{S_P}$ is the pullback of all three fundamental forms F_Q^{ij}. However, as long as S_P is not one of the following configurations (illustrated in Figure II.3)

- S_P is smooth and there is a camera center such that the two lines passing through it, each pass through one other camera center.
- S_P is a cone and two camera centers lie on the same line, but neither of the two lies at the vertex

then three copies of S_P satisfy the conditions of Proposition II.5.11, and hence form a compatible triple, coming from a triple of collinear cameras Q_i.

Proposition II.6.3. *Let P_1, P_2, P_3 be three cameras, and let $S_P^{12}, S_P^{13}, S_P^{23}$ be a compatible triple of quadrics. If the three quadrics are all equal to the same irreducible quadric S_P, the set of critical points X is the strict transform of a rational curve of degree at most four passing through all camera centers. The conjugate configuration is the strict transform of a curve of one degree less, not passing through any camera centers (plus the strict transform of the line passing through the camera centers, if S_P is a cone).*

Remark II.6.4. Note that the two cases in Figure II.3 are exactly the two where S_P does not contain a rational curve passing through all three camera centers

Proof. Since the three cameras Q_i on the other side have to be collinear, our critical configuration does not consist of all the points on the strict transform

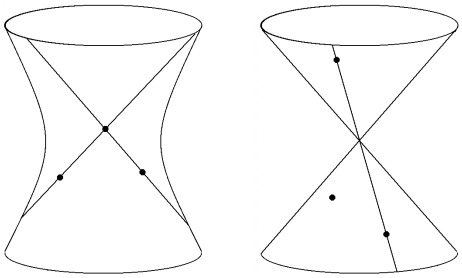


Figure II.3: If the camera centers are positioned as shown, the three quadrics (all three equal) do not constitute a compatible triple.

of S_P. By Proposition II.5.22, the critical points on $\widetilde{S_P}$ form a codimension 1 subvariety, the strict transform of a curve $C_P \subset S_P$.

The three cameras Q_i lie on a line, so by Proposition II.5.6, the ideal of their image in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ is generated by three bilinear and three trilinear forms $(F_Q^{12}, F_Q^{13}, F_Q^{23}, T_1, T_2, T_3)$. The fundamental forms pull back to quadrics S_P^{ij} through two camera centers, while the trilinear forms pull back to cubic surfaces through all three camera centers. Hence the curve C_P is the intersection of S_P with three cubic surfaces, all containing the three camera centers. It follows that C_P is of type (a, b, c_1, c_2) (recall Definition II.4.13) with $a, b \leq 3$ and $c_1, c_2 \geq 1$.

The curve C_P lies on all three quadrics S_P^{ij} and its type (a, b, c_i, c_j) depends on which quadric we consider since it is defined by the choice of permissible lines and the number of times it intersects two of the three camera centers. The permissible lines lie in the same family on all three quadrics, so a and b are independent of the choice of quadric. Furthermore, the total degree of the conjugate curve C_Q depends (by Proposition II.4.14) on the multiplicity of C_P in the camera centers. Since the curve C_Q is the same regardless of which quadric we consider, it follows that C_P needs to have the same multiplicity in all three camera centers, the same is true for C_Q.

If we assume that the three camera centers p_i are not collinear, the three quadrics S_Q^{ij} on the other side cannot be the same quadric. It follows that the conjugate curve $C_Q \subset S_Q^{12} \cap S_Q^{13} \cap S_Q^{23}$ is of type (a', b', c_1', c_2') with $a', b' \leq 2$. If, on the other hand, the camera centers p_i are collinear, the resulting curve is still the intersection of quadrics and cubics, and so needs to satisfy $a', b' \leq 3$.

Furthermore, if S_P is smooth, the curve C_Q cannot pass through any of the three camera centers q_i, since this would imply that all three quadrics S_Q contain the line spanned by the camera centers, which (by Proposition II.4.10) violates our condition that S_P is smooth.

On the other hand, if S_P is a cone, then the three quadrics S_Q^{ij} must (by Proposition II.4.10) contain the line l_Q spanned by the camera centers. However, the whole line l_Q is conjugate to a single point, namely the vertex of S_P (by Proposition II.4.12). So in the case where S_P is a cone, C_Q is the union of l_Q and some other curve.

This leaves only a finite number of options for what kind of curve C_P can be. Finally, the curves should satisfy the result in Proposition II.4.14: that if C_P is of type (a, b, c_1, c_2), then C_Q is of type $(a, a+b-c_1-c_2, a-c_2, a-c_1)$. We can now check every possible type for C_P and remove all cases where a' or b' are either negative or greater than 3, the cases where c_i' is negative, and the cases where the multiplicity in the camera centers is greater than what the degree of the curve allows (for instance, a curve of degree 1 can not have multiplicity 2 in a camera center).

This leaves only three options, listed in Table II.2. Note that if S_P is a cone, then the curve on the other side will also include the line l_Q in addition to the component given in the table. ■

Curve on S_P	Curve on S_Q
(1,3,1,1)	(1,2,0,0)
(1,2,1,1)	(1,1,0,0)
(1,1,1,1)	(1,0,0,0)

Table II.2: Recall that a curve of type (a, b, c_1, c_2) intersects the family containing the permissible lines a times, the lines in the other family b times, and passes c_i times through the camera center p_i.

Remark II.6.5. This proves that any critical configuration where the three quadrics on one side coincide has to be one of the types in Table II.2. In Section II.6.2 we prove the converse, that any configuration of such type is critical. The general case is the rational quartic on one side, and a twisted cubic on the other, while the two other configurations appear as degenerates of this one.

II.6.1.2 Two planes

Proposition II.6.6. *When the three quadrics S_P^{ij} are all equal to the same two planes, the set of critical points is the union of a plane and the strict transform of a conic passing through all three camera centers. Conversely, any configuration where the points lie on the union of a plane and the strict transform of a conic curve passing through the camera centers is a critical configuration. Its conjugate is of the same type (plane + conic curve).*

Proof. Let the three quadrics S_P^{ij} all be equal to the same two planes S_P, Theorem II.4.5 states that all three cameras need to lie in the same plane, with any number of them lying on the intersection of both. In this case, unlike the previous one, there exist six lines satisfying the conditions in Theorem II.5.12 as long as not all three cameras lie on the intersection of the two planes. An example is given in Figure II.4. Hence, the three cameras Q_i on the other side can be chosen to not be collinear.

Since the three cameras Q_i on the other side can be assumed to not be collinear, Theorem II.5.20 proves that the plane not containing all camera centers is part of the set of critical points. Furthermore, Proposition II.5.22 proves that there is a curve C_P in the plane with the camera centers whose strict transform is also part of this set. By Proposition II.5.6, the set of critical points is the strict transform of the intersection of the two planes with a cubic surface passing through the three camera centers, so the curve C_P is a conic passing through the three camera centers.

For the converse, we want to determine the conic curve C whose strict transform lies in the set of critical points. We already know that C_P passes through the three camera centers. Furthermore, the intersection of the two permissible lines $g^j_{P_i}$ and $g^j_{P_k}$ (denote it by a_j) must also lie in the set of critical points since it has a conjugate point lying on the exceptional divisor E_{q_j}. This gives us three more points on the conic. Usually, there is no conic passing through six given points. In this case, however, the six points $p_1, p_2, p_3, a_1, a_2, a_3$ along with the six lines $g^j_{P_i}$ span a hexagon whose opposite sides ($g^j_{P_i}$ and $g^i_{P_j}$) meet at three points which lie on a line (the intersection of the two planes) so by the Braikenridge–Maclaurin theorem [CG67, p.76] the six points $p_1, p_2, p_3, a_1, a_2, a_3$ lie on a conic.

Finally, for any conic C_P through p_1, p_2, p_3 one may choose the six lines $g^j_{P_i}$ in such a way that the three points a_i lie on C_P *and* each pair of lines form a permissible pair. This can be done by picking a_1 to be any[1] point on C_P and then taking a_2 to be the intersection of $g^2_{P_1}$ with C_P. Since we can get any conic by making the appropriate choice of permissible lines, this proves that any plane + conic configuration is critical. ■

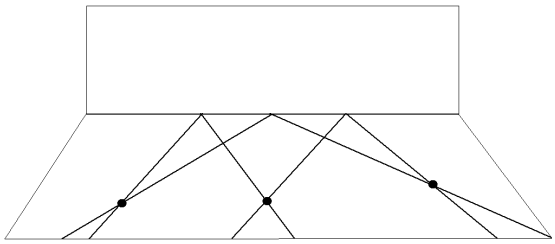


Figure II.4: One example of six lines satisfying the conditions in Theorem II.5.12

II.6.1.3 Single plane

With this, there is only one way left that the three quadrics S^{ij}_P can intersect in a surface, namely if they are all reducible, and contain only one common plane. In this case, the set of critical points is at most a plane plus a line. Hence, this

[1] not equal to a camera center and not lying in the intersection of the two planes

is not a maximal critical configuration, as it is always a subconfiguration of the plane + conic configuration in Proposition II.6.6.

II.6.2 Quadrics intersecting in a curve

Curves that appear as the intersection of quadrics have at most degree 4. Moreover, any curve that lies on the union of a plane and a conic passing through the camera centers is critical by Proposition II.6.6, which further reduces the number of curves we need to check. We analyze all curves appearing as the intersection of quadrics sorted by the degree of their highest degree component. In Section II.6.2.1, we analyze the curves where the highest degree component is of degree 4, covering the elliptic quartic as well as the singular rational quartics. Section II.6.2.2 covers the curves containing a twisted cubic, while Section II.6.2.3 covers the curves containing no components of degree higher than 2.

Lemma II.6.7. *Let S_P^{ij} be a triple of quadrics in $\mathbb{P}^3$ intersecting in a curve C_P. Let $\widetilde{S_P^{ij}}$ be the surface we get in $\mathrm{Bl}_{\boldsymbol{P}}(\mathbb{P}^3)$ when taking the strict transform under the blow-up in p_i, p_j and the total transform under the final blow-up. Then the intersection $\widetilde{S_P^{12}} \cap \widetilde{S_P^{13}} \cap \widetilde{S_P^{23}}$ is the strict transform of C_P.*

Proof. The statement is trivial when C_P does not pass through a camera center. If C_P does pass through a camera center, any tangent to C_P in this camera center must also be a tangent to each of the quadrics S_P^{ij}, it follows that the intersection of the $\widetilde{S_P^{ij}}$ is the strict transform of C_P. ∎

While it is not true in general that the intersection of strict transforms is the strict transform of the intersections, Lemma II.6.7 tells us that this is the case for three quadrics intersecting in a curve. This allows us to work in $\mathbb{P}^3$, considering all curves appearing as intersections of three quadrics. The critical configurations will then consist of points lying on the strict transforms of whichever curves we have in $\mathbb{P}^3$.

For most curves, the main issue is to find a compatible (in the non-collinear sense) triple of quadrics containing it. The issue of having to remove a point, line, or plane (as per Theorem II.5.20) is less relevant since the curves will usually not contain any such component. To find compatible triples of quadrics, the following lemmas are useful:

Lemma II.6.8. *Let P_1, P_2, P_3 be a triple of cameras, and let S_P^{12}, S_P^{13}, S_P^{23} be a triple of critical quadrics, each passing through two camera centers (indicated by the superscript). Then the three quadrics are compatible if*

A) *The intersection of S_P^{ij} and S_P^{ik} does not contain a line passing through p_i.*

B) *There exists a point x lying on all three quadrics, as well as three distinct lines l_{12}, l_{13}, l_{23} passing through x, satisfying*

1. $l_{ij} \subset S_P^{ij}$.

2. $p_i \in \text{span}(l_{ij}, l_{ik})$.
3. $l_{ij} \notin \text{span}(l_{ik}, l_{jk})$.
4. $p_i, p_j \notin l_{ij}$.

Proof. Given three quadrics S_P^{ij}, assume there exists a point x and three lines l_{ij} satisfying the conditions in the lemma above. For each quadric S_P^{ij}, pick two lines $g_{P_i}^j$ and $g_{P_j}^i$ such that they both intersect l_{ij}, and form a permissible pair. The only case where such a choice is not possible is if the quadric S_P^{ij} contains the line spanned by p_i and p_j, and the line l_{ij} intersects this line, but this would violate either condition 2, 3 or 4.

This gives us six lines g_P, two on each quadric, and two through each camera center. Condition A) ensures that the two lines through each camera center are distinct, so they span a plane. By construction, the plane spanned by l_{ij} and l_{ik} is the same as the one spanned by $g_{P_i}^j$ and $g_{P_i}^k$. This means that the lines g_P satisfy the conditions in Theorem II.5.12. It follows that the three quadrics are compatible. ■

Lemma II.6.9. *Let P_1, P_2, P_3 be a triple of cameras, and let S_P^{12}, S_P^{13}, S_P^{23} be a triple of critical quadrics. If there exists a line L lying on all three quadrics, but not passing through the camera centers such that the intersection of S_P^{ij} and S_P^{ik} does not contain a line passing through p_i and intersecting L, then the quadrics form a compatible triple.*

Proof. On each quadric S_P^{ij}, take $g_{P_i}^j$ to be the line through p_i intersecting L. By the same arguments as above, this gives us 6 lines satisfying Theorem II.5.12. ■

Remark II.6.10. In the two lemmas above, the lines l_{ij} and L all intersect the permissible lines, meaning they are of type $(1, 0, 0, 0)$ on their respective quadrics.

Remark II.6.11. The two lemmas above have the condition that the intersection of two quadrics does not contain a line passing through their common camera center. This is to ensure that the two lines $g_{P_i}^j$ and $g_{P_i}^k$ span a plane, which is a necessary condition for the quadrics to be compatible (in the non-collinear sense). This is a weak condition, and in the following sections, whenever we apply either of the two lemmas above, the cases where two lines through a camera center coincide can easily be avoided, if such cases exist at all. To keep the arguments relatively short, the verification of this fact is left to the reader.

II.6.2.1 Quartic curves

We begin with the case where the three quadrics all lie in the same pencil, that is, where the intersection of all three is equal to the intersection of any two. In this case, the intersection is a quartic curve, passing through at least two of the camera centers. If the curve is irreducible, it is either a smooth elliptic quartic curve or a singular rational quartic (cusp or node). The cases where the curve is reducible are covered in Sections II.6.2.2 and II.6.2.3.

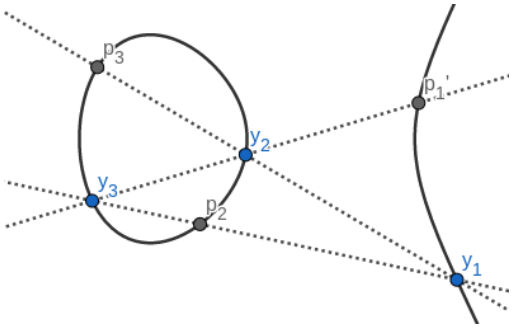


Figure II.5: Given three points p_1', p_2', p_3' on a smooth cubic plane curve, there exist three lines forming a triangle with vertices y_1, y_2, y_3 such that each line contains exactly one point.

Proposition II.6.12. *Any configuration consisting of three cameras along with a set of points lying on the strict transform of an elliptic quartic curve passing through the camera centers is critical. The same is the case if the points lie on the strict transform of a singular quartic curve passing through the camera centers, as long as neither camera center lies on the singular point. In both cases, they have conjugates that are curves of the same type.*

Proof. A quartic curve (elliptic or singular) is contained in a pencil of quadrics. This leaves 1 degree of freedom when choosing quadrics. In particular, for each line secant to the curve, there is a unique quadric in the pencil that contains this line.

Denote the quartic curve by C, and let x be a smooth point on C not lying in the plane spanned by p_1, p_2, p_3. Projecting from the point x, we get a cubic curve C' in the plane, along with three distinct points p_1', p_2', p_3' on the curve (the projection of the camera centers).

Assume there exist three real points y_1, y_2, y_3 on this curve, different from the p_i', such that p_i' lies on the line $\overline{y_j y_k}$ for $i \neq j \neq k$ (see Figure II.5).

The preimages of the three points y_i are three lines through x, secant to C. Denote the preimage of y_i by l_{jk}, and let S_P^{ij} be the unique quadric containing C and the secant l_{ij}. Then the point x along with the three lines l_{12}, l_{13}, l_{23} satisfy the conditions in Lemma II.6.8, so the three quadrics $S_P^{12}, S_P^{13}, S_P^{23}$ constitute a compatible triple. Since the curve C is the intersection of three compatible quadrics, the three cameras along with the strict transform of C constitute a critical configuration.

Now we need only prove that three such points y_i exist. Given the three points p_i' on C', define a map

$$f : C' \dashrightarrow C'$$

as follows: for each smooth point a, the lines $\overline{p_1' a}$ and $\overline{p_2' a}$ intersect the cubic in two new points, which we denote by b_1 and b_2 respectively. The line $\overline{b_1 b_2}$

intersects C in a third point, let this point be $f(a)$ (see Figure II.6). This construction yields a rational map that extends to a morphism by considering $\overline{p_i' a}$ to be the tangent line to C in p_i' if $a = p_i'$ (this only gives a morphism in the smooth case, in the singular it remains undefined in the singular point).

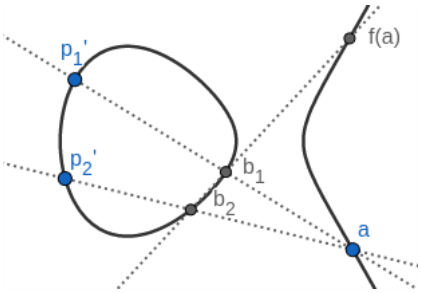


Figure II.6: Illustration of the map f on a smooth cubic curve

If there exists a point a such that $f(a) = p_3'$, we can take y_1, y_2, y_3 to be the points a, b_1, b_2, since these satisfy the conditions above. The map f can be expressed as

$$f(a) = p_1' + p_2' + 2a$$

where "+" is the sum under the usual group law on an elliptic curve (and a similarly defined sum on the singular one). The fiber over p_3' consists of all points a satisfying

$$2a = p_3' - p_1' - p_2' \tag{II.7}$$

The point a is determined up to 2-torsion, so in the smooth case, the map f is $4:1$, in the singular case the map is $2:1$ and $1:1$ if the curve is a node or cusp respectively. This means there exists four, two, or one suitable triples y_1, y_2, y_3 respectively. Still, we need to show that at least one is real and satisfying $y_i \neq p_j'$.

In the elliptic quartic case, the only case where we have no real solutions for (II.7) is if C' consists of two connected components and the point $p_3' - p_1' - p_2'$ lies on the component not containing the origin (only in this case do no real tangents pass through $p_3' - p_1' - p_2'$). Since $p_3' - p_1' - p_2'$ lies on the component of C' that contains an odd number of cameras, we can ensure that there is a real solution by taking our projection center x to lie on whichever component of C that has an odd number of camera centers.

In the nodal case, it might also happen that neither of the two solutions is real. Like in the smooth case, this can be avoided by choosing the initial projection center on the quartic curve to lie on the side of the singularity that contains an odd number of cameras. In the cuspidal case, the point a is uniquely determined and necessarily real since complex solutions would appear as conjugate pairs.

This ensures that there exists a real solution. Next, given a solution to (II.7), we take

$$\begin{aligned} y_1 &= p_2' - a, \\ y_2 &= p_1' - a, \\ y_3 &= a, \end{aligned}$$

so the only way $y_i = p_j'$ is a solution is if $3p_j' - p_i' + p_k' = 0$. $y_i = p_i'$ is only a solution if $p_i' + p_j' + p_k' = 0$. There is a finite number of choices of x where this is the case, so we can again avoid the issue by picking an appropriate x.

In all these cases, the curve C (a quadric passing through all camera centers) is of type $(2, 2, 1, 1)$, so by Proposition II.4.14, its conjugate is also of type $(2, 2, 1, 1)$. ■

Remark II.6.13. The proof above shows that nodal or cuspidal quartic curves are critical by constructing a conjugate configuration that is of the same type, nodal or cuspidal curves respectively. The nodal and cuspidal quartic curves also have another family of conjugates (the elliptic ones do not). Unlike the elliptic quartic case, the singular curves are rational, this means they appear as critical configurations when the three quadrics S_P^{ij} coincide as a cone (they are of type (1,3,1,1) in Table II.2). This is proven in Proposition II.6.18, the conjugates are twisted cubics with a secant or tangent line respectively.

In the case where the elliptic quartic only passes through two of the camera centers, the curve is of type $(2, 2, 1, 0)$ on two of the quadrics and $(2, 2, 1, 1)$ on the last. If such a configuration was critical, the conjugate curve would have to be of type $(2, 3, 2, 1)$ on two of the quadrics and $(2, 2, 1, 1)$ on the last, but a curve of degree 5 does not appear as the intersection of quadrics, so this can not be a critical configuration. A similar problem appears if the curve is singular and one of the camera centers lies on the singular point

II.6.2.2 The twisted cubic

Having covered the quartic curves, we move one degree lower, to the cubics. Plane cubics do not appear as the intersection of quadrics, so the only irreducible cubic curve that might be critical is the twisted cubic curve. Unlike the case with the elliptic quartic, we no longer require all camera centers to lie on the curve. We consider the possible configurations of cameras one by one.

Proposition II.6.14. *Any configuration consisting of three cameras along with a set of points all lying on the strict transform of a twisted cubic passing through the camera centers is critical. The conjugate consists of points on the strict transform of a conic curve not passing through the camera centers.*

Proof. Given a twisted cubic, there is a three-dimensional family of quadrics containing it. This leaves us 2 degrees of freedom for each of the quadrics. In other words, for each line intersecting the curve, there is a unique quadric

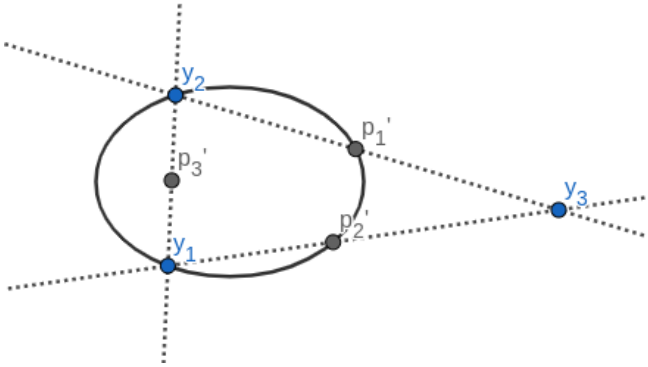


Figure II.7: Given three points p_1', p_2', p_3', and a smooth conic curve passing through two of them, there exists a triangle with vertices y_1, y_2, y_3 such that each line contains exactly one point and the points y_1 and y_2 lie on the conic.

containing both the cubic and the line (unless the line is a secant, in which case a whole pencil of quadrics contains it).

Take a line L, secant to C, and not passing through the camera centers. There exists a pencil of quadrics containing both C and L. By Lemma II.6.9, any three distinct quadrics in this pencil form a compatible triple of quadrics. The line L has bidegree $(1,0)$ and is a secant to C, so it intersects C twice. It follows that the twisted cubic is of type $(1,2,1,1)$, so the conjugate is of type $(1,1,0,0)$, a conic not passing through the camera centers. ■

Proposition II.6.15. *Any configuration consisting of three non-collinear cameras along with a set of points lying on the strict transform of a twisted cubic passing through two of the camera centers is critical. The conjugate is a curve of the same type.*

Proof. Denote the three camera centers by p_1, p_2, p_3, and the twisted cubic by C. We assume, without loss of generality, that the camera center not lying on the twisted cubic is p_3.

Since we need the quadrics to contain C, this gives us 2 degrees of freedom for each of the quadrics. However, two of the quadrics, S_P^{13} and S_P^{23}, also need to contain p_3, leaving us with only 1 degree of freedom.

Let x be a point on C not lying in the plane spanned by p_1, p_2, p_3, and not lying on the secant line passing through p_3. Projecting from the point x, we get a smooth conic curve C' in the plane, along with three distinct points p_1', p_2', p_3' where p_3' does not lie on the curve (illustrated in Figure II.7).

We want to find three points y_i such that p_i' lies on $\overline{y_j y_k}$. Furthermore, since two of the quadrics only have 1 degree of freedom, we need y_1 and y_2 to lie on C'. Many choices satisfying this is possible, one for each line through p_3' (one is illustrated in Figure II.7). Again, the preimages of the three points y_i are three lines through x. Denote the preimage of y_i by l_{jk}, and let S_P^{ij} be the

unique quadric containing C and the line l_{ij} (and in the case of S_P^{13} and S_P^{23}, also containing p_3).

Then the point x, along with the three lines l_{ij} satisfies the conditions in Lemma II.6.8, so the three quadrics $S_P^{12}, S_P^{13}, S_P^{23}$ constitute a compatible triple. Hence the configuration is critical.

By construction of the quadrics, the twisted cubic C is of type $(1, 2, 1, 0)$ on the quadrics S_P^{13} and S_P^{23}, while it is of type $(2, 1, 1, 1)$ on S_P^{12}. The conjugate configuration is of the same type, a twisted cubic passing through exactly two of the camera centers, of type $(1, 2, 1, 0)$ on two of the quadrics, and type $(2, 1, 1, 1)$ on the third. ■

If we take a twisted cubic passing through only one camera center, we end up with only 1 degree of freedom for two of the quadrics and 0 for the last. Moreover, a twisted cubic not passing through any camera center fixes all three quadrics. With so few degrees of freedom, one can typically not find a compatible triple of quadrics, so neither of these configurations is critical−that is, unless the three camera centers happen to be collinear. To prove this, we first need the following (seemingly unrelated) lemma:

Lemma II.6.16. *Let $C \in \mathbb{P}^3$ be a twisted cubic, let l be a line in $\mathbb{P}^3$, not secant to C, and let p be a fixed point on l not lying on the twisted cubic. There exists a point $x \in C$ such that for every point $q \in l$, such that $q \neq p$ and $q \notin C$, the unique quadric surface containing C, p, and q, also contains the line $\overline{xq}$.*

Proof. Let s denote the unique secant line of C passing through p. The plane spanned by s and l intersects C in 3 points. Two of these points lie on s, the last one does not. We denote this one by x.

In the pencil of quadrics containing C and p, each quadric also contains the secant s. For any point q as above, the quadric containing C, p, and q, intersects the line $\overline{xq}$ in three points, x, q, and a third point lying on s. Since it contains three points on $\overline{xq}$, it contains the whole line. ■

Proposition II.6.17. *Any configuration consisting of three collinear cameras along with a set of points lying on the strict transform of a twisted cubic not intersecting the line with the camera centers is critical. The conjugate is a rational quartic curve containing the three camera centers.*

Proof. Let C denote the twisted cubic, let p_i denote the camera centers, and let L denote the line spanned by the camera centers.

Each quadric needs to contain C, as well as two camera centers, which means all three quadrics are fixed. However, we show that even without any freedom to choose the quadrics, the triple is still compatible.

Let S_P^{ij} be the unique quadric containing C, p_i, and p_j. If S_P^{ij} is smooth there are two lines through p_i. One is a secant to C, the other line intersects C exactly once; denote the latter line by $g_{P_i}^j$. If S_P^{ij} is a cone, there is only one line through p_i, in this case, we take this one to be $g_{P_i}^j$. Repeating this process for

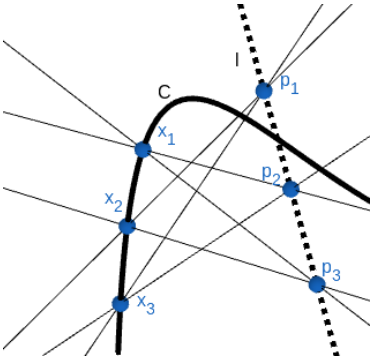


Figure II.8: The twisted cubic C and the line l. The line $g^j_{p_i}$ is the line spanned by p_i and x_j.

each pair of cameras on each of the three quadrics, we get 6 lines, two on each quadric, and two through each camera center.

By Lemma II.6.16, the lines $g^k_{P_i}$ and $g^k_{P_j}$ intersect C at the same point, which we denote by x_k (see Figure II.8). For now, assume that the two lines $g^j_{p_i}$ and $g^k_{p_i}$ through each p_i are distinct, and denote their spanned plane by Π_i.

The plane Π_1 intersects S^{12}_P in two lines. The first line is $g^2_{p_1}$, whereas the second is the unique line on S^{12}_P that is secant to C *and* passes through x_3. Denote the second line by l_{12}. The plane Π_2 intersects S^{12}_P in two lines as well, the first being $g^1_{p_2}$, and the latter being l_{12}. Repeating this construction with the two other quadrics, we get two other lines l_{13} and l_{23}. The three lines l_{ij} along with their point of intersection x satisfy the conditions for Lemma II.6.8, so the quadrics $S^{12}_P, S^{13}_P, S^{23}_P$ do indeed form a compatible triple.

Since the twisted cubic is of type $(1,2,0,0)$, its conjugate is of type $(1,3,1,1)$, a rational quartic curve passing through all three camera centers. This is one of the curves from Proposition II.6.3, appearing when all the quadrics S^{ij}_Q on the other side coincide.

In the construction above, the pair of lines $g^j_{P_i}$ through each camera center may coincide. In this case, the three points x_i also coincide, and the three quadrics S^{ij}_P are all cones. By Proposition II.5.11 the quadrics are compatible in this case also, and the camera centers on the rational quartic on the other side are collinear. ■

In the case where the line containing the cameras intersects the twisted cubic, but is not a secant or tangent, the three quadrics S^{ij}_P are all equal. In this case, the only way the lines $g^j_{P_i}$ can be permissible is if they are all secant to the twisted cubic, but in this case, the twisted cubic is of type $(2,1,0,0)$ (or type $(3,1,1,1)$ if you include the line) and neither of these appear as critical configurations when the three quadrics intersect. Hence no such critical configurations exist.

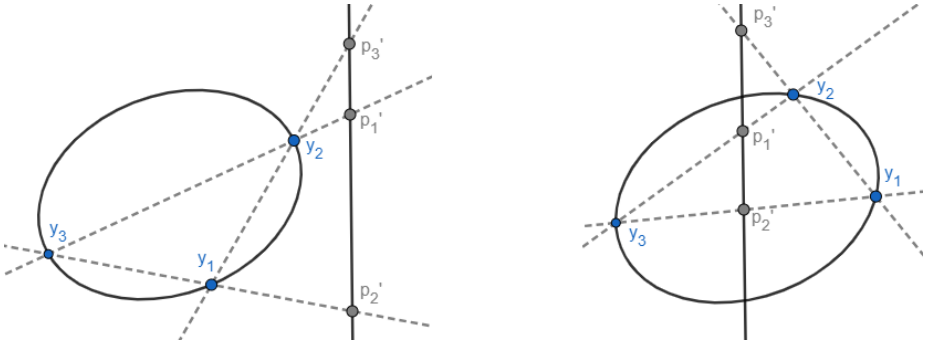


Figure II.9: By our choice of x, zero (left) or two (right) camera centers will lie "inside" the conic, in both cases there exists a triangle y_1, y_2, y_3 with vertices on the conic such that p_i' lies on $\overline{y_j y_k}$.

Proposition II.6.18. *Any configuration consisting of three collinear cameras along with a set of points lying on the strict transform of a twisted cubic having the line with the camera centers as a secant or tangent is critical. The conjugate is the strict transform of a singular rational quartic curve passing through the three camera centers, a nodal curve if the line with the cameras is secant, and a cuspidal curve if the line is tangent.*

Proof. Let C denote the twisted cubic, and let L denote the line spanned by the camera centers. Since the line L containing the cameras is a secant (or tangent) to C, any quadric containing C and two cameras contains all of L. The union of a twisted cubic and a secant (or tangent) line is contained in a pencil of quadrics, leaving us 1 degree of freedom when choosing S_P^{ij}. In particular, for each line secant to $C \cup L$, there is a unique quadric containing $C \cup L$ and the secant.

Let $x \in C$ be a point not lying on L, chosen in the following way:

- If L intersects C in two real, distinct points, these two points cut both C and L into two connected parts, one that intersects the plane at infinity, and one that does not. One part contains an even number of cameras, let x lie on the corresponding part of C.
- If L is a tangent to C, or if the two intersection points are not real, let x be any point on C not lying on L.

Projecting from x, we get a conic curve C' in the plane, along with a line L' containing the projections p_1', p_2', p_3' of the camera centers.

We can choose coordinates in such a way that C' is an ellipse. By choosing x as above, we have ensured that among p_1', p_2', p_3', an even number lie "inside" C'. Without loss of generality, we can assume that p_3' lies "outside" C' (see Figure II.9). Let y_3 be a point on C', the line spanned by y_3 and p_1' intersects C' in a second point, we denote this by y_2. Similarly, we denote by y_1 the second point where the line spanned by y_3 and p_1' intersects C'. As y_3 moves along the

conic, the line $\overline{y_1 y_2}$ hits every point on L' "outside" C' twice, fix y_3 to be one of the two points where $\overline{y_1 y_2}$ passes through p'_3(see Figure II.9).

The preimages of the three points y_i are three lines through x. Denote the preimage of y_i by l_{jk}, and let S_P^{ij} be the unique quadric containing $C \cup L$ and the line l_{ij}. Then the point x, along with the three lines l_{ij} satisfies the conditions in Lemma II.6.8, so the three quadrics $S_P^{12}, S_P^{13}, S_P^{23}$ constitute a compatible triple. Hence the configuration is critical.

The three quadrics S_Q^{ij} on the other side are all equal to the same cone. The twisted cubic is of type $(1, 2, 0, 0)$, so its conjugate is of type $(1, 3, 1, 1)$, a rational quartic curve with a singularity in the vertex of the cone. These are the curves discussed in Proposition II.6.12. Note that the line L is also part of the critical locus, with its conjugate being the vertex of the cone on the other side. ■

II.6.2.3 Curves of lower degree

We are now left with curves appearing as the intersection of quadrics where all components are of degree at most 2. Most of these are subsets of the "plane + conic" configuration, and hence critical, having conjugates that are of the same type (see Proposition II.6.6). There are also some that do not appear as subsets of the "plane + conic" configuration, these are degeneration of the cases already discussed above and can be proven to be critical using proofs similar to the non-degenerate cases, these are:

- Union of two conics containing the camera centers, not all camera centers on one conic (degenerate elliptic quartic)
- Union of conic and two intersecting lines containing the camera centers, one or two cameras lying on the conic (degenerate elliptic quartic)
- Union of four lines containing the camera centers, each line intersecting exactly two others (degenerate elliptic quartic)
- Union of conic and a line intersecting the conic once, camera centers either collinear or at least two lying in the union (degenerate twisted cubic)
- Union of three lines, one line intersecting the other two, camera centers either collinear or at least two lying on the lines (degenerate twisted cubic)

Moreover, among the configurations that *are* subsets of the "plane + conic" configuration, some also appear as the intersection of non-reducible quadrics, these have other conjugates that are not of the same type. For the sake of completeness, we give a summary of them here. No proof is given, but the reader can verify that the curves lie on the intersection of three compatible irreducible quadrics. This task should be fairly straightforward since there are many degrees of freedom for the quadrics when the curves are of a low degree.

1. Two intersecting lines on one side, and two disjoint lines on the other. The number of cameras lying on the two lines is equal on both sides.

2. Cameras lying on a conic curve. The conjugate is a line not passing through the camera centers.

3. Two lines not passing through the camera centers. This one has two conjugates, the first is three cameras lying on the union of 3 lines (one intersecting the other two). The second conjugate consists of points lying on the union of a conic and a line, and the cameras lying on the conic. This triple of conjugates is a degeneration of the "twisted cubic $\leftrightarrow$ conic" configuration shown in Proposition II.6.14.

II.6.3 Quadrics intersecting in a finite number of points

Lastly, we have the case where three quadrics intersect in a finite number of points. This is the "general" case in the sense that with two general triples of cameras, the set of critical points on each side is finite. Indeed, on each side, we get a compatible triple of quadrics, whose intersection is 8 points (counted with multiplicity and including possible complex points), 7 of these points lie in the set of critical points, whereas the last one does not (recall Lemma II.5.19).

Conversely: given a set of seven points X and three cameras P_i, when is (P_1, P_2, P_3, X) a critical configuration? Since 9 points are enough to fix a quadric, there is generally only one triple of quadrics S_P^{ij} such that $p_i, p_j \in S_P^{ij}$ and $X \in \widetilde{S_P^{ij}}$ where $\widetilde{S_P^{ij}}$ is the strict transform of S_P^{ij} under the blow-up of p_i, p_j and the total transform under the final blow-up (in other words, $\widetilde{S_P^{ij}}$ belongs to the class $2H - E_i - E_j$). Note that this is generally the case even if one (or more) of the points in X happen to lie on an exceptional divisor over a camera center, say p_1. In this case, the quadric S_P^{12} is fixed by the constraints of having to contain $\pi(X)$, p_1, p_2 as well as having a certain line through p_1 as a tangent, likewise for S_P^{13}, the quadric S_P^{23} is fixed to be the one containing $\pi(X)$ and all three camera centers.

So generally the triple of quadrics will be fixed. Now IF this triple of quadrics is compatible, it comes with six permissible lines, spanning three planes. The three planes intersect in one of the 8 points lying on the intersection of the three quadrics. The configuration (P_1, P_2, P_3, X) is critical if and only if the triple of quadrics is compatible *and* the three planes do not intersect in one of the points in X (by Theorem II.5.20). This gives us the compatibility condition for 7 general points.

There exist certain positions of points and camera centers such that they do not fix all three quadrics. In this case, the configuration is still critical as long as there exists a suitable triple of quadrics, the points and camera centers do not need to fix the quadrics. The conditions are summarized in Proposition II.6.19.

Note that if S_P^{ij} is not fixed, this is because there exists a (possibly degenerate or reducible) elliptic quartic curve that passes through p_i, p_j and whose strict transform contains all of X. In the previous section we have shown that if the curve also passes through the final camera, the whole curve is part of a critical

configuration, so the seven points along with the cameras will be a critical (but not maximal) configuration in this case.

Proposition II.6.19. *Let P_1, P_2, P_3 be three cameras, and let $X \subset \mathrm{Bl}_{\boldsymbol{P}}(\mathbb{P}^3)$ be a set consisting of seven points. Then (P_1, P_2, P_3, X) is critical if and only if there exists three quadrics S_P^{ij} satisfying the following conditions:*

II.6.19.1. *S_P^{ij} contains the two camera centers p_i, p_j.*

II.6.19.2. *$X \subset \widetilde{S_P^{12}} \cap \widetilde{S_P^{13}} \cap \widetilde{S_P^{23}}$*

II.6.19.3. *$S_P^{12}, S_P^{13}, S_P^{23}$ is compatible.*

II.6.19.4. *The intersection of the three planes spanned by the six permissible lines $g_{P_i}^j$ does not lie in $\pi(X)$.*

Remark II.6.20. A sextuple of permissible lines $g_{P_i}^j$ only exists if the quadrics are compatible, so the first two conditions need to be fulfilled for the third to make sense.

For the sake of completeness, we examine what happens if X does not consist of exactly 7 points. For 8 or more points, a configuration is critical if and only if it is a subconfiguration of one of the critical configurations mentioned earlier in the section, i.e. if the points lie on a critical curve or surface. For 6 or fewer points, the configuration is always critical [MS98], but not maximal, since any configuration is contained in a critical configuration consisting of at least 7 points. In particular, the generic 6-point configuration is contained in exactly 3 critical 7-point configurations.

The critical configurations for three views are summarized in Figure II.1 on page 39. For illustration purposes, the blow-up is not shown, the actual configurations consist of points lying on the strict transform of the varieties depicted in Figure II.1.

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Paper III

Compatibility of Fundamental Matrices for Complete Viewing Graphs

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Code is available at https://github.com/mabraate/Compatibility-of-Fundamental-Matrices-for-Complete-Viewing-Graphs.

Abstract

This paper studies the problem of recovering cameras from a set of fundamental matrices. A set of fundamental matrices is said to be compatible if a set of cameras exists for which they are the fundamental matrices. We focus on the complete graph, where fundamental matrices for each pair of cameras are given. Previous work has established necessary and sufficient conditions for compatibility as rank and eigenvalue conditions on the n-view fundamental matrix obtained by concatenating the individual fundamental matrices. In this work, we show that the eigenvalue condition is redundant in the generic and collinear cases. We provide explicit homogeneous polynomials that describe necessary and sufficient conditions for compatibility in terms of the fundamental matrices and their epipoles. In this direction, we find that quadruple-wise compatibility is enough to ensure global compatibility for any number of cameras. We demonstrate that for four cameras, compatibility is generically described by triple-wise conditions and one additional equation involving all fundamental matrices.

Introduction

The problem of finding camera matrices that correspond to a given set of fundamental matrices is crucial in 3D reconstructions from 2D images. Typically, multiview structure-from-motion pipelines start by estimating fundamental matrices from point correspondences, with early methods for such estimations dating back to the 1990s and new methods still being developed today [RK18; TM97; Tor95; XZ96]. However, these methods usually only estimate a subset of

all possible fundamental matrices between cameras. To describe this incomplete set of fundamental matrices, viewing graphs are often used [LW03].

In this paper, we focus on understanding the conditions under which a reconstruction of n cameras can be obtained given complete knowledge of $\binom{n}{2}$ fundamental matrices, but we also give a result for general graphs at the end. Here, a *camera* refers to a full-rank 3×4 matrix, and the *fundamental matrix* of two cameras P_1 and P_2 with distinct kernels is a 3×3 rank-2 matrix that encodes all point correspondences between them. For any given rank-2 3×3 matrix F^{12}, there exists a pair of cameras P_1 and P_2 for which F^{12} is the fundamental matrix, this pair is unique up to global projective transformation. However, for a set of $\binom{n}{2}$ rank-2 3×3 matrices F^{ij}, where $n > 2$, it is not guaranteed that there exist cameras $P_1, \dots, P_n$ such that F^{ij} is the fundamental matrix of P_i and P_j for each i, j. Following the notation of [HZ04] we say that the set F^{ij} is *compatible* if such cameras do exist. Note that some recent literature uses the term *consistent* instead [Kas+19b].

Finding necessary and sufficient conditions for compatibility of fundamental matrices has practical applications as well as theoretical ones. [Kas+19b] proposes an algorithm for projective structure-from-motion that employs their necessary and sufficient condition for compatibility. The algorithm is designed to handle collections of measured fundamental matrices, both complete and partial, and aims to find camera matrices that minimize a global algebraic error for the given set of matrices. As for theoretical purposes, [Brå21; Brå24; HK07] uses necessary and sufficient conditions for compatibility to give a classification of critical configurations.

In the case of $n = 3$, a classical result [HZ04, Section 15.4] provides triple-wise constraints on F^{12}, F^{13}, F^{23} in terms of the fundamental matrices and their epipoles, where the i-th epipole in the j-th image is defined as $e^i_j := \ker F^{ij}$. For non-collinear cameras, [Kas+19b, Theorem 1] provides necessary and sufficient conditions for compatibility for any n. These conditions rely on the eigenvalues and rank of the n-view fundamental matrix, which is obtained by stacking all fundamental matrices into a $3n \times 3n$ matrix. In the follow-up work, [Gei+20, Theorem 2] arrives at a similar condition in the collinear case. Both methods rely on fixing a correct scaling of each matrix and are therefore not projectively well-defined, nor are the conditions expressed in terms of the fundamental matrices and their epipoles, as in the $n = 3$ case.

The contributions of this paper include giving explicit homogeneous polynomials that provide necessary and sufficient conditions for the compatibility of fundamental matrices in the case of complete graphs. The paper is structure as follows. In Section III.2, we introduce the fundamental action, a key tool in simplifying the problem of finding compatibility conditions. In Section III.3, for the case of $n = 4$, we establish that a set of six fundamental matrices admits a reconstruction of camera matrices with linearly independent centers only if the triple-wise constraints and one additional polynomial equation involving all six fundamental matrices and their epipoles are satisfied. We also demonstrate, using the computer algebra system `Macaulay2` [GS20], that the eigenvalue conditions from [Gei+20; Kas+19b] are superfluous in the generic case and in the case

where all epipoles in each image coincide. Section III.4 presents a necessary and sufficient condition for compatibility for any viewing graph via a cycle condition, similar to cycle-based formulations of parallel rigidity that appear in the calibrated case. Finally, in Section III.5, we discuss the image of the fundamental map and prove a first result on this topic.

We approach compatibility of fundamental matrices from an algebraic point of view, i.e., we aim to describe constraints through algebraic equations and polynomial equations using techniques and software from applied algebraic geometry. This approach to questions in computer vision has a long standing tradition [APT19; Duf+19; HÅ97; KK22; THP15].

Related work

History. The problem of determining whether a set of fundamental matrices is compatible has a curious history. [HZ04] provided a necessary and sufficient triple-wise condition for the compatibility of three fundamental matrices F^{12}, F^{13} and F^{23} arising from three cameras with non-collinear centers. In 2007, the paper [HK07, Theorem 2.2] claimed that this condition was sufficient for compatibility even in the case of cameras with collinear centers, a claim that we show to be false in Example III.3.3. During the next decade, few advances were made in understanding compatibility. Over time, a belief seemed to develop that triple-wise compatibility was enough to ensure global compatibility. In fact, articles such as [Swe+15, Section 2.1] claimed this to be true, based on a faulty proof provided in [RPP11]. In 2018, [TOP18, Section 3.3] pointed out that the proof in [RPP11] fails in some cases, but still agreed that the result holds for complete graphs. Example III.3.7 shows that this is not the case by providing a counterexample.

Essential matrices. In the context of *uncalibrated cameras*, which are defined as full-rank 3×4 matrices, this work, as well as [Kas+19b], provide necessary and sufficient conditions for compatibility of fundamental matrices. However, camera matrices are often assumed to be *calibrated*, represented in the form of $[R|t]$ for a rotation matrix R and a translation vector t. The corresponding fundamental matrices are called *essential matrices*. In [Kas+19a], the authors build upon their previous work and provide a necessary and sufficient condition for compatibility of essential matrices, in terms of the n-view essential matrix obtained by stacking all essential matrices into a larger matrix. This condition is then used to recover a consistent set of essential matrices, given a partial set of measured essential matrices. In [Mar20], Martyushev provides a necessary and sufficient condition for compatibility of three essential matrices.

Solvability. There has been extensive research on the topic of solvability of viewing graphs in computer vision, as evidenced by various studies such as [Arr+21; Arr+22; LW03; NCC11; RPP11; THP15; TOP18]. A viewing graph is considered *solvable* if, given a generic set of cameras, their fundamental matrices

have a unique solution in terms of cameras up to global projective transformation. Recently, [Arr+21] proposed a new formulation of solvability and developed an effective algorithm for testing it.

The primary distinction between solvability and compatibility lies in the fact that, in the latter, the existence of cameras that correspond to a set of fundamental matrices is not assumed to exist. Moreover, compatibility has mostly been studied for graphs where each possible fundamental matrix is given, whereas papers on solvability study viewing graphs without such restrictions.

Furthermore, solvability has been investigated in the case of calibrated cameras, where it is known that the solvable graphs are precisely those that are parallel rigid [OS15; SLK16].

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III.1 Preliminaries

In this section we recall established notation and results, as well as concepts of algebraic geometry in Section III.1.1. We work over the real numbers, although all results in this paper either directly hold in the complex case or can be reformulated to do so. Where slight adjustments have to be made over the complex numbers, we make a remark.

Let $\mathbb{R}^n$ denote the set of real vectors with n coordinates, we call this *affine space*. Let $\mathbb{P}^{n-1}$ denote its projectivization. We write $\mathbb{R}^{n\times m}$ to denote the set of real $n \times m$ matrices, and we write $\mathbb{P}^{n\times m}$ to denote the set of real projective $n \times m$ matrices.

We define a rational map (this notion is formally defined in Section III.1.1)

$$\psi : \mathbb{P}^{3\times 4} \times \mathbb{P}^{3\times 4} \dashrightarrow \mathbb{P}^{3\times 3}, \tag{III.1}$$

as follows. Given a pair of 3×4 matrices P_1 and P_2 (defined up to scale), let $\mathbf{x}$ and $\mathbf{y}$ be two 3×1 vectors. The determinant

$$\det \begin{bmatrix} P_1 & \mathbf{x} & 0 \\ P_2 & 0 & \mathbf{y} \end{bmatrix} \tag{III.2}$$

is a bilinear polynomial with in $\mathbf{x}$ and $\mathbf{y}$, meaning there is a matrix F^{12} (defined up to scale) such that (III.2) can be written as $\mathbf{x}^T F^{12} \mathbf{y}$. We define $\psi(P_1, P_2)$ to be this 3×3 matrix. This map is undefined, i.e. $\psi(P_1, P_2) = 0$, precisely when $\ker P_1 \cap \ker P_2 \neq \{0\}$.

We refer to rank-2 3×3 matrices as *fundamental matrices* (either in $\mathbb{R}^{3\times 3}$ or $\mathbb{P}^{3\times 3}$) and we refer to rank-3 3×4 matrices as *cameras* (either in $\mathbb{R}^{3\times 4}$ or $\mathbb{P}^{3\times 4}$).

The *center* of a camera P is its kernel $\ker P$. Before we list a set of well-known results, partly found in [HZ04, Section 9], we recall that GL_n denotes the set of invertible $n \times n$ matrices and that PGL_n is its projectivization.

Proposition III.1.1.

III.1.1.1. *$\psi(P_1, P_2)$ is of rank at most 2, and it attains this rank if P_1, P_2 are cameras with distinct centers;*

III.1.1.2. *for any fundamental matrix F^{12}, there exist two cameras P_1, P_2 such that F^{12} is their fundamental matrix. All other cameras C_1, C_2 with fundamental matrix F^{12} satisfy $C_1 = P_1 H, C_2 = P_2 H$ for some $H \in \mathrm{PGL}_4$;*

III.1.1.3. *$\psi(P_2, P_1) = \psi(P_1, P_2)^T$;*

III.1.1.4. *if F^{12} is the fundamental matrix of P_1, P_2, then $\ker F^{12} = P_2 \ker(P_1)$;*

III.1.1.5. *for cameras P_1, P_2, we have $F^{12} = \psi(P_1, P_2)$ if and only if $P_1^T F^{12} P_2$ is a skew-symmetric matrix.*

We say that a set of fundamental matrices $\{F^{ij}\}$ is *compatible* if there are cameras $P_1, \ldots, P_n$ such that $F^{ij} = \psi(P_i, P_j)$. The cameras $P_1, \ldots, P_n$ are called a *solution* to F^{ij}. We mostly focus on complete viewing graphs, i.e. when $\{F^{ij}\}$ contains all $\binom{n}{2}$ fundamental matrices for n indices. Still, in Section III.4, we provide a result that holds not only in this setting, but for any viewing graph.

We define the i-th *epipole* e^i_j in the j-th image to be an affine representative of $\ker F^{ij}$. By Proposition III.1.1 4., e^i_j is the image of the i-th camera center taken by the j-th camera.

Lemma III.1.2 ([HZ04, Section 15.4])**.** *Let $\{F^{12}, F^{13}, F^{23}\}$ be compatible. There is a unique solution if and only if the two epipoles in each image are distinct.*

Although fundamental matrices and epipoles are only defined up to scale, i.e. as elements in projective space, we always assume for convenience that we are given affine representatives of them and that the representatives of fundamental matrices satisfy $(F^{ij})^T = F^{ji}$, unless otherwise is specified.

Given a fixed set of fundamental matrices F^{ij}, we point out that there is a rather simple method of finding possible solutions in terms of cameras by first using F^{12} to recover P_1, P_2 and then using Lemma III.1.2 with matrices $\{F^{12}, F^{1i}, F^{2i}\}$ to recover the remaining P_i (a detailed algorithm can be found in [HK07, Section 6.1]). Finding explicit equations in terms of the fundamental matrices and epipoles for compatability is however more difficult, and is the subject of this paper.

III.1.1 Methods of algebraic geometry

For this paper, it is helpful to understand saturation and elimination of *ideals*. We refer the reader to [Gat19] for the basics on algebraic geometry and [Cox+94]

for a detailed study of these topics. Consider a field k and its polynomial ring $k[x] = k[x_1, \dots, x_m]$; the set of all polynomials with coefficients in k. That $k[x]$ is a *ring* means that addition and multiplication of polynomials satisfy a certain set of axioms that we don't list here. An *ideal* I of a ring R is an additive subgroup that is closed under multiplication of elements in R.

Let $f_1, \dots, f_s \in k[x]$ be polynomials. They generate an ideal of $k[x]$ as follows:

$$\langle f_1, \dots, f_s \rangle := \Big\{ \sum g_i f_i : g_i \in k[x] \Big\} \subseteq k[x]. \tag{III.3}$$

From the geometric point of view, an ideal in a polynomial ring defines a *variety* $\mathcal{V}$ as the zero set of all polynomials in the ideal. In other words,

$$\mathcal{V}(I) := \big\{ x \in k^m : f(x) = 0 \; \forall f \in I \big\}. \tag{III.4}$$

The Zariski closure $\overline{U}$ of a set $U \subseteq k^m$ is the smallest variety X that contains U.

The goal of saturation is to remove unwanted components from a variety. Let I, J be ideals. The *saturation* of I with respect to J is

$$I : J^\infty := \big\{ f \in k[x] : \forall g \in J, \; \exists N \in \mathbb{N} \quad \text{such that } fg^N \in I \big\}. \tag{III.5}$$

It follows from definition that $I : J^\infty = I : (I + J)^\infty$, where $I + J = \{g + f : g \in I, f \in I\}$, and therefore we may assume without restriction that $I \subseteq J$.

Theorem III.1.3 ([Cox+94, p. 203])**.** *Let $\mathcal{V}(J), \mathcal{V}(I)$ be two varieties over any field k. Then*

$$\overline{\mathcal{V}(I) \setminus \mathcal{V}(J)} \subseteq \mathcal{V}(I : J^\infty). \tag{III.6}$$

The *elimination* of variables $x_1, \dots, x_l$ from an ideal $I \subseteq k[x]$ is the intersection

$$I \cap k[x_{l+1}, \dots, x_m]. \tag{III.7}$$

Given $(x_1, \dots, x_n) \in \mathcal{V}(I)$, we have that $(x_{l+1}, \dots, x_n) \in \mathcal{V}(I \cap k[x_{l+1}, \dots, x_m])$, because any f in Equation (III.7) also lies in I. In this way, elimination of variables gives us conditions on the projection of $\mathcal{V}(I)$ away from the first l coordinates.

In Section III.3, we use the symbolic programming language `Macaulay2` [GS20] to symbolically saturate ideals and eliminate variables in the ring $\mathbb{Q}[x]$. In our study, all polynomials have rational coefficients, i.e. are elements of $\mathbb{Q}[x]$. However, our varieties lie in real space. For saturation and elimination, it may matter in which ring the operations are performed in. In `Macaulay2` all such operations happen inside $\mathbb{Q}[x]$, and we therefore prove the following lemma for clarity.

Lemma III.1.4. *Let I, J be ideals in $\mathbb{R}[x]$ generated by elements of $\mathbb{Q}[x]$. Write $I_Q, J_Q \subseteq \mathbb{Q}[x]$ for the ideals defined as the intersections $I \cap \mathbb{Q}[x], J \cap \mathbb{Q}[x]$, respectively. If $y \in \mathbb{R}^m$ lies in $\mathcal{V}(I) \setminus \mathcal{V}(J)$, then $f(y) = 0$ for every f in the saturation $I_Q : J_Q^\infty$ performed inside the ring $\mathbb{Q}[x]$.*

Hence saturation in $\mathbb{Q}[x]$ tell us something also for the real numbers. The statement and proof works the same if $\mathbb{R}[x]$ is replaced by $\mathbb{C}[x]$.

Proof. By Theorem III.1.3, $y \in \mathcal{V}(I : J^\infty)$. It suffices to show that $I_Q : J_Q^\infty \subseteq I : J^\infty$, since then we have $\mathcal{V}(I : J^\infty) \subseteq \mathcal{V}(I_Q : J_Q^\infty)$ over the real numbers. Let $f \in I_Q : J_Q^\infty$. Then $f \in \mathbb{Q}[x]$ and for every $g \in J_Q$, there is an N such that $fg^N \in I_Q$. Let $g_1, \ldots, g_k \in \mathbb{Q}[x]$ generate J and J_Q. Let N_i denote an integer such that $fg_i^{N_i} \in I_Q$. Now take any $g \in J$. We can write $g = \sum_{i=1}^k h_i g_i$ for some $h_i \in \mathbb{R}[x]$. There is an integer N depending on k and N_i such that each term of g^N is divisible by some $g_i^{N_i}$ and $fg^N \in I_Q$. For such N, we can write $g^N = \sum_{i=1}^k h_i' g_i^{N_i}$ for some $h_i' \in \mathbb{R}[x]$. Then, since $fg_i^{N_i} \in I_Q$, we must have that $fg^N \in I$. This shows that inclusion $I_Q : J_Q^\infty \subseteq I : J^\infty$ and we are done. ■

In the main body of the text, the term rational map was used, which we now define. A variety $\mathcal{V}$ is called *irreducible* if it cannot be written as a union of two proper varieties, meaning that for two subvarieties X, Y of $\mathcal{V}$, the equality $\mathcal{V} = X \cup Y$ implies $\mathcal{V} = X$ or $\mathcal{V} = Y$. A *rational map* f between projective varieties X and Y, with X irreducible, is defined on a *Zariski open* set of X, which is a set that can be written $X \setminus Y$ for a proper subvariety $Y \subseteq X$. A rational map between X and Y is written

$$f : X \dashrightarrow Y. \tag{III.8}$$

III.2 The Fundamental Action

In this section, we formally introduce the fundamental action, a key tool in simplifying the problem of finding compatibility conditions. GL_3^n (or equivalently PGL_3^n) acts on a set of fundamental matrices $\left\{F^{ij}\right\}$ by

$$\left\{F^{ij}\right\} \mapsto \left\{H_i^T F^{ij} H_j\right\}. \tag{III.9}$$

We call this the *fundamental action* of GL_3^n. The main appeal of this action is that we can use it to simplify a set of fundamental matrices, without affecting compatibility.

Proposition III.2.1. *Let $\left\{F^{ij}\right\}$ be a set of fundamental matrices. Let P_i be a solution to $\left\{F^{ij}\right\}$. For any $(H_1, \ldots, H_n, H) \in \mathrm{PGL}_3^n \times \mathrm{PGL}_4$, we have,*

$$\psi(H_i^{-1} P_i H, H_j^{-1} P_j H) = H_i^T \psi(P_i, P_j) H_j. \tag{III.10}$$

In particular, $\left\{F^{ij}\right\}$ is compatible if and only if $\left\{G^{ij}\right\}$ is compatible, where $G^{ij} := H_i^T F^{ij} H_j$.

Proof. It is a standard fact that the action of $H \in \mathrm{PGL}_4$ in Equation (III.10) does not change the fundamental matrix, so we may set $H = I$. Consider the following equality up to scaling,

$$\det \begin{bmatrix} H_i^{-1} P_i & x_i & 0 \\ H_j^{-1} P_j & 0 & x_j \end{bmatrix} = \det \begin{bmatrix} P_i & H_i x_i & 0 \\ P_j & 0 & H_j x_j \end{bmatrix}. \tag{III.11}$$

Writing these expressions in terms of fundamental matrices, we get exactly Equation (III.10). ■

The fundamental action gives rise to an equivalence relation. For compatible fundamental matrices, the equivalence classes turn out to be the equivalence classes of n points in $\mathbb{P}^3$ under PGL_4.

Proposition III.2.2. *Let $\{F^{ij}\}$ and $\{G^{ij}\}$ be two sets of compatible fundamental matrices. They are equivalent under fundamental action if and only if they have solutions whose camera centers are equivalent under PGL_4.*

For the proof we need the following lemma:

Lemma III.2.3 ([HZ04, Result 22.1])**.** *Let P and P' be two camera matrices with the same center. Then there exists $H \in \mathrm{PGL}_3$ such that $P' = HP$.*

Proof of Proposition III.2.2.
⇒) Let $G^{ij} = H_i^T F^{ij} H_j$. If $P_1, \dots, P_n$ is a solution to $\{F^{ij}\}$, then by Proposition III.2.1, $H_1^{-1}P_1, \dots, H_n^{-1}P_n$ is a solution to G^{ij}, which have the same centers as $P_1, \dots, P_n$.
⇐) Let $P_1, \dots, P_n$ be a solution to $\{F^{ij}\}$ with centers c_i and $P'_1, \dots, P'_n$ a solution to $\{G^{ij}\}$ with centers c'_i such that $c'_i = H^{-1}c_i$ for some $H \in \mathrm{PGL}_4$. By Lemma III.2.3, there are $H_i \in \mathrm{PGL}_3$ such that $P'_i = H_i P_i H$, since P'_i and $P_i H$ have the same center $H^{-1}c_i$. Then by Proposition III.2.1, $\{F^{ij}\}$ and $\{G^{ij}\}$ are equivalent under fundamental action. ■

In this paper, quantities of the form $\mathbf{e}_{sijt} := (e_i^s)^T F^{ij} e_j^t$, called *epipolar numbers*, are important (see Theorems III.3.2 and III.3.8). The epipolar numbers are invariant under the fundamental action:

Lemma III.2.4. *Let $\{F^{ij}\}$ be a set of fundamental matrices with epipoles $\{e_j^i\}$. Let $H_i \in \mathrm{GL}_3^n$ and consider the fundamental matrices $G^{ij} := H_i^T F^{ij} H_j$, whose epipoles are $h_j^i = H_j^{-1} e_j^i$. Then*

$$(e_i^s)^T F^{ij} e_j^t = (h_i^s)^T G^{ij} h_j^t. \tag{III.12}$$

Proof. The equality follows directly by the definitions of G^{ij} and h_j^i. ■

We have the following geometrical interpretation of the epipolar numbers.

Lemma III.2.5. *Let $\{F^{ij}\}$ be set of compatible fundamental matrices that include F^{si}, F^{ij} and F^{jt}. We have $\mathbf{e}_{sijt} = 0$ if and only if the centers c_s, c_i, c_j and c_t of any solution are coplanar.*

The *back-projected line* of an image point x for a camera P is the line in $\mathbb{P}^3$ of all points that are projected by P to x. This line contains the center of P.

Proof. Let $P_1, \dots, P_n$ be a solution to $\{F^{ij}\}$. Let $L_{i,s}$ be the back-projected line of e_i^s and $L_{j,t}$ the back-projected line of e_j^t. Then $e_i^s F^{ij} e_j^t = 0$ means precisely that the back-projected lines $L_{i,s}$ and $L_{j,t}$ meet in a point. Therefore, $L_{i,s}$ and

$L_{j,t}$ together span a plane unless they are the same line. In either case, all centers lie in this span, since $L_{i,s}$ contains c_i and c_s, and $L_{j,t}$ contains c_j and c_t. The other direction follows similarly. ■

It follows from the lemma that putting any of the two indices s, i, j, t equal, the epipolar number is zero. In particular, $\mathbf{e}_{sijs}$ is always zero for compatible fundamental matrices, because three centers are always in a plane.

III.3 Compatibility for Complete Graphs

We begin by giving our main results for complete graphs, that is, the case where all the fundamental matrices are known. The main contribution of this paper is providing explicit, algebraic conditions for compatibility expressed in terms of the fundamental matrices and their epipoles for any number of views. Let K_n denote the complete graph on n nodes.

In Section III.3.1, we deal with K_3 graphs and recall the triple-wise conditions. We also state a result for the collinear case. In Section III.3.2 we find necessary and sufficient constraints for compatibility in the case of K_4. In Section III.3.3 we prove that quadruple-wise compatbility implies global compatibility. Finally, in Section III.3.4 we state that the eigenvalue condition from the theorem of Kasten et. al. is redundant in the generic and collinear cases.

Remark III.3.1. In this section, we work only with real numbers, because it allows us to give polynomials equations using the standard inner product and norm on $\mathbb{R}^3$. However, all of our statements in Section III.3.1 and Section III.3.2 can be extended to the complex numbers.

III.3.1 K_3

The case of three fundamental matrices is fairly straightforward. We have two possible configurations for the three camera centers; they either all lie on a line, or they do not.

Theorem III.3.2 ([HZ04, Section 15.4])**.** *Let F^{12}, F^{13}, F^{23} be fundamental matrices. There exist non-collinear cameras P_1, P_2, P_3 such that $F^{ij} = \psi(P_i, P_j)$ if and only if*

$$e_1^2 \neq e_1^3, \quad e_2^1 \neq e_2^3, \quad e_3^1 \neq e_3^2, \tag{III.13}$$

and

$$(e_1^3)^T F^{12} e_2^3 = (e_1^2)^T F^{13} e_3^2 = (e_2^1)^T F^{23} e_3^1 = 0. \tag{III.14}$$

If P_1, P_2, P_3 are cameras with collinear centers, then it follows that $P_i(\ker P_j) = P_i(\ker P_k)$ for all distinct i, j, k. This implies that for the corresponding fundamental matrices F^{12}, F^{13}, F^{23}, we have $e_j^i = e_j^k$ for all distinct i, j, k. However, contrary to what is claimed in [HK07], the conditions in Equation (III.14) are not enough in this case:

Example III.3.3. Consider the fundamental matrices:

$$F^{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F^{13} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad F^{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \tag{III.15}$$

whose epipoles are all equal to $[1, 0, 0]$. These six matrices satisfy the conditions in Equation (III.14). However, no solution of cameras P_1, P_2, P_3 exist for which F^{12}, F^{13}, F^{23} are the fundamental matrices. This can be checked for instance via the algorithm described at the end of Section III.1. ◇

Given a vector $t \in \mathbb{R}^3$, we define

$$[t]_\times = \begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix}. \tag{III.16}$$

Then with respect to the cross product $\times$ on $\mathbb{R}^3 \times \mathbb{R}^3$, we have $t \times u = [t]_\times u$. To the best of our knowledge, the following result does not appear in the literature:

Proposition III.3.4. *Let F^{12}, F^{13}, F^{23} be fundamental matrices. There exist collinear cameras P_1, P_2, P_3 such that $F^{ij} = \psi(P_i, P_j)$ if and only if*

$$e_1^2 = e_1^3, \quad e_2^1 = e_2^3, \quad e_3^1 = e_3^2, \tag{III.17}$$

and (up to scaling)

$$(F^{12})^T [e_1^2]_\times F^{13} = F^{23}. \tag{III.18}$$

The conditions of Theorem III.3.2 and Proposition III.3.4 are called the *triple-wise conditions.*

Remark III.3.5. When we in the proofs below write "it can be verified that" or "it can be checked that" in relation to the shape of fundamental matrices, we have checked this fact in `Macaulay2`.

Proof. Recall that the epipole e_j^i equals $P_j(\ker(P_i))$. It follows that if a solution to F^{12}, F^{13}, F^{23} consists of collinear cameras, then Equation (III.17) must be satisfied. Conversely, if Equation (III.17) is satisfied, any solution must consist of collinear camera centers.

We begin by simplifying the problem using the fundamental action. Let

$$H_i = \begin{bmatrix} e_i^k & \mathbf{x}_i & \mathbf{y}_i \end{bmatrix}, \tag{III.19}$$

for any $k \neq i$ and $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^3$ such that the determinant is non-zero, meaning H_i is invertible. We get a new triple of fundamental matrices

$$G^{ij} = H_i^T F^{ij} H_j. \tag{III.20}$$

Write h^i_j for the epipoles of G^{ij}. By the fact that $H_j^{-1}e^i_j$ spans $\ker G^{ij}$ we have $h^i_j = H_j^{-1}e^i_j$ (up to scaling). By construction of H_j, we then have:

$$\begin{aligned} h^2_1 &= [1,0,0], \quad h^1_2 = [1,0,0], \quad h^1_3 = [1,0,0], \\ h^3_1 &= [1,0,0], \quad h^3_2 = [1,0,0], \quad h^2_3 = [1,0,0]. \end{aligned} \tag{III.21}$$

Since the epipoles span the kernels of G^{ij}, we conclude that G^{ij} take the following form

$$G^{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{12} & b_{12} \\ 0 & c_{12} & d_{12} \end{bmatrix}, \quad G^{13} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{13} & b_{13} \\ 0 & c_{13} & d_{13} \end{bmatrix}, \quad G^{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{23} & b_{23} \\ 0 & c_{23} & d_{23} \end{bmatrix}, \tag{III.22}$$

for some $a_{ij}, b_{ij}, c_{ij}, d_{ij} \in \mathbb{R}$ making them rank-2.

We next find conditions on triplets of cameras P_1, P_2, P_3 with collinear centers whose fundamental matrices are of the form given by Equation (III.22). We may up to PGL_4 action assume that the center of P_1 is $[1,0,0,0]$, the center of P_2 is $[0,1,0,0]$ and the center of P_3 is $[1,1,0,0]$. Fix P_1 to be

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{III.23}$$

Using the fact that $e^i_j = P_j(\ker P_i)$, we find that P_2 and P_3 must take the following form:

$$P_2 = \begin{bmatrix} 1 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}, P_3 = \begin{bmatrix} 1 & -1 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}. \tag{III.24}$$

One can check that the two right-most elements of the first rows of P_2 and P_3 do not affect the fundamental matrices. In particular, if G^{ij} are compatible, then one solution must be

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & \alpha_3 & \alpha_4 \end{bmatrix}, P_3 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & \beta_1 & \beta_2 \\ 0 & 0 & \beta_3 & \beta_4 \end{bmatrix}, \tag{III.25}$$

for α_i and β_i such that $\alpha_1\alpha_4 - \alpha_2\alpha_3 \neq 0$ and $\beta_1\beta_4 - \beta_2\beta_3 \neq 0$. Given such cameras, the fundamental matrices are calculated as

$$\begin{aligned} \psi(P_1,P_2) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha_3 & \alpha_1 \\ 0 & -\alpha_4 & \alpha_2 \end{bmatrix}, \psi(P_1,P_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\beta_3 & \beta_1 \\ 0 & -\beta_4 & \beta_2 \end{bmatrix}, \\ \psi(P_2,P_3) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha_4\beta_3 + \alpha_3\beta_4 & \alpha_4\beta_1 - \alpha_3\beta_2 \\ 0 & \alpha_2\beta_3 - \alpha_1\beta_4 & -\alpha_2\beta_1 + \alpha_1\beta_2 \end{bmatrix}. \end{aligned} \tag{III.26}$$

Define the $\star$ operator on 2×2 matrices as

$$\begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} \star \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix} := \begin{bmatrix} v_3 w_1 - v_1 w_3 & v_3 w_2 - v_1 w_4 \\ v_4 w_1 - v_2 w_3 & v_4 w_2 - v_2 w_4 \end{bmatrix} = \begin{bmatrix} v_3 & v_1 \\ v_4 & v_2 \end{bmatrix} \begin{bmatrix} w_1 & w_2 \\ -w_3 & -w_4 \end{bmatrix}. \tag{III.27}$$

Then, by Equation (III.26), G^{ij} on the form Equation (III.26) are compatible if and only if (up to scaling) we have

$$\begin{bmatrix} a_{12} & b_{12} \\ c_{12} & d_{12} \end{bmatrix} \star \begin{bmatrix} a_{13} & b_{13} \\ c_{13} & d_{13} \end{bmatrix} = \begin{bmatrix} a_{23} & b_{23} \\ c_{23} & d_{23} \end{bmatrix}. \tag{III.28}$$

By the construction of our fundamental action, we have

$$\begin{aligned} a_{ij} &= [0,1,0] G^{ij} [0,1,0]^T = \mathbf{x}_i^T F^{ij} \mathbf{x}_j, \quad b_{ij} = [0,1,0] G^{ij} [0,0,1]^T = \mathbf{x}_i^T F^{ij} \mathbf{y}_j, \\ c_{ij} &= [0,0,1] G^{ij} [0,1,0]^T = \mathbf{y}_i^T F^{ij} \mathbf{x}_j, \quad d_{ij} = [0,0,1] G^{ij} [0,0,1]^T = \mathbf{y}_i^T F^{ij} \mathbf{y}_j. \end{aligned} \tag{III.29}$$

In the below, and throughout this section, we skip the transpose notation and write for instance $\mathbf{x}_i F^{ij} \mathbf{x}_j$ instead of $\mathbf{x}_i^T F^{ij} \mathbf{x}_j$. We get

$$\begin{bmatrix} \mathbf{x}_1 F^{12} \mathbf{x}_2 & \mathbf{x}_1 F^{12} \mathbf{y}_2 \\ \mathbf{y}_1 F^{12} \mathbf{x}_2 & \mathbf{y}_1 F^{12} \mathbf{y}_2 \end{bmatrix} \star \begin{bmatrix} \mathbf{x}_1 F^{13} \mathbf{x}_3 & \mathbf{x}_1 F^{13} \mathbf{y}_3 \\ \mathbf{y}_1 F^{13} \mathbf{x}_3 & \mathbf{y}_1 F^{13} \mathbf{y}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_2 F^{23} \mathbf{x}_3 & \mathbf{x}_2 F^{23} \mathbf{y}_3 \\ \mathbf{y}_2 F^{23} \mathbf{x}_3 & \mathbf{y}_2 F^{23} \mathbf{y}_3 \end{bmatrix}. \tag{III.30}$$

However,

$$\begin{bmatrix} \mathbf{x}_i F^{ij} \mathbf{x}_j & \mathbf{x}_i F^{ij} \mathbf{y}_j \\ \mathbf{y}_i F^{ij} \mathbf{x}_j & \mathbf{y}_i F^{ij} \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_i^T \\ \mathbf{y}_i^T \end{bmatrix} F^{ij} \begin{bmatrix} \mathbf{x}_j & \mathbf{y}_j \end{bmatrix}, \tag{III.31}$$

and therefore,

$$\begin{aligned} &\begin{bmatrix} \mathbf{x}_1 F^{12} \mathbf{x}_2 & \mathbf{x}_1 F^{12} \mathbf{y}_2 \\ \mathbf{y}_1 F^{12} \mathbf{x}_2 & \mathbf{y}_1 F^{12} \mathbf{y}_2 \end{bmatrix} \star \begin{bmatrix} \mathbf{x}_1 F^{13} \mathbf{x}_3 & \mathbf{x}_1 F^{13} \mathbf{y}_3 \\ \mathbf{y}_1 F^{13} \mathbf{x}_3 & \mathbf{y}_1 F^{13} \mathbf{y}_3 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{x}_2^T \\ \mathbf{y}_2^T \end{bmatrix} F^{21} \begin{bmatrix} \mathbf{y}_1 & \mathbf{x}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1^T \\ -\mathbf{y}_1^T \end{bmatrix} F^{13} \begin{bmatrix} \mathbf{x}_3 & \mathbf{y}_3 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{x}_2^T \\ \mathbf{y}_2^T \end{bmatrix} F^{23} \begin{bmatrix} \mathbf{x}_3 & \mathbf{y}_3 \end{bmatrix}. \end{aligned} \tag{III.32}$$

Since this holds for generic choices of $\mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_3, \mathbf{y}_3$, we conclude that, projectively,

$$F^{21} \begin{bmatrix} \mathbf{y}_1 & \mathbf{x}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1^T \\ -\mathbf{y}_1^T \end{bmatrix} F^{13} = F^{23}, \tag{III.33}$$

for all $\mathbf{x}_1, \mathbf{y}_1$ such that $[e_1^2 \ \mathbf{x}_1 \ \mathbf{y}_1]$ is invertible. Further,

$$\begin{bmatrix} \mathbf{y}_1 & \mathbf{x}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1^T \\ -\mathbf{y}_1^T \end{bmatrix} \tag{III.34}$$

is skew-symmetric and equals $[\ell]_\times$ for $\ell = \mathbf{x}_1 \times \mathbf{y}_1 \in \mathbb{R}^3$. Then choosing $\mathbf{x}_1, \mathbf{y}_1$ such that $\ell = e_1^2$, we have over the real numbers that $[e_1^2\ \mathbf{x}_1\ \mathbf{y}_1]$ is full-rank. In other words,

$$F^{21}[e_1^2]_\times F^{13} = F^{23} \tag{III.35}$$

is a necessary and sufficient condition for compatibility. ■

Remark III.3.6. In the complex setting, it does not always suffice to put $\ell = e_1^2$, because it could be the case that $(e_1^2)^T e_1^2 = 0$. Then ℓ should be any vector such that $\ell^T e_1^2 \neq 0$.

III.3.2 K_4

We start this section with a counterexample to the previous belief that triple-wise compatibility is enough to ensure full compatibility.

Example III.3.7. Consider the fundamental matrices:

$$\begin{aligned} F^{12} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad F^{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad F^{14} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ F^{23} &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad F^{24} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F^{34} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{III.36}$$

with epipoles:

$$\begin{aligned} &e_1^2 = [1,0,0], \quad e_2^1 = [1,0,0], \quad e_3^1 = [1,0,0], \quad e_4^1 = [1,0,0], \\ &e_1^3 = [0,1,0], \quad e_2^3 = [0,1,0], \quad e_3^2 = [0,1,0], \quad e_4^2 = [0,1,0], \\ &e_1^4 = [0,0,1], \quad e_2^4 = [0,0,1], \quad e_3^4 = [0,0,1], \quad e_4^3 = [0,0,1]. \end{aligned} \tag{III.37}$$

It can easily be verified that these six matrices satisfy the conditions in Theorem III.3.2. Nonetheless, no solution exists. Any attempt to find four cameras will end up matching at most five of the six fundamental matrices. We will soon see that this is because the sextuple does not satisfy the conditions in Theorem III.3.8. ◇

Before we get to the main results, we list the possible configurations of camera centers in the case of four cameras (six fundamental matrices). These are illustrated in Figure III.1. By Proposition III.2.2, these correspond to the equivalence classes of compatible fundamental matrices. Each of these will be recognizable from the epipoles e_i^j:

Case 1: Cameras are in generic position, meaning no plane contains all four centers. Epipoles are in generic position, meaning in each image, the three epipoles do not lie on a line.

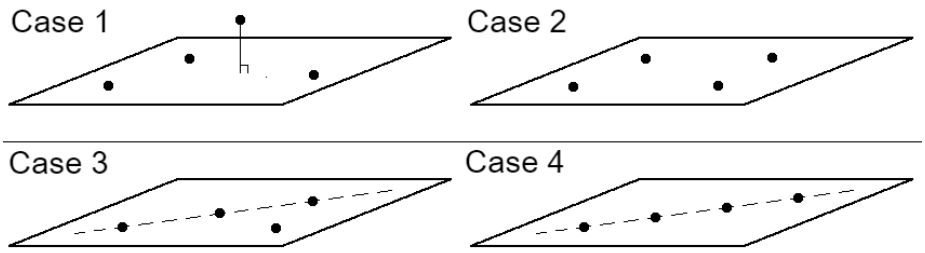

Figure III.1: Illustration of the 4 cases.

Case 2: All camera centers lie in the same plane, but no three lie on a line. In each image, the three epipoles are distinct and lie on a line.

Case 3: Precisely three camera centers lie on a line. In the three corresponding images, the epipoles corresponding to the other two cameras are equal, with the third one different from these two. In the final image, the three epipoles are distinct and lie on a line.

Case 4: All four camera centers lie on a line. In each image, the three epipoles coincide.

These are the only possible configurations of four cameras, so any compatible sextuple $\left\{F^{ij}\right\}$ must have its epipoles in one of the configurations above. If we have, for instance, collinear epipoles in one image, but not all, the fundamental matrices can not be compatible. In Cases 3 and 4, the configuration of the epipoles together with the triple-wise conditions from Section III.3.1 ensure compatibility. This is not true for Cases 1 and 2; here we need additional constraints. We cover all cases in sequence. We recall the epipolar numbers: $\mathbf{e}_{sijt} = (e_i^s)^T F^{ij} e_j^t$.

Theorem III.3.8 (Case 1)**.** *Let $\left\{F^{ij}\right\}$ be a sextuple of fundamental matrices such that the three epipoles in each image do not lie on a line. Then $\left\{F^{ij}\right\}$ is compatible if and only if the triple-wise conditions hold and*

$$\mathbf{e}_{4123}\mathbf{e}_{2134}\mathbf{e}_{3142}\mathbf{e}_{4231}\mathbf{e}_{1243}\mathbf{e}_{2341} = \mathbf{e}_{3124}\mathbf{e}_{4132}\mathbf{e}_{2143}\mathbf{e}_{1234}\mathbf{e}_{3241}\mathbf{e}_{1342}. \tag{III.38}$$

Remark III.3.9. The condition that the epipoles in each image do not lie on a line is equivalent to all epipolar number $\mathbf{e}_{ijkl}$ being non-zero for distinct i, j, k, l. In Cases 2, 3 and 4, the three epipoles in each image lie on a line. This is equivalent to all epipolar numbers $\mathbf{e}_{ijkl}$ being zero for distinct i, j, k, l.

Proof. The triple-wise conditions are necessary for compatibility, so we assume that they are satisfied and prove that in this case compatibility is equivalent to Equation (III.38) being satisfied. We begin by simplifying the problem. Let

$$H_i = \begin{bmatrix} e_i^j & e_i^k & e_i^l \end{bmatrix}. \tag{III.39}$$

This 3×3 matrix is of full-rank and takes the three coordinate points to the three epipoles in the i-th image. Using this as our fundamental action, we get a new sextuple of fundamental matrices

$$G^{ij} = H_i^T F^{ij} H_j. \tag{III.40}$$

Since the fundamental action preserves compatibility, the sextuple $\{G^{ij}\}$ is compatible if and only if $\{F^{ij}\}$ is. Note that the epipoles of G^{ij}, denoted by h_j^i, are:

$$\begin{aligned} &h_1^2 = [1,0,0], \quad h_2^1 = [1,0,0], \quad h_3^1 = [1,0,0], \quad h_4^1 = [1,0,0], \\ &h_1^3 = [0,1,0], \quad h_2^3 = [0,1,0], \quad h_3^2 = [0,1,0], \quad h_4^2 = [0,1,0], \\ &h_1^4 = [0,0,1], \quad h_2^4 = [0,0,1], \quad h_3^4 = [0,0,1], \quad h_4^3 = [0,0,1]. \end{aligned} \tag{III.41}$$

Moreover, since G^{ij} satisfy the triple-wise conditions (we assumed F^{ij} did, and these are preserved under fundamental action), it follows that the six matrices must be on the form:

$$\begin{aligned} G^{12} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x_{12} \\ 0 & y_{12} & 0 \end{bmatrix}, \quad G^{13} = \begin{bmatrix} 0 & 0 & x_{13} \\ 0 & 0 & 0 \\ 0 & y_{13} & 0 \end{bmatrix}, \quad G^{14} = \begin{bmatrix} 0 & 0 & x_{14} \\ 0 & y_{14} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ G^{23} &= \begin{bmatrix} 0 & 0 & x_{23} \\ 0 & 0 & 0 \\ y_{23} & 0 & 0 \end{bmatrix}, \quad G^{24} = \begin{bmatrix} 0 & 0 & x_{24} \\ y_{24} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G^{34} = \begin{bmatrix} 0 & x_{34} & 0 \\ y_{34} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{III.42}$$

The sextuple $\{G^{ij}\}$ is compatible if and only if there exists a reconstruction consisting of 4 cameras P_i. Since the epipoles do not lie on a line, any such reconstruction must have 4 linearly independent centers. We are free to choose coordinates in $\mathbb{P}^3$ without affecting compatibility, so we take the four camera centers (assuming cameras exist) to be the four unit vectors. Furthermore, we know that the epipoles satisfy

$$h_i^j = P_i(\ker(P_j)). \tag{III.43}$$

So if $\{G^{ij}\}$ has a reconstruction $\{P_i\}$, it must be on the form:

$$\begin{aligned} P_1 &= \begin{bmatrix} 0 & \alpha_1^1 & 0 & 0 \\ 0 & 0 & \alpha_1^2 & 0 \\ 0 & 0 & 0 & \alpha_1^3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} \alpha_2^1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2^2 & 0 \\ 0 & 0 & 0 & \alpha_2^3 \end{bmatrix}, \\ P_3 &= \begin{bmatrix} \alpha_3^1 & 0 & 0 & 0 \\ 0 & \alpha_3^2 & 0 & 0 \\ 0 & 0 & 0 & \alpha_3^3 \end{bmatrix}, \quad P_4 = \begin{bmatrix} \alpha_4^1 & 0 & 0 & 0 \\ 0 & \alpha_4^2 & 0 & 0 \\ 0 & 0 & \alpha_4^3 & 0 \end{bmatrix}, \end{aligned} \tag{III.44}$$

where α_i^j are scalars. Since the fundamental matrices are of rank-2 and the cameras are rank-3, all the α_i^j, as well as the x_{ij} and y_{ij} are non-zero. Computing

the fundamental matrices of these four cameras, and setting them equal to the G^{ij}, we get the following six equations:

$$\begin{aligned} x_{12}\alpha_1^2\alpha_2^3 &= -y_{12}\alpha_1^3\alpha_2^2, \quad x_{13}\alpha_1^1\alpha_3^3 = -y_{13}\alpha_1^3\alpha_3^2, \quad x_{14}\alpha_1^1\alpha_4^3 = -y_{14}\alpha_1^2\alpha_4^2, \\ x_{23}\alpha_2^1\alpha_3^3 &= -y_{23}\alpha_2^3\alpha_3^1, \quad x_{24}\alpha_2^1\alpha_4^3 = -y_{24}\alpha_2^2\alpha_4^1, \quad x_{34}\alpha_3^1\alpha_4^2 = -y_{34}\alpha_3^2\alpha_4^1. \end{aligned} \tag{III.45}$$

Eliminating the variables α_i^j, we are left with a single polynomial,

$$x_{12}y_{13}x_{14}x_{23}y_{24}x_{34} - y_{12}x_{13}y_{14}y_{23}x_{24}y_{34} = 0. \tag{III.46}$$

This tells us that Equation (III.45) implies Equation (III.46), and we are left to argue that if x_{ij}, y_{ij} are non-zero numbers such that Equation (III.46) holds, then there are non-zero α_i^j such that Equation (III.45) holds. Note that we can assume $\alpha_1^j = 1$ by PGL_4 action and that $\alpha_i^1 = 1$ by scaling. Writing $\lambda_{ij} = x_{ij}/y_{ij}$, we then aim to find non-zero α_i^j such that

$$\begin{aligned} \lambda_{12}\alpha_2^3 &= \alpha_2^2, \quad \lambda_{13}\alpha_3^3 = \alpha_3^2, \quad \lambda_{14}\alpha_4^3 = \alpha_4^2, \\ \lambda_{23}\alpha_3^3 &= \alpha_2^3, \quad \lambda_{24}\alpha_4^3 = \alpha_2^2, \quad \lambda_{34}\alpha_4^2 = \alpha_3^2. \end{aligned} \tag{III.47}$$

It is clear that we can find non-zero α_i^j that solve the first five equations. However, this is enough because using $\lambda_{12}\lambda_{14}\lambda_{23}\lambda_{34} = \lambda_{13}\lambda_{24}$, the sixth equation $\lambda_{34}\alpha_4^2 = \alpha_3^2$ is implied by the other five through substitution.

It follows that the set $\{G^{ij}\}$ is compatible if and only if Equation (III.46) is satisfied. Finally, we can express the x_{ij} and y_{ij} in terms of F^{ij} and e_i^j, for instance we have

$$x_{12} = (h_1^3)^T G^{12} h_2^4 = (h_1^3)^T H_1^T F^{12} H_2 h_2^4 = (e_1^3)^T F^{12} e_2^4. \tag{III.48}$$

Making these substitutions for all the x_{ij} and y_{ij}, we get Equation (III.38). ■

Theorem III.3.10 (Case 2)**.** *Let $\{F^{ij}\}$ be a sextuple of fundamental matrices whose epipoles in each image are distinct and lie on a line. Then $\{F^{ij}\}$ is compatible if and only if the triple-wise conditions hold,*

$$\langle F^{jk}e_k^i, F^{jl}e_l^i\rangle\langle F^{kj}e_j^i, F^{kl}e_l^i\rangle\langle F^{lj}e_j^i, F^{lk}e_k^i\rangle + \|F^{lj}e_j^i\|^2\|F^{jk}e_k^i\|^2\|F^{kl}e_l^i\|^2 = 0 \tag{III.49}$$

for all distinct i, j, k, l satisfying $j < k < l$, and if for $\boldsymbol{x}_i = F^{ij}e_j^l$ with $j < k < l$, we have

$$\begin{aligned} -\frac{e_2^3F^{24}\boldsymbol{x}_4}{\boldsymbol{x}_2F^{24}e_4^1}\frac{\boldsymbol{x}_1F^{12}\boldsymbol{x}_2}{\boldsymbol{x}_1F^{12}e_2^3} + \frac{e_3^2F^{34}\boldsymbol{x}_4}{\boldsymbol{x}_3F^{34}e_4^1}\frac{\boldsymbol{x}_1F^{13}\boldsymbol{x}_3}{\boldsymbol{x}_1F^{13}e_3^2} + \frac{e_3^1F^{34}\boldsymbol{x}_4}{\boldsymbol{x}_3F^{34}e_4^1}\frac{\boldsymbol{x}_2F^{23}\boldsymbol{x}_3}{\boldsymbol{x}_2F^{23}e_3^1} + \\ + \frac{e_3^2F^{34}\boldsymbol{x}_4}{e_1^2F^{14}\boldsymbol{x}_4}\frac{e_1^2F^{13}\boldsymbol{x}_3}{\boldsymbol{x}_1F^{13}e_3^2}\frac{\boldsymbol{x}_1F^{14}\boldsymbol{x}_4}{\boldsymbol{x}_3F^{34}e_4^1} + \frac{\boldsymbol{x}_2F^{24}\boldsymbol{x}_4}{\boldsymbol{x}_2F^{24}e_4^1} - \frac{\boldsymbol{x}_3F^{34}\boldsymbol{x}_4}{\boldsymbol{x}_3F^{34}e_4^1} = 0. \end{aligned} \tag{III.50}$$

Remark III.3.11. As Equation (III.50) is already oversaturated with sub/superscript, we are omitting the transpose symbol from these equations. It is to be understood that the 3-vectors $\mathbf{x}_i$ and e_i^j are column-vectors when directly right of a fundamental matrix, and row-vectors when to the left.

Proof. Like in the previous proof, we begin by assuming the triple-wise conditions are satisfied. The three epipoles in each image lie on a line and therefore we fix a scaling such that for each i we have $e_i^l = e_i^j + e_i^k$, where $l > k > j$. Let

$$H_i = \begin{bmatrix} e_i^j & e_i^k & \mathbf{x}_i \end{bmatrix}. \tag{III.51}$$

Note that $(e_i^j)^T \mathbf{x}_i$ and $(e_i^k)^T \mathbf{x}_i$ for $\mathbf{x}_i$ in the statement are both zero, so H_i is of full-rank. Using this as our fundamental action, we get a new sextuple of fundamental matrices

$$G^{ij} = H_i^T F^{ij} H_j. \tag{III.52}$$

Since the fundamental action preserves compatibility, the sextuple $\left\{G^{ij}\right\}$ is compatible if and only if $\left\{F^{ij}\right\}$ is. Note that the epipoles of G^{ij} are as follows:

$$\begin{aligned} &h_1^2 = [1,0,0], \quad h_2^1 = [1,0,0], \quad h_3^1 = [1,0,0], \quad h_4^1 = [1,0,0], \\ &h_1^3 = [0,1,0], \quad h_2^3 = [0,1,0], \quad h_3^2 = [0,1,0], \quad h_4^2 = [0,1,0], \\ &h_1^4 = [1,1,0], \quad h_2^4 = [1,1,0], \quad h_3^4 = [1,1,0], \quad h_4^3 = [1,1,0]. \end{aligned} \tag{III.53}$$

With these epipoles and the fact that the G^{ij} satisfy the triple-wise conditions (we assumed F^{ij} did, and these are preserved under fundamental action), it follows that the six matrices must be on the form:

$$\begin{aligned} G^{12} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x_{12} \\ 0 & y_{12} & z_{12} \end{bmatrix}, \quad G^{13} = \begin{bmatrix} 0 & 0 & x_{13} \\ 0 & 0 & 0 \\ 0 & y_{13} & z_{13} \end{bmatrix}, \quad G^{14} = \begin{bmatrix} 0 & 0 & x_{14} \\ 0 & 0 & -x_{14} \\ 0 & y_{14} & z_{14} \end{bmatrix}, \\ G^{23} &= \begin{bmatrix} 0 & 0 & x_{23} \\ 0 & 0 & 0 \\ y_{23} & 0 & z_{23} \end{bmatrix}, \quad G^{24} = \begin{bmatrix} 0 & 0 & x_{24} \\ 0 & 0 & -x_{24} \\ y_{24} & 0 & z_{24} \end{bmatrix}, \quad G^{34} = \begin{bmatrix} 0 & 0 & x_{34} \\ 0 & 0 & -x_{34} \\ y_{34} & -y_{34} & z_{34} \end{bmatrix}. \end{aligned} \tag{III.54}$$

The sextuple $\left\{G^{ij}\right\}$ is compatible if and only if there exists a reconstruction consisting of 4 cameras P_i with centers that lie in a plane, but no three collinear, since the three epipoles are collinear in each image. We are free to choose coordinates in $\mathbb{P}^3$ without changing the fundamental matrices, so we take the four camera centers (assuming they exist) to be $[1,0,0,0]$, $[0,1,0,0]$, $[0,0,1,0]$, and $[1,1,1,0]$. Furthermore, by the definition of the epipole, we know that the epipoles satisfy

$$h_i^j = P_i(\ker(P_j)). \tag{III.55}$$

So if $\left\{G^{ij}\right\}$ has a reconstruction $\{P_i\}$, it must be on the form:

$$\begin{aligned} P_1 &= \begin{bmatrix} 0 & 1 & 0 & \alpha_1^1 \\ 0 & 0 & 1 & \alpha_1^2 \\ 0 & 0 & 0 & \alpha_1^3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 0 & \alpha_2^1 \\ 0 & 0 & 1 & \alpha_2^2 \\ 0 & 0 & 0 & \alpha_2^3 \end{bmatrix}, \\ P_3 &= \begin{bmatrix} 1 & 0 & 0 & \alpha_3^1 \\ 0 & 1 & 0 & \alpha_3^2 \\ 0 & 0 & 0 & \alpha_3^3 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 1 & 0 & -1 & \alpha_4^1 \\ 0 & 1 & -1 & \alpha_4^2 \\ 0 & 0 & 0 & \alpha_4^3 \end{bmatrix}, \end{aligned} \tag{III.56}$$

where the α_i^j are scalars. Since the fundamental matrices are of rank 2 and the cameras of rank 3, the four scalars α_i^3, as well as all the x_{ij} and y_{ij} are non-zero. Computing the fundamental matrices of these four cameras, and setting them equal to the G^{ij}, we get the following set of equations:

$$\begin{aligned} \frac{x_{12}}{y_{12}} &= -\frac{\alpha_1^3}{\alpha_2^3}, & \frac{x_{13}}{y_{13}} &= -\frac{\alpha_1^3}{\alpha_3^3}, & \frac{x_{14}}{y_{14}} &= -\frac{\alpha_1^3}{\alpha_4^3}, \\ \frac{x_{23}}{y_{23}} &= -\frac{\alpha_2^3}{\alpha_3^3}, & \frac{x_{24}}{y_{24}} &= -\frac{\alpha_2^3}{\alpha_4^3}, & \frac{x_{34}}{y_{34}} &= -\frac{\alpha_3^3}{\alpha_4^3}, \end{aligned} \tag{III.57}$$

and

$$\begin{aligned} \frac{z_{12}}{y_{12}} &= \frac{\alpha_1^2 - \alpha_2^2}{\alpha_2^3}, & \frac{z_{13}}{y_{13}} &= \frac{\alpha_1^1 - \alpha_3^2}{\alpha_3^3}, & \frac{z_{14}}{y_{14}} &= \frac{\alpha_1^1 - \alpha_1^2 - \alpha_4^2}{\alpha_4^3}, \\ \frac{z_{23}}{y_{23}} &= \frac{\alpha_2^1 - \alpha_3^1}{\alpha_3^3}, & \frac{z_{24}}{y_{24}} &= \frac{\alpha_2^1 - \alpha_2^2 - \alpha_4^1}{\alpha_4^3}, & \frac{z_{34}}{y_{34}} &= \frac{\alpha_3^1 + \alpha_4^2 - \alpha_3^2 - \alpha_4^1}{\alpha_4^3}. \end{aligned} \tag{III.58}$$

Eliminating the α_i^j from these equations gives us the following constraints:

$$x_{jk}x_{kl}y_{jl} + y_{jk}y_{kl}x_{jl} = 0 \quad \forall j < k < l, \tag{III.59}$$

and

$$\frac{x_{24}}{y_{24}}\frac{z_{12}}{y_{12}} - \frac{x_{34}}{y_{34}}\frac{z_{13}}{y_{13}} + \frac{x_{34}}{y_{34}}\frac{z_{23}}{y_{23}} - \frac{z_{14}}{y_{14}} + \frac{z_{24}}{y_{24}} - \frac{z_{34}}{y_{34}} = 0. \tag{III.60}$$

As in the proof of Theorem III.3.8, the fundamental matrices are compatible if and only if Equations (III.59) and (III.60) are satisfied. Let k be the smallest index satisfying $k \neq i, j$, then we can write

$$\begin{aligned} x_{ij} &= e_i^k F^{ij} \mathbf{x}_j, \\ y_{ij} &= \mathbf{x}_i F^{ij} e_j^k, \\ z_{ij} &= \mathbf{x}_i F^{ij} \mathbf{x}_j. \end{aligned} \tag{III.61}$$

With the substitution $\mathbf{x}_i = F^{ij} e_j^l$ in Equation (III.59), we get:

$$\begin{aligned} & y_{jl}x_{jk}x_{kl} + x_{jl}y_{jk}y_{kl} \\ =& (\mathbf{x}_j F^{jl} e_l^i)(e_j^i F^{jk} \mathbf{x}_k)(e_k^i F^{kl} \mathbf{x}_l) + (e_j^i F^{jl} \mathbf{x}_l)(\mathbf{x}_j F^{jk} e_k^i)(\mathbf{x}_k F^{kl} e_l^i) \\ =& (e_k^i F^{kj} F^{jl} e_l^i)(e_j^i F^{jk} F^{kl} e_l^i)(e_j^i F^{jl} F^{lk} e_k^i) + (e_j^i F^{jl} F^{lj} e_j^i)(e_k^i F^{kj} F^{jk} e_k^i)(e_l^i F^{lk} F^{kl} e_l^i), \\ =& \langle F^{jk} e_k^i, F^{jl} e_l^i \rangle \langle F^{kj} e_j^i, F^{kl} e_l^i \rangle \langle F^{lj} e_j^i, F^{lk} e_k^i \rangle + \|F^{lj} e_j^i\|^2 \|F^{jk} e_k^i\|^2 \|F^{kl} e_l^i\|^2 = 0, \end{aligned} \tag{III.62}$$

hence we arrive at Equation (III.49). In Equation (III.60), we use Equation (III.59) to substitute

$$-\frac{1}{y_{14}} = \frac{x_{13}x_{34}}{x_{14}y_{13}y_{34}} \tag{III.63}$$

and then plug in $\mathbf{x}_i = F^{ij} e_j^l$ (we do this step to get a homogeneous equation in every fundamental matrix and epipole). This gives us Equation (III.50). ■

Remark III.3.12. In the complex setting, we cannot always put $\mathbf{x}_i = F^{ij}e_j^l$ in Theorem III.3.10, because there is no longer any guarantee that this makes H_i invertible. For fixed complex F^{ij}, one can check if they are compatible in Case 2 instead by choosing any $\mathbf{x}_i$ that make H_i invertible. The same principle applies in Case 3.

Theorem III.3.13 (Case 3)**.** *Let $\{F^{ij}\}$ be a sextuple of fundamental matrices such that*

$$e_1^2 = e_1^3 \neq e_1^4,\ e_2^1 = e_2^3 \neq e_2^4,\ e_3^1 = e_3^2 \neq e_3^4, \tag{III.64}$$

and e_4^1, e_4^2, e_4^3 are distinct and lie on a line. Then $\{F^{ij}\}$ is compatible if and only if each triple is compatible.

Proof. Like in the two previous proofs, we begin by assuming the triple-wise conditions are satisfied, since we know them to be necessary. Fix a scaling such that $e_4^3 = e_4^1 + e_4^2$. Let

$$H_i = \begin{bmatrix} e_i^j & e_i^l & \mathbf{x}_i \end{bmatrix}, \quad H_4 = \begin{bmatrix} e_4^1 & e_4^2 & \mathbf{x}_4 \end{bmatrix} \tag{III.65}$$

for $i = 1, 2, 3$ and $j < k < l$, and

$$G^{ij} = H_i^T F^{ij} H_j. \tag{III.66}$$

Let $\mathbf{x}_i = F^{ij}e_j^l$ with $j < k < l$ for $i = 1, 2, 3$. Since all epipolar numbers are zero in this case, $(e_i^j)^T\mathbf{x}_i$ and $(e_i^l)^T\mathbf{x}_i$ are both zero. It follows that H_i is full-rank for $i = 1, 2, 3$. Let $\mathbf{x}_4$ be such that H_4 is full-rank. The fundamental matrices G^{ij} are compatible if and only if F^{ij} are. Note that the epipoles of G^{ij} are:

$$\begin{aligned}
h_1^2 &= [1,0,0], & \underline{h}_2^1 &= [1,0,0], & h_3^1 &= [1,0,0], & \underline{h}_4^1 &= [1,0,0],\\
h_1^3 &= [1,0,0], & \underline{h}_2^3 &= [1,0,0], & h_3^2 &= [1,0,0], & h_4^2 &= [0,1,0],\\
\underline{h}_1^4 &= [0,1,0], & \underline{h}_2^4 &= [0,1,0], & \underline{h}_3^4 &= [0,1,0], & \underline{h}_4^3 &= [1,1,0].
\end{aligned}$$

With these epipoles and the fact that the G^{ij} satisfy the triple-wise conditions (preserved under fundamental action), it follows that the six matrices must be on the form:

$$\begin{aligned}
G^{12} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x_{12} \\ 0 & y_{12} & z_{12} \end{bmatrix}, \quad G^{13} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x_{13} \\ 0 & y_{13} & z_{13} \end{bmatrix}, \quad G^{14} = \begin{bmatrix} 0 & 0 & x_{14} \\ 0 & 0 & 0 \\ 0 & y_{14} & z_{14} \end{bmatrix},\\
G^{23} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x_{23} \\ 0 & y_{23} & z_{23} \end{bmatrix}, \quad G^{24} = \begin{bmatrix} 0 & 0 & x_{24} \\ 0 & 0 & 0 \\ y_{24} & 0 & z_{24} \end{bmatrix}, \quad G^{34} = \begin{bmatrix} 0 & 0 & x_{34} \\ 0 & 0 & 0 \\ y_{34} & -y_{34} & z_{34} \end{bmatrix}.
\end{aligned} \tag{III.67}$$

The sextuple $\{G^{ij}\}$ is compatible if and only if there exists a reconstruction consisting of 4 cameras P_i with the centers of P_1, P_2, P_3 lying on a line that

does not contain the center of P_4. To see this, note that the three epipoles in each image are collinear, implying that any reconstruction must consist of cameras with coplanar centers. Furthermore, since two epipoles coincide in the first three images, the centers of P_1, P_2, P_3 must lie on a line. We are free to choose coordinates in $\mathbb{P}^3$ without changing the fundamental matrices, so we take the four camera centers (assuming they exist) to be $[1,0,0,0]$, $[0,1,0,0]$, $[1,1,0,0]$, and $[0,0,1,0]$. We recall that the epipoles satisfy

$$h_i^j = P_i(\ker(P_j)). \tag{III.68}$$

So if $\{G^{ij}\}$ has a reconstruction $\{P_i\}$, it must be on the form:

$$\begin{aligned} P_1 &= \begin{bmatrix} 0 & 1 & 0 & \alpha_1^1 \\ 0 & 0 & \beta_1 & \alpha_1^2 \\ 0 & 0 & 0 & \alpha_1^3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 0 & \alpha_2^1 \\ 0 & 0 & \beta_2 & \alpha_2^2 \\ 0 & 0 & 0 & \alpha_2^3 \end{bmatrix}, \\ P_3 &= \begin{bmatrix} 1 & -1 & 0 & \alpha_3^1 \\ 0 & 0 & \beta_3 & \alpha_3^2 \\ 0 & 0 & 0 & \alpha_3^3 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 1 & 0 & 0 & \alpha_4^1 \\ 0 & 1 & 0 & \alpha_4^2 \\ 0 & 0 & 0 & \alpha_4^3 \end{bmatrix}. \end{aligned} \tag{III.69}$$

where the β_i, α_i^j are scalars. Since the fundamental matrices are rank-2 and the cameras rank-3, the four scalars α_i^3, as well as all the β_i, x_{ij} and y_{ij} are non-zero. Computing the fundamental matrices of these four cameras, and setting them equal to the G^{ij}, we get after elimination the following two equations:

$$\begin{aligned} x_{12}x_{23}y_{13} + x_{13}y_{12}y_{23} &= 0, \\ \frac{x_{23}}{y_{23}}\frac{z_{12}}{y_{12}} + \frac{z_{13}}{y_{13}} - \frac{z_{23}}{y_{23}} &= 0. \end{aligned} \tag{III.70}$$

Similarly to the proofs of Cases 1 and 2, Equation (III.70) are equivalent to G^{ij} being compatible. We next observe that these equations precisely describe that G^{12}, G^{13} and G^{23} are compatible. Indeed, we have seen in the proof of Proposition III.3.4 that for compatibility we must have (up to scale)

$$G^{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -y_{12}x_{13} \\ 0 & x_{12}y_{13} & x_{12}z_{13} - x_{13}z_{12} \end{bmatrix}. \tag{III.71}$$

This is equivalent to

$$\operatorname{rank} \begin{bmatrix} -y_{12}x_{13} & x_{23} \\ x_{12}z_{13} - x_{13}z_{12} & z_{23} \\ x_{12}y_{13} & y_{23} \end{bmatrix} = 1 \tag{III.72}$$

Setting the 2×2 minors of this 3×2 matrix to zero, we get a polynomial system equivalent to Equation (III.70), finishing the proof. ■

Theorem III.3.14 (Case 4)**.** *Let $\{F^{ij}\}$ be a sextuple of fundamental matrices such that in each image, all three epipoles coincide. Then $\{F^{ij}\}$ is compatible if and only if each triple is compatible.*

This result is a direct consequence of Theorem III.3.15, proven in the next subsection.

III.3.3 K_n

For the case of more than 4 cameras, it turns out that quadruple-wise compatibility is sufficient to ensure global compatibility.

Theorem III.3.15. *Let $\{F^{ij}\}$ be a complete set of $\binom{n}{2}$, $n \geq 4$, fundamental matrices such that for all i, j, k, l, the sextuple $F^{ij}, F^{ik}, F^{jk}, F^{il}, F^{jl}, F^{kl}$ is compatible. Then $\{F^{ij}\}$ is compatible.*

Moreover, if all epipoles in each image coincide, then triple-wise compatibility implies that $\{F^{ij}\}$ is compatible. The reconstruction in this case will be a set of cameras whose centers all lie on a line.

In the non-collinear case, we actually don't need to assume that all sextuples are compatible. It suffices that there is a sextuple $F^{12}, F^{13}, F^{14}, F^{23}, F^{24}, F^{34}$ that is compatible with a solution of cameras P_1, P_2, P_3, P_4 such that the line spanned by the centers of P_1, P_2 do not contain the centers of P_3, P_4, and that each sextuple of fundamental matrices corresponding to indices $\{1, 2, 3, i\}$ and $\{1, 2, 4, i\}$ for $i \geq 5$ are compatible. This is what we show in the proof below.

Proof. We start with the collinear case. As in the proof of Proposition III.3.4, it suffices to prove the statement for fundamental matrices

$$G^{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{ij} & b_{ij} \\ 0 & c_{ij} & d_{ij} \end{bmatrix}. \tag{III.73}$$

By the compatibility of $\{G^{1i}, G^{1j}, G^{ij}\}$, we have by Proposition III.3.4 that

$$G^{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_{1i}a_{1j} - a_{1i}c_{1j} & c_{1i}b_{1j} - a_{1i}d_{1j} \\ 0 & d_{1i}a_{1j} - b_{1i}c_{1j} & d_{1i}b_{1j} - b_{1i}d_{1j} \end{bmatrix}, \tag{III.74}$$

for all $i, j \neq 1$. It can be verified that the following cameras P_i form a reconstruction of these fundamental matrices:

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad P_i = \begin{bmatrix} \gamma_i & 1 & 0 & 0 \\ 0 & 0 & b_{1i} & d_{1i} \\ 0 & 0 & -a_{1i} & -c_{1i} \end{bmatrix}, \forall i \neq 1, \tag{III.75}$$

where $\gamma_i \neq 0$ are distinct numbers. Hence the $\binom{n}{2}$-tuple is compatible whenever each triple is compatible. We also observe that all cameras have a center lying on the line $[\lambda_1, \lambda_2, 0, 0]$.

Now assume that in some image, not all epipoles coincide. We prove the theorem for the case $n = 5$ and note that the principle extends to any n.

Consider a sextuple $S_{1234} = \{F^{12}, F^{13}, F^{14}, F^{23}, F^{24}, F^{34}\}$, where in some image, not all epipoles coincides. Let P_1, P_2, P_3, P_4 be a solution and without loss of generality assume that the line spanned by the centers of P_1, P_2 do not contain the centers of P_3, P_4. Let P'_1, P'_2, P'_3, P_5 be a solution to $S_{1235} =$

$\{F^{12}, F^{13}, F^{15}, F^{23}, F^{25}, F^{35}\}$. By Lemma III.1.2, we have that P_1, P_2, P_3 and P_1', P_2', P_3' differ by PGL_4, and we may therefore take them to be equal.

It remains to prove that F^{45} is the fundamental matrix of P_4, P_5. For this we note that either 1) P_1, P_2, P_5 or 2) P_1, P_3, P_5 are not collinear cameras, since P_1, P_2, P_3 are not collinear. In the first case 1), consider the tuple $S_{1245} = \{F^{12}, F^{14}, F^{15}, F^{24}, F^{25}, F^{45}\}$ with solution $P_1'', P_2'', P_4'', P_5''$. By Lemma III.1.2, the overlap between S_{1235} and S_{1245} imply that we can via PGL_4 action assume $P_1'' = P_1, P_2'' = P_2, P_5'' = P_5$, and the overlap between S_{1234} and S_{1245} imply that we can also assume $P_4'' = P_4$, since P_1, P_2, P_4 are not collinear. But since F^{45} is the fundamental matrix of P_4'', P_5'' we conclude that it is also the fundamental matrix of P_4, P_5. In the second case 2) the argument is analogous when we consider S_{1345} instead of S_{1245}. ■

While uniqueness is not the focus of this paper, we give the following useful theorem on the complete graph:

Proposition III.3.16. *A compatible set of $\binom{n}{2}$ fundamental matrices has a unique solution up to action by* $\mathrm{PGL}(4)$*, unless all the epipoles in each image are equal.*

Proof. If the set of fundamental matrices is compatible, and the epipoles in each image are not all equal, we know that there exists a reconstruction consisting of n cameras, not all lying on a line. It follows from the Sylvester-Gallai theorem [AZ99, Chapter 11] that there will always be at least two cameras P_1, P_2 such that the line spanned by their camera centers does not contain any other camera centers. By Lemma III.1.2, a triple of compatible fundamental matrices has a unique solution if the two epipoles in each image are distinct, or equivalently if their reconstruction consists of three non-collinear cameras. Up to projective transformation, we can uniquely recover P_1, P_2 from F^{12}, which fixes coordinates in $\mathbb{P}^3$. All other cameras P_i are then uniquely determined by the triple F^{12}, F^{1i}, F^{2i}. Since this uniquely determines all cameras (up to global projective transformation), the fundamental matrices F^{ij} can only have one solution.

Conversely, in the case that all epipoles in each image are equal, the constructed solution of cameras P_i in the proof of Theorem III.3.15 shows that there is no unique reconstruction of the centers (up to global projective transformation). This is because the choice of distinct numbers γ_i, was arbitrary. ■

III.3.4 n-view matrices

The compatibility of $\binom{n}{2}$ fundamental matrices F^{ij} was also studied in [Gei+20; Kas+19b] and we recall their results below. Given a set of $\binom{n}{2}$ fundamental

matrices F^{ij}, the *n-view fundamental matrix* is the $3n \times 3n$ symmetric matrix

$$\mathbf{F} := \begin{bmatrix} 0 & F^{12} & \cdots & F^{1n} \\ F^{21} & 0 & \cdots & F^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ F^{n1} & F^{n2} & \cdots & 0 \end{bmatrix}. \tag{III.76}$$

Theorem III.3.17 (Theorem 1 of [Kas+19b], Theorem 2 of [Gei+20])**.** *Let* $\{F^{ij}\}$ *be a complete set of* $\binom{n}{2}$ *real fundamental matrices, where* $n \geq 3$*. Then* $\{F^{ij}\}$ *is compatible with a solution of real cameras whose centers are not all collinear if and only if there exist non-zero scalars* $\lambda_{ij} = \lambda_{ji}$ *such that:*

III.3.17.1. *the n-view fundamental matrix* $\mathbf{F} = (\lambda_{ij}F^{ij})_{ij}$ *is rank-6 and has exactly three positive and three negative eigenvalues;*

III.3.17.2. *the* $3 \times 3n$ *and* $3n \times 3$ *block rows and block columns of* $\mathbf{F}$ *are all of rank 3.*

Further, $\{F^{ij}\}$ *is compatible with a solution of real cameras whose centers are all collinear if and only if there exist non-zero scalars* $\lambda_{ij} = \lambda_{ji}$ *such that:*

III.3.17.1. *the n-view fundamental matrix* $\mathbf{F} = (\lambda_{ij}F^{ij})_{ij}$ *is rank-4 and has exactly two positive and two negative eigenvalues;*

III.3.17.2. *the* $3 \times 3n$ *and* $3n \times 3$ *block rows and block columns of* $\mathbf{F}$ *are all of rank 2.*

Our work regarding the K_3 and K_4 cases can be used to improve on this result by showing that the eigenvalue condition can be dropped in the cases below.

Theorem III.3.18. *In the collinear case of Theorem III.3.17, the eigenvalue condition can be dropped. In the non-collinear case, the eigenvalue condition can be dropped if in each image, no three epipoles lie on a line.*

Proof. The structure of the proof is as follows. We prove in detail the when $n = 3$ and sketch $n = 4$ for Case 1. The `Macaulay2` code used in all these settings is attached. Then, we use Theorem III.3.15 to argue that the general setting is implied by these case studies.

We start with $n = 3$ in the collinear setting. Let F^{ij} be three fundamental matrices for which there exists a scaling λ such that

$$\begin{bmatrix} 0 & F^{12} & F^{13} \\ F^{21} & 0 & \lambda F^{23} \\ F^{31} & \lambda F^{32} & 0 \end{bmatrix} \tag{III.77}$$

is rank-4 and the 3×6 and 6×3 block rows and colums are rank-2. Note that we don't need to scale F^{12} and F^{21} or F^{13} and F^{31}, because scaling each row and each column does not change the rank of the 3-view matrix, so we may

choose their scalings to be 1 without loss of generality. By the latter condition, F^{12} and F^{13} must have the same epipoles. We can say even more, namely that

$$e_1^2 = e_1^3, \quad e_2^1 = e_2^3, \quad e_3^1 = e_3^2. \tag{III.78}$$

As in the proof of Proposition III.3.4, this assumption allows us to assume via fundamental action F^{ij} take the form G^{ij} of Equation (III.22). We work in the polynomial ring $R = \mathbb{Q}[a_{ij}, b_{ij}, c_{ij}, d_{ij}, \lambda]$, where $1 \leq i < j \leq 3$ consider the following 3-view matrix:

$$\mathbf{G}(\lambda) := \begin{bmatrix} 0 & G^{12} & G^{13} \\ G^{21} & 0 & \lambda G^{23} \\ G^{31} & \lambda G^{32} & 0 \end{bmatrix}. \tag{III.79}$$

The rank of $\mathbf{G}(\lambda)$ is at most 4 if and only if all 5×5 minors of $\mathbf{G}(\lambda)$ vanish and we therefore consider the ideal I_{minors} in R defined by the 5×5 minors of $\mathbf{G}(\lambda)$. Since we don't want solutions with $\lambda = 0$ or $\text{rank} G^{ij} < 2$, we saturate I_{minors} with respect to the ideals $I_\lambda = \langle \lambda \rangle$ and $I_{ij} = \langle a_{ij}d_{ij} - b_{ij}c_{ij} \rangle$. After this is done in `Macaulay2`, we get a new ideal I_{rank} in R with nine generators.

Write $G^{ij'}$ for the matrices we get by removing the first row and column from G^{ij}. Recall that G^{ij} as in Equation (III.22) are compatible if and only if they are rank-2 and up to scaling, $G^{12'} \star G^{13'} = G^{23'}$, i.e. Equation (III.28) holds. As in the proof of Theorem III.3.10, this equality is described by vectorizing $G^{12'} \star G^{13'}$ and $G^{23'}$, putting them into a 4×2 matrix and setting the rank to 1. By doing this we get 6 2×2 minors and we let J_{red} in R be the ideal generated by these equations. Note that this ideal is reducible, as shown by the command `primaryDecomposition` in `Macaulay2`. One component consists of rank-deficient tuples $G^{ij'}$ and we call the other component $J_\star$. In particular, any tuple of rank-2 matrices $G^{ij'}$ satisfy Equation (III.28) (up to scale) if and only if they satisfy the conditions of $J_\star$.

By Lemma III.1.4, if G^{ij} are rank-2, on the form Equation (III.22), and there exists $\lambda \neq 0$ with $\mathbf{G}(\lambda)$ rank-4, then the entries of G^{ij} satisfy the equations of I_{rank}. In `Macaulay2` we see that the ideals I_{rank} and $J_\star$ are equal. It follows that G^{ij} satisfy the equations of $J_\star$. By the above, this implies that G^{ij} are compatible, showing that the eigenvalue condition was not needed for compatibility.

For $n = 3$ in the non-collinear setting, we choose a fundamental action

$$H_1 = \begin{bmatrix} e_1^2 & e_1^3 & \mathbf{x}_1 \end{bmatrix}, H_2 = \begin{bmatrix} e_2^1 & e_2^3 & \mathbf{x}_2 \end{bmatrix}, H_3 \quad = \begin{bmatrix} e_3^1 & e_3^2 & \mathbf{x}_3 \end{bmatrix}, \tag{III.80}$$

for $\mathbf{x}_i$ making H_i full-rank. Using this as our fundamental action, we get a new sextuple of fundamental matrices

$$G^{ij} = H_i^T F^{ij} H_j. \tag{III.81}$$

The sextuple $\{G^{ij}\}$ is compatible if and only if $\{F^{ij}\}$ is. Note that the epipoles of G^{ij}, denoted by h_j^i, are:

$$\begin{aligned} h_1^2 &= [1,0,0], \quad h_2^1 = [1,0,0], \quad h_3^1 = [1,0,0], \\ h_1^3 &= [0,1,0], \quad h_2^3 = [0,1,0], \quad h_3^2 = [0,1,0]. \end{aligned} \tag{III.82}$$

The three matrices must be on the form:

$$G^{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_{12} & y_{12} \\ 0 & z_{12} & w_{12} \end{bmatrix}, \quad G^{13} = \begin{bmatrix} 0 & x_{13} & y_{13} \\ 0 & 0 & 0 \\ 0 & y_{13} & z_{13} \end{bmatrix}, \quad G^{23} = \begin{bmatrix} x_{23} & 0 & y_{23} \\ 0 & 0 & 0 \\ z_{23} & 0 & w_{23} \end{bmatrix}. \tag{III.83}$$

We work in the polynomial ring $R = \mathbb{Q}[x_{ij}, y_{ij}, z_{ij}, w_{ij}, \lambda]$, where $1 \leq i < j \leq 3$ consider the following 3-view matrix:

$$\mathbf{G}(\lambda) := \begin{bmatrix} 0 & G^{12} & G^{13} \\ G^{21} & 0 & \lambda G^{23} \\ G^{31} & \lambda G^{32} & 0 \end{bmatrix}. \tag{III.84}$$

The corresponding I_{rank}, defined analogously to the collinear case, equals $\langle x_{12}, x_{13}, x_{23} \rangle$. This means $\mathbf{G}(\lambda)$ being rank-6 for a $\lambda \neq 0$ implies $x_{12} = 0, x_{13} = 0, x_{23} = 0$.

As in the proof of Theorem III.3.8, if there is a solution of cameras P_i with non-collinear centers to Equation (III.83), then we may choose them to be

$$P_1 = \begin{bmatrix} 0 & \alpha_1^1 & 0 & * \\ 0 & 0 & \alpha_1^2 & * \\ 0 & 0 & 0 & \alpha_1^3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} \alpha_2^1 & 0 & 0 & * \\ 0 & 0 & \alpha_2^2 & * \\ 0 & 0 & 0 & \alpha_2^3 \end{bmatrix}, \quad P_3 = \begin{bmatrix} \alpha_3^1 & 0 & 0 & * \\ 0 & \alpha_3^2 & 0 & * \\ 0 & 0 & 0 & \alpha_3^3 \end{bmatrix}, \tag{III.85}$$

where α_i^j are non-zero scalars, and $*$ are some other (possible zero) scalars. Computing the fundamental matrices of these four cameras, one can check that by the degrees of freedom of the cameras established in Equation (III.85), any triple of fundamental matrices on the form Equation (III.83) with $x_{ij} = 0$ has a solution with cameras on the form Equation (III.85). It follows that if there is a non-zero scalar λ for which Equation (III.84) is rank-6, then the triples of fundamental matrices G^{ij} are compatible, which is sufficient.

In the setting of $n = 4$ in Case 1, we use the same ideas and therefore only sketch the proofs. Start with a 4-view matrix $\mathbf{F}$ that is rank-6 and with block rows and columns of rank-3 as in Theorem III.3.17. Then take any sub 3-view matrix $\mathbf{F}'$. It is at most rank-6. However, since the epipoles in each image are all distinct, all its block rows and columns must be rank-3. This is only possible if $\mathbf{F}'$ is at least rank-6. Now we can apply the above to see that the three fundamental matrices of this 3-view matrix are compatible. In other words, we have triple-wise compatibility. Then we can assume the fundamental matrices to be of the form Equation (III.42) and look at the ideal generated by the 7×7 minors given such matrices with indeterminate entries. Here we scale G^{23}, G^{24}, G^{34} with $\lambda_1, \lambda_2, \lambda_3$, respectively. After saturation of λ_i and rank-deficienly loci, and after elimination of λ_i, we get in each case an ideal that we call I_{rank}. This ideal in each case describes the same conditions as the ideal generated by Equation (III.46). This means that the rank condition implies compatibility.

Now we move on to general values of n. First, in the general collinear case, let F^{ij} be fundamental matrices for which there are scalars λ_{ij} such that the

n-view matrix $\mathbf{F} = (\lambda_{ij} F^{ij})$ is rank-4 and whose $3 \times 3n$ and $3n \times 3$ block rows and columns are rank-2. By Theorem III.3.15, it suffices to show triple-wise compatibility. Take any 3-view submatrix $\mathbf{F}'$. It is at most rank-4 and its block rows and columns at most rank-2. But since the fundamental matrices are rank-2, the block rows and columns must be at least rank-2 and it follows that the 3-view matrix itself is at least rank-4. Therefore triple-wise compatibility follows from an earlier step of this proof. By similar logic, if the n-view matrix $\mathbf{F}$ instead is rank-6 with block rows and columns of rank 3, then this also applies for any sub 4-view matrix $\mathbf{F}'$, since we assumed that any three epipoles in each image do not lie on a line. In particular, we are then in Case 1 and by the above, we have quadruple-wise compatibility. By Theorem III.3.15, this suffices. ■

III.4 The Cycle Theorem

Although the focus of this paper has been on complete graphs, in this section we state the cycle theorem, which holds for all graphs. We use this theorem to give an alternative derivation of necessary conditions for compatibility from Section III.3. We consider sets of fundamental matrices $\{F^{ij}\}$, where the index pairs (ij) are a subset of all $\binom{n}{2}$ possible ones. Let $\mathcal{G} = (V, E)$ denote the corresponding graph, where V is the set of indices and E the set of pairs of indices for which there is a fundamental matrix in our set. The definitions of compatibility and solution extend naturally to this setting.

The theorem below gives a necessary and sufficient condition for when a set of fundamental matrices are compatible using the cycle condition for any graph $\mathcal{G}$. Recall that a directed cycle C of a graph is a closed path, i.e. a path that starts and ends at the same vertex. Let $E(C)$ denote its directed edges.

Theorem III.4.1. *Let $\{F^{ij}\}$ be a set of fundamental matrices with corresponding graph $\mathcal{G}$. $\{F^{ij}\}$ is compatible if and only if there are matrices $H_i \in \mathrm{GL}_3$ and scalars $\lambda_{ij} = \lambda_{ji} \neq 0$ such that $G^{ij} := \lambda_{ij} H_i^T F^{ij} H_j$ satisfy*

$$\sum_{(ij) \in E(C)} G^{ij} = 0, \text{ for each directed cycle } C \text{ of } \mathcal{G}. \tag{III.86}$$

In particular, any set of 3×3 rank-2 matrices G^{ij} satisfying the cycle condition *Equation (III.86) are the fundamental matrices of some set of cameras.*

This theorem is very similar to the result [AF18, Proposition 5], which appears in the context of parallel rigidity and is relevant for the solvability of essential matrices. Observe that the cycles of length two in Equation (III.86) imply that G^{ij} are skew-symmetric.

Proof of Theorem III.4.1, direction $\Rightarrow$. Let $\{F^{ij}\}$ be a compatible set of fundamental matrices, with a solution of cameras P_i. By right action of $H \in \mathrm{GL}_4$, we may assume that the centers of these cameras has a non-zero last coordinate. Then the first three vectors must be linearly independent and the cameras can be written $[H_i | v^{(i)}]$, where $H_i \in \mathrm{GL}_3$ and $v^{(i)} \in \mathbb{R}^3$. By left multiplication with

H_i^{-1}, we may further assume that all cameras are of the form $C_i = [I|t^{(i)}]$, where $t^{(i)} \in \mathbb{R}^3$. Recall that definition of $[t]_\times$ for $t \in \mathbb{R}^3$ from Section III.3.1. One can check that the fundamental matrix of C_i and C_j is

$$[t^{(j)}]_\times - [t^{(i)}]_\times = [t^{(j)} - t^{(i)}]_\times \in \mathbb{R}^{3\times 3}, \tag{III.87}$$

and we call these skew-symmetric matrices G^{ij}. Note that G^{ij} are scalings of $H_i^T F^{ij} H_j$. If we sum G^{ij} for (ij) in a cycle C, we must get 0 by Equation (III.87). ■

For the other direction, we need a lemma.

Lemma III.4.2. *Let $\mathcal{G}$ be a connected graph and T any spanning tree subgraph. Then there is a sequence $T^i \subseteq \mathcal{G}$ such that*

$$T = T^0 \subseteq \cdots \subseteq T^k = \mathcal{G}, \tag{III.88}$$

where T^{i+1} contains exactly one more edge than T^i and this edge is part of a cycle of T^{i+1}.

Proof. We get T^{k-1} from T^k by removing an edge of T^k that is not in T. We repeat this process until we reach T^0. To see that this suffices, assume by contradiction that the edge removed from T^{i+1} is not part of a cycle of T^{i+1}. Then T^i would have to be disconnected. This implies that T cannot be connected, which is a contradiction. ■

Proof of Theorem III.4.1, direction $\Leftarrow$. We find a set of cameras C_i such that $\psi(C_i, C_j)$ equals G^{ij} for every edge of $\mathcal{G}$. Since F^{ij} and G^{ij} are equivalent under fundamental action, this is enough. We may without restriction assume that $\mathcal{G}$ is connected with n nodes. Since G^{ij} are skew-symmetric and rank-2, there are non-zero $g^{ij} \in \mathbb{R}^3$ such that $G^{ij} = [g^{ij}]_\times$. The cycle condition is then equivalent to

$$\sum_{(ij)\in E(C)} g^{ij} = 0, \text{ for each directed cycle } C \text{ of } \mathcal{G}. \tag{III.89}$$

Let T be a spanning tree subgraph of $\mathcal{G}$.

Fix $i = 1$ and let $t^{(1)} = 0 \in \mathbb{R}^3$. To any node v in T, there is a unique path with no repeated vertices from 1 to v in T, since T is a tree. Let $\sigma_{u,v} = \{u = i_1, i_2, \ldots, i_k = v\}$ denote the unique path between two vertices u, v of T. For $i > 1$, define

$$t^{(v)} := \sum_{(ij)\in\sigma_{1,v}} g^{ij}. \tag{III.90}$$

This gives us cameras $C_i = [I|t^{(i)}]$ for each $i = 1, \ldots, m$. We must check that $G^{ij} = \psi(C_i, C_j)$ for every edge of $\mathcal{G}$. Recall that for cameras on this form, $\psi(C_i, C_j) = [t^{(j)} - t^{(i)}]_\times$. If (ij) is an edge of T, then $t^{(j)} - t^{(i)} = g^{ij}$ by construction, which shows $G^{ij} = \psi(C_i, C_j)$. For (ij) that are not edges of T, we

proceed as follows. Consider the sequence T^i of Lemma III.4.2. We proceed via induction to show that $G^{ij} = \psi(C_i, C_j)$ for every edge of T^l for any l. The base case $T^0 = T$ is already done. Assume that C_i satisfy $G^{ij} = \psi(C_i, C_j)$ for all edges of T^l. In T^{l+1}, there is precisely one new edge (ij) and that edge is part of a cycle C of T^{l+1}. Using Equation (III.90), we get after some cancellation for some vertex u of the cycle that

$$\psi(C_i, C_j) = [t^{(j)} - t^{(i)}]_\times \tag{III.91}$$

$$= \sum_{(st)\in\sigma_{u,j}} [g^{st}]_\times - \sum_{(st)\in\sigma_{u,i}} [g^{st}]_\times. \tag{III.92}$$

Since G^{ij} are skew-symmetric by the conditions of the 2-cycles, $g^{ji} = -g^{ij}$. Therefore we get

$$\psi(C_i, C_j) = \sum_{(st)\in\sigma_{i,j}} [g^{st}]_\times. \tag{III.93}$$

However, by the cycle condition for the cycle C, this equals $[g^{ij}]_\times$, which shows $G^{ij} = \psi(C_i, C_j)$ for every edge in T^{l+1} and completes the induction. ■

For the rest of this section, we apply the cycle theorem to find conditions that must hold for compatible fundamental matrices. For instance, let G^{12}, G^{13} and G^{23} be fundamental matrices satisfying the cycle condition. By the 2-cycles, we can write $G^{ij} = [g^{ij}]_\times$ for some $g^{ij} \in \mathbb{R}^3$. Letting $\{h^i_j\}$ be the epipoles of $\left\{G^{ij}\right\}$ defined as

$$h^i_j := (g^{ij}_1, g^{ij}_2, g^{ij}_3)^T, \tag{III.94}$$

one can check that

$$(h^1_2)^T G^{23} h^1_3 = \det[g^{12}\ g^{23}\ g^{31}]. \tag{III.95}$$

Therefore, $g^{12} + g^{23} + g^{31} = 0$ implies $(h^1_2)^T G^{23} h^1_3 = 0$. Now if F^{12}, F^{13} and F^{23} are compatible fundamental matrices, then by the cycle theorem there is a scaling and fundamental action such that $G^{ij} = \lambda_{ij} H_i^T F^{ij} H_j$ satisfy the cycle condition. This means that F^{ij} must satisfy $e^1_2 F^{23} e^1_3 = 0$, hence giving us Equation (III.14).

We next sketch an argument for why the n-view fundamental matrix $\mathbf{F}$ (see Section III.3) for compatible $\left\{F^{ij}\right\}$ is at most rank 6 given appropiate scalings. For the sake of simplicity assume $n = 4$, but note that the below principle directly extends to any n. Let $\left\{G^{ij}\right\}$ be six fundamental matrices satisfying the cycle condition. Consider the 4-view matrix $\mathbf{G} = (G^{ij})_{ij}$. Subtracting the first row of $\mathbf{G}$ from the other rows, we have

$$\mathbf{G} = \begin{bmatrix} 0 & G^{12} & G^{13} & G^{14} \\ G^{21} & 0 & G^{23} & G^{24} \\ G^{31} & G^{32} & 0 & G^{34} \\ G^{41} & G^{42} & G^{43} & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & G^{12} & G^{13} & G^{14} \\ -G^{12} & -G^{12} & -G^{12} & -G^{12} \\ -G^{13} & -G^{13} & -G^{13} & -G^{13} \\ -G^{14} & -G^{14} & -G^{14} & -G^{14} \end{bmatrix}, \tag{III.96}$$

where $\sim$ denotes equivalence under Gaussian eliminiation. The rank of the first three rows of Equation (III.96) is at most 3, and the rank of the last nine rows is the rank of the first three columns of Equation (III.96), which is at most 3. In total, the matrix is of rank at most 6. Now if $\{F^{ij}\}$ is a set of compatible fundamental matrices, there is a scaling and fundamental action such that $G^{ij} = \lambda_{ij} H_i^T F^{ij} H_j$ satisfy the cycle condition. Define the n-view fundamental matrix $\mathbf{F} = (\lambda_{ij} F^{ij})_{ij}$. Since the rank of a matrix is invariant under conjugation, the above shows that rank $\mathbf{F} \leq 6$.

Finally, we use the cycle theorem to give alternative proof that Equation (III.38) is necessary to ensure compatibility. Let $\{G^{ij}\}$ be 6 skew-symmetric matrices. Again, write $G^{ij} = [g^{ij}]_\times$ and let $\lambda_{ij} = \lambda_{ji} \neq 0$ be scalars such that $\lambda_{ij} G^{ij}$ satisfy the cycle condition. Then

$$\lambda_{kl} g^{kl} = -\lambda_{jk} g^{jk} - \lambda_{ij} g^{ij} - \lambda_{li} g^{li}, \tag{III.97}$$

for all indices $i, j, k, l \in \{1, 2, 3, 4\}$ and it follows that

$$\begin{aligned} &\det[\lambda_{ij} g^{ij} \ \lambda_{jk} g^{jk} \ \lambda_{kl} g^{kl}] \\ =&\det[\lambda_{ij} g^{ij} \ \lambda_{jk} g^{jk} \ -\lambda_{li} g^{li}] \\ =&-\det[\lambda_{li} g^{li} \ \lambda_{ij} g^{ij} \ \lambda_{jk} g^{jk}]. \end{aligned} \tag{III.98}$$

Factoring out the constants, and with h_j^i defined as in Equation (III.94), we get

$$\lambda_{ij}\lambda_{jk}\lambda_{ki} (h_j^i)^T G^{jk} h_k^l = -\lambda_{li}\lambda_{ij}\lambda_{jk} (h_i^l)^T G^{ij} h_j^k. \tag{III.99}$$

Assuming that all epipolar numbers $(h_j^i)^T G^{jk} h_k^l$ are non-zero, and recalling that λ_{ij} are non-zero, we find

$$\frac{(h_j^i)^T G^{jk} h_k^l}{(h_i^l)^T G^{ij} h_j^k} = -\frac{\lambda_{li}}{\lambda_{ki}}. \tag{III.100}$$

Further, using $\lambda_{ij} = \lambda_{ji}$,

$$\frac{\lambda_{31}}{\lambda_{21}} \frac{\lambda_{12}}{\lambda_{32}} \frac{\lambda_{23}}{\lambda_{43}} \frac{\lambda_{34}}{\lambda_{24}} \frac{\lambda_{24}}{\lambda_{14}} \frac{\lambda_{41}}{\lambda_{31}} = 1. \tag{III.101}$$

Combining Equations (III.100) and (III.101), we get Equation (III.38) for $\{\lambda_{ij} G^{ij}\}$. Now if we start with a set of six compatible fundamental matrices $\{F^{ij}\}$, then by Theorem III.4.1, there is a fundamental action such that $G^{ij} = H_i^T F^{ij} H_j$ are skew-symmetric and there are scalars λ_{ij} making the cycle condition hold for $\{\lambda_{ij} G^{ij}\}$. Then Equation (III.38) holds for $\{G^{ij}\}$ and by the invariance of the epipolar numbers under fundamental action, we get Equation (III.38) for $\{F^{ij}\}$.

III.5 Image of the Fundamental Map

Related to the study of the constraints satisfied by compatible fundamental matrices, is the image of the *fundamental map* given a viewing graph $\mathcal{G} = (V, E)$:

$$\Psi_{\mathcal{G}} : (\mathbb{P}^{3\times 4})^m \dashrightarrow (\mathbb{P}^{3\times 3})^E, \tag{III.102}$$

$$(P_1, \dots, P_m) \mapsto (\psi(P_i, P_j))_{(ij)\in E}. \tag{III.103}$$

The fundamental map sends real projective camera matrices to a set of corresponding fundamental matrices. We define the *viewing graph variety* $\mathcal{V}_{\mathcal{G}}$ to be the *Zariski closure* of the image $\mathrm{Im}\Psi_{\mathcal{G}}$. By Chevalley's theorem, in this case the Zariski closure is equal to the Euclidean closure [MS21, Theorem 4.19].

A natural question from the algebraic geometry point of view is if this variety is described by the constraints we proposed in Section III.3, in the complete graph case. We prove that that is not the case, and leave it is as an open problem to describe the viewing graph variety precisely.

Proposition III.5.1. *The viewing graph variety of K_n for $n \geq 3$ is a proper subset of the variety in $(\mathbb{P}^{3\times 3})^{\binom{n}{2}}$ defined by the $3\binom{n}{3}$ triple-wise constraints and the $\binom{n}{4}$ quadruple-wise constraints of Theorem III.3.8.*

Note that strictly speaking, e^i_j is not a polynomial in the entries of F^{ij}, because there is no way to write a generator of the left kernel of a matrix X as a polynomial expression that works for every 3×3 matrix of rank-2. However, one can for instance turn the expression $(e^s_i)^T F^{ij} e^s_j = 0$ into a polynomial system in F^{si}, F^{ij} and F^{sj} by defining the epipoles on affine patches of the fundamental matrices, which we don't explain here in further detail. In any case, for a rank-1 3×3 matrices, the epipoles are understood as the 0 vector.

Proof. We do the proof for K_3, but note that our counterexample below can be directly extended to any K_n.

The Euclidean closure of the set of three camera matrices $(P_1, P_2, P_3) \in (\mathbb{P}^{3\times 4})^3$ of different centers is all of $(\mathbb{P}^{3\times 4})^3$. Then since $\mathcal{V}_{K_3}$ is the Euclidean closure of $\mathrm{Im}\Psi_{K_3}$, any of its elements can be arbitrarily approximated by the image of full-rank cameras. We give an example showing that the triple-wise constraints are not enough to describe $\mathcal{V}_{K_3}$ by finding an element that cannot be approximated in the way described above. Consider the following example:

$$F^{12} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, F^{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}. \tag{III.104}$$

We can assume that $P_1 = [I|0]$ and $P_2 = [I|(0; 0; -1)]$. Then the following are the only options for P_3:

$$P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1+b & c & d \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{III.105}$$

for any a, b, c, d such that $d \neq 0$. We get that

$$F^{23} = \psi(P_2, P_3) = \begin{bmatrix} a & -1 & c+d \\ b+1 & 0 & 0 \\ -d & 0 & 0 \end{bmatrix}. \tag{III.106}$$

This matrix is rank-2 if and only if $d \neq 0$ or $b \neq -1$. Any such choice gives a triplet satisfying the triple-wise conditions. Also F^{12}, F^{13}, S^{23} satisfy the triple-wise constraints, where

$$S^{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{III.107}$$

because the epipole of a rank 1 matrix is 0.

Now any arbitrarily small perturbance of F^{12} and F^{13} leads to an arbitrarily small change in the choice of F^{23} from Equation (III.106). But no small perturbance of Equation (III.106) equals S^{23}, which shows that F^{12}, F^{13}, S^{23} does not lie in $\mathcal{V}_{K_3}$ and we are done. ■

III.6 Conclusion

This paper provided explicit polynomial contraints as necessary and sufficient conditions for $\binom{n}{2}$ fundamental matrices to be compatible. These polynomials were expressed in terms of the fundamental matrices and their epipoles, and are projectively well-defined, i.e. homogeneous. As a consequence of our work, the previously established necessary and sufficient condition [Kas+19b] can be simplified by dropping the eigenvalue condition in certain cases. Our main tool was to define and use the fundamental action of sets of fundamental matrices. We gave a necessary and sufficient condition for compatibility that applied not only to complete graphs, but to any viewing graph. We used it to give an alternative derivation of necessary conditions for compatibility. In the final section, we introduced the viewing graph variety and gave a first result in the case of complete graphs.

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Paper IV

A Classification of Critical Configurations for any Number of Projective Views

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Abstract

Structure from motion is the process of recovering information about cameras and 3D scene from a set of images. Generally, in a noise-free setting, all information can be uniquely recovered if enough images and image points are provided. There are, however, certain cases where unique recovery is impossible, even in theory; these are called *critical configurations*. We use a recently developed algebraic approach to classify all critical configurations for any number of projective cameras. We show that they form well-known algebraic varieties, such as quadric surfaces and curves of degree at most 4. This paper also improves upon earlier results both by finding previously unknown critical configurations and by showing that some configurations previously believed to be critical are in fact not.

IV.1 Introduction

One of the oldest and most well-studied problems in computer vision is that of *structure from motion*, where given a set of 2-dimensional images the task is to reconstruct a scene in 3-space and find the camera positions in this scene. Early works on this problem predate even the first computers, with works going back to Taylor and Lagrange [Stu11]. Over time, many techniques have been developed for solving these problems for varying camera models and under different assumptions on the space and image points [Har97; HZ04; May93; MF92]. In general, when assuming noiseless images and with enough images and enough points in each image, one can uniquely recover all information about the original scene. There are, however, some configurations of cameras and points whose 3D information can not be uniquely recovered from the images. These are called *critical configurations*.

Much work has been done to understand critical configurations for various settings [Ber+20; BT07; Buc88; Buc92; Duf+19; Duf+20; Har00; HK02; HK07; KBP08; KHA01; LAO17], with results dating as far back as 1941 [Kra41].

While interesting from a purely theoretical viewpoint, critical configurations also play a part in practical applications. In real-life reconstruction problems, critical configurations rarely show up when enough data is available (noise alone should be enough to avoid the critical configurations), nonetheless, it has been shown that as the configurations approach the critical ones, reconstruction algorithms can become less stable [BTB07; HK07; LF94]. Moreover, when doing reconstruction with little data available, the chances of running into critical configurations greatly increase.

We study the critical configurations for projective cameras observing points using the approach from [Brå23a; Brå23b]. The idea is as follows (full details in Section IV.3): A set of n cameras $\mathbf{P}$ defines a rational map $\phi_{\mathbf{P}} \colon \mathbb{P}^3 \dashrightarrow (\mathbb{P}^2)^n$. A set of n cameras $\mathbf{P}$ and a set of points $X \subset \mathbb{P}^3$ is referred to as a *configuration*. A configuration is critical if there exists another set of cameras $\mathbf{Q}$ and points Y such that $\phi_{\mathbf{P}}(X) = \phi_{\mathbf{Q}}(Y)$. The map $\phi_{\mathbf{P}}$ takes $\mathbb{P}^3$ to a variety in $(\mathbb{P}^2)^n$ called the *multi-view variety*. Since X and Y both map into the intersection of the multi-view varieties of $\phi_{\mathbf{P}}$ and $\phi_{\mathbf{Q}}$, one can classify critical configurations by classifying all possible intersections of multi-view varieties.

Using this approach, we give a complete classification of the critical configurations for any number of projective views. An earlier classification appears in [HK07], using a different approach. While [HK07] laid the groundwork and served as an inspiration for this paper, the classification it gives is incomplete. It is missing some of the critical configurations for three or more views, and more importantly, the main theorem for critical configurations for four or more views does not hold. This is explained in more detail in section Section IV.8.1, where we give a counterexample.

A complete classification for the case of one and two views is given in [Brå23b], and a classification for three views is given in [Brå23a], although the latter is missing at least one critical configuration. For the sake of completeness, the results from these two works are included in this paper, sometimes with new proofs. The classification of the critical configurations for two views can be found in Theorem IV.5.12 and Figure IV.2, for three views in Theorem IV.7.25 and Figure IV.9, and for four or more views in Theorem IV.8.18.

Section IV.2 introduces notation and the main concepts and gives some preliminary results. Section IV.3 describes the approach used for the classification of critical configurations. Section IV.4 briefly covers the one-view case before the two-view case is covered in Section IV.5. Here we introduce the fundamental matrix and show that for two views, the critical configurations lie on quadric surfaces. Sections IV.6 and IV.7 cover the case of three cameras showing that the critical configurations for three views tend to lie on curves of degree at most four (see Figure IV.9). Lastly, Section IV.8 shows that most of these configurations remain critical when more cameras are added, thus completing the classification.

IV.2 Background

As this paper deals with a very similar topic as [Brå23a; Brå23b], this section is largely the same as Section 2 in those works. We refer the reader to [Cox+94] for the basics on ideals and varieties and to [HZ04] for the basics on algebraic vision and multi-view geometry.

Let $\mathbb{C}$ denote the complex numbers, and let $\mathbb{P}^n$ denote the projective space over the vector space $\mathbb{C}^{n+1}$. Projection from a point $p \in \mathbb{P}^3$ is a linear map

$$P\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2.$$

We refer to such a projection and its projection center p as a *camera* and its *camera center* in $\mathbb{P}^3$[1]. Following this theme, we refer to points in $\mathbb{P}^3$ as *space points* and points in $\mathbb{P}^2$ as *image points*. Similarly, $\mathbb{P}^2$ will be referred to as an *image*.

Once coordinates are chosen in $\mathbb{P}^3$ and $\mathbb{P}^2$, a camera can be represented by a 3×4 matrix of full rank called the *camera matrix*. The camera center is then given as the nullspace (or kernel) of the matrix. For the most part, we make no distinction between a camera and its camera matrix, referring to both simply as cameras. We use the *real projective pinhole camera model*, meaning that we require a camera matrix to be of full rank and to have only real entries.

Remark IV.2.1. Throughout the paper, whenever we talk about cameras it is to be understood that a choice of coordinates has been made, both on the images and 3-space, so that each camera comes with a camera matrix.

Remark IV.2.2. Throughout the paper, when dealing with multiple cameras, we always assume that all camera centers are distinct.

Definition IV.2.3. Given an n-tuple of cameras $\mathbf{P} = (P_1, \dots, P_n)$, with camera centers $p_1, \dots, p_n$. We define the *joint camera map* to be the map

$$\begin{aligned}\phi_{\mathbf{P}} = P_1 \times \cdots \times P_n\colon \mathbb{P}^3 &\dashrightarrow (\mathbb{P}^2)^n,\\ x &\mapsto (P_1(x), \dots, P_n(x)).\end{aligned}$$

Lemma IV.2.4. *Let $(P_1, \dots, P_n)$ be a tuple of $n \geq 2$ cameras. If the camera centers do not all lie on a line, the joint camera map $\phi_{\mathbf{P}}$ is injective. If the camera centers all lie on a line, the joint camera map sends this line to a point and is injective everywhere else.*

Proof. Let $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{P}^2)^n$ be a point in the image of $\phi_{\mathbf{P}}$. The preimage of $\mathbf{x}$ is the intersection $\bigcap_{i=1}^{n} l_{P_i}$ where $l_{P_i} = \overline{P_i^{-1}(x_i)}$ is a line through p_i. As such, they intersect in a single point unless they are all equal to a line passing through all camera centers. Hence the joint camera map is injective outside of a potential line spanned by the camera centers. ∎

[1] Following established terminology, we use the words *camera* and *view* interchangeably

Definition IV.2.5. We call the closure of the image of the joint camera map $\phi_{\mathbf{P}}$ the *multi-view variety* of $P_1, \dots, P_n$, and denote it by $\mathcal{M}_P$. The set of all multi-homogeneous polynomials vanishing on $\mathcal{M}_P$ is an ideal that we denote as the *multi-view ideal*.

Notation. While it follows from Lemma IV.2.4 that the multi-view variety is always irreducible, we use the term *variety* to also include reducible algebraic sets.

Theorem IV.2.6 ([APT21, Theorem 3.7])**.** *The ideal of the multi-view variety is generated only by bilinear and trilinear polynomials. In particular, it is generated by the determinant of*

$$\begin{bmatrix} P_i' & \mathbf{x} & \mathbf{0} \\ P_j' & \mathbf{0} & \mathbf{y} \end{bmatrix},$$

for each pair of cameras, and the 7×7 *minors of*

$$\begin{bmatrix} P_i' & \mathbf{x} & \mathbf{0} & \mathbf{0} \\ P_j' & \mathbf{0} & \mathbf{y} & \mathbf{0} \\ P_k' & \mathbf{0} & \mathbf{0} & \mathbf{z} \end{bmatrix},$$

for each triple. Here $\mathbf{x}$*,* $\mathbf{y}$ *and* $\mathbf{z}$ *are the* 3×1 *vectors with variables in the* i*-th,* j*-th and* k*-th image respectively.*

Definition IV.2.7. Given a set of points $S \subset (\mathbb{P}^2)^n$, a *reconstruction* of S is a configuration of cameras and points $(P_1, \dots, P_n, X)$ such that $S = \phi_{\mathbf{P}}(X)$ where $\phi_{\mathbf{P}}$ is the joint camera map of the cameras $P_1, \dots, P_n$.

Remark IV.2.8. By Definition IV.2.7 it is clear that for a set S to have a reconstruction, it needs to lie in the image of some joint camera map. Following established conventions for the study of critical configurations, we slacken this requirement somewhat by considering a set to have a reconstruction as long as it lies in the *closure* of the image of some joint camera map, in other words, in a multi-view variety. If S lies in the closure, but not the image, at least one of the points $x \in X$ in the reconstruction will be a camera center.

Definition IV.2.9. Given a configuration of cameras and points $(P_1, \dots, P_n, X)$, we refer to $\phi_{\mathbf{P}}(X) \subset (\mathbb{P}^2)^n$ as the *images* of $(P_1, \dots, P_n, X)$.

Remark IV.2.10. Every configuration of cameras and points is a reconstruction of its images, so every configuration (of cameras and points) is a reconstruction and vice versa.

Given a set of image points $S \subset (\mathbb{P}^2)^n$ as well as a reconstruction $(P_1, \dots, P_n, X)$ of S, any scaling, rotation, translation, or more generally, any real projective transformation of $(P_1, \dots, P_n, X)$ does not change the images, giving rise to a large family of reconstructions of S. However, we are not interested in differentiating between these reconstructions.

Definition IV.2.11. Given a set of points $S \subset (\mathbb{P}^2)^n$, let $(P_1, \dots, P_n, X)$ and $(Q_1, \dots, Q_n, Y)$ be two reconstructions of S. The two reconstructions are considered *equivalent* if there exists an element $H \in \mathrm{PGL}(4)$, such that

$$\begin{aligned} H(X) &= Y, \\ P_i H^{-1} &= Q_i, \quad \forall i. \end{aligned}$$

From now on, whenever we talk about a configuration of cameras and points, it is to be understood as unique up to action by PGL(4), and two configurations will be considered different only if they are nonequivalent. As such, we consider a reconstruction to be unique if it is unique up to action by PGL(4).

Definition IV.2.12. Two configurations of cameras and points $(P_1, \dots, P_n, X)$ and $(Q_1, \dots, Q_n, Y)$ are called *conjugate configurations* if they are nonequivalent reconstructions of the same set. That is, if they satisfy $\phi_{\mathbf{P}}(X) = \phi_{\mathbf{Q}}(Y)$, but not the conditions in Definition IV.2.11. Pairs of points $(x, y) \in X \times Y$ are called *conjugate points* if $\phi_{\mathbf{P}}(x) = \phi_{\mathbf{Q}}(y)$.

Definition IV.2.13. A configuration of cameras and points $(P_1, \dots, P_n, X)$ is said to be a *critical configuration* if it has at least one conjugate configuration. A critical configuration $(P_1, \dots, P_n, X)$ is said to be *maximal* if there exists no critical configuration $(P_1, \dots, P_n, X')$ such that $X \subsetneq X'$.

Hence, a configuration is critical if and only if the images it produces do not have a unique reconstruction.

Remark IV.2.14. Various definitions of critical configurations exist. For instance, [Kra41] considers the cone with two cameras on the same line to be critical, while it fails to be critical by our definition. [Brå23a; Brå23b] define it using the blow-up, which, when blowing back down, is equivalent to our definition. Our definition is the same as the one in [HK07].

Definition IV.2.15. Let $\mathbf{P}$ and $\mathbf{Q}$ be two n-tuples of cameras. Define

$$I = \left\{ (x, y) \in \mathbb{P}^3 \times \mathbb{P}^3 \mid \phi_{\mathbf{P}}(x) = \phi_{\mathbf{Q}}(y) \right\}.$$

The projection of I to each coordinate gives us two varieties, X and Y, where we denote X as the *set of critical points of* $\mathbf{P}$ *with respect to* $\mathbf{Q}$, and similarly for Y.

This definition is motivated by the following fact:

Proposition IV.2.16. *Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be two n-tuples of cameras such that there is no $H \in \mathrm{PGL}(4)$ satisfying $P_i H^{-1} = Q_i$ for all i. Let X be the critical points of P with respect to Q and conversely for Y. Then $(\boldsymbol{P}, X)$ is a critical configuration, with $(\boldsymbol{Q}, Y)$ as its conjugate. Furthermore, $(\boldsymbol{P}, X)$ is maximal with respect to $\boldsymbol{Q}$ in the sense that if there exists a critical configuration $(\boldsymbol{P}, X')$ with $X \subsetneq X'$, then its conjugate consists of cameras different from $\boldsymbol{Q}$.*

Proof. (2.12 in [Brå23b]): It follows from Definition IV.2.15 that for each point $x \in X$, we have a conjugate point $y \in Y$. Hence the two configurations have

the same images. Inequivalence follows from the fact that the cameras lie in different orbits under PGL(4), so the second point in Definition IV.2.11 can not be satisfied. Hence they are both critical configurations, conjugate to one another.

The (partial) maximality follows from the fact that if we add a point x_0 to X that does not lie in the set of critical points, there is (by Definition IV.2.15) no point y_0 such that $\phi_{\mathbf{P}}(x_0) = \phi_{\mathbf{Q}}(y_0)$. ■

Sections IV.5, IV.7 and IV.8 classify all maximal critical configurations for 2, 3, and 4+ views respectively. The reason we focus primarily on the maximal ones is that every critical configuration is contained in a maximal one and (when working with more than one camera) the converse is true as well, any subconfiguration of a critical configuration is itself critical.

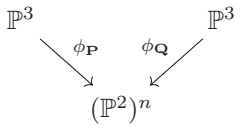
We conclude this section with a final, useful property of critical configurations, namely that the only property of the cameras we need to consider when exploring critical configurations is the position of their camera centers (i.e. change of coordinates in the images does not affect criticality).

Proposition IV.2.17 ([HK07, Proposition 3.7])**.** *Let $(P_1, \dots, P_n)$ be n cameras with centers $p_1, \dots, p_n$, and let $(P_1, \dots, P_n, X)$ be a critical configuration. If $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ is a set of cameras sharing the same camera centers, the configuration $(\mathcal{P}_1, \dots, \mathcal{P}_n, X)$ is critical as well.*

Proof. Since P_i and $\mathcal{P}_i$ share the same camera center and the camera center determines the camera uniquely up to a choice of coordinates, there exists some $H_i \in \mathrm{PGL}(3)$ such that $\mathcal{P}_i = H_i P_i$. Let $(Q_1, \dots, Q_n, Y)$ be a conjugate to $(P_1, \dots, P_n, X)$. Then $(H_1 Q_1, \dots, H_n Q_n, Y)$ is a conjugate to $(H_1 P_1, \dots, H_n P_n, X) = (\mathcal{P}_1, \dots, \mathcal{P}_n, X)$, so this configuration is critical as well. ■

IV.3 Approach

Consider a critical configuration $(P_1, \dots, P_n, X)$. Since it is critical, there exists a conjugate configuration $(Q_1, \dots, Q_n, Y)$ giving the same images in $(\mathbb{P}^2)^n$. The two sets of cameras define two joint-camera maps $\phi_{\mathbf{P}}$ and $\phi_{\mathbf{Q}}$.



The image of X lies in the multi-view variety $\mathcal{M}_P \subseteq (\mathbb{P}^2)^n$ and the image of Y lies in $\mathcal{M}_Q$. As such, the two sets of points X and Y lie in such a way that they both map (with their respective maps) into the intersection of the two multi-view varieties $\mathcal{M}_P \cap \mathcal{M}_Q$.

Remark IV.3.1. If two sets of cameras the second equation in Definition IV.2.11 their multi-view varieties are equal. Hence the multi-view varieties being different (which is the case we study throughout the paper) is enough to ensure that the two configurations are inequivalent.

The two multi-view varieties $\mathcal{M}_P$ and $\mathcal{M}_Q$ are irreducible 3-folds, so their intersection is either a surface, a curve, or a set of points. Taking the preimage under $\phi_{\mathbf{P}}$, $\mathcal{M}_P \cap \mathcal{M}_Q$ pulls back to a surface, curve, or point set in $\mathbb{P}^3$, which is exactly the set of critical points of $\mathbf{P}$ with respect to $\mathbf{Q}$. As such, *classifying all maximal critical configurations can be done by classifying all possible intersections between two multi-view varieties.* The configuration of points in $\mathbb{P}^3$ can be found by taking the pullback of $\mathcal{M}_P \cap \mathcal{M}_Q$ under $\phi_{\mathbf{P}}$.

Proposition IV.3.2. *Let $(P_1, \dots, P_n, X)$ be a maximal critical configuration. Then X is the intersection of quadric surfaces S^{ij} containing p_i, p_j, one such surface for each pair of cameras, and cubic surfaces S^{ijk} containing p_i, p_j, p_k, one for each triple of cameras.*

Proof. Since $(P_1, \dots, P_n, X)$ is a maximal critical configuration, X is the set of critical points of P with respect to some other set of cameras Q (from the conjugate configuration). Then X is the pullback of $\mathcal{M}_P \cap \mathcal{M}_Q$ under $\phi_{\mathbf{P}}$. Since $\phi_{\mathbf{P}}$ is the product of linear maps, it pulls a polynomial of multidegree $(d_1, \dots, d_n)$ in $(\mathbb{P}^2)^n$ back to a polynomial of degree $d_1 + \cdots + d_n$ with multiplicity d_i in the camera center p_i in $\mathbb{P}^3$. By Theorem IV.2.6, the ideal of the multi-view variety $\mathcal{M}_Q$ is generated only by bilinear and trilinear polynomials, one for each pair/triple of cameras. The pullback of $\mathcal{M}_P$ is all of $\mathbb{P}^3$, the pullback of each bilinear polynomial in the ideal of $\mathcal{M}_Q$ is a quadric surface S^{ij} and the pullback of each trilinear polynomial is a cubic surface. Thus X is the intersection of such surfaces. ∎

IV.4 Critical configurations for one view

Reconstruction of a 3D scene from the image of one projective camera is generally considered impossible, so most papers start with the two-view case. Still, for the sake of completeness, we give a summary of the critical configurations for one camera.

Let P be a camera, and let p be its camera center, we then have the joint camera map:

$$\phi_P \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2.$$

For any point $x \in \mathbb{P}^2$, the fiber over x is a line through p, so no point can be uniquely recovered. From this, one might assume that every configuration with one camera is critical. However, this is only the case if our configuration consists of sufficiently many points.

Theorem IV.4.1. *A configuration of one point and one camera is never critical. A configuration of one camera and $n > 1$ points is critical if and only if the camera center along with the n points span a space of dimension less than n.*

Proof. For the first part, note that up to a projective transformation, there exists only one configuration of one point and one camera. In other words, any configuration of one point and one camera can be taken to any other such configuration by simply changing coordinates. By Definition IV.2.11 this makes them equivalent, which means that only one reconstruction exists.

The same turns out to be the case if the configuration is such that the camera center along with the $n > 1$ points span a space of dimension n. If they do then the points and camera center lie in general position. However, for four or fewer points in $\mathbb{P}^3$ (fewer than three space points + one camera center), there exists only one configuration (up to action with PGL(4)) where all points are in general position. Hence any configuration where the camera center along with the n points span a space of dimension n can not be critical. This can only happen for $n \leq 3$ since $\mathbb{P}^3$ itself is of dimension 3.

Now it only remains to show that a configuration is critical if the camera center along with the $n > 1$ points span a space of dimension less than n. Indeed, if the points along with the camera center span a space of dimension less than n, the image points span a space of dimension $m < n - 1$. Then there are at least two nonequivalent reconstructions; one reconstruction where the points span a space of dimension m not containing the camera center and one where they span a space of dimension $m + 1$ which contains the camera center. For instance, if one has three image points lying on a line, there are two reconstructions: one where the three space points span a plane containing the camera center, and one where the three space points lie on a line. ■

IV.5 Critical configurations for two views

IV.5.1 The fundamental matrix

We start the study of the case of two cameras P_1 and P_2 by understanding their multi-view variety. The two cameras define the joint camera map:

$$\phi_P \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2.$$

By Lemma IV.2.4, the multi-view variety is an irreducible singular 3-fold in $(\mathbb{P}^2)^n$. In particular, for two views, it is a 3-fold in $\mathbb{P}^2 \times \mathbb{P}^2$, so it is described by a single bilinear polynomial. This polynomial can be represented[2] by a 3×3 matrix called the *fundamental matrix* of P_1, P_2, which we denote F_P. See [HZ04, Section 9.2] for a geometric construction of the fundamental matrix. We use the construction in [APT21] where the fundamental matrix (or rather, its polynomial representation) is given as the determinant of a 6×6 matrix.

Definition IV.5.1. The *epipoles* $e^j_{P_i}$ are the image points we get by mapping the j-th camera center to the i-th image

$$e^j_{P_i} = P_i(p_j).$$

[2] For every bilinear polynomial $f(\mathbf{x},\mathbf{y})$ there exists a matrix F such that $f(\mathbf{x},\mathbf{y}) = \mathbf{x}^T F \mathbf{y}$.

Proposition IV.5.2 (The fundamental matrix)**.** *[HZ04, Sections 9.2 and 17.1] For two cameras P_1, P_2, the multi-view variety $\mathcal{M}_P \subset \mathbb{P}^2 \times \mathbb{P}^2$ is the vanishing locus of a single, bilinear, rank 2[3] polynomial, given as follows:*

$$\mathbf{x}^T F_P \mathbf{y} := \det \begin{bmatrix} P_i & \mathbf{x} & \mathbf{0} \\ P_j & \mathbf{0} & \mathbf{y} \end{bmatrix},$$

where $\mathbf{x}$ and $\mathbf{y}$ are the variables in the first and second image respectively. The left and right nullspaces of F_P are the two epipoles $e^2_{P_1}$ and $e^1_{P_2}$ respectively

Proof. Starting with the second claim, observe that setting $\mathbf{x} = e^2_{P_1}$ ensures that the left 6×5 block of the matrix above is of rank 4, meaning the 6×6 matrix has determinant 0. The same happens if one sets $\mathbf{y} = e^1_{P_2}$. Hence

$$(e^2_{P_1})^T F_P = F_P e^1_{P_2} = 0. \tag{IV.1}$$

For the first claim, it follows from Lemma IV.2.4 that the multi-view variety for two cameras is an irreducible 3-fold. Hence it is generated by a single polynomial. Let $(\mathbf{x}, \mathbf{y})$ be a generic point in the multi-view variety $\mathcal{M}_P$. Then there exists a point $\mathbf{X} \in \mathbb{P}^3$ such that $P_1(\mathbf{X}) = \lambda_1 \mathbf{x}$ and $P_2(\mathbf{X}) = \lambda_2 \mathbf{y}$, hence

$$\begin{bmatrix} P_i & \mathbf{x} & \mathbf{0} \\ P_j & \mathbf{0} & \mathbf{y} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ -\lambda_1 \\ -\lambda_2 \end{bmatrix} = 0.$$

Since the matrix on the left has a non-zero nullspace, the determinant $\mathbf{x}^T F_P \mathbf{y}$ has to vanish on $(\mathbf{x}, \mathbf{y})$. Now we need only show that the determinant is irreducible to prove that $\mathbf{x}^T F_P \mathbf{y}$ generates the multi-view ideal. Irreducibility follows from the fact that F_P is of rank 2 (since its nullspace is just a point), whereas a reducible bilinear polynomial has a matrix representation that is always of rank 1 since in this case, the matrix is just the product of two vectors, each representing one of the two factors. ■

Remark IV.5.3. Since the entries in the camera matrix are real, so are the entries in the fundamental matrix.

For each pair of cameras, we get a fundamental matrix, which is a 3×3 matrix of rank 2. The following result states that the converse is also true, i.e. that any 3×3 matrix of rank 2 is the fundamental matrix for some pair of cameras.

Theorem IV.5.4. *There is a $1:1$ correspondence between 3×3 matrices of rank 2 (defined up to scale), and pairs of cameras P_1, P_2 (up to action by $\mathrm{PGL}(4)$).*

Proof. By Proposition IV.5.2, the fundamental matrix of two cameras is a 3×3 matrix of rank 2. The converse follows from Theorem 9.13. in [HZ04]. ■

With these results, we can move on to classifying all the critical configurations for two views. We start with a special type of critical configuration:

[3] The rank of a bilinear polynomial is the rank of its matrix representation.

IV.5.2 Trivial critical configurations

Definition IV.5.5. A configuration $(P_1, \dots, P_n, X)$ is said to be a *non-trivial critical configuration* if it has a conjugate configuration $(Q_1, \dots, Q_n, Y)$ satisfying

$$F_P^{ij} \neq F_Q^{ij}, \quad \forall i, j \leq n. \tag{IV.2}$$

Here F_P^{ij} denotes the fundamental matrix of P_i, P_j and likewise for F_Q^{ij}.

Critical configurations not satisfying this property exist, they are called trivial. If a critical configuration is trivial, then by Theorem IV.5.4 this means that, after a change of coordinates, $Q_i = P_i$ and $Q_j = P_j$. Since the cameras are the same, Lemma IV.2.4 tells us that the sets X and Y are equal, with the possible exception of any points lying on the line spanned by the two camera centers. It is a well-known fact that no number of cameras can differentiate between points lying on a line containing all the camera centers, hence the name "trivial".

A classification of the trivial critical configurations for any number of views can be found in [HK07] Section 4, or in [HZ04] Chapter 22. The focus of this paper is the non-trivial critical configurations, so for the remainder of the paper, when we talk about critical configurations, it is to be understood that they satisfy Equation (IV.2). This, in turn, ensures that the second item of Definition IV.2.11 is never satisfied, so there is no need to prove inequivalence between each configuration and its conjugates.

IV.5.3 Critical configurations for two views

Let us consider a non-trivial critical configuration (P_1, P_2, X). Since it is critical, there exists a conjugate configuration (Q_1, Q_2, Y) giving the same images in $\mathbb{P}^2 \times \mathbb{P}^2$. So the points X lie on the pullback $\phi_P^{-1}(\mathcal{M}_Q)$, which is a quadric surface S_P. Similarly, Y lies on a quadric surface S_Q. Since the multi-view variety $\mathcal{M}_Q$ is described by the fundamental matrix F_Q, we can compute the matrix of S_P (and S_Q):

$$\begin{aligned} S_P &= P_1^T F_Q P_2, \\ S_Q &= Q_1^T F_P Q_2. \end{aligned} \tag{IV.3}$$

The two quadrics have the following properties:

Lemma IV.5.6 (Lemma 5.10 in [HK07])**.**

IV.5.6.1. *The quadric S_P contains the camera centers p_1, p_2.*

IV.5.6.2. *The quadric S_P is ruled (contains real lines).*

Proof. A point $x \in \mathbb{P}^3$ lies on S_P if and only if $x^T S_P x = 0$. Since $P_i(p_i) = 0$, the camera centers lie on S_P. For the second item, recall that $F_Q e^1_{P_2} = 0$, so the line $g^1_{P_2} = \overline{P_2^{-1}(e^1_{P_2})}$ lies on S_P. ∎

Up to action by PGL(4), there are only four quadrics containing lines. These are illustrated in Figure IV.1.

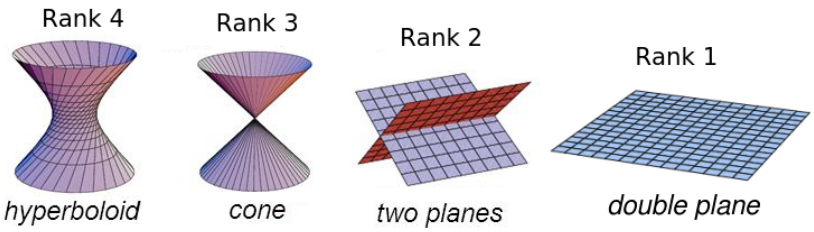


Figure IV.1: All types of real ruled quadrics.

The discussion so far can be summarized as follows:

Theorem IV.5.7 (Lemma 5.10 in [HK07])**.** *Let (P_1, P_2, X) be a non-trivial critical configuration. Then there exists a ruled quadric S_P passing through the camera centers p_1, p_2 containing X.*

Theorem IV.5.7 tells us that all non-trivial critical configurations have their points and camera centers lying on a ruled quadric. The converse, however, is not always true. For instance, we will soon show that cameras and camera centers all lying on a cone is not a critical configuration if the cone contains the line spanned by the two camera centers (see Figure IV.2). Let us instead give a partial converse:

Lemma IV.5.8. *Let (P_1, P_2, X) be a configuration of cameras and points such that X lies on a ruled quadric S_P that passes through both the camera centers. Then for each real 3×3 (defined up to scale) matrix F_Q of rank 2 such that:*

$$S_P = P_1^T F_Q P_2,$$

there exists a conjugate configuration to (P_1, P_2, X).

Proof. Assume such a matrix F_Q exists. By Theorem IV.5.4 there exists a pair of cameras (Q_1, Q_2) such that F_Q is their fundamental matrix. Since X lies on S_P, we have

$$\phi_P(X) \subset \mathcal{M}_Q.$$

Then for every point $x \in X$ we can find a point y such that

$$\phi_P(x) = \phi_Q(y).$$

Let Y be the set of these points y. Then (Q_1, Q_2, Y) is conjugate to (P_1, P_2, X). This can be repeated for each bilinear polynomial of rank 2, giving unique conjugate configurations. ∎

The problem of determining which configurations are critical is now reduced to finding out which quadrics are the pullbacks of real bilinear polynomials of rank 2.

Let $\mathbb{P}^{3\times 3}$ denote the space of 3×3 matrices defined up to scale. We have that $\mathbb{P}^{3\times 3}$ is isomorphic to $\mathbb{P}^8$. The fundamental matrix F_P of the pair (P_1, P_2) is an element in $\mathbb{P}^{3\times 3}$.

Lemma IV.5.9. *There is a* $1:1$ *correspondence between the set of real quadrics in* $\mathbb{P}^3$ *passing through* p_1, p_2*, and the set of real lines in* $\mathbb{P}^{3\times 3}$ *passing through* F_P*.*

Proof. Let $L \subset \mathbb{P}^{3\times 3}$ be a line through F_P, and let $F_0 \neq F_P$ be some point on L. Every point $F \neq F_P$ on L can be written as $F_0 + \lambda F_P$ for some $\lambda \in \mathbb{R}$. We then have:

$$\begin{aligned} P_1^T F P_2 &= P_1^T F_0 P_2 + \lambda P_1^T F_P P_2^T, \\ &= P_1^T F_0 P_2 + \lambda \cdot 0, \\ &= P_1^T F_0 P_2. \end{aligned}$$

Hence, the matrix F pulls back to the same quadric for all points $F \neq F_P$ on L.

For the $1:1$, note that a quadric S passing through p_1 and p_2 has seven degrees of freedom, the same as a line in $\mathbb{P}^{3\times 3}$ through a fixed point. ■

Using this $1:1$ correspondence and Lemma IV.5.8, the problem has been reduced to determining which quadrics correspond to lines in $\mathbb{P}^{3\times 3}$ containing at least one viable polynomial of rank 2. Let $\mathbb{P}_2^{3\times 3} \subset \mathbb{P}^{3\times 3}$ be the set of 3×3 matrices with rank at most 2. $\mathbb{P}_2^{3\times 3}$ is a hypersurface of degree 3, so a generic line L contains two polynomials of rank 2 in addition to F_P. There are also other possibilities, listed in the tables below[4].

We start with the cases where the line L corresponding to S_P has a finite number of intersections with the rank 2 locus $\mathbb{P}_2^{3\times 3}$.

Intersection points	S_P
All three intersection points are distinct real points	A smooth ruled quadric, camera centers not on a line
Two intersection points F_P and F_Q, $L \cap \mathbb{P}_2^{3\times 3}$ has multiplicity 2 at F_P	A smooth ruled quadric, camera centers lie on a line
Two intersection points F_P and F_Q, $L \cap \mathbb{P}_2^{3\times 3}$ has multiplicity 2 at F_Q	A cone, two camera centers not on a line, neither camera on a vertex

Table IV.1: The cases where L intersects the rank 2 locus finitely many times.

Remark IV.5.10. In the published version of [Brå23b], there is an error in Table IV.1 (this error has been corrected here and in the arXiv version). In the published version, the two bottom columns are reversed.

[4]The full computations behind these tables can be found in [Brå23b]

These are the cases where we have at least one real rank 2 matrix $F \in L$ different from F_P. There are, however, some cases where there are no viable matrices:

Intersection points	S_P
The two other intersection points are complex conjugates	A smooth non-ruled quadric
The two other intersection points are of rank 1	Union of two planes, camera centers in different planes
The two intersection points are equal to F_P	A cone, both camera centers on a line, neither camera center at the vertex

Table IV.2: The cases where there are no viable matrices.

In the case where L is contained in $\mathbb{P}_2^{3\times 3}$, all the matrices on L are of rank 2 (with the possible exception of at most 2 that can be of rank 1). As such, rather than looking at where the intersections are, we look at the epipoles of the matrices in L:

Epipoles	S_P
All matrices have different epipoles	Two planes, camera centers lying in same plane
All matrices share the same right epipole, the left epipoles trace a line[5]	Two planes, one camera center on the intersection of the planes
All matrices share the same right epipole, the left epipoles trace a conic[5]	Cone, one camera center at the vertex
All matrices share the same right and left epipole	Two (possibly complex) planes, both camera centers lying on the intersection of the planes
All matrices share the same right and left epipole AND the two rank 1 matrices on L coincide	Double plane (as a set it is equal to a plane, but every point has multiplicity 2)

Table IV.3: The cases where L lies in the rank 2 locus.

With this, we know exactly which quadrics form critical configurations. These are summarized in Table IV.4. To distinguish these from other ruled quadrics we give the following definition:

Definition IV.5.11. Let P_1, P_2 be two cameras. A quadric S is said to be a *critical quadric* (with respect to P_1, P_2) if S and the camera centers p_1, p_2 form one of the configurations in Table IV.4 (illustrated in Figure IV.2).

Theorem IV.5.12. *A configuration (P_1, P_2, X) is non-trivially critical if and only if the two camera centers and all the points X lie on a critical quadric.*

[5]The statement also holds if we swap "left" and "right".

Quadric S_P	**Conjugate quadric**	**Conjugates**
Smooth ruled quadric, camera centers not on a line	Same	2
Smooth ruled quadric, camera centers on a line	Cone, camera centers not on a line	1
Cone, camera centers not on a line	Smooth ruled quadric, camera centers on a line	1
Cone, one camera center at vertex, other one not	Same	∞
Two planes, camera centers in the same plane	Same	∞
Two planes, one camera center on the singular line	Same	∞
Two planes, camera centers on the singular line[6]	Same	∞
A double plane, camera centers in the plane	Same	∞

Table IV.4: A list of all possible non-trivial critical configurations for two views, and their conjugates, as well as the number of conjugates for each configuration.

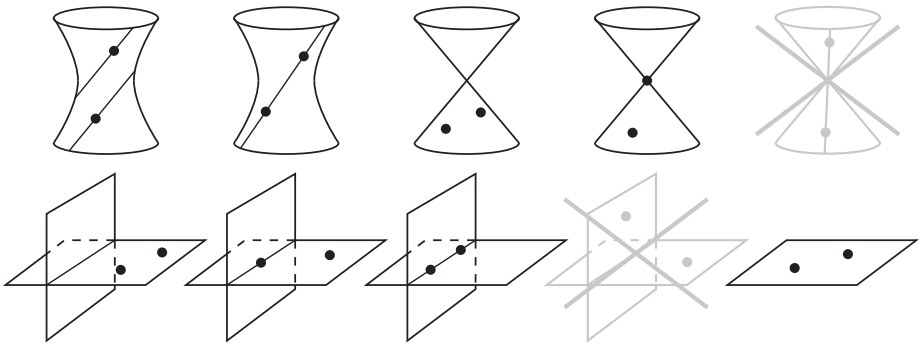


Figure IV.2: Illustration of all configurations of two points on a ruled quadric. Apart from the two marked with crosses, all these configurations are critical.

Recall that by Definition IV.2.11, we required two conjugate configurations to not be projectively equivalent. Yet by Theorem IV.5.12, most critical configurations have conjugates that are of the same type. Now, while there is

[6]This includes the case where the two planes are complex conjugates. In this case, the only real part of the quadric is the line spanned by the camera centers.

indeed some $H \in \mathrm{PGL}(4)$ taking any smooth quadric S_P to any other smooth quadric S_Q, we will soon show that the map taking a point in S_P to its conjugate on S_Q certainly does not lie in PGL(4). In the next subsections, we describe the map taking a point to its conjugate, to show that it is not a projective transformation.

IV.5.4 Epipolar lines

Before we can describe the map taking a point to its conjugate, we need to point out a special pair of lines on S_P. Given two pairs of cameras (P_1, P_2) and (Q_1, Q_2), let S_P be the quadric $P_1^T F_Q P_2$ and define

$$\begin{aligned} g^2_{P_1} &= \overline{P_1^{-1}(e^2_{Q_1})} = \overline{P_1^{-1}(Q_1(q_2))}, \\ g^1_{P_2} &= \overline{P_2^{-1}(e^1_{Q_2})} = \overline{P_2^{-1}(Q_2(q_1))}. \end{aligned} \tag{IV.4}$$

The lines $g^2_{P_1}$ and $g^1_{P_2}$ both lie on S_P. They are the pullback of the epipoles $e^j_{Q_i}$ from the other set of cameras, so we call them *epipolar lines*.

Epipolar lines are key in understanding the relation between points on S_P and its conjugate S_Q. They also play an important role in the study of critical configurations for more than 2 cameras, so let us give a brief analysis of these lines.

Lemma IV.5.13 (Lemma 5.10, Definition 5.11 in [HK07])**.**

IV.5.13.1. *The line $g^j_{P_i}$ lies on S_P and passes through p_i.*

IV.5.13.2. *Any point lying on both $g^2_{P_1}$ and $g^1_{P_2}$ is a singular point on S_P.*

IV.5.13.3. *Any point in the singular locus of S_P that lies on one of the lines also lies on the other.*

IV.5.13.4. *If S_P is the union of two planes, $g^2_{P_1}$ and $g^1_{P_2}$ lie in the same plane.*

The first two properties are taken from Lemma 5.10 in [HK07]. The last two are neither stated nor proven in [HK07]. Nevertheless, the authors seem to have been aware of all four properties.

Proof.

1. The line $g^j_{P_i}$ is defined as the preimage of a point in the i-th image, every such preimage is a line through p_i. Next, let $x \in g^1_{P_2}$, then

$$\begin{aligned} x^T S_P x &= x^T P_1^T F_Q P_2 x, \\ &= x^T P_1^T F_Q e^1_{Q_2}, \\ &= 0, \end{aligned}$$

   so the line lies on S_P.

2. A point x is a singular point on S_P if it satisfies $(S_P + S_P^T)x = 0$, let $x \in g^2_{P_1} \cap g^1_{P_2}$, then

$$\begin{aligned}(S_P + S_P^T)x &= P_1^T F_Q P_2 x + x^T P_1^T F_Q P_2, \\ &= P_1^T F_Q e^1_{Q_2} + (e^2_{Q_1})^T F_Q P_2, \\ &= 0 + 0.\end{aligned}$$

3. If $x \in g^2_{P_1}$ lies in the singular locus of S_P, then we have

$$\begin{aligned}(S_P + S_P^T)x &= (P_1^T F_Q P_2 + P_2^T F_Q^T P_1)x, \\ &= P_1^T F_Q P_2 x + (e^2_{Q_1})^T F_Q P_2, \\ &= P_1^T F_Q P_2 x.\end{aligned}$$

However, since x lies in the singular locus of S_P, this expression is equal to zero. Since all camera matrices are of full rank, the only way we can get zero is if $x = p_2$ or if $P_2 x = e^1_{Q_2}$. In either case, it follows that x lies on $g^1_{P_2}$ as well.

4. Assume there exist cameras P_1, P_2, Q_1, Q_2 such that S_P is the union of two planes and the epipolar lines $g^j_{P_i}$ lie in different planes. When the quadric S_P consists of two planes, one of the planes, which we denote by Π, will (by Theorem IV.5.12) contain both camera centers. As such, the only way that the epipolar lines can lie in different planes is if one of the camera centers, say p_2, lies on the intersection of the two planes and the other does not (if both lie on the intersection, then by 3., the epipolar lines are both equal to the intersection of the two planes).

   Recall that the quadric S_P and its conjugate S_Q (also two planes) are both pullbacks of the surface $\mathcal{M}_P \cap \mathcal{M}_Q \subset \mathbb{P}^2 \times \mathbb{P}^2$. The map ϕ_P takes the plane Π to the product of two lines in $\mathbb{P}^2 \times \mathbb{P}^2$. The line in the first image passes through the epipole $e^2_{Q_1}$, whereas the line in the second image does not pass through the epipole $e^1_{Q_2}$ (this is because Π contains one of the epipolar lines but not the other). The problem is now that neither of the planes on S_Q can map to the product of these two lines since any such plane would have to be both

   a) a plane passing through q_2 but not through q_1 (because in the second image, the line does not pass through $e^1_{Q_2}$) and

   b) a plane passing through both q_1 and q_2 (because the plane maps to a line in both images),

   which gives us a contradiction. It follows that there are no P_1, P_2, Q_1, Q_2 such that S_P is the union of two planes and the epipolar lines $g^j_{P_i}$ lie in different planes. So whenever S_P is the union of two planes, the epipolar lines lie in the same plane. ■

Definition IV.5.14. Let S_P be a quadric surface and let p_1, p_2 be two distinct points on S_P. A pair of lines $g^2_{P_1}, g^1_{P_2}$ is called *permissible* if $g^2_{P_1}, g^1_{P_2}$ satisfy the four conditions in Lemma IV.5.13.

Hence epipolar lines are always permissible. The next result shows that for each pair of permissible lines on S_P, we can find a conjugate configuration making them the epipolar lines.

Proposition IV.5.15. *Let P_1, P_2 be two cameras, and let S_P be a quadric passing through their camera centers. The configuration (P_1, P_2, S_P) is critical if and only if S_P contains a permissible pair of lines. Furthermore, if p_1, p_2 do not both lie in the singular locus of S_P, then for each pair of permissible lines on S_P, there is a unique fundamental matrix F_Q such that $S_P = P_1^T F_Q P_2$ and such that the two permissible lines are the epipolar lines on S_P.*

Proof. The first part can be proven by comparing the quadrics in Table IV.4 to the set of quadrics containing the required lines, and noting that they are the same. We leave this to the reader.

The second part is immediate for the three cases where there is a finite number of conjugates since the number of conjugates is equal to the number of pairs of permissible lines and each conjugate comes with its unique choice of lines. For the remaining quadrics, (cone and two planes), we observe from Tables IV.1 to IV.3 on pages 126–127 that each quadric has a one-dimensional family of conjugates, each conjugate coming with a pair of epipolar lines. These epipolar lines are unique, as long as the cameras do not both lie in the singular locus. This one-dimensional family matches exactly the one-dimensional family of pairs of permissible lines on these quadrics. ■

IV.5.5 Maps between quadrics

Let us now have a closer look at the map ψ taking a point to its conjugate. Given two pairs of cameras P_1, P_2 and Q_1, Q_2 with camera centers p_1, p_2 and q_1, q_2 respectively, let the quadrics S_P and S_Q and the epipolar lines g be defined as before (Equations (IV.3) and (IV.4)).

Let ψ_P be the map taking a point on S_P to its conjugate:

$$\begin{aligned} \psi_P \colon S_P &\dashrightarrow S_Q, \\ x &\mapsto l_1 \cap l_2, \end{aligned}$$

where l_i is the line $\overline{Q_i^{-1}(P_i(x))}$. This map is not well-defined for all points on S_P, since some points might have more than one conjugate (see 2. and 3. in Lemma IV.5.16).

Lemma IV.5.16. *The map ψ_P has the following properties:*

IV.5.16.1. *The conjugate to any point on the epipolar line $g^j_{P_i}$, apart from the camera center p_i itself, is the camera center q_j.*

IV.5.16.2. *The conjugate to the camera center p_j is the epipolar line $g^j_{Q_i}$*

IV.5.16.3. *The conjugate to the intersection of the two epipolar lines $g^2_{P_1}$ and $g^1_{P_2}$ (if it exists) is the line spanned by the two camera centers q_1, q_2.*

IV.5.16.4. *The map ψ_P is a bijective map everywhere else.*

Proof.

1. Let $x \in g^j_{P_i}$. Then l_i is the line spanned by the two camera centers q_1 and q_2, while l_j is a line through q_j. Thus the two lines intersect in q_j.
2. Same argument as in 1. but starting with the line $g^j_{Q_i}$ instead.
3. Let $x \in g^2_{P_1} \cap g^2_{P_1}$. Then l_1 and l_2 are both equal to the line spanned by the two camera centers q_1 and q_2, so any point on this line is conjugate to x.
4. If x does not lie on either epipolar line, the two lines l_1 and l_2 are distinct, and meet in a point y that is not a camera center. For any such point y, the lines l_1 and l_2 are the only pair of such lines meeting in this point (since l_i passes through q_i). Hence x only has y as its conjugate and vice versa. ■

Remark IV.5.17. When the two pairs of cameras P_1, P_2 and Q_1, Q_2 are known, one can explicitly compute the map ψ_P as a rational function. For some representatives of a general point $X \in S_P$ and its conjugate $\psi_P(X) \in S_Q$, we have

$$\underbrace{\begin{pmatrix} Q_1 & P_1X & 0 \\ Q_2 & 0 & P_2X \end{pmatrix}}_{A} \cdot \begin{pmatrix} \psi_P(X) \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} P_1 & Q_1\psi_P(X) & 0 \\ P_2 & 0 & Q_2\psi_P(X) \end{pmatrix} \cdot \begin{pmatrix} X \\ 1 \\ 1 \end{pmatrix} = 0.$$

Using the adjugate of A, we can get an expression for ψ_P as a rational function in the homogeneous coordinates of X.

With these results in place, we can give two formulas for how ψ_P acts on the curves on S_P:

Definition IV.5.18. Let S_P be a smooth quadric or cone and let C_P be a curve on S_P. We say that C_P is of type (a, b, c_1, c_2), where c_i is the multiplicity of C_P in the camera center p_i and where a and b is:

- If S_P is smooth, a is the number of times C_P intersects a generic line in the same family as the epipolar lines $g^j_{P_i}$, and b is the number of times it intersects the lines in the other family, (meaning (a, b) is the bidegree of C_P).
- If S_P is a cone, a is the number of times C_P intersects each line outside of the vertex, and b the number of times it intersects each line.

Proposition IV.5.19 ([HK07, Lemma 8.32],[Brå23b, Proposition 5.7])**.** *Let S_P be a smooth quadric or a cone, and let $C_P \subset S_P$ be a curve of type (a, b, c_1, c_2), such that C_P does not contain either of the epipolar lines. Then the conjugate curve $C_Q \subset S_Q$ is of type $(a, a+b-c_1-c_2, a-c_2, a-c_1)$.*

The action of ψ_P on the lines on a quadric/cone is illustrated in Figures IV.3 and IV.4.

Proof. For a full proof using intersection theory, see the proof in [Brå23b]. Here we give a shorter and more intuitive version of the proof. Let the conjugate curve C_Q be of type (a', b', c_1', c_2'). The curve C_P intersects $g^2_{P_1}$ a times, with c_1 of these intersection point being in the camera center p_1. By Lemma IV.5.16 Item IV.5.16.1, every point on $g^j_{P_i}$ (apart from the camera center itself) is mapped to q_2. It follows that $c_2' = a - c_1$, and likewise for c_1'. Since this argument goes both ways, we get that $c_2 = a' - c_1' = a' - (a - c_2)$. It follows that $a' = a$.

To find an expression for b', we consider how ψ_P treats various lines on S_P. We consider lines on S_P different from the epipolar lines. So by Lemma IV.5.16 Item IV.5.16.4, their conjugate is a curve on S_Q. Consider first a line through p_1 different from $g^2_{P_1}$. This maps to a point in the first image, and a line in the second. Hence its conjugate is a line on S_Q passing through q_1 (type (1,0,1,0)). Next, we consider a line that intersects the epipolar lines but does not pass through any camera centers. In each image, this line appears as a line passing through $e^j_{Q_i}$. Back-projecting these lines using the cameras Q, we get two planes, both passing through the two camera centers q_1, q_2. The intersection of these two planes with S_Q is a conic curve passing through both camera centers (type (1,1,1,1)). Lastly, if we take a line on S_P in the same family as the epipolar lines, we get a line in each image that does not pass through any epipoles. Back-projecting these lines using the cameras Q, we get two distinct planes, whose intersection is a line not intersecting the epipolar lines (type (0,1,0,0)). To summarize, the map ψ_P maps:

$$(1,0,1,0) \mapsto (1,0,1,0), \qquad (1,0,0,1) \mapsto (1,0,0,1),$$
$$(1,0,0,0) \mapsto (1,1,1,1), \qquad (0,1,0,0) \mapsto (0,1,0,0).$$

Due to ψ_P being a rational map, we have $b' = \lambda_a a + \lambda_b b + \lambda_{c_1} c_1 + \lambda_{c_2} c_2$. Solving for λ using the equations above, we get $b' = a + b - c_1 - c_2$. ∎

Definition IV.5.20. Let S_P be two planes with two camera centers not both lying on the intersection and let C_P be a curve on S_P. We say that C_P is of type (a, b, c_0, c_1, c_2), where a is the degree of the curve in the plane with the epipolar lines, b the degree of the curve in the other plane, c_0 is the multiplicity of C_P in the intersection of the epipolar lines $g^j_{P_i}$ and c_1, c_2 is the multiplicity in the camera centers p_1, p_2 respectively.

Proposition IV.5.21 ([Brå23b, Proposition 5.9])**.** *Let S_P be two planes with two camera centers not both lying on the intersection and let $C_P \subset S_P$ be a curve of type (a, b, c_0, c_1, c_2) such that C_P does not contain either of the epipolar lines or*

the line spanned by the camera centers. Then the conjugate curve $C_Q \subset S_Q$ is of type $(2a - c_0 - c_1 - c_2, b, a - c_1 - c_2, a - c_0 - c_2, a - c_0 - c_1)$.

Proof. See [Brå23b] for a fully detailed proof. We give a shorthand version here. Let the conjugate curve C_Q be of type $(a', b', c_0', c_1', c_2')$, and denote the intersection of the two epipolar lines on S_P and the two on S_Q by x_P and x_Q respectively. The curve C_P intersects $g^2_{P_1}$ a times, with c_1 of these intersection point being in the camera center p_1 and c_0 of these being in x_P. Every point on $g^j_{P_i}$ (apart from p_1 and x_P) is mapped to q_2. It follows that $c_2' = a - c_0 - c_1$, and likewise for c_1'. Moreover, C_P intersects the line $\overline{p_1 p_2}$ a times, with $c_1 + c_2$ of these intersection points being camera centers. By Lemma IV.5.16 Item IV.5.16.3 the conjugate to all these intersection points is the point x_Q on S_Q, so $c_0' = a - c_1 - c_2$. The formula goes both ways, so we get that

$$\begin{aligned} c_0 &= a' - c_1' - c_2', \\ &= a' - (a - c_0 - c_2) - (a - c_0 - c_1), \\ &= a' - 2a + 2c_0 + c_1 + c_2. \end{aligned}$$

It follows that $a' = 2a - c_0 - c_1 - c_2$. Lastly, since the map ψ_P acts as an isomorphism on the plane not containing the camera centers, we have $b' = b$. ■

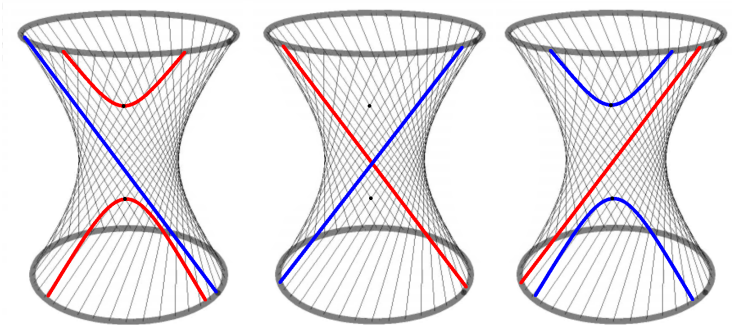


Figure IV.3: Illustration of the map taking a point to its conjugate. The smooth quadric has two conjugates (original in the middle). The left is the one we get if we take the lines in the blue family to be the epipolar lines, whereas the one on the right is the one we get if we choose the red. In both cases the lines in the family with the epipolar lines are preserved, whereas the lines in the other family are mapped to conic curves.

IV.6 Preliminaries for the three-view case

We start with the observation that any critical configuration for three views is critical for each pair of views.

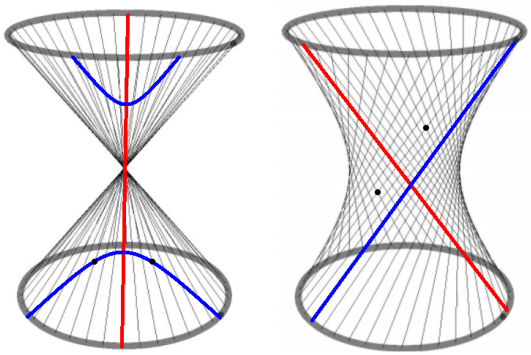


Figure IV.4: Illustration of the map taking a point to its conjugate. The line spanned by the camera centers (right) is mapped to the vertex on the cone. The other lines in this family map to conics. Lines in the other family are preserved.

Theorem IV.6.1. *Let $(P_1, \dots, P_n, X)$ be a non-trivial critical configuration, and let $\mathcal{P} \subset \{P_1, \dots, P_n\}$ be a subset of at least two cameras. Then $(\mathcal{P}, X)$ is a critical configuration.*

Proof. Since $(P_1, \dots, P_n, X)$ is a critical configuration, there exists a conjugate configuration $(Q_1, \dots, Q_n, Y)$. Each camera P_i has a conjugate camera Q_i. Let $\mathcal{Q} \subset \{Q_1, \dots, Q_n\}$ be the set of cameras conjugate to $\mathcal{P}$. Then $(\mathcal{P}, X)$ and $(\mathcal{Q}, Y)$ produce the same images. By the non-triviality, these two configurations are inequivalent and hence conjugate to one another. ■

Definition IV.6.2. Let $(P_1, \dots, P_n, X)$ be a configuration of cameras and points, and let $\left\{S_P^{ij}\right\}$ be a set of $\binom{n}{2}$ quadrics. The configuration $(P_1, \dots, P_n, X)$ is said to *lie on the intersection of the quadrics* $\left\{S_P^{ij}\right\}$ if each quadric S_P^{ij} contains the camera centers p_i, p_j, and X.

Corollary IV.6.3. *For $n \geq 2$ views, the critical configurations always lie on the intersection of $\binom{n}{2}$ critical quadrics.*

Proof. This follows from Theorem IV.6.1 along with the fact that for two views, the critical configurations lie on critical quadrics. It also follows directly from Proposition IV.3.2. ■

In particular, for the three-view case, the critical configurations lie on the intersection of three critical quadrics. The task of classifying the critical configurations for three views, however, is not as simple as just classifying the possible intersections of three quadrics. Even if a configuration is critical for each pair of views, it need not be critical for all three views. There are two issues, namely the *compatibility of the quadrics*, and the *ambiguous points*. Before we

tackle these issues, let us state a result on the generators of the multi-view ideal for three views.

Lemma IV.6.4 ([Brå23a, Proposition 5.6])**.** *For the three non-collinear cameras, the multi-view ideal is generated by three bilinear and one trilinear polynomial.*[7]

Proof. A full proof, that also shows that one needs three trilinear polynomials in the collinear case, can be found in [Brå23a]. We give a shorthand version here.

The fact that it takes three bilinear polynomials (the fundamental matrices) is simply an application of Theorem IV.2.6 to the case of three cameras. Next, we show that the ideal contains a ten-dimensional family of trilinear polynomials. There is a 27-dimensional family of trilinear polynomials on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, which pulls back to a 17-dimensional family of cubics passing through the three camera centers (20 cubics, less three for the constraint of passing through the camera centers). Hence a 10-dimensional family of trilinear polynomials pull back to the zero polynomial and thus lie in the multi-view ideal.

Each of the bilinear polynomials generates three trilinear polynomials in the ideal, by multiplying it with one of the three variables from the last $\mathbb{P}^2$, for instance multiplying $\mathbf{x}^T F_P^{12} \mathbf{y}$ with z_1, z_2 and z_3 will give three trilinear polynomials. As such, one only needs one final trilinear polynomial to generate the ideal in the general case, that is, in the case where the cameras are not collinear. ■

IV.6.1 Compatible triples of quadrics

In the two-view case, any fundamental matrix (3×3 matrix of rank 2) comes from some pair of cameras (Theorem IV.5.4). In the three-view case, however, three arbitrary fundamental matrices will generally not come from some triple of cameras. This motivates the following definition:

Definition IV.6.5. Let $\left\{ F_Q^{ij} \mid 1 \leq i \leq j \leq n \right\}$ be a set of $\binom{n}{2}$ fundamental matrices. The matrices are said to be *compatible* if there exist n cameras $Q_1, ..., Q_n$ such that F_Q^{ij} is the fundamental matrix of Q_i, Q_j.

Definition IV.6.6. Given n cameras $P_1, ..., P_n$ we say a set of quadrics $\left\{ S_P^{ij} \right\}$ is *compatible* (with respect to the cameras P_i), if there exists a compatible set of fundamental matrices F_Q^{ij} such that

$$S_P^{ij} = P_i^T F_Q^{ij} P_j.$$

For a configuration (P_1, P_2, P_3, X) to be critical, it is not enough that it lies on the intersection of three critical quadrics, the three quadrics also need to be compatible. Only then can we find three conjugate cameras Q_1, Q_2, Q_3 which in turn can be used to construct a conjugate configuration. A detailed study of

[7] In the collinear case, it is generated by three bilinear and three trilinear polynomials, but this is harder to prove. See [Brå23a, Proposition 5.6] for proof.

compatible fundamental matrixes can be found in [BR23; HZ04]. Recall that the *epipoles* $e^j_{Q_i}$ are the unique points satisfying

$$(e^2_{P_1})^T F_P = F_P e^1_{P_2} = 0. \tag{IV.1}$$

Theorem IV.6.7 ([BR23, Proposition 5.8])**.** *Let F^{12}_Q, F^{13}_Q, F^{23}_Q be a triple of fundamental matrices, and let $e^j_{Q_i}$ be their epipoles. Then the triple is compatible and comes from a collinear triple of cameras if and only if*

$$e^2_{Q_1} = e^3_{Q_1}, \quad e^1_{Q_2} = e^3_{Q_2}, \quad e^1_{Q_3} = e^2_{Q_3}, \tag{IV.5}$$

and

$$(F^{12}_Q)^T [e^2_{Q_1}]_\times F^{13}_Q = F^{23}_Q.^{8} \tag{IV.6}$$

Theorem IV.6.8 ([HZ04, Section 15.4])**.** *Let F^{12}_Q, F^{13}_Q, F^{23}_Q be a triple of rank 2 fundamental matrices, and let $e^j_{Q_i}$ be their epipoles. Then the triple is compatible and comes from a non-collinear triple of cameras if and only if*

$$e^2_{Q_1} \neq e^3_{Q_1}, \quad e^1_{Q_2} \neq e^3_{Q_2}, \quad e^1_{Q_3} \neq e^2_{Q_3}, \tag{IV.7}$$

and

$$(e^3_{Q_1})^T F^{12}_Q e^3_{Q_2} = (e^2_{Q_1})^T F^{13}_Q e^2_{Q_3} = (e^1_{Q_2})^T F^{23}_Q e^1_{Q_3} = 0. \tag{IV.8}$$

Since we are working with quadrics in $\mathbb{P}^3$, we want to translate these conditions on the fundamental matrices, to conditions on the quadrics. The remainder of this subsection is used to give two results stating these conditions. We start with the collinear case, that of Theorem IV.6.7 (clarification: this is NOT the case where the camera centers p_i are collinear, but the case where the camera centers q_i of the conjugate configuration are collinear).

Proposition IV.6.9 (**Collinear case**)**.** *Let P_1, P_2, P_3 be a triple of cameras, and let S^{12}_P, S^{13}_P, S^{23}_P be a triple of critical quadrics. Then the three quadrics come from a triple of fundamental matrices that satisfy the conditions in Theorem IV.6.7, if and only if there exist three lines L_1, L_2, L_3 satisfying:*

IV.6.9.1. *$p_i \in L_i$.*

IV.6.9.2. *L_i and L_j form a permissible pair on S^{ij}_P.*

IV.6.9.3. *$L_i \subseteq S^{ij}_P \cap S^{ik}_P$.*

IV.6.9.4. *Any point $x \in S^{12}_P \cap S^{13}_P$ not lying on L_1 also lies on S^{23}_P.*

[8] Here $[a]_\times$ denotes the unique 3×3 matrix such that $[a]_\times b = a \times b$ for all b

Proof. ($\Rightarrow$): Let S_P^{12}, S_P^{13}, S_P^{23} be a compatible (in the collinear sense) triple of critical quadrics. This means they come from a triple of fundamental matrices satisfying the conditions in Theorem IV.6.7. By the first part of IV.5.15, each quadric contains a pair of permissible lines, the pullback of the epipoles. By Equation (IV.5) the two epipoles coincide in each image, so the two epipolar lines passing through p_i have to coincide as well. This gives us three lines satisfying the first three conditions in Proposition IV.6.9.

Next, let x be any point in $S_P^{12} \cap S_P^{13}$ not lying on L_1, and denote $x_i = P_i(x)$. Since x lies on S_P^{1i}, $F_Q^{1i}x_i$ is the unique line in the first image spanned by x_1 and the epipole $e_1^2 = e_1^3$. Now, since $F_Q^{12}x_2$ and $F_Q^{13}x_3$ define the same line, we have

$$\begin{aligned} 0 &= x_2^T (F_Q^{12})^T [e_{Q_1}^2]_\times F_Q^{13} x_3, \\ &= x_2^T F_Q^{23} x_3, \\ &= x^T P_2'^T F_Q^{23} P_3' x, \\ &= x^T S_P^{23} x. \end{aligned}$$

It follows that x lies on S_P^{23} as well

($\Leftarrow$): Let S_P^{ij} be three quadrics satisfying the conditions above. Since L_i and L_j form a permissible pair on S_P^{ij}, there exists (by Proposition IV.5.15) fundamental matrices F_Q^{ij} such that $S_P^{ij} = P_i^T F_Q^{ij} P_j$, and the lines L_i, L_j are the epipolar lines on S_P^{ij}. Since the line L_i is the preimage of both the epipole $e_{Q_i}^j$ and the epipole $e_{Q_i}^k$, the two epipoles are equal. Hence, the epipoles satisfy Equation (IV.5).

Let $S_P^{23'}$ denote the quadric defined by the matrix

$$P_2'^T F_Q^{12T} [e_{Q_1}^2]_\times F_Q^{13} P_1.$$

For the next part, we want to show that the $S_P^{23'} = S_P^{23}$. If so, the three quadrics S_P^{ij} come from the three fundamental matrices F_Q^{12}, F_Q^{13} and $F_Q^{12T}[e_{Q_1}^2]_\times F_Q^{13}$, since this triple of fundamental matrices is compatible, so is the triple of quadrics. By assumption, the set $(S_P^{12} \cap S_P^{13}) - L_1$ lies on all three quadrics. If the quadrics do not contain a common surface, this set is a (possibly reducible) cubic curve C. The curve C lies on both S_P^{23} and $S_P^{23'}$. Moreover, since $F_Q^{12T}[e_{Q_1}^2]_\times F_Q^{13}$ has the same left and right nullspaces as F_Q^{23}, $S_P^{23'}$ must also contain the two lines L_2, L_3. The union of a cubic curve with two lines uniquely determines a quadric, so we have $S_P^{23} = S_P^{23'}$. A similar argument can be made if the quadrics contain a common surface, and in particular, if they are equal. ∎

Among other things, this shows that as long as the cameras are not in the position shown in Figure IV.5, three copies of the same quadric form a compatible triple. For the remainder of the section, we assume that compatible triples of quadrics come from triples of fundamental matrices that satisfy the non-collinear conditions (Equations (IV.7) and (IV.8)), rather than Equations (IV.5) and (IV.6).

Theorem IV.6.10 (**General case**). *Let P_1, P_2, P_3 be a triple of cameras, and let S_P^{12}, S_P^{13}, S_P^{23} be a triple of critical quadrics. Then the three quadrics come from a triple of fundamental matrices that satisfy the conditions in Theorem IV.6.8, if and only if there exist six lines $g_{P_i}^j$ such that:*

IV.6.10.1. *$g_{P_i}^j$ and $g_{P_j}^i$ form a permissible pair on S_P^{ij} (see Definition IV.5.14).*

IV.6.10.2. *The intersection of the plane spanned by $g_{P_i}^j$ and $g_{P_i}^k$ with the plane spanned by $g_{P_j}^i$ and $g_{P_j}^k$ lies on S_P^{ij}.*

Proof. ($\Rightarrow$): If the triple of quadrics is compatible, they come from a compatible triple of fundamental matrices. Such quadrics always come with two epipolar lines, which are always permissible. Hence, the first part is satisfied.

Among the six epipolar lines, there are 2 lines on each quadric and 2 through each camera center. By (IV.7), for each camera center, the two lines passing through it are distinct. Hence the two lines through each camera center p_i span a plane, which we denote by Π_i.

Let y be any point lying in the intersection of Π_i and Π_j. Recall that epipolar lines satisfy $P_i(g_{P_i}^j) = e_{Q_i}^j$. We then have

$$\begin{aligned} P_i y &= \alpha e_{Q_i}^j + \beta e_{Q_i}^k, \\ P_j y &= \gamma e_{Q_j}^i + \delta e_{Q_j}^k. \end{aligned}$$

for some $[\alpha : \beta], [\gamma : \delta] \in \mathbb{P}^1$. Hence

$$\begin{aligned} y^T S_P^{ij} y &= y^T P_i^T F_Q^{ij} P_j y \\ &= (\alpha e_{Q_i}^j + \beta e_{Q_i}^k)^T F_Q^{ij} (\gamma e_{Q_j}^i + \delta e_{Q_j}^k) \\ &= (\beta e_{Q_i}^k)^T F_Q^{ij} (\delta e_{Q_j}^k) \\ &= 0, \end{aligned}$$

(final equality follows from (IV.8)) which proves that y lies on S_P^{ij}. Hence S_P^{ij} contains the whole intersection of Π_i, Π_j.

($\Leftarrow$): For the converse, assume that the quadrics contain six lines satisfying the two conditions. Since 1. is satisfied, there exists a F_Q^{ij} such that S_P^{ij} is the pullback of F_Q^{ij}, and the lines $g_{P_i}^j$ and $g_{P_j}^i$ are its epipolar lines (by Proposition IV.5.15). Denote the plane spanned by $g_{P_i}^j$ and $g_{P_i}^k$ by Π_i.

Let x and y be the two points where $\Pi_i \cap \Pi_j$ intersects $g_{P_i}^k$ and $g_{P_j}^k$ respectively (in the case where $\Pi_i \cap \Pi_j$ is a plane, let x and y be any two points lying on $g_{P_i}^k$ and $g_{P_j}^k$ respectively). If x and y are equal, we get

$$\begin{aligned} 0 &= x^T S_P^{ij} x \\ &= x^T P_i^T F_Q^{ij} P_j y \\ &= (e_{Q_i}^k)^T F_Q^{ij} e_{Q_j}^k), \end{aligned}$$

proving that F_Q^{ij} satisfies (IV.8). On the other hand, if they are different, they span a line lying in $\Pi_i \cap \Pi_j$, which in turn lies in S_P^{ij}. In other words:

$$\alpha x + \beta y \in S_P^{ij}$$

for all $[\alpha : \beta] \in \mathbb{P}^1$. Now

$$\begin{aligned} P_i x &= e^k_{Q_i} & P_j x &= \gamma' e^i_{Q_j} + \delta' e^k_{Q_j} \\ P_i y &= \gamma e^j_{Q_i} + \delta e^k_{Q_i} & P_j y &= e^k_{Q_j} \end{aligned}$$

for some $[\gamma : \delta], [\gamma' : \delta'] \in \mathbb{P}^1$, so

$$\begin{aligned} 0 &= (\alpha x + \beta y)^T S_P^{ij} (\alpha x + \beta y) \\ &= (P_i(\alpha x + \beta y))^T F_Q^{ij} P_j (\alpha x + \beta y) \\ &= (\alpha e^k_{Q_i} + \beta(\gamma e^j_{Q_i} + \delta e^k_{Q_i}))^T F_Q^{ij} (\alpha(\gamma' e^i_{Q_j} + \delta' e^k_{Q_j}) + \beta e^k_{Q_j}) \\ &= (\alpha e^k_{Q_i} + \beta\delta e^k_{Q_i})^T F_Q^{ij} (\alpha\delta' e^k_{Q_j} + \beta e^k_{Q_j}) \\ &= (\alpha + \beta\delta)(\alpha\delta' + \beta)(e^k_{Q_i})^T F_Q^{ij} e^k_{Q_j}. \end{aligned}$$

This holds for all $[\alpha : \beta] \in \mathbb{P}^1$. In particular, we can take $[\alpha : \beta]$ to be such that $(\alpha + \beta\delta)(\alpha\delta' + \beta) \neq 0$, proving that F_Q^{ij} satisfies (IV.8). Hence, S_P^{12}, S_P^{13}, S_P^{23} form a compatible triple of quadrics in this case also. ■

IV.6.2 Ambiguous points

Having dealt with the first issue, the fact that not all triples of quadrics provide three conjugate cameras Q_1, Q_2, Q_3, we move on to the second issue, namely that even if a configuration (P_1, P_2, P_3, X) lies on the intersection of three compatible, critical quadrics, it might not be a critical configuration.

By Proposition IV.3.2, the set of critical points is the intersection of three quadrics and some number of cubic surfaces. Hence, there might be some points in the intersection $S_P^{12} \cap S_P^{13} \cap S_P^{23}$ which are not critical.

Proposition IV.6.11. *Given two triples of cameras (P_1, P_2, P_3) and (Q_1, Q_2, Q_3), and a point $x \in S_P^{12} \cap S_P^{13} \cap S_P^{23}$. The three lines*

$$l'_{Q_i} = \overline{Q_i^{-1}(P_i x)} \subset \mathbb{P}^3$$

either have a point in common or all lie in the same plane.

Proof. For any point $x \in S_P^{12}$ there exists a point y (its conjugate with respect to the two cameras Q_1, Q_2) satisfying $P_1 x = Q_2 y$ and $P_2 x = Q_2 y$. It follows that the two lines l'_{Q_1} and l'_{Q_2} intersect. So if $x \in S_P^{12} \cap S_P^{13} \cap S_P^{23}$ each pair of lines l'_{Q_i} will intersect. This leaves two options, the l'_{Q_i} are either three lines lying in the same plane, or they have at least one point in common. ■

Definition IV.6.12. Given two triples of cameras, the intersection $S_P^{12} \cap S_P^{13} \cap S_P^{23}$ is the union of two subvarieties: the critical points, for which l'_{Q_i} pass through a common point, and the *ambiguous points*, for which the lines l_{Q_i} are coplanar.

Remark IV.6.13. The subvarieties are not always disjoint, three lines may all lie in the same plane *and* have a point in common.

The issue of these ambiguous points is well-known, and descriptions can be found in [HZ04, Section 15.3], and, for a more reconstruction-oriented angle, in [HK07, Section 6]. In this section, we recall a key result by Hartley and Kahl [HK07, Theorem 6.17], although approached from a more algebraic setting; we also provide a few additional results on the ambiguous points in the case where the cameras are collinear.

IV.6.2.1 The non-collinear case

Lemma IV.6.14. *Let P_1, P_2, P_3, and Q_1, Q_2, Q_3 be two triples of cameras, such that the camera centers q_i are not collinear. Then the ambiguous points on $S_P^{12} \cap S_P^{13} \cap S_P^{23}$ are the intersection of three planes, one through each camera center, where the plane through the i-th camera center is:*

$$\begin{aligned}\Pi_i &= \operatorname{span}(g^j_{P_i}, g^k_{P_i}) \\ &= \operatorname{span}(P_i^{-1}(e^j_{Q_i}), P_i^{-1}(e^k_{Q_i})) \\ &= \operatorname{span}(P_i^{-1}(Q_i(q_j)), P_i^{-1}(Q_i(q_k))\end{aligned}$$

Proof. The ambiguous points on $S_P^{12} \cap S_P^{13} \cap S_P^{23}$ correspond exactly to where the three lines l'_{Q_i} all lie in the same plane. Since l'_{Q_i} passes through q_i and the three camera centers are not collinear, the three lines l'_{Q_i} lie in the same plane only if they lie in the plane Π_Q spanned by the camera centers q_i. In each image, $Q_i(\Pi_Q)$ is the line spanned by the two epipoles $e^j_{Q_i}, e^k_{Q_i}$, so $P_i^{-1}(Q_i(\Pi_Q))$ is the plane Π_i spanned by $g^j_{P_i}$ and $g^k_{P_i}$. The ambiguous points are the intersection of these three planes. ∎

Theorem IV.6.15 ([HK07, Theorem 6.17])**.** *Let P_1, P_2, P_3 be a triple of cameras, let S_P^{12}, S_P^{13}, S_P^{23} be a compatible (in the non-collinear sense) triple of quadrics, and let X be their intersection. The compatible triple of quadrics comes with six permissible lines, spanning three planes. Let x be the intersection of these three planes. Then the cameras P_i along with the points in $\overline{X - x}$ constitute a critical configuration.*

Proof. Since the three quadrics S_P^{ij} are compatible, there exist three non-collinear cameras Q_1, Q_2, Q_3 giving rise to these quadrics, so we can find a conjugate configuration. By Lemma IV.6.14, the ambiguous points on X are exactly the points x, so $(P_1, P_2, P_3, X - x)$ is a critical configuration. By continuity, so is $(P_1, P_2, P_3, \overline{X - x})$. ∎

Remark IV.6.16. In the case where x is a plane, it might happen that the configuration $(P_1, P_2, P_3, \overline{X - x})$ is not maximal, as the intersection of the critical and ambiguous points need not lie in the closure of $X - x$. For instance, in the case where the three quadrics are all equal to the union of two planes, x is the plane containing the camera centers. $(P_1, P_2, P_3, \overline{X - x})$ is not a maximal critical configuration in this case, as there is a conic curve in x which also lies in the set of critical points (see Proposition IV.7.6).

Theorem IV.6.15 tells us all we need to classify the (non-collinear) critical configurations for three views. We need only check in what ways three compatible quadrics can intersect, and then remove a point/line/plane if needed. In particular, if the intersection $S_P^{12} \cap S_P^{13} \cap S_P^{23}$ does not contain a plane, a line, or any isolated points, the whole intersection is part of a critical configuration.

IV.6.2.2 The collinear case

We now consider the other case, where (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) are two triples of cameras, such that the camera centers q_1, q_2, q_3 lie on a line. In this case, the ambiguous points form all of $S_P^{12} \cap S_P^{13} \cap S_P^{23}$. Indeed, whenever the lines l'_{Q_i} intersect in a point, the three lines are coplanar, since the camera centers q_i lie on a line.

Since the set of critical points is a subset of the ambiguous points, it is not enough to simply remove a point, line, or plane. It follows that Theorem IV.6.15 does not hold in the case where the cameras on the other side are collinear. Unlike the non-collinear case, the permissible lines on the quadrics S_P^{ij} are not enough to determine the critical points either. We do, however, have the following result, which holds for both the collinear and non-collinear cases:

Proposition IV.6.17. *The intersection of the critical points with the ambiguous points forms a subvariety of the latter of codimension at most one*[9].

Proof. Given three cameras Q_1, Q_2, Q_3, consider the variety

$$A = V(F_Q^{12}, F_Q^{13}, F_Q^{23}) \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2.$$

For each point $(x_1, x_2, x_3) \in A$, the three lines $l'_{Q_i} = \overline{Q_i^{-1}(P_i x)} \subset \mathbb{P}^3$ either lie in the same plane or have a common point, or both. In the family of triples of coplanar lines, one through each camera center, requiring the third line to pass through the intersection of the first two is (generally) a linear constraint. Hence the triples of coplanar lines all having at least one point in common form a codimension one subvariety of this family. To get to $S_P^{12} \cap S_P^{13} \cap S_P^{23}$ however, we need to intersect A with the multi-view variety $\mathcal{M}_P$. This intersection is a point, line, or plane (as discussed before), in the event of a line or plane, the critical points form a codimension one subvariety of the line/plane. If the intersection is a single point, this point may or may not be critical, we will see both happen in the following section. ■

[9] exactly one if we assume that the ambiguous points are not contained in the set of critical points

IV.7 Critical configurations for three views

In this section, we classify all the critical configurations for three views. By Corollary IV.6.3, and the discussion in Section IV.6.1, any critical configuration for three views lies on the intersection of three compatible quadrics. Furthermore, any such intersection is critical, with the possible exception of some number of ambiguous points (usually a point, line, or plane). As such, our approach is to consider all possible intersections of three quadrics, and for each case check whether it exists as the intersection of a compatible triple, and if so, remove any potential ambiguous points. This gives us all critical configurations for three views. We start with the cases where the quadrics intersect in a surface (Section IV.7.1), then a curve (Section IV.7.2), and finally in a finite number of points (Section IV.7.3).

Remark IV.7.1. The intersection of quadrics may sometimes contain non-reduced components[10]. One such example is three lines through a single point, with one line having multiplicity 2, which appears as the intersection of three reducible quadrics. However, since the definition of critical configurations is a set-theoretical one, it does not see multiplicity of components. As such, we consider all components to be reduced.

Remark IV.7.2. While the critical configurations for two views tend to have a finite number of conjugates, this is generally not the case for three views. The only critical configurations for three views that have a finite number of conjugates are the ones consisting of six or seven points. The rest all have an infinite number of conjugates (although the dimension of the family of conjugates may vary). The proofs usually revolve around constructing a single conjugate, but the choices made in the construction should make it clear that many conjugates exist when different choices are made.

IV.7.1 Quadrics intersecting in a surface

In the two-view case, the points in a maximal critical configuration always form a surface. We begin by checking whether any such configurations remain critical in the three-view case. In particular, we want to examine whether there exist any critical configurations where the points form a surface. There are two ways three quadrics can intersect in a surface, either by all being equal or by all being reducible and sharing a common plane.

IV.7.1.1 The smooth quadric/cone

We start with the case where the quadrics S_P^{ij} are all the same smooth quadric or cone (denoted simply S_P). In this case, it is impossible to find six lines satisfying the conditions in Theorem IV.6.10, so one cannot find a non-collinear conjugate triple of cameras Q_1, Q_2, Q_3. However, as long as S_P is not one of the following configurations (illustrated in Figure IV.5)

[10] Components with multiplicity greater than one

- S_P is smooth and there is a camera center such that the two lines passing through it, each pass through one other camera center.
- S_P is a cone and two camera centers lie on the same line, but neither of the two lies at the vertex

then three copies of S_P satisfy the conditions of Proposition IV.6.9, and hence form a compatible triple, coming from a triple of collinear cameras Q_i.

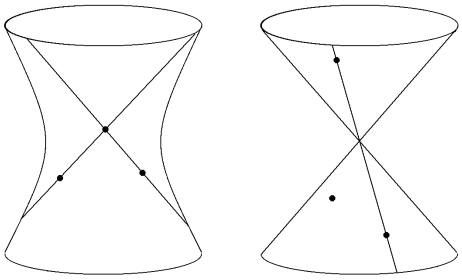


Figure IV.5: If the camera centers are positioned as shown, the three quadrics (all three equal) do not constitute a compatible triple.

Proposition IV.7.3. *Let P_1, P_2, P_3 be three cameras, and let $S_P^{12}, S_P^{13}, S_P^{23}$ be a compatible triple of quadrics. If the three quadrics are all equal to the same irreducible quadric S_P, the set of critical points X is a rational curve of degree at most four passing through all camera centers. The conjugate configuration is a curve of one degree less, not passing through any camera centers (plus the line passing through the camera centers, if S_P is a cone).*

Remark IV.7.4. The two cases in Figure IV.5 are exactly the two where S_P does not contain a smooth rational curve passing through all three camera centers.

Proof. Since the three cameras Q_i on the other side have to be collinear, our critical configuration does not consist of all the points on S_P. By Proposition IV.6.17, the critical points on S_P form a codimension one subvariety, that is, a curve $C_P \subset S_P$.

By Proposition IV.3.2, the set of critical points is the intersection of three quadric and some number of cubic surfaces. The three quadrics are all equal to S_P, so C_P is the intersection of S_P with one or more cubic surfaces, all of which contain all three camera centers. It follows that C_P is of type (a, b, c_1, c_2) (recall Definition IV.5.18) with $a, b \leq 3$ and $c_1, c_2 \geq 1$.

The curve C_P lies on all three quadrics S_P^{ij} and its type (a, b, c_i, c_j) depends on which quadric we consider since the multiplicity in each camera center could be different. However, the total degree of the conjugate curve C_Q depends (by Proposition IV.5.19) on the multiplicity of C_P in the camera centers. Since the curve C_Q is the same regardless of which quadric we consider, it follows that

C_P needs to have the same multiplicity in all three camera centers, the same is true for C_Q.

If we assume that the three camera centers p_i are not collinear, the three quadrics S_Q^{ij} on the other side cannot be the same quadric. It follows that the conjugate curve $C_Q \subset S_Q^{12} \cap S_Q^{13} \cap S_Q^{23}$ is of type (a', b', c_1', c_2') with $a', b' \leq 2$. If, on the other hand, the camera centers p_i are collinear, the resulting curve is still the intersection of quadrics and cubics, and so needs to satisfy $a', b' \leq 3$.

Furthermore, if S_P is smooth, the curve C_Q cannot pass through any of the three camera centers q_i, since this would imply that all three quadrics S_Q contain the line spanned by the camera centers, which (by Table IV.4) violates our condition that S_P is smooth.

On the other hand, if S_P is a cone, then the three quadrics S_Q^{ij} must (by Table IV.4) contain the line l_Q spanned by the camera centers. However, the whole line l_Q is conjugate to a single point, namely the vertex of S_P (by Lemma IV.5.16). So in the case where S_P is a cone, C_Q is the union of l_Q and some other curve with $a', b' \leq 2$.

This leaves only a finite number of options for what kind of curve C_P can be. Finally, the curves should satisfy the result in Proposition IV.5.19: that if C_P is of type (a, b, c_1, c_2), then C_Q is of type $(a, a+b-c_1-c_2, a-c_2, a-c_1)$. We can now check every possible type for C_P and remove all cases where a' or b' are either negative or greater than 3, the cases where c_i' is negative, and the cases where the multiplicity in the camera centers is greater than what the degree of the curve allows (for instance, a curve of degree one can not have multiplicity 2 in a camera center).

This leaves only three options, listed in Table IV.5. Note that if S_P is a cone, then the curve on the other side will also include the line l_Q in addition to the component given in the table. ■

Curve on S_P	Curve on S_Q
(1,3,1,1)	(1,2,0,0)
(1,2,1,1)	(1,1,0,0)
(1,1,1,1)	(1,0,0,0)

Table IV.5: Recall that a curve of type (a, b, c_1, c_2) intersects the family containing the epipolar lines a times, the lines in the other family b times, and passes c_i times through the camera center p_i.

Remark IV.7.5. This proves that any critical configuration where the three quadrics on one side coincide has to be one of the types in Table IV.5. In Section IV.7.2 we prove the converse, that any configuration of such type is critical. The general case is the rational quartic [11] on one side, and a twisted cubic on the other, while the two other configurations appear as degenerates of this one.

[11] degree four curve

IV.7.1.2 Two planes

Proposition IV.7.6. *When the three quadrics S_P^{ij} are all equal to the same two planes, the set of critical points is the union of a plane and a conic passing through all three camera centers. Conversely, any configuration where the points lie on the union of a plane and a conic curve passing through the camera centers is a critical configuration. Its conjugate is of the same type (plane + conic curve).*

Proof. Let the three quadrics S_P^{ij} all be equal to the same two planes S_P. By Theorem IV.5.12, all three camera centers need to lie in the same plane, with any number of them lying on the intersection of both. In this case, unlike the previous one, there exist six lines satisfying the conditions in Theorem IV.6.10 as long as not all three camera centers lie on the intersection of the two planes. An example is given in Figure IV.6. Hence, the three cameras Q_i on the other side can be chosen to not be collinear.

Since the three cameras Q_i on the other side can be assumed to not be collinear, Theorem IV.6.15 proves that the plane not containing all camera centers is part of the set of critical points. Furthermore, Proposition IV.6.17 proves that there is a curve C_P in the plane with the camera centers which is also part of this set. Since the cameras Q_i are not collinear, Lemma IV.6.4 tells us that the set of critical points is the intersection of the two planes with a single cubic surface passing through the three camera centers, so the curve C_P is the union of a conic passing through the three camera centers and the line where the two planes intersect.

For the converse, we want to determine the conic curve C in the set of critical points. We already know that C_P passes through the three camera centers. Furthermore, the intersection of the two permissible lines $g^j_{P_i}$ and $g^j_{P_k}$ (denote it by a_j) also lies in the set of critical points since it has a conjugate point, namely q_j. This gives us three more points on the conic. Usually, there is no conic passing through six given points. In this case, however, the six points $p_1, p_2, p_3, a_1, a_2, a_3$ along with the six lines $g^j_{P_i}$ span a hexagon whose opposite sides ($g^j_{P_i}$ and $g^i_{P_j}$) meet at three points which lie on a line (the intersection of the two planes) so by the Braikenridge–Maclaurin theorem [CG67, p.76] the six points $p_1, p_2, p_3, a_1, a_2, a_3$ lie on a conic.

Finally, for any conic C_P through p_1, p_2, p_3 one may choose the six lines $g^j_{P_i}$ in such a way that the three points a_i lie on C_P *and* each pair of lines form a permissible pair. This can be done by picking a_1 to be any[12] point on C_P and then taking a_2 to be the intersection of $g^2_{P_1}$ with C_P. Since we can get any conic by making the appropriate choice of permissible lines, this proves that any plane + conic configuration is critical. ∎

[12] not equal to a camera center and not lying in the intersection of the two planes

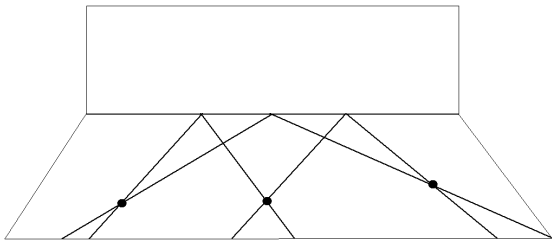


Figure IV.6: One example of six lines satisfying the conditions in Theorem IV.6.10

IV.7.1.3 Single plane

With this, there is only one way left that the three quadrics S_P^{ij} can intersect in a surface, namely if they are all reducible, and contain only one common plane. In this case, the set of critical points is at most a plane plus a line. Hence, this is not a maximal critical configuration, as it is always a subconfiguration of the plane + conic configuration in Proposition IV.7.6.

IV.7.2 Quadrics intersecting in a curve

Curves that appear as the intersection of quadrics have at most degree 4. Moreover, any curve that lies on the union of a plane and a conic passing through the camera centers is critical by Proposition IV.7.6, which further reduces the number of curves we need to check. We analyze all curves appearing as the intersection of quadrics sorted by the degree of their highest degree component. In Section IV.7.2.1, we analyze the curves where the highest degree component is of degree 4, covering the elliptic quartic as well as the singular rational quartics. Section IV.7.2.2 covers the curves containing a twisted cubic, while Section IV.7.2.3 covers the curves containing no components of degree higher than 2.

For most curves, the main issue is to find a compatible (in the non-collinear sense) triple of quadrics containing it. The issue of having to remove a plane, line, or isolated point (as per Theorem IV.6.15) is less relevant since the curves will usually not contain any such component. To find compatible triples of quadrics, the following lemmas are useful:

Lemma IV.7.7. *Let P_1, P_2, P_3 be a triple of cameras, and let S_P^{12}, S_P^{13}, S_P^{23} be a triple of critical quadrics, each passing through two camera centers (indicated by the superscript). Then the three quadrics are compatible if*

A) *The intersection of S_P^{ij} and S_P^{ik} does not contain a line passing through p_i.*

B) *There exists a point x lying on all three quadrics, as well as three distinct lines l_{12}, l_{13}, l_{23} passing through x, satisfying*

1. $l_{ij} \subset S_P^{ij}$.

2. $p_i \in \operatorname{span}(l_{ij}, l_{ik})$.
3. $l_{ij} \notin \operatorname{span}(l_{ik}, l_{jk})$.
4. $p_i, p_j \notin l_{ij}$.

Proof. Given three quadrics S_P^{ij}, assume there exists a point x and three lines l_{ij} satisfying the four conditions. For each quadric S_P^{ij}, pick two lines $g_{P_i}^j$ and $g_{P_j}^i$ such that they both intersect l_{ij}, and form a permissible pair. The only case where such a choice is not possible is if the quadric S_P^{ij} contains the line spanned by p_i and p_j, and the line l_{ij} intersects this line, but this would violate either condition 2, 3 or 4.

This gives us six lines g_P, two on each quadric, and two through each camera center. Condition A) ensures that the two lines through each camera center are distinct, so they span a plane. By construction, the plane spanned by l_{ij} and l_{ik} is the same as the one spanned by $g_{P_i}^j$ and $g_{P_i}^k$. This means that the lines g_P satisfy the conditions in Theorem IV.6.10. It follows that the three quadrics are compatible. ■

Lemma IV.7.8. *Let P_1, P_2, P_3 be a triple of cameras, and let S_P^{12}, S_P^{13}, S_P^{23} be a triple of critical quadrics. If there exists a line L lying on all three quadrics, but not passing through the camera centers such that the intersection of S_P^{ij} and S_P^{ik} does not contain a line passing through p_i and intersecting L, then the quadrics form a compatible triple.*

Proof. On each quadric S_P^{ij}, take $g_{P_i}^j$ to be the line through p_i intersecting L. By the same arguments as in the proof of Lemma IV.7.7, this gives us six lines satisfying Theorem IV.6.10. ■

Remark IV.7.9. In Lemmas IV.7.7 and IV.7.8, the lines l_{ij} and L all intersect the permissible lines, meaning they are of type $(1, 0, 0, 0)$ on their respective quadrics.

Remark IV.7.10. Lemmas IV.7.7 and IV.7.8 have the condition that the intersection of two quadrics does not contain a line passing through their common camera center. This is to ensure that the two lines $g_{P_i}^j$ and $g_{P_i}^k$ span a plane, which is a necessary condition for the quadrics to be compatible (in the non-collinear sense). This is an open condition, and in the following sections, whenever we apply either of the two lemmas, the cases where two lines through a camera center coincide can easily be avoided, if such cases exist at all. To keep the arguments relatively short, the verification of this fact is left to the reader.

IV.7.2.1 Quartic curves

We begin with the case where the intersection of the quadrics is a quartic curve. This happens when the quadrics lie in the same pencil[13], that is, when the intersection of the three quadrics is equal to the intersection of any two of them.

[13] pencil: a one-dimensional family of objects

Remark IV.7.11. Not every quartic curve appears as the intersection of a pencil of quadrics, for instance, four skew lines do not. The ones that do are the elliptic quartic curve, the singular rational quartic (cusp or node), the union of a twisted cubic and a secant[14] line, and the union of two (possibly reducible) conics intersecting in two (possibly complex) points. When we talk about quartics in the rest of this section, it is to be understood that we mean quartics appearing as the intersection of quadrics.

As it turns out, a set of three cameras and any number of points lying on a quartic curve C form a critical configuration if every camera center lies on C, but is not a singular point on C. We give a different proof of this statement than what appears in [Brå23a], instead giving one similar to that in [HK07] since this will more easily generalize to the case of 4+ views. To do this we first introduce the notion of quasi-parallel lines.

Definition IV.7.12. Given a quartic curve C, two secants L_1 and L_2 to C are called *quasi-parallel* if they lie on the same quadric in the pencil of quadrics containing C, and form a permissible pair on this quadric. We use the notation $L_1 \parallel L_2$ when L_1 is quasi-parallel to L_2.

Lemma IV.7.13 ([HK07, Theorem 8.21])**.** *Let $p_1, p_2, p_3, y_1, y_2, y_3$ be six distinct points in the non-singular locus of a quartic curve C, such that $\overline{p_1y_2} \parallel \overline{p_2y_1}$ and such that $\overline{p_1y_3} \parallel \overline{p_3y_1}$. Then $\overline{p_2y_3} \parallel \overline{p_3y_2}$.*

Proof. Project from a generic point x on C, if a plane contains more than one of the lines $\overline{p_iy_j}$, we pick a point x not lying in this plane. This projection gives us a plane (possibly reducible) cubic curve C' along with six points $p_1', p_2', p_3', y_1', y_2', y_3'$ on C'. The secants $\overline{p_iy_j}$ and $\overline{p_jy_i}$ are only quasi-parallel if the two lines $\overline{p_i'y_j'}$ and $\overline{p_j'y_i'}$ intersect in a point on C'. Since $\overline{p_2'y_1'}$ intersects $\overline{p_1'y_2'}$ in a point on C', and the same is the case for $\overline{p_1'y_3'}$ and $\overline{p_3'y_1'}$, the Cayley–Bacharach theorem [Bac86] (see Figure IV.7) states that $\overline{p_2'y_3'}$ intersects $\overline{p_3'y_2'}$. This means that $\overline{p_2y_3} \parallel \overline{p_3y_2}$, thus completing the proof. ∎

Lemma IV.7.14. *Let C be a quartic curve, let $p_1, \dots, p_n \in C$ be camera centers, and let $y_1, \dots, y_n \in C$ be points such that $\overline{p_1y_i} \parallel \overline{p_iy_1}$ for all $i \leq n$. Let S_P^{ij} be the unique quadric containing C and the line $\overline{p_iy_j}$, then for all $i, j, k \leq n$ the triple $S_P^{ij}, S_P^{ik}, S_P^{jk}$ is compatible.*

Proof. By Lemma IV.7.13, $\overline{p_1y_i} \parallel \overline{p_iy_1}$ for all i, implies $\overline{p_iy_j} \parallel \overline{p_jy_i}$ for all i, j. Let S_P^{ij} be the unique quadric containing C and having $\overline{p_iy_j} \parallel \overline{p_jy_i}$ as its epipolar lines. Since y_k lies on C, and hence on S_P^{ij}, we have

$$\begin{aligned} e_{Q_i}^k F_Q^{ij} e_{Q_j}^k &= y_k^T P_i^T F_Q^{ij} P_j y_k \\ &= y_k^T S_P^{ij} y_k, \\ &= 0, \end{aligned}$$

[14] secant to a curve: a line spanned by two points on the curve

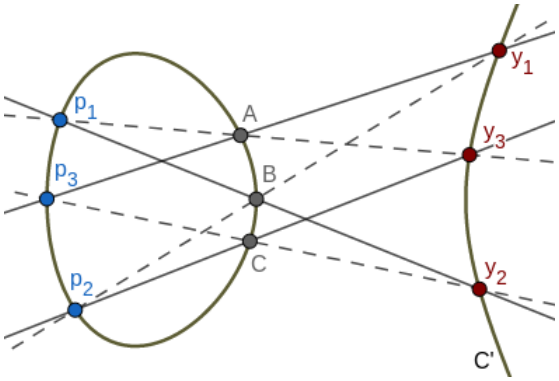


Figure IV.7: Illustration of the Cayley–Bacharach theorem. Given 9 points appearing as the intersection of two cubics, any cubic containing 8 of them also contains the ninth. This figure shows three cubics, two of which are reducible (the union of three lines).

Which by Theorem IV.6.8 means that each triple of quadrics is compatible. ■

Proposition IV.7.15. *Let C be a quartic curve appearing as the intersection of a pencil of quadrics. Let P_1, P_2, P_3 be three cameras whose centers are non-singular points on C. Then (P_1, P_2, P_3, C) is a critical configuration. The conjugate is a quartic curve passing through the camera centers.*

Proof. Let y_1 be any smooth point on C not lying in the plane spanned by the camera centers. Let S_P^{12} denote the unique quadric containing C and the secant $\overline{p_2 y_1}$. The quadric S_P^{12} contains a unique line L through p_1 such that l and $\overline{p_2 y_1}$ form a permissible pair. The line L intersects C in two points, one being p_1, we denote the other one by y_2. We construct S_P^{13} and y_3 in a similar fashion, using p_3 in place of p_2. Let S_P^{23} be the quadric containing C and the line $\overline{p_2 y_3}$. By construction, y_1, y_2, y_3 satisfy the conditions in Lemma IV.7.14, proving that the three quadrics S_P^{ij} are compatible, and thus, that the configuration is critical, with the possible exception of some ambiguous points.

By Lemma IV.6.14, the ambiguous points form a plane, line or an isolated point lying on C. In particular, they are the intersection of the three planes

$$\Pi_i = \operatorname{span}(p_i, y_j, y_k) \qquad \forall i \neq j \neq k.$$

The quartic curve C can never contain a plane or an isolated point, and in most cases, it does not contain any line L either. If it does, one can always choose the y_i such that the three planes Π_i do not intersect in L. Hence avoiding any trouble with ambiguous points.

Assuming at least one of the quadrics S_P^{ij} is irreducible, the curve C (a quadric passing through all camera centers) is of type $(2, 2, 1, 1)$ on S_P^{ij}, so by Proposition IV.5.19, its conjugate is also of type $(2, 2, 1, 1)$. If none of the

quadrics are irreducible, it is of type $(2,2,1,1,1)$, in which case the conjugate, $(2,2,1,1,1)$ is still of the same type. ■

In the construction in the proof of Proposition IV.7.15, we take y_1 to not lie in the plane spanned by the camera centers, which will generally ensure that y_2, y_3 do not lie in this plane either, this in turn means that the conjugate configuration consists of points on a curve isomomorphic to C. There is, however, one case where it is unavoidable that at least one of the y_i lie in the plane spanned by the camera centers, this is in the case where the curve C contains a line L passing through at least two of the camera centers. In this case, the conjugate curve is singular, with the singular point being conjugate to the entire line L (recall Lemma IV.5.16, Item IV.5.16.3). Moreover, if the line L passes through *exactly* two of the camera centers, say p_1, p_2, then the conjugate configuration has the conjugate to the final camera, in this case, q_3, lying in the singular locus. We prove any such configuration is critical:

Proposition IV.7.16. *Let C be a singular quartic curve appearing as the intersection of a pencil of quadrics. Let P_1, P_2, P_3 be three cameras whose centers lie on C and with the camera center of P_3 lying in the singular locus of C. Then (P_1, P_2, P_3, C) is a critical configuration. The conjugate is a reducible curve consisting of a line L through the conjugate camera centers q_1, q_2 and a twisted cubic curve through q_3, having L as a secant.*

Proof. Let S_P^{12} be the unique cone containing C and having a vertex at p_3. We take the epipolar lines $g^2_{P_1}$ and $g^1_{P_2}$ to be the lines $\overline{p_1 p_2}$ and $\overline{p_2 p_3}$, these form a permissible pair on S_P^{12}. Next, consider the set of planes containing the line spanned by p_1, p_2, each plane intersects C in two, usually distinct, points. There are however three planes where the two intersection points coincide, one where they are both equal to p_3 and two where they coincide as a point on C distinct from any camera center, denote either of these points by y_3. Let S_P^{13} and S_P^{23} both be equal to the unique quadric containing both the secants $\overline{p_1 y_3}$ and $\overline{p_2 y_3}$ (by construction of y_3, these secants lie on the same quadric). Let the epipolar line $g^3_{P_i}$ be the line $\overline{p_i y_3}$ and let the epipolar lines $g^i_{P_3}$ be such that $g^3_{P_i}$ and $g^i_{P_3}$ form a permissible pair on S_P^{i3}. Since each pair of epipolar lines $e^k_{P_i}, e^k_{P_j}$ intersect in a point on S_P^{ij}, we get that $(e^k_{Q_i})F_Q^{ij}e^k_{Q_j} = 0$, hence proving that the quadrics are compatible (by Theorem IV.6.8). This in turn means that the configuration (P_1, P_2, P_3, C) is critical.

The second half of the statement follows from Proposition IV.5.19 along with Lemma IV.5.16, Item IV.5.16.3. ■

We round off the quartic curve section by proving that the curves in

Lemma IV.7.17. *The critical configurations in Propositions IV.7.15 and IV.7.16 are the only critical configurations where the points lie on a quartic curve.*

Proof. By Theorem IV.6.1 any quartic curve that is part of a critical configuration must lie on the intersection of three critical quadrics. Any quartic curve appearing

as the intersection of three different quadrics where each quadric contains two camera centers contains all three camera centers. The only way for such a quartic curve to not contain all camera centers is if two of the quadrics are equal. Then the curve only contains two camera centers. In this case the curve is of type $(2, 2, 1, 0)$ on the two equal quadrics and $(2, 2, 1, 1)$ on the last. If such a configuration was critical, the conjugate curve would have to be of type $(2, 3, 2, 1)$ on two of the quadrics and $(2, 2, 1, 1)$ on the last, but a curve of degree five does not appear as the intersection of quadrics, so this can not be a critical configuration. ■

IV.7.2.2 The twisted cubic

Having covered the quartic curves, we move one degree lower, to the cubics. Plane cubics do not appear as the intersection of quadrics, so the only irreducible cubic curve that might be critical is the twisted cubic curve. The results in Propositions IV.7.15 and IV.7.16 already prove that any configuration of points and cameras lying on the union of a twisted cubic and a secant line are critical, with a conjugate of the same type. There are, however, some cases where the conjugate is not of the same type, we show these here. We also give a critical configuration that is not a subconfiguration of the quartic curve ones.

Proposition IV.7.18. *Any configuration consisting of three cameras along with a set of points lying on a twisted cubic C passing through two of the camera centers is critical. The same is the case if C passes through only one camera center, as long as the two remaining camera centers lie on a secant to C.*

Proof. Both configurations consist of cameras and points lying on the union of a twisted cubic + secant line, so by Proposition IV.7.15, they are critical. ■

Remark IV.7.19. Neither of the configurations in Proposition IV.7.18 appear in [HK07], the latter does not appear in [Brå23a] either, demonstrating that both previous classifications are incomplete.

Proposition IV.7.20. *Any configuration consisting of three cameras and any number of points all lying on a twisted cubic is critical. A conjugate consists of points on a conic curve not passing through the camera centers.*

Proof. Given a twisted cubic, there is a two-dimensional family of quadrics containing it. This leaves us 2 degrees of freedom when choosing each of the quadrics. In other words, for each line intersecting the cubic, there is a unique quadric containing both the cubic and the line (unless the line is a secant, in which case a whole pencil of quadrics contains it).

Take a line L, secant to C, and not passing through the camera centers. There exists a pencil of quadrics containing both C and L. By Lemma IV.7.8, any three distinct quadrics in this pencil form a compatible triple of quadrics. The line L has bidegree $(1, 0)$ and is a secant to C, so it intersects C twice. It follows that the twisted cubic is of type $(1, 2, 1, 1)$, so the conjugate is of type $(1, 1, 0, 0)$, a conic not passing through the camera centers. ■

Proposition IV.7.20 shows that three cameras on a twisted cubic also has other conjugates than the ones in IV.7.15. As for the case where the camera centers do not all lie on the twisted cubic, these will generally only be critical if the cameras not on the twisted cubic all lie on the same secant line. There is, however, one exception:

Proposition IV.7.21. *Any configuration consisting of three collinear cameras along with a set of points lying on a twisted cubic not intersecting the line with the camera centers is critical. The conjugate is a rational quartic curve containing the three camera centers.*

To prove this, we first give this, seemingly unrelated, lemma:

Lemma IV.7.22. *Let $C \in \mathbb{P}^3$ be a twisted cubic, let l be a line in $\mathbb{P}^3$, not secant to C, and let p be a fixed point on l not lying on the twisted cubic. There exists a point $x \in C$ such that for every point $q \in l$, such that $q \neq p$ and $q \notin C$, the unique quadric surface containing C, p, and q, also contains the line $\overline{xq}$.*

Proof. Let s denote the unique secant line of C passing through p. The plane spanned by s and l intersects C in three points. Two of these points lie on s, the last one does not. We denote this one by x.

In the pencil of quadrics containing C and p, each quadric also contains the secant s. For any point q as above, the quadric containing C, p, and q, intersects the line $\overline{xq}$ in three points, x, q, and a third point lying on s. Since it contains three points on $\overline{xq}$, it contains the whole line. ∎

Proof of Proposition IV.7.21. Let C denote the twisted cubic, let p_i denote the camera centers, and let L denote the line spanned by the camera centers.

Let S_P^{ij} be the unique quadric containing C, p_i, and p_j. If S_P^{ij} is smooth there are two lines through p_i. One is a secant to C, the other line intersects C exactly once; denote the latter line by $g_{P_i}^j$. If S_P^{ij} is a cone, there is only one line through p_i, in this case, we take this one to be $g_{P_i}^j$. Repeating this process for each pair of cameras on each of the three quadrics, we get 6 lines, two on each quadric, and two through each camera center.

By Lemma IV.7.22, the lines $g_{P_i}^k$ and $g_{P_j}^k$ intersect C at the same point, which we denote by x_k (see Figure IV.8). Since x_k lies on S_P^{ij}, we have:

$$\begin{aligned} x_k^T S_P^{ij} x_k &= x_k^T P_i^T F_Q^{ij} P_j x_k, \\ &= e_{Q_i}^k F_Q^{ij} e_{Q_j}^k, \\ &= 0. \end{aligned}$$

Meaning the quadrics $S_P^{12}, S_P^{13}, S_P^{23}$ do indeed form a compatible triple.

Since the twisted cubic is of type $(1, 2, 0, 0)$, its conjugate is of type $(1, 3, 1, 1)$, a rational quartic curve passing through all three camera centers. This is one of the curves from Proposition IV.7.3, appearing when all the quadrics S_Q^{ij} on the other side coincide.

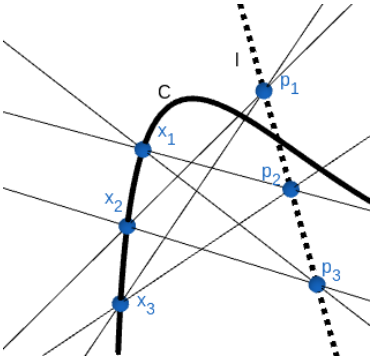


Figure IV.8: The twisted cubic C and the line l. The line $g^j_{p_i}$ is the line spanned by p_i and x_j.

In this construction, the pair of lines $g^j_{P_i}$ through each camera center may coincide. In this case, the three points x_i also coincide, and the three quadrics S^{ij}_P are all cones. By Proposition IV.6.9 the quadrics are compatible in this case also, and the camera centers on the rational quartic on the other side are collinear. ■

In the case where the line containing the cameras intersects the twisted cubic, but is not a secant or tangent, the three quadrics S^{ij}_P all have to be equal. In this case, the only way the lines $g^j_{P_i}$ can be permissible is if they are all secant to the twisted cubic, but in this case, the twisted cubic is of type $(2, 1, 0, 0)$ (or type $(3, 1, 1, 1)$ if you include the line) and neither of these appear as critical configurations when the three quadrics intersect. Hence no such critical configurations exist.

IV.7.2.3 Curves of lower degree

We are now left with curves appearing as the intersection of quadrics where all components are of degree at most 2. These are all either subsets of degenerate quartic curves or degenerate quartic curves themselves, or subsets of the "plane + conic" configuration. Hence critical in all three cases, having conjugates that are of the same type (see Propositions IV.7.6 and IV.7.15).

However, among the configurations that *are* subsets of the "plane + conic" configuration or the degenerate quartic configuration, some also have conjugates that are not of the same type, as was the case for three cameras on a twisted cubic. This happens because for curves of low degree one has more freedom when choosing the quadrics, so more conjugates may appear. For the sake of completeness, we give a summary of them here. No proof is given, but the reader can verify that the curves lie on the intersection of three compatible irreducible quadrics. This task should be fairly straightforward when the curves are of a low degree.

1. Two intersecting lines on one side, and two disjoint lines on the other. The number of cameras lying on the two lines is equal on both sides.

2. Cameras lying on a conic curve. The conjugate is a line not passing through the camera centers.

3. Two lines not passing through the camera centers. This one has two conjugates, the first is three cameras lying on the union of three lines (one intersecting the other two). The second conjugate consists of points lying on the union of a conic and a line, and the cameras lying on the conic. This triple of conjugates is a degeneration of the "twisted cubic $\leftrightarrow$ conic" configuration shown in Proposition IV.7.20.

IV.7.3 Quadrics intersecting in a finite number of points

Lastly, we have the case where three quadrics intersect in a finite number of points. This is the "general" case in the sense that with two general triples of cameras, the set of critical points on each side is finite. Indeed, on each side, we get a compatible triple of quadrics, whose intersection is 8 points (counted with multiplicity and including possible complex points), seven of these points lie in the set of critical points, whereas the last one is an ambiguous point (recall Lemma IV.6.14).

Conversely: given a set of seven points X and three cameras P_i, when is (P_1, P_2, P_3, X) a critical configuration? Since 9 points[15] are enough to fix a quadric, there is generally only one triple of quadrics S_P^{ij} such that $p_i, p_j, X \in S_P^{ij}$. Now IF this triple of quadrics is compatible, it comes with six epipolar lines, spanning three planes. The three planes intersect in one of the 8 points lying on the intersection of the three quadrics. This point is, by Lemma IV.6.14, not critical. As such, the configuration (P_1, P_2, P_3, X) is critical if and only if the triple of quadrics is compatible *and* the three planes do not intersect in one of the points in X (by Theorem IV.6.15). This gives us the compatibility condition for seven general points.

Proposition IV.7.23. *Let P_1, P_2, P_3 be three cameras, and let X be a set consisting of seven points. Then (P_1, P_2, P_3, X) is critical if and only if there exists three quadrics S_P^{ij} satisfying:*

IV.7.23.1. $p_i, p_j, X \in S_P^{ij}$.

IV.7.23.2. $S_P^{12}, S_P^{13}, S_P^{23}$ *is compatible.*

IV.7.23.3. *The intersection of the three planes spanned by the six epipolar lines $g_{P_i}^j$ does not lie in X.*

Remark IV.7.24. A sextuple of epipolar lines $g_{P_i}^j$ only exists if the quadrics are compatible, so the first two conditions need to be satisfied for the third to make sense.

[15] 7 points and two camera centers.

There are, of course, certain positions of seven points and three camera centers such that they do not fix all three quadrics, for instance when points and camera centers coincide. If S_P^{ij} is not fixed, this is because there exists a (possibly degenerate or reducible) elliptic quartic curve that passes through p_i, p_j and X. In the previous section we have shown that if the curve also passes through the final camera, the whole curve is part of a critical configuration, so the seven points along with the cameras is a critical (but not maximal) configuration in this case.

For the sake of completeness, we examine what happens if X does not consist of exactly seven points. For eight or more points, a configuration is critical if and only if it is a subconfiguration of one of the critical configurations mentioned earlier in the section, i.e. if the points lie on a critical curve or surface. For six or fewer points, the configuration is always critical [MS98], but not maximal, since any configuration is contained in a critical configuration consisting of at least seven points. In particular, the generic 6-point configuration is contained in exactly three critical 7-point configurations.

This completes the classification of the critical configurations for three views:

Theorem IV.7.25. *A configuration of three projective cameras P and any number of points X form a critical configuration if and only if the points X form one of the surfaces/curves/point sets described in Figure IV.9, or if they form a subset of one of these.*

IV.8 More than three views

IV.8.1 A counterexample

By Theorem IV.6.1, any critical configuration for four or more views has to be critical for each triple of views. Contrary to what is claimed in [HK07, Theorem 9.34], however, the converse is not true. Although a configuration is critical for each triple of views, the whole configuration might not be critical. The simplest counterexample is that of four cameras and six points in general position.

Theorem IV.8.1 ([MS98])**.** *A set of six points and any number of cameras is a critical configuration if and only if the six points and all the cameras all lie on a smooth ruled quadric.*

While six points and three cameras in general position always lie on a smooth quadric, this is not true for four cameras. So while such a configuration is critical for each triple of views, it is not critical for all four views.

The proof of [HK07, Theorem 9.34] fails on two points

1. Given a configuration (P_1, P_2, P_3, P_4, X) which is critical for each triple of views, there is a compatible triple of quadrics $S_P^{12}, S_P^{13}, S_P^{23}$ containing p_1, p_2, p_3 and X, and there is a compatible triple of quadrics $S_P^{12'}, S_P^{14'}, S_P^{24'}$ containing p_1, p_2, p_4 and X. There is however, no reason why the quadric

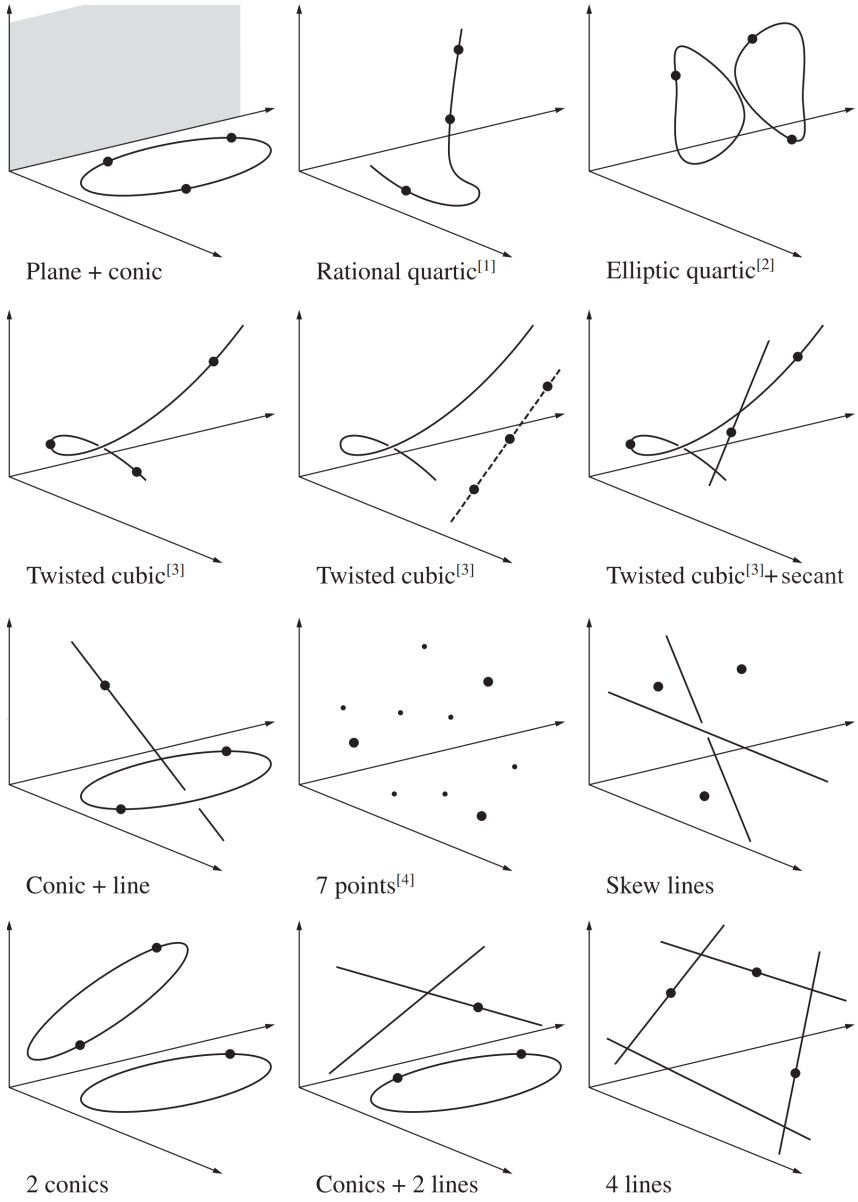


Figure IV.9: Critical configurations for three views.
[1] The rational quartic may have a node or cusp.
[2] The elliptic quartic may degenerate to the union of a twisted cubic and a secant line, or to the union of two (possibly reducible) conics intersecting in two points.
[3] The twisted cubic may degenerate to the union of a conic and a line, or to three lines.
[4] The 7-point configuration must satisfy the conditions in Proposition IV.7.23.

S_P^{12} should equal $S_P^{12'}$, which seems to be assumed in [HK07]. This is what gives rise to the 6-point counterexample.

2. Until the recent results in [BR23], it was commonly believed that a sextuple of quadrics $S_P^{12}, S_P^{13}, S_P^{23}, S_P^{14}, S_P^{24}, S_P^{34}$ was compatible as long as each triple was compatible. As it turns out, this is not the case, and additional conditions are needed to ensure compatibility. Since a critical configuration for four views is the intersection of a compatible sextuple of quadrics, we need to check some additional conditions that do not appear in the three-view case.

As such, the case of four cameras needs to be handled on its own and does not follow directly from the three-view case. That being said, while the proof of [HK07, Theorem 9.34] is flawed, the result itself is not far from being true. As we will soon show, the only three-view critical configurations that fail to remain critical when we add cameras turn out to be the 6- and 7-point configurations.

IV.8.2 Preliminaries

Since any critical configuration for four or more views has to be critical for each triple of views, any critical configurations for four or more views has to be either among the ones in Figure IV.9, or has to be the proper intersection of the ones in Figure IV.9 with some number of quadric and cubic surfaces. Intersecting any of the critical configurations in Figure IV.9 with a surface, however, leaves either a finite set of points or something that is already in Figure IV.9. As such, in order to classify the critical configurations for four views, one needs only check which of the critical configurations from the three-view case remain critical when one adds a fourth camera. We cover these cases one by one, in the same order as in Section IV.7, but using more computational proofs than in the previous section. For the most part, we either prove compatibility by explicitly giving a conjugate, or by checking that the quadrics come from a set of fundamental matrices satisfying:

Proposition IV.8.2 ([BR23, Theorem 3.6])**.** *Let $\{F^{ij}\}$ be a sextuple of fundamental matrices such that the three epipoles in each image do not lie on a line. Then $\{F^{ij}\}$ is compatible if and only if the triple-wise conditions hold and*

$$\begin{aligned}&(e_1^4)^T F^{12} e_2^3 (e_1^2)^T F^{13} e_3^4 (e_1^3)^T F^{14} e_4^2 (e_2^4)^T F^{23} e_3^1 (e_2^1)^T F^{24} e_4^3 (e_3^2)^T F^{34} e_4^1 \\ =&(e_1^3)^T F^{12} e_2^4 (e_1^4)^T F^{13} e_3^2 (e_1^2)^T F^{14} e_4^3 (e_2^1)^T F^{23} e_3^4 (e_2^3)^T F^{24} e_4^1 (e_3^1)^T F^{34} e_4^2.\end{aligned} \tag{IV.9}$$

The case of more than four cameras is further simplified by the following fact:

Proposition IV.8.3 ([BR23, Theorem 3.12])**.** *Let $\{F^{ij}\}$ be a complete set of $\binom{n}{2}$ fundamental matrices such that for all i, j, k the triple F^{ij}, F^{ik}, F^{jk} is compatible, and such that for all i, j, k, l, the sextuple $F^{ij}, F^{ik}, F^{jk}, F^{il}, F^{jl}, F^{kl}$ is compatible. Then $\{F^{ij}\}$ is compatible.*

Moreover, if all epipoles in each image coincide, then triple-wise compatibility implies that $\{F^{ij}\}$ *is compatible. The reconstruction in this case is a set of cameras whose centers all lie on a line.*

Corollary IV.8.4. *For* $n \geq 4$*, an* $\binom{n}{2}$*-tuple of quadrics is compatible if and only if each sixtuple* $(S^{ij}, S^{ik}, S^{jk}, S^{il}, S^{jl}, S^{kl})$ *is.*

Moreover, if the cameras on the other side are all collinear, an $\binom{n}{2}$*-tuple of quadrics is compatible if and only if each triple* S^{ij}, S^{ik}, S^{jk} *is.*

Proof. By definition, a set of quadrics is compatible if and only if it is the pullback of a set of compatible fundamental matrices. So this result follows immediately from Proposition IV.8.3. ∎

Corollary IV.8.5. *Any configuration that is critical for each quadruple of views is critical.*

Lemma IV.8.6. *Let* $\{P_i\}$ *be a set of four cameras, and let* $\left\{S_P^{ij}\right\}$ *be a set of six irreducible critical quadrics such that each triple of quadrics* $S_P^{ij}, S_P^{ik}, S_P^{jk}$ *is compatible (in the non-collinear sense). If we fix five of the quadrics and let the final one vary over all quadrics in space, then*

IV.8.6.1. *The final quadric can move in a linear family without breaking triple-wise compatibility.*

IV.8.6.2. *There is exactly one quadric in the pencil such that the whole sextuple of quadrics is compatible.*

Proof. Each irreducible critical quadric S_P^{ij} is the pullback of a unique fundamental matrix F_Q^{ij}. The set of quadrics is compatible if it is the pullback of a set of compatible fundamental matrices. Fixing five of the quadrics corresponds to fixing five of the fundamental matrices. Let the final quadric (the one we do not fix) be the quadric S_P^{34}, the pullback of F_Q^{34}.

1. A triple of fundamental matrices is compatible if and only if

$$e_{Q_i}^k F_Q^{ij} e_{Q_j}^k = 0$$

for all triples of indices. The matrix F_P^{34} appears in two triples of cameras, namely 134 and 234. For each of these triples, triple-wise compatibility gives three conditions on F_P^{34} for a total of 6. Since a fundamental matrix has seven degrees of freedom, the triple-wise conditions restrict F_P^{34} to a pencil.

2. By Proposition IV.8.2, a sextuple of fundamental matrices is compatible if each triple is compatible and it satisfies

$$\begin{aligned}&(e_1^4)^T F^{12} e_2^3 (e_1^2)^T F^{13} \underline{e_3^4} (e_1^3)^T F^{14} e_4^2 (e_2^4)^T F^{23} e_3^1 (e_2^1)^T F^{24} \underline{e_4^3} (e_3^2)^T \underline{F^{34}} e_4^1 \\ =&(e_1^3)^T F^{12} e_2^4 (e_1^4)^T F^{13} e_3^2 (e_1^2)^T F^{14} \underline{e_4^3} (e_2^1)^T F^{23} \underline{e_3^4} (e_3^2)^T F^{24} e_4^1 (e_3^1)^T \underline{F^{34}} e_4^2.\end{aligned} \tag{IV.9}$$

For ease of reading, the variables that are not fixed by fixing the first five quadrics are underlined. Moreover, the fact that F_Q^{34} has to satisfy triple-wise compatibility with the triples 134 and 234 fixes its left and right nullspaces $e^4_{Q_3}$ and $e^3_{Q_4}$. This leaves F_Q^{34} as the only non-fixed variable in Equation (IV.9), turning it into a linear equation in the entries of F_Q^{34}. This in turn means that there is exactly one value of F_Q^{34} satisfying Equation (IV.9) and hence exactly one quadric making the sextuple compatible. ■

IV.8.3 Critical configurations

Proposition IV.8.7. *A set of points X lying on the union of a plane and a conic curve together with any number of cameras whose centers lie on that same conic form a critical configuration.*

Proof. By Corollary IV.8.4, it suffices to show that this configuration is critical for four cameras. Let X be the union of a plane Π and a conic curve C and let $P_1, \dots, P_4$ be a set of four cameras whose centers all lie on the conic curve. Since we are free to choose coordinates in $\mathbb{P}^3$ without affecting criticality, we can take the four camera centers to be

$$p_1 = \left[1:0:0:0\right], \qquad p_3 = \left[0:0:1:0\right],$$
$$p_2 = \left[0:1:0:0\right], \qquad p_4 = \left[1:1:1:0\right].$$

By also requiring the plane Π to pass through the point $[0:0:0:1]$, we have fixed the coordinate frame in $\mathbb{P}^3$. Moreover, the only property of cameras that affects criticality is their camera centers, so we can choose coordinates in each image so that the four cameras are:

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad P_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

When X is the union of a plane Π and a conic C, there is only one quadric containing X, namely the reducible quadric consisting of the two planes Π and Π_C where Π_C denotes the plane containing C. Denote the points in $\mathbb{P}^3$ by $[x_1 : x_2 : x_3 : x_4]$. We have chosen coordinates so that Π_C is the plane where $x_4 = 0$. Assuming Π does not pass through p_3 (which can always be avoided through a relabelling of the cameras) it will be the plane where $x_3 - \lambda_1 x_1 + \lambda_2 x_2 = 0$ for some $(\lambda_1, \lambda_2) \in \mathbb{R}^2$. Likewise, the conic C will be given by $\mu_1 x_2(x_1 - x_3) + \mu_2 x_3(x_1 - x_2) = 0$ for some $[\mu_1 : \mu_2] \in \mathbb{P}^1$.

The quadrics S_P^{ij} must all be taken to be equal to the quadric consisting of the two planes Π and Π_C. Moreover, the twelve epipolar lines $g^j_{P_i}$ must be chosen in such a way that:

1. $g^j_{P_i}$ passes through p_i and lies in Π_C.
2. $g^j_{P_i}$ and $g^i_{P_j}$ intersect in a point on Π.
3. $g^k_{P_i}$ and $g^k_{P_j}$ intersect in a point on C.

Once such a choice of lines is made (a one-dimensional family of valid choices exists), one can compute the fundamental matrices F^{ij}_Q by letting F^{ij}_Q be the unique fundamental matrix that pulls back to S^{ij}_P (under the joint camera map of P_i, P_j) and whose epipoles pull back to $g^j_{P_i}$ and $g^j_{P_j}$ (under that same map). If the matrices F^{ij}_Q are then compatible, and in this particular case, they always are, one can recover the conjugate cameras Q_i, and then find the conjugate points Y by triangulation, thus constructing a conjugate configuration. One example of such a conjugate configuration consists of the cameras

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} \lambda_2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \lambda_1 & 0 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} -\lambda_2\mu_1 & \mu_1 + \lambda_1\mu_1 + \lambda_1\mu_2 & 0 & \mu_1 + \mu_2 \\ \lambda_1\mu_1 & \lambda_1\mu_2 & 0 & \mu_2 \\ 0 & 0 & \lambda_1\mu_1 & 0 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} 0 & 0 & \lambda_1\mu_1 & 0 \\ -\mu_1(\lambda_1 + \lambda_2) & \mu_1 & 0 & 0 \\ \lambda_1\mu_1 & \lambda_1\mu_2 & 0 & \mu_1 + \mu_2 \end{bmatrix},$$

where the conjugate to any point

$$\begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 x_1 + \lambda_2 x_2 \\ x_4 \end{bmatrix} \in \Pi \text{ is the point } \begin{bmatrix} x_2 \\ \lambda_1 x_1 + \lambda_2 x_2 \\ x_4 \\ -\lambda_1^2 x_1 - \lambda_1\lambda_2 x_2 \end{bmatrix},$$

and the conjugate to any point

$$\begin{bmatrix} (\mu_1 + \mu_2)x_2x_3 \\ (\mu_1 x_2 + \mu_2 x_3)x_2 \\ (\mu_1 x_2 + \mu_2 x_3)x_3 \\ 0 \end{bmatrix} \in C \text{ is the point } \begin{bmatrix} (\lambda_1 + \lambda_2)x_2^2 \\ (\lambda_1 + \lambda_2)x_2x_3 \\ 0 \\ -(x_3 - \lambda_2 x_2)(\mu_1 x_2 + \mu_2 x_3) \end{bmatrix}.$$

■

Curves of degree four

Proposition IV.8.8 ([HK07, Corollary 8.31])**.** *Any configuration of points and cameras all lying on a smooth rational quartic curve is critical. So is any configuration of points all lying on a twisted cubic and cameras all lying on a line not intersecting the cubic. These two configurations are conjugate to one another.*

Proof. We have shown that for three views, these configurations are both critical and conjugate to one another. Since they form a conjugate pair, it suffices to show that one of them remains critical when we add more cameras. Consider a rational quartic curve with any number of cameras all lying on the curve. For each triple of cameras this is a critical configuration with the camera centers of the conjugate configuration all lying on a line. By the second half of Corollary IV.8.4 however, triple-wise compatibility is sufficient in the case where the conjugate cameras all lie on a line, which is the case here. Hence the whole configuration is critical. ■

Proposition IV.8.9. *Any configuration of points and cameras all lying on a singular rational quartic curve is critical. So is any configuration of points all lying on a twisted cubic and cameras all lying on a line secant to the cubic: These two configurations are conjugate to one another.*

Proof. This is proven the same way as Proposition IV.8.8. ■

Next, we show that any configuration consisting of points and cameras all lying on a quartic curve C appearing as the intersection of a pencil of quadrics is critical as long as no camera center lies on a singular point on C. We begin by giving three useful lemmas for proving criticality in these cases.

Proposition IV.8.10. *A set of any number of cameras and points all lying on an irreducible elliptic or singular quartic curve is critical as long as no camera center is a singular point on C.*

Proof. By Corollary IV.8.4, we need only prove the result for four cameras. Denote the quartic curve by C. A conjugate configuration can be constructed in the following way:

Using the same construction as in the three-view case, let y_1 be any regular point on C such that it does not lie in a plane spanned by any three camera centers. In the pencil of quadrics containing C, there is exactly one, S_P^{12}, containing the secant line $\overline{p_2 y_1}$. This quadric contains a unique line L through p_1 such that $L \parallel \overline{p_2 y_1}$. L intersects C in two points, one is p_1, denote the other one by y_2. In a similar fashion, we can get the two remaining points y_3 and y_4. These are constructed in such a way that the conditions of Lemma IV.7.14 are satisfied, so C appears as the intersection of six quadrics S_P^{ij} where each triple is compatible.

Next, we need to show that the entire sextuple is compatible. This can be done by verifying that Equation (IV.9) is satisfied. As it turns out, this is always the case, regardless of which point we take to be the initial y_1. While this verification is fairly straightforward for any explicitly given C, proving this in the general case is a rather lengthy computation. Instead, we prove that it is critical by giving the conjugate configuration for a representative example (the conjugate is constructed using the algorithm described above). This is done in the appendix. ■

Proposition IV.8.11. *Let C be the union of a twisted cubic and a secant line. A set of cameras and points all lying on C is critical.*

Proof. By Corollary IV.8.4, we need only prove the result for four cameras. Moreover, the case where all cameras lie on the secant line is already covered by Proposition IV.8.9, so we need only cover the cases where 1,2,3, or all 4 cameras lie on the twisted cubic. The approach generally follows the irreducible case, but we need to take a bit more care in which points we pick as the y_i:

1. If at most one camera lies on the secant line: By a relabelling, we can always take the camera centers p_1, p_2, p_3 to lie on the cubic. Let y_1 be a generic point on the twisted cubic. For each camera center $p_i \neq p_1$, let S_P^{1i} be the unique quadric containing the line $\overline{p_i y_1}$. The quadric S_P^{1i} contains a unique line through p_1 quasi-parallel to $\overline{p_i y_1}$. This line intersects the C in two points, p_i and a point which we label y_i. This construction gives us four points y_i satisfying the conditions in Lemma IV.7.14, hence proving that each triple of quadrics is compatible. The final part of the proof, showing that the entire sextuple is indeed compatible, is left to the appendix.

2. If two or more cameras lie on the secant line: This situation is somewhat special since the conjugate curve C_Q is singular and every point on the secant line on C is conjugate to the singular point on C_Q. In this case, like in case 1., for each camera p_i on the twisted cubic, we must take the point y_i to lie on the secant line. But since the points y_1 map to camera centers on C_Q, and since the secant line collapses to a singular point on C_Q, this means that more than one of the conjugate cameras Q_i, Q_j may lie in the singular point, which in turn means that one can not recover any fundamental matrices F_Q^{ij} or quadric S_P^{ij}. Nonetheless, one can construct such a conjugate.

   Let the camera center p_1 lie on the twisted cubic, and let p_4 lie on the secant line. Let y_1 be any generic point on the secant line, and let y_4 be any generic point on the twisted cubic. Let S_P^{i4} be the unique quadric containing C and the line $\overline{p_i y_4}$, and let S_P^{1j} be the unique quadric containing C and the line $\overline{p_i y_i}$ (S_P^{14} contains both). The quadrics S_P^{1i} and S_P^{i4} will each contain a line quasi-parallel to $\overline{p_i y_1}$ and $\overline{p_i y_4}$ respectively. These two lines intersect in a point which we denote by y_i. This gives us four points y_i satisfying Lemma IV.7.14 (illustrated in Figure IV.10), so each triple is compatible. Proving that the entire sextuple is compatible, is left to the appendix. ∎

Proposition IV.8.12. *Let C be the union of two (possibly reducible) conics appearing as the intersection of two quadrics. Any set of cameras and points all lying on C is critical.*

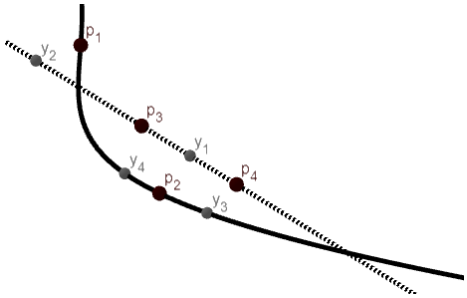


Figure IV.10: One possible choice of four points y_i

Proof. As before, we just need to prove this for four cameras. This result covers several cases, which we cover one by one. For all cases, denote the conic containing p_1 by C_1 and the other conic by C_2

1. In the event that all four cameras lie on the same conic, this is a subcase of the plane+conic configuration and as such, is critical.

2. In the event that three of the cameras lie on the same conic, and the final one (say, p_1) does not, we do as follows:

   Let y_1 be a generic point on C_1, not lying in the plane containing C_2. Let S_P^{1i} be the unique quadric containing C_1 and C_2 as well as the secant $\overline{p_i y_1}$. Chosen this way, the quadric S_P^{1i} contains a unique line L_i through p_1 such that L_i and $\overline{p_i y_1}$ form a permissible pair. Denote the intersection point of L_i with C_2 by y_i. The points p_i and y_i now satisfy the conditions in Lemma IV.7.14, proving triple-wise compatibility. The proof that the whole quadruple is compatible is in the appendix.

3. In the event that there are two camera centers on each conic, we do as follows (let p_3, p_4 be the two camera centers on C_2):

   First, take S_P^{12} to be the reducible quadric consisting of the two planes containing C_1 and C_2. Let y_1 be a generic point on C_1, not lying in the plane containing C_2 (if C_1 is reducible, let y_1 be a generic point on the component not containing p_1) and let y_2 be the unique point on C such that the lines $\overline{p_1 y_2}$ and $\overline{p_2 y_1}$ form a permissible pair on S_P^{12} (i.e. they intersect in a singular point on S_P^{12}). Next, let S_P^{13} be the unique quadric containing C_1, C_2, and the line $\overline{p_j y_1}$. The quadric S_P^{13} contains a unique line L such that L and $\overline{p_3 y_1}$ form a permissible pair on S_P^{13}. L intersects C_2 in a point which we denote by y_3. In a similar fashion, we can construct a point y_4. The points p_i and y_i now satisfy the conditions in Lemma IV.7.14, proving triple-wise compatibility. The proof that the whole quadruple is compatible is in the appendix. ■

Curves of degree three

Proposition IV.8.13. *A configuration consisting of any number of points and cameras all lying on a (possibly reducible) twisted cubic is critical. Its conjugate consists of points all lying on a conic not passing through the camera centers.*

Proof. By Corollary IV.8.4, we need only prove this for the four camera case. Let $\mathbf{P}$ be a set of four cameras, and let X be a set of points all lying on a smooth twisted cubic C passing through all camera centers. Let L be a secant to the twisted cubic not passing through any camera centers. By Lemma IV.7.8, any three distinct quadrics containing both C and L form a compatible triple. Fix five of the quadrics S_P^{ij} (all but $S_P^{34)}$) to be five distinct quadrics all containing both C and L. Then by the first item of Lemma IV.8.6, the final quadric can move along a linear pencil without breaking triple-wise compatibility. This is exactly the pencil of quadrics containing C and L. Moreover, by the second item of Lemma IV.8.6, there is one quadric in this pencil which would make the whole sextuple compatible. Let S_P^{34} be this quadric. Then we have a compatible sextuple of quadrics all containing X, making $(\mathbf{P}, X)$ a critical configuration.

The same argument holds in the cases where the twisted cubic degenerates to the union of a conic and a line, or to three lines. In this case, take the line L to be a secant such that $C \cup L$ is a degenerate elliptic quartic curve. That is:

1. If C is the union of a line and a conic, let L be a generic line intersecting both.

2. If C is the union of three lines, two of these lines do not intersect. Let L be a generic line intersecting these two lines.

The remaining proof is as in the smooth case. ■

Proposition IV.8.14. *A configuration consisting of any number of cameras, along with points all lying on a twisted cubic passing through all but one camera center is critical.*

Proof. By Corollary IV.8.4 it is sufficient to prove the statement for the four camera case. Denote the twisted cubic by C_P and let p_4 be the camera center not lying on C_P. By changing coordinates, one can take C_P to be the curve of points on the form

$$[\alpha^3 : \alpha^2\beta : \alpha\beta^2 : \beta^3] \text{ for } [\alpha : \beta] \in \mathbb{P}^1$$

and take the four camera centers to be

$$p_1 = \left[1 : 0 : 0 : 0\right], \qquad p_3 = \left[0 : 0 : 1 : 0\right],$$
$$p_2 = \left[0 : 1 : 0 : 0\right], \qquad p_4 = \left[x_1 : x_2 : x_3 : 1\right]^{16}.$$

By Proposition IV.2.17, the only property of cameras that affects criticality is their camera centers. So we can choose coordinates in each image so that the four cameras are:

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \qquad P_4 = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -x_2 \\ 0 & 0 & 1 & -x_3 \end{bmatrix}.$$

This configuration is critical. One example of a conjugate consists of cameras:

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 1 \\ 0 & -1 & 2 & -1 \end{bmatrix},$$
$$Q_3 = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \qquad Q_4 = \begin{bmatrix} 1 & x_1 & -x_1 & x_1+1 \\ 0 & x_2 & -x_2 & x_2-1 \\ 0 & x_3+1 & -x_3 & x_3+1 \end{bmatrix}.$$

The conjugate to any point

$$[\alpha^3 : \alpha^2\beta : \alpha\beta^2 : \beta^3] \in C_P$$

is the point

$$[-\alpha^2(\alpha-\beta) : \alpha\beta(\alpha-\beta) : -\beta^2(\alpha-\beta) : -\alpha^2\beta] \in C_Q.$$

■

Finite point sets

Lemma IV.8.15. *Let $X \subset \mathbb{P}^3$ be a set of seven points in general position and let P_1, P_2 be two cameras such that there is only one (possibly reducible) quadric containing p_1, p_2 and X. Then there exists at most one camera P_3 such that (P_1, P_2, P_3, X) is a critical configuration.*

Proof. Given X and the cameras P_1, P_2, we want to find all cameras P_3 such that (P_1, P_2, P_3, X) is a critical configuration. If (P_1, P_2, P_3, X) is a critical configuration, then by Proposition IV.7.23 there exists three planes Π_i, one through each camera center p_i, such that their intersection is the point x_0 (uniquely determined by X). Moreover, X must appear as the intersection of three compatible quadrics S_P^{ij}.

[16] By choosing coordinates for C_P and the first three camera centers, the coordinate frame is fixed. In this example, we assume that p_4 does not lie on the plane at infinity in this coordinate frame. If it does, one can amend this by relabelling the cameras P_1, P_2, P_3

Let S_P^{12} be the unique quadric containing X, p_1, p_2. This quadric comes with a pair of permissible lines, $g^2_{P_1}$ and $g^1_{P_2}$, one through each camera center. These are enough to fix the two planes Π_1, Π_2. The line $g^3_{P_1}$ must pass through p_1 and lie in the plane Π_1. In the pencil of quadrics containing X and p_1 there are only two quadrics containing such lines, one is S_P^{12}, we take S_P^{13} to be the other one. In a similar fashion, there is only one choice for the quadric S_P^{23}.

With this, we have fixed the two lines $g^3_{P_1}$ and $g^3_{P_2}$ as well. The quadric S_P^{13} contains only one line through x_0 intersecting $g^3_{P_1}$, and similarly S_P^{23} contains only one line through x_0 intersecting $g^3_{P_2}$. These two lines span the third plane Π_3. The camera center p_3 must lie on this plane and on the two quadrics S_P^{13}, S_P^{23}, but not on the lines $g^2_{P_1}$ or $g^3_{P_1}$. Hence, p_3 is uniquely determined, and, up to a choice of coordinates, so is P_3. ■

Proposition IV.8.16. *A configuration of seven points and four or more cameras is never a maximal critical configuration.*

Proof. Let $P_1, \dots, P_4$ be four cameras and let $X \subset \mathbb{P}^3$ be a set of 7 points. Assume there exists at least one pair of cameras, for instance, P_1, P_2 such that there is only one quadric containing the 9 points p_1, p_2, X. By Lemma IV.8.15, there exists only one camera P_{x} such that (P_1, P_2, P_3, X) is a critical configuration. It follows that both $P_3 = P_4 = P_{\mathrm{x}}$ for this to be a critical configuration, violating our assumption of distinct cameras.

On the other hand, if there is no pair of cameras such that the two camera centers and the seven points fix the quadric, then all cameras and all of X lies on the intersection of a pencil of quadrics, that is, some quartic curve. In this case, the configuration is not maximal. ■

Proposition IV.8.17. *A set of six points and any number of cameras is critical if there is a smooth critical quadric containing all the points and all the cameras. A set of five or fewer points and two or more cameras is always critical.*

Proof. The case of six points is covered in [MS98]. As for the the case of five or fewer points, we make use of the fact that with five fixed points in space, there are many inequivalent choices of cameras that produce the same image in $\mathbb{P}^2$. Assume that among the five points, no four lie in the same plane. One can choose a coordinate frame so that the five points are:

$$x_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad x_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad x_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad x_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \quad x_5 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

Assuming no camera lies in the plane at infinity, we can choose coordinates in each image such that all cameras are of the form

$$P_i = \begin{bmatrix}1 & 0 & 0 & \alpha_i^1\\ 0 & 1 & 0 & \alpha_i^2\\ 0 & 0 & 1 & \alpha_i^3\end{bmatrix}.$$[17]

Recall that the choice of coordinates in space does not affect criticality, so it suffices to show that this configuration is critical. Let $y_i = x_i$, and let

$$Q_i = \begin{bmatrix} \lambda_i(\alpha_i^1 + 1) - \alpha_i^1 & 0 & 0 & \alpha_i^1 \\ 0 & \lambda_i(\alpha_i^2 + 1) - \alpha_i^2 & 0 & \alpha_i^2 \\ 0 & 0 & \lambda_i(\alpha_i^3 + 1) - \alpha_i^3 & \alpha_i^3 \end{bmatrix},$$

for some $\lambda_i \in \mathbb{R}$. Then the cameras Q_i along with points y_i produce the same images as the cameras P_i along with the points x_i. Since we are free to choose the λ_i however we like, this gives us a large family of conjugates, hence proving that the configuration is critical. In a similar fashion (not given here) one can construct conjugates in the cases where one or more camera lies at the plane at infinity and in the case where there are four or more points in the same plane. ■

Proposition IV.8.17 covers the last of the three view cases, thus completing our classification. To summarize:

Theorem IV.8.18. *A configuration of four or more cameras P and seven or more points X is critical if and only if each subconfiguration of three cameras and all points X is critical. A configuration of four or more cameras and six points is critical if and only if there is a smooth critical quadric containing all the points and all the cameras. A configuration of four or more cameras and five or fewer points is always critical.*

IV.9 Conclusion

This paper completes the classification of all critical configurations for any number of projective cameras, showing that these configurations always have to lie on the intersection of ruled quadrics, and showing exactly which such intersections are critical. By solving the 4+ camera case left unfinished in [Brå23a; Brå23b; HK07], it completes a long story starting with Krames work from 1941. The idea of classifying critical configurations by classifying intersections of multi-view varieties is not restricted to projective cameras, and we believe that the techniques demonstrated here can be used for other camera models in the future.

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[17]The superscript in α_i^j is an index, not an exponent.

Appendix IV.A

Some proofs in the appendix rely on MATLAB code for computations, this code can be found at https://github.com/mabraate/Critical-Configurations

Proof of Proposition IV.8.10. We start by choosing y_1 to be such that there exists at least one quadric containing C but not containing both the line $\overline{x_1p_2}$ and the line $\overline{x_1p_3}$ (only a few points y_1 fail to fulfill this criterion, and the configuration is critical even if it does not). Assume for starters that the four camera centers do not lie in a common plane, then one can choose coordinates such that $y_1 = [1:1:1:1]$ and such that the four camera centers are

$$p_1 = \begin{bmatrix}1:0:0:0\end{bmatrix}, \qquad p_3 = \begin{bmatrix}0:0:1:0\end{bmatrix},$$
$$p_2 = \begin{bmatrix}0:1:0:0\end{bmatrix}, \qquad p_4 = \begin{bmatrix}0:0:0:1\end{bmatrix},$$

By Proposition IV.2.17, the only property of cameras that affects criticality is their camera centers, so we can choose coordinates in each image so that the four cameras are:

$$P_1 = \begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \qquad P_2 = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},$$
$$P_3 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix}, \qquad P_4 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}.$$

Next, since the quadrics S_P^{12} and S_P^{13} contain all the camera centers, the point y_1, and the secant line $\overline{p_2y_1}$ and $\overline{p_3y_1}$ respectively, they are on the form

$$S_P^{12} = \begin{bmatrix}0 & a_1 & a_2 & a_3\\ a_1 & 0 & a_4 & -a_1-a_4\\ a_2 & a_4 & 0 & -a_2-a_3\\ a_3 & -a_1-a_4 & -a_2-a_3 & 0\end{bmatrix}, \qquad S_P^{13} = \begin{bmatrix}0 & b_1 & b_2 & b_3\\ b_1 & 0 & b_4 & -b_1-b_3\\ b_2 & b_4 & 0 & -b_2-b_4\\ b_3 & -b_1-b_3 & -b_2-b_4 & 0\end{bmatrix}.$$

The remaining four quadrics can be written as a linear combination of these two. Moreover, every irreducible elliptic or singular quartic curve passing through the four camera centers and y_1 is the intersection of two quadrics on this form. This configuration turns out to be critical, with its conjugate consisting of the four cameras:

$$Q_1 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}, \quad Q_2 = \begin{bmatrix}-a_1-2a_4 & -a_2 & a_3 & 1\\ 2a_1+a_4 & 2a_2 & a_3 & 1\\ a_4-a_1 & -a_2 & -2a_3 & 1\end{bmatrix},$$

$$Q_3 = \begin{bmatrix}(a_1b_3+2a_1b_4-3a_4b_1\\ -a_4b_3+a_4b_4) & (a_2b_3+2a_2b_4-3a_4b_2\\ +3a_3b_4-3a_4b_4) & -a_3(b_3-b_4) & b_4-b_3\\ (a_1b_3-3a_3b_1+2a_1b_4\\ -a_4b_3+a_4b_4) & a_2b_3-3a_3b_2+2a_2b_4 & -a_3(b_3-b_4) & b_4-b_3\\ (a_1b_3+2a_1b_4-3a_4b_1\\ -a_4b_3+a_4b_4) & a_2b_3+2a_2b_4-3a_4b_2 & 2a_3b_3+a_3b_4-3a_4b_3 & b_4-b_3\end{bmatrix},$$

$$Q_4 = \begin{bmatrix}(a_1b_3+2a_1b_4-3a_4b_1\\ -a_4b_3+a_4b_4) & a_2(b_3-b_4) & (3a_2b_1-3a_1b_2+3a_3b_1\\ -3a_1b_4+3a_2b_3+2a_3b_3\\ -3a_4b_2+a_3b_4-3a_4b_4) & b_4-b_3\\ (3a_2b_1-3a_1b_2+a_1b_3\\ -a_1b_4-a_4b_3+a_4b_4) & a_2(b_3-b_4) & (3a_2b_3-3a_3b_2\\ +2a_3b_3-2a_3b_4) & b_4-b_3\\ (a_1b_3+2a_1b_4-3a_4b_1\\ -a_4b_3+a_4b_4) & (3a_2b_1-3a_1b_2+a_2b_3\\ +2a_2b_4-3a_4b_2) & (3a_3b_1-3a_1b_3+2a_3b_3\\ +a_3b_4-3a_4b_3) & b_4-b_3\end{bmatrix},$$

and points lying at the intersection of the two quadrics S_Q^{12} and $S_Q^{13} = S_Q^{13'} + S_Q^{13'T}$, where

$$S_Q^{12} = \begin{bmatrix} 0 & a_1-a_4 & 2a_1+a_4 & 0 \\ a_1-a_4 & 2a_2 & 2a_2+2a_3 & -1 \\ 2a_1+a_4 & 2a_2+2a_3 & 2a_3 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$S_Q^{13'} = \begin{bmatrix} (6a_4b_1 - 4a_1b_4 - 2a_1b_3 \\ +2a_4b_3 - 2a_4b_4) & 0 & 0 & 0 \\ 3a_4b_2-2a_2b_4-a_2b_3 & 0 & 0 & 0 \\ (a_1b_3 - 3a_3b_1 + 2a_1b_4 - 2a_3b_3 \\ -a_3b_4 + 2a_4b_3 + a_4b_4) & a_2b_3-3a_3b_2+2a_2b_4 & -2a_3(b_3-b_4) & 0 \\ b_3-b_4 & 0 & b_4-b_3 & 0 \end{bmatrix}.$$

In particular, the conjugate to a point

$$x_P = \begin{bmatrix} a_1wy+a_4wy+a_2wz+a_3wz-a_4yz \\ y(a_3w+a_1y+a_2z) \\ z(a_3w+a_1y+a_2z) \\ w(a_3w+a_1y+a_2z) \end{bmatrix}$$

on S_P^{12} is the point

$$y_Q = \begin{bmatrix} y(w-z) \\ z(w-z) \\ w(w-z) \\ a_4yz-a_2z^2-2a_1wy-a_4wy-2a_2wz-2a_3wz-a_1yz-a_3w^2 \end{bmatrix}$$

on S_Q^{12}. The pair (x, y) satisfies $P_i x = Q_i y$ for all i whenever x also lies on S_P^{13}, in which case the conjugate point y lies on S_Q^{13}.

This completes the case where the four camera centers do not lie in a common plane. The case where they do can be covered in a similar way (though not repeated here) by choosing coordinates such that $y_1 = [0:0:0:1]$ and such that the camera centers are

$$p_1 = \begin{bmatrix} 1:0:0:0 \end{bmatrix}, \qquad p_3 = \begin{bmatrix} 0:0:1:0 \end{bmatrix},$$
$$p_2 = \begin{bmatrix} 0:1:0:0 \end{bmatrix}, \qquad p_4 = \begin{bmatrix} 1:1:1:0 \end{bmatrix},$$

and then repeating the process from the non-coplanar case. ■

Proof of Proposition IV.8.11. By a change of coordinates, one can always take the twisted cubic to be on the form $[s^3 : s^2t : st^2 : t^3]$ for $[s : t] \in \mathbb{P}^1$. Moreover, one can fix three points on the twisted cubic. By taking the two intersection points with the secant to be $[1:0:0:0]$ and $[0:0:0:1]$, the secant line is on the form $[u:0:0:v]$ for $[u:v] \in \mathbb{P}^1$. We also fix a point y_i on the twisted cubic, distinct from all camera centers, to be the point $[1,1,1,1]$. Finally, by a change of coordinates in each image, we can take the cameras to be:

$$P_i = \begin{bmatrix} 1,0,0,-a_i^3 \\ 0,1,0,-a_i^2 \\ 0,0,1,-a_1 \end{bmatrix}, \quad P_j = \begin{bmatrix} 1,0,0,b_j \\ 0,1,0,0 \\ 0,0,1,0 \end{bmatrix}.$$

for p_i lying on the cubic and p_j lying on the secant line.

Continued proof of Item 1, no cameras on the secant:
We take $y_1 = [1:1:1:1]$, the remaining y_i turn out to be

$$y_i = [a_i^3 : a_1 a_i^2 : a_1^2 a_i : a_1^3].$$

Taking S_P^{ij} to be the quadric containing C and $\overline{p_i y_j}$, the quadrics turn out to be

$$S_P^{1i} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & -a_i \\ 1 & 0 & 2a_i & 0 \\ 0 & -a_i & 0 & 0 \end{bmatrix}, \qquad S_P^{ij} = \begin{bmatrix} 0 & 0 & a_1 & 0 \\ 0 & -2a_1 & 0 & -a_i a_j \\ a_1 & 0 & 2a_i a_j & 0 \\ 0 & -a_i a_j & 0 & 0 \end{bmatrix}, \tag{IV.10}$$

for $i, j \neq 1$. These come from the fundamental matrices

$$F_Q^{1i} = \begin{bmatrix} 0 & -a_1 & a_1(a_i+1) \\ a_1 & (a_1-a_i)(a_1-1) & -a_1^2 a_i - a_1^2 + a_1 a_i - a_i^2 - a_i \\ -a_1^2 - a_i & a_1^2 a_i + a_1^2 - a_1 a_i + a_i^2 + a_i & -a_i(a_1-a_i)(a_1-1) \end{bmatrix},$$

$$F_Q^{ij} = \begin{bmatrix} 0 & a_1^2 & -a_1(a_i+a_1 a_j) \\ -a_1^2 & -a_1(a_i-a_j)(a_1-1) & a_1^2 a_i a_j + a_1 a_i^2 - a_1 a_i a_j + a_1 a_j^2 + a_i a_j \\ a_1(a_j+a_1 a_i) & -a_1^2 a_i a_j - a_1 a_i^2 + a_1 a_i a_j - a_1 a_j^2 - a_i a_j & a_i a_j (a_i - a_j)(a_1 - 1) \end{bmatrix},$$

which satisfy the condition in Proposition IV.8.2[18], hence proving that the sextuple is compatible.

Continued proof of Item 1, one camera on the secant:
We take the camera center on the secant to be p_4, and let $y_1 = [1:1:1:1]$. In this case, $y_i = [a_i^3 : a_1 a_i^2 : a_1^2 a_i : a_1^3]$ for $i \in \{2, 3\}$ and $y_4 = [-a_1 b_4 : 0 : 0 : 1]$. Taking S_P^{ij} to be the quadric containing C and $\overline{p_i y_j}$, the quadrics turn out to be the same as in Equation (IV.10). These come from a set of fundamental matrices where $F_Q^{12}, F_Q^{13}, F_Q^{23}$ are as in Equation (IV.10), and

$$F_Q^{14} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1-a_1 & a_1+b_4 \\ a_1 & -a_1-b_4 & -b_4(a_1-1) \end{bmatrix},$$

$$F_Q^{i4} = \begin{bmatrix} 0 & a_1 a_i & -a_i^2 \\ -a_1 a_i & -a_i^2(a_1-1) & b4a_1^2 + a_i^3 \\ a_1 a_i^2 & -b_4 a_1^2 - a_i^3 & -a_1 a_i b_4 (a_1 - 1) \end{bmatrix},$$

which also satisfy the condition in Proposition IV.8.2[18], hence proving that the sextuple is compatible.

Continued proof of Item 2, two or more cameras on the secant:
In this case we may have two or three cameras on the secant line and 2 or 1, respectively, on the twisted cubic. We prove both cases at once by showing that the 5-view configuration with three cameras on the secant and two on the cubic is critical. We do this by explicitly giving a conjugate configuration:

Let p_1, p_2 be the camera centers on the twisted cubic, and let $y_3 = [1:1:1:1]$. By the construction described in Item 2, we get a conjugate configuration

[18] See attached code.

consisting of the cameras[18]:

$$Q_1 = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0 \end{bmatrix}, Q_2 = \begin{bmatrix} 1 & a_2-a_1 & -a_2(a_1-a_2) & 0\\ 0 & 1 & a_2-a_1 & 0\\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} -a_1-1 & (a_1+1)^2-1 & a_1-a_1(a_1+1)+1 & -1\\ a_1^2+a_1+1 & -(a_1+1)(a_1^2+a_1+1) & a_1(a_1^2+a_1+1) & 0\\ 0 & a_1^2+a_1+1 & -(a_1+1)(a_1^2+a_1+1) & 0 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} b_3+a_1b_3+a_1^2b_3-a_1^2b_4 & \begin{matrix}-a_1(b_3+b_4+a_1b_3+\\ a_1^2b_3-a_1^2b_4)\end{matrix} & \begin{matrix}b_4(b_3-b_4+a_1b_3-a_1b_4+\\ a_1^2b_3-a_1^2b_4+a_1^2-1)\end{matrix} & b_4\\ -b_4(a_1^2+a_1+1) & (b_3+a_1b_4)(a_1^2+a_1+1) & -a_1b_3(a_1^2+a_1+1) & 0\\ 0 & -b_4(a_1^2+a_1+1) & (b_3+a_1b_4)(a_1^2+a_1+1) & 0 \end{bmatrix},$$

$$Q_5 = \begin{bmatrix} b_3+a_1b_3+a_1^2b_3-a_1^2b_5 & \begin{matrix}-a_1(b_3+b_5+a_1b_3+\\ a_1^2b_3-a_1^2b_5)\end{matrix} & \begin{matrix}b_5(b_3-b_5+a_1b_3-a_1b_5+\\ a_1^2b_3-a_1^2b_5+a_1^2-1)\end{matrix} & b_5\\ -b_5(a_1^2+a_1+1) & (b_3+a_1b_5)(a_1^2+a_1+1) & -a_1b_3(a_1^2+a_1+1) & 0\\ 0 & -b_5(a_1^2+a_1+1) & (b_3+a_1b_5)(a_1^2+a_1+1) & 0 \end{bmatrix},$$

and the points lying on the curve:

$$\begin{bmatrix} st(a_1^2t^2+a_1st+s^2)\\ st^2(s+a_1t)\\ st^3\\ -(s-t)(a_1^2s^3+b_3a_1^2t^3+a_1s^3+a_1s^2t+b_3a_1t^3+s^3+s^2t+st^2+b_3t^3) \end{bmatrix},$$

This is a quartic curve with a singularity in $[0:0:0:1]$. ■

Proof of Proposition IV.8.12. If the four cameras all lie in the same plane, this is a subconfiguration of the plane+conic configuration, which we have already proven to be critical. As such, we assume that the four camera centers do not lie in the same plane, so we can (with the right choice of coordinates) take them to be:

$$P_1 = \begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 1&0&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1 \end{bmatrix}, \qquad P_4 = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0 \end{bmatrix}.$$

Continued proof of Item 2 (the case of three cameras on one conic):

Let $[x_1 : x_2 : x_3 : x_4]$ denote a generic point in $\mathbb{P}^3$. The two planes spanned by the two conics are the planes where $x_4 = 0$ (the plane containing p_1, p_2, p_3) and $a_1x_1 + a_2x_2 + a_3x_3 = 0$ for some $(a_1, a_2, a_3) \in \mathbb{R}^3$ (the plane containing p_4). Since we are free to take y_1 to be any point on the conic through the first three camera centers, we take $y_1 = [1 : 1 : 1 : 0]$[19]. Now, every possible union of two conics lying in these planes and containing the four camera centers and y_1 is given as the intersection of two quadrics on the form:

$$S_P^{12} = \begin{bmatrix} 0&0&0&a_1\\0&0&0&a_2\\0&0&0&a_3\\-a_1&-a_2&-a_3&0 \end{bmatrix}, \qquad S_P^{14} = \begin{bmatrix} 0&b_1&b_2&b_3\\b_1&0&-b_1-b_2&b_4\\b_2&-b_1-b_2&0&-b_3-b_4\\b_3&b_4&-b_3-b_4&0 \end{bmatrix}.$$

[19] One can make a change of coordinates such that y_1 lies on the conic while fixing the camera centers and the two planes.

Using the construction described in the main text, we take the remaining four quadrics to be:

$$
\begin{aligned}
S_P^{13} &= S_P^{23} = S_P^{12} \\
S_P^{24} &= (a_1b_1b_4 - a_2b_2b_3 - a_2b_2b_4 + a_3b_1b_4)S_P^{12} + a_1a_2b_1b_2S_P^{14} \\
S_P^{34} &= (a_1b_2b_3 + a_1b_2b_4 + a_2b_2b_3 + a_2b_2b_4 - a_3b_1b_4)S_P^{12} - a_1a_3b_1S_P^{14}
\end{aligned}
$$

Then these six quadrics come from the fundamental matrices:

$$
F_Q^{12} = \begin{bmatrix} 0 & 0 & a_2 \\ 0 & 0 & a_1{+}a_3 \\ a_1 & -a_1 & 0 \end{bmatrix}, \quad
F_Q^{13} = \begin{bmatrix} 0 & 0 & a_1{+}a_2 \\ 0 & 0 & a_3 \\ a_1 & -a_1 & 0 \end{bmatrix}, \quad
F_Q^{14} = \begin{bmatrix} b_1 & 0 & -b_1 \\ b_2 & -b_2 & 0 \\ b_3 & b_4 & -b_3{-}b_4 \end{bmatrix},
$$

$$
F_Q^{23} = \begin{bmatrix} 0 & 0 & a_1b_2{+}a_2b_2{-}a_3b_1 \\ 0 & 0 & a_3(b_1{+}b_2) \\ a_1b_1{-}a_2b_2{+}a_3b_1 & a_2(b_1{+}b_2) & 0 \end{bmatrix}
$$

$$
F_Q^{24} = \begin{bmatrix} 0 & a_1a_2b_1b_2 & a_1b_1b_2(a_1{+}a_3) \\ -a_1b_2(a_1b_1{-}a_2b_2{+}a_3b_1) & -a_1a_2b_2(b_1{+}b_2) & 0 \\ a_1b_4(a_1b_1{-}a_2b_2{+}a_3b_1) & a_2(a_1b_1b_4 + a_1b_2b_4 - a_2b_2b_3 - a_2b_2b_4 + a_3b_1b_4) & -(a_1 + a_3)(a_2b_2b_3 + a_2b_2b_4 - a_3b_1b_4) \end{bmatrix},
$$

$$
F_Q^{34} = \begin{bmatrix} 0 & a_1b_1b_2(a_1{+}a_2) & a_1a_3b_1b_2 \\ -a_1b_1(a_1b_2{+}a_2b_2{-}a_3b_1) & 0 & -a_1a_3b_1(b_1{+}b_2) \\ -a_1(b_3 + b_4)(a_1b_2 + a_2b_2 - a_3b_1) & -(a_1 + a_2)(a_2b_2b_3 + a_2b_2b_4 - a_3b_1b_4) & -a_3(a_1b_1b_3 + a_1b_1b_4 + a_1b_2b_3 + a_1b_2b_4 + a_2b_2b_3 + a_2b_2b_4 - a_3b_1b_4) \end{bmatrix},
$$

which can be verified to be compatible (see attached code).

Continued proof of Item 3 (the case of two cameras on each conic):

Let $[x_1 : x_2 : x_3 : x_4]$ denote a generic point in $\mathbb{P}^3$. We can take the two planes spanned by the two conics to be the planes where $x_3 = x_4$ (the plane containing p_1, p_2) and $a_1x_1 = -a_2x_2$ (the plane containing p_3, p_4)[20]. Now, every possible union of two conics lying in these planes and containing the four camera centers and y_1 is given as the intersection of two quadrics on the form:

$$
S_P^{12} = \begin{bmatrix} 0 & 0 & a_1 & -a_1 \\ 0 & 0 & a_2 & -a_2 \\ a_1 & a_2 & 0 & 0 \\ -a_1 & -a_2 & 0 & 0 \end{bmatrix}, \qquad
S_P^{13} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & b_4 & -b_1{-}b_3 \\ b_2 & b_4 & 0 & -b_2{-}b_4 \\ b_3 & -b_1{-}b_3 & -b_2{-}b_4 & 0 \end{bmatrix}.
$$

Using the construction described in the main text, we take the remaining four quadrics to be:

$$
\begin{aligned}
S_P^{23} &= (a_2b_2b_3 - a_1b_1b_4 + a_2b_3b_4)S_P^{12} + (-a_1a_2(b_3 - b_4))S_P^{13} \\
S_P^{14} &= (b_1 + b_2 + b_4)S_P^{12} + (a_1 + a_2)S_P^{13} \\
S_P^{24} &= (a_1b_1 + a_1b_3 - a_2b_2)(a_2b_2 - a_1b_1 + a_2b_4)S_P^{12} - a_1a_2(a_1 + a_2)(b_3 - b_4)S_P^{13} \\
S_P^{34} &= S_P^{12}
\end{aligned}
$$

[20] Here we assume that if any camera center lies in both planes, it has to be either p_3 or p_4. We can always ensure that this is the case by a relabelling of the cameras

Then these six quadrics come from the fundamental matrices:

$$F_Q^{12} = \begin{bmatrix} 0 & a_4 & -a_4 \\ a_2 & 0 & -a_2 \\ -a_2 & a_2 & 0 \end{bmatrix}, \quad F_Q^{13} = \begin{bmatrix} b_1 & 0 & -b_1 \\ b_2 & b_4 & -b_2-b_4 \\ b_3 & -b_3 & 0 \end{bmatrix},$$

$$F_Q^{23} = \begin{bmatrix} 0 & -a_2a_4b_1(b_3-b_4) & -a_2^2b_1(b_3-b_4) \\ a_2b_4(a_4b_2-a_2b_1+a_4b_3) & \begin{matrix} a_4(a_2b_4^2 - a_2b_1b_4 \\ -a_2b_3b_4 + a_4b_2b_3 + a_4b_3b_4) \end{matrix} & a_2a_4(b_2+b_4)(b_3-b_4) \\ -a_2b_3(a_4b_2-a_2b_1+a_4b_3) & \begin{matrix} -a_4b_3(a_2b_4 - a_2b_3 - a_2b_1 \\ +a_4b_2 + a_4b_4) \end{matrix} & 0 \end{bmatrix},$$

$$F_Q^{14} = \begin{bmatrix} b_1(a_2+a_4) & 0 & -b_1(a_2+a_4) \\ a_4b_2-a_2b_4-a_2b_1 & a_2b_1+a_2b_4-a_4b_2 & 0 \\ a_2b_1+a_2b_2+a_2b_3+a_2b_4+a_4b_3 & a_4b_2-a_2b_3-a_2b_1-a_4b_3+a_4b_4 & -(a_2+a_4)(b_2+b_4) \end{bmatrix},$$

$$F_Q^{24} = \begin{bmatrix} 0 & a_2a_4b_1(a_2+a_4)(b_3-b_4) & a_2^2b_1(a_2+a_4)(b_3-b_4) \\ \begin{matrix} -a_2(a_2b_1 + a_2b_4 \\ -a_4b_2)(a_4b_2 \\ -a_2b_1 + a_4b_3) \end{matrix} & \begin{matrix} -\ a_4(a_2b_1 + a_2b_4 - a_4b_2)(a_2b_4 \\ -a_2b_3 - a_2b_1 + a_4b_2 + a_4b_4) \end{matrix} & 0 \\ \begin{matrix} a_2(a_4b_2 - a_2b_1 \\ +a_4b_3)(a_2b_1 + a_2b_3 \\ -a_4b_2 + a_4b_3 - a_4b_4) \end{matrix} & \begin{matrix} -a_4(a_2^2b_1^2 + 2a_2^2b_1b_3 - b_4a_2^2b_1 \\ +a_2^2b_3^2 - b_4a_2^2b_3 - 2a_2a_4b_1b_2 \\ +a_2a_4b_1b_3 - 2b_4a_2a_4b_1 - a_2a_4b_2b_3 \\ +a_2a_4b_3^2 - 2b_4a_2a_4b_3 + a_4^2b_2^2 + b_4a_4^2b_2) \end{matrix} & \begin{matrix} -a_2a_4(a_2 + a_4)(b_2 \\ +b_4)(b_3 - b_4) \end{matrix} \end{bmatrix},$$

$$F_Q^{34} = \begin{bmatrix} 0 & -a_2b_1 & -a_2(b_2+b_4) \\ a_2b_1 & 0 & -a_4(b_2+b_4) \\ a_2(b_2+b_4) & a_4(b_2+b_4) & 0 \end{bmatrix},$$

which can be verified to be compatible (see attached code). ■

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