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ON SURFACES IN PROJECTIVE 4-SPACE

by

ALF BJØRN AURE

UNIVERSITETET I OSLO
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P.B. 1063 BLINDERN, 0316 OSLO

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INTRODUCTION

This paper deals with the classification of smooth surfaces in $\mathbb{P}^4$.

The first topic we study is the surfaces on cubic hypersurfaces, and the main result is the classification of the smooth surfaces on cubic hypersurfaces with isolated singularities (Theorem 2.1.1). The second topic is the smooth surfaces in $\mathbb{P}^4$ without apparent triple points. We show that such a surface is either an elliptic quintic scroll or it is contained in a quadric hypersurface (Theorem 3.0.2). This result was conjectured by C. Peskine. For not necessarily smooth surfaces in general, this question has been studied by Severi ([Se]).

Chapter 1 consists mainly of a listing of known results about surfaces (in $\mathbb{P}^4$) that will be used in the rest of this paper. However, Proposition 1.2.4 should be emphasized. It gives a bound for the sectional genus of a surface not contained in a hypersurface of given degree.

Chapter 2 is a treatment of the question about the smooth surfaces on the cubic hypersurfaces. In §1 we give a complete classification when the cubic hypersurfaces have isolated singularities. I have not succeeded with the classification when the cubic hypersurfaces have non-isolated singularities; §2 of Chapter 2 is a discussion of some approaches to this problem.

Chapter 3 gives the classification of smooth surfaces without apparent triple points.

Appendix A deals with the classification of the curves of "nearly maximal" genus on a cubic Del Pezzo surface in $\mathbb{P}^3$. The results are needed in Chapter 2. I have chosen to do this separately in an Appendix because of the "technical nature".

Appendix B consists of a proof of the fact that the only smooth non-degenerate scrolls in $\mathbb{P}^4$ are those of degree 3 or 5. The proof is a revised version of one given by Lanteri ([L]).

Most of this paper has developed from various attempts to prove the Theorem concerning non-apparent triple points. The proof we present in Chapter 3 uses a formula about foursecant-lines that was seen to apply first at a late stage. However, considering the other approaches there are interesting open questions:

- (1) Does there exist any smooth surface in $\mathbb{P}^4$ with $\chi < 0$ (necessarily ruled)?

Or more generally:

- (2) What are the irregular smooth surfaces in $\mathbb{P}^4$?

An answer to the difficult question (2) would have many implications to the classification problem (see for instance Theorem 2.1.15).

- (3) What are the smooth surfaces in $\mathbb{P}^4$ that contain a 1-dimensional family of plane curves (besides Veronese surfaces, elliptic quintic scrolls, and surfaces on quadric hypersurfaces)?

An answer to question (3) when the plane curves have degree 3 or more, would yield another proof of Theorem 3.0.2, by Severi's own method. It is also related to question (1); since the scrolls in $\mathbb{P}^4$ are known, what are the surfaces ruled in conic sections?

I would like to thank all persons in the group of Algebraic Geometry at the University of Oslo for the stimulating atmosphere. In particular I am grateful to C. Peskine for his suggestions and advice.

CHAPTER 1

Preliminaries

§0. Notation and conventions

The ground field k , is assumed to be algebraically closed of characteristic zero.

For a scheme X , let $\mathcal{O}_X$ denote its structure sheaf. If $Y \subseteq X$ is a subscheme, then $I_{Y|X}$ is the ideal sheaf of Y in X . If X is a projective space, then we simply write I_Y when the meaning is clear.

If F is a sheaf on X , then we let $H^i(F) = H^i(X; F)$ and $h^i(F) = \dim_k H^i(F)$. If $X \subseteq \mathbb{P}^N$, then $F(n) = F \otimes_{\mathcal{O}_{\mathbb{P}^N}} \mathcal{O}_{\mathbb{P}^N}(n)$. The Euler-Poincaré characteristic of F is $\chi(F) = \sum_{i \geq 0} (-1)^i h^i(F)$ (whenever meaningful).

$\text{Sing}(X)$ denotes the singular locus of X .

If J is an ideal in a polynomial ring, then $V(J)$ denotes the set of zeros (set-theoretically).

A surface S , is a projective variety of dimension 2 and unless otherwise stated, assumed to be smooth.

If D is a divisor on S , then $\mathcal{O}_S(D)$ is the corresponding linebundle.

$P(D)$ = the arithmetic genus of D .

K denotes a canonical divisor on S and

$$\omega_S = \mathcal{O}_S(K).$$

$$\chi = \chi(\mathcal{O}_S)$$

$q = h^1(\mathcal{O}_S)$; the irregularity of S

$$P_g = h^2(\mathcal{O}_S).$$

If S is embedded in a projective space $\mathbb{P}^N$, then we let

H = hyperplane section of S ; $\mathcal{O}_S(H) = \mathcal{O}_S(1)$.

$\pi = p(H)$; the geometric genus of a general hyperplane section
(the sectional genus).

$d = \deg(S) = H^2$; the degree of S .

Furthermore, S is said to be non-degenerate if it is not contained in a proper subspace of $\mathbb{P}^N$.

§1.

First of all we give a list of well known results that will be used frequently in this paper.

Let C be a divisor on a smooth surface S .

The Riemann-Roch Theorem:

$$\chi(\mathcal{O}_S(C)) = \chi + \frac{1}{2}C \cdot (C - K)$$

The adjunction formula:

$$2p(C) - 2 = C \cdot (C + K)$$

Then we have the Theorem par excellence to be used on surfaces in $\mathbb{P}^4$:

Severi's Theorem:

The only non-degenerate surfaces in $\mathbb{P}^5$ with isolated singularities that may be projected isomorphically to $\mathbb{P}^4$ are cones and Veronese surfaces.

This is the version proven in [M, p.72], the original proof can be found in [Se, p.42].

In this paper we will mostly be concerned about surfaces in $\mathbb{P}^4$ that are not contained in a quadric hypersurface. To illustrate the usefulness of Severi's Theorem, we give an obvious consequence:

Corollary 1.1.1:

Let S be a smooth non-degenerate surface in $\mathbb{P}^4$, and suppose S is not a Veronese surface. If a hyperplane section of S is contained in a quadric surface, then this surface is the hyperplane section of a quadric hypersurface that contains S .

Proof:

Severi's Theorem says that $h^0(\mathcal{O}_S(1)) = h^0(\mathcal{O}_{\mathbb{P}^4}(1)) = 5$, hence $H^1(I_S(1)) = 0$. Let C denote the hyperplane section of S . From the induced exact sequence

$$0 \rightarrow I_S(1) \rightarrow I_S(2) \rightarrow I_C(2) \rightarrow 0,$$

we find $H^0(I_S(2)) = H^0(I_C(2))$, and we are done.

The next two results are also classical. They can be found in [S.R, Ch IX], where they are given in terms of another set of projective characters than ours.

The c_2 -formula:

The characters of a smooth surface in $\mathbb{P}^4$ satisfy

$$d^2 - 10d - 5H.K - 2K^2 + 12\chi = 0$$

We have chosen to name it the c_2 -formula because it is based on the fact that the second Chern class of the normal bundle of a surface in $\mathbb{P}^4$ is d^2 . For a proof see [H, p.434].

The Jacobi formula:

Suppose a smooth surface S is contained in a hypersurface V of degree s in $\mathbb{P}^4$. If $\text{Sing}(V) \cap S$ is finite, then

$$\pi = 1 + \frac{d}{2} \left(\frac{d}{s} - 4 + s \right) - \frac{\mu}{2s} ,$$

where μ is the degree of the finite scheme defined by the restriction of the Jacobi ideal of V to S .

Proof:

Let $\mathcal{J} = I_S/\mathbb{P}^4$, and let F be a homogeneous polynomial of degree s defining V . Then F defines a global section of the conormal bundle of S in $\mathbb{P}^4$ twisted by s ; $\mathcal{J}/\mathcal{J}^2(s)$. If $T_{\mathbb{P}^4}$ is the tangent bundle of $\mathbb{P}^4$, then the natural surjection $T_{\mathbb{P}^4} \otimes \mathcal{O}_S \rightarrow N$ (N the normal bundle of $S \subset \mathbb{P}^4$), shows that the vanishing ideal of this section is the restriction of the Jacobi ideal. It implies clearly

$$\mu = c_2(\mathcal{J}/\mathcal{J}^2(s)) = c_2(N(-s)) ,$$

where c_2 denotes (the degree of) the second Chern class.

Using standard formulas for Chern numbers, we have

$$\mu = c_2(N) - c_1(N) \cdot sH + (sH)^2 .$$

If we combine the tangent bundle sequence

$$0 \rightarrow T_S \rightarrow T_{\mathbb{P}^4} \otimes \mathcal{O}_S \rightarrow N \rightarrow 0 ,$$

and the Euler sequence for $\mathbb{P}^4$

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(1)^5 \rightarrow T_{\mathbb{P}^4} \rightarrow 0 ,$$

we get $c_1(N)=5H-c_1(T_S)=5H+K$.

If we use $c_2(N)=d^2$ (see the c_2 -formula), $H^2=d$, and the adjunction formula on H ; $H.K=2\pi-2-d$, we get

$$\mu = d^2 - 5sd - s(2\pi-2-d) + s^2d$$

which yields the formula stated.

We will end this section by mentioning a result that gives a useful criterion for a smooth surface ($[So], [VdV]$).

Theorem 1.1.2 (Sommese-Van de Ven)

For a smooth surface in a projective space the linear system $|H+K|$ is generated by its global sections except for scrolls, Veronese surfaces and planes.

One knows that the only scrolls in $\mathbb{P}^4$ are the cubic rational scrolls and the elliptic quintic scrolls. (A proof can be found in Appendix B.)

We will not be able to draw full advantage of this Theorem in the sequel, but it will turn out to be natural to connect this result to the discussion in Chapter 2 about surfaces on cubic hypersurfaces in $\mathbb{P}^4$.

§2.

One method in the study of surfaces in $\mathbb{P}^4$ is to examine a hyperplane section and use the rich theory about curves in $\mathbb{P}^3$. However, this method has clearly its limitations because we lose what is extraordinary when we pass from a surface in $\mathbb{P}^4$ to a curve in $\mathbb{P}^3$. Nevertheless, some properties might as well be studied at the "hyperplane section level", as we shall see in this paragraph.

Roth shows the following ([Ro, p.152]), by using Severi's Theorem and a result of Bertini.

Proposition 1.2.1 (Roth):

Let S be a smooth non-degenerate surface of degree d in $\mathbb{P}^4$. Suppose a general hyperplane section of S is contained in a surface of degree s , where $s^2 < d$. Then this surface is the hyperplane section of a hypersurface of degree s in $\mathbb{P}^4$ that contains S .

Remark: This Proposition is only of interest when $s > 3$; for $s=2$ we have a more precise statement in Corollary 1.1.1, where we do not restrict to a general hyperplane section.

We need this result only when $s=3$, in this case the contribution of Bertini is simple and we give Roth's proof:

Let $H_0 = V(x_0)$ and $H_1 = V(x_1)$ be taken as general hyperplanes. For $i=0,1$ let $C_i = S \cap H_i$ be contained in the degenerate surface S_i of degree s , where we can assume S_i irreducible. Then the d points in $C_0 \cap C_1$ are in the intersection between the curves

$S_0 \cap H_1$ and $S_1 \cap H_0$, both of degree s and lying in the plane $H_0 \cap H_1$. But we can assume that one of these curves is irreducible, and since $s^2 < d$, these curves have to be equal by Bezout's Theorem. Let $F_{01} = F_{01}(x_2, x_3, x_4)$ define this curve, and let $F_0 = F_0(x_1, x_2, x_3, x_4)$ and $F_1 = F_1(x_0, x_2, x_3, x_4)$ define S_0 and S_1 respectively. If we adjust F_0 and F_1 (multiply with scalars), then we get a form of degree s

$$F = F_0 + F_1 - F_{01}$$

such that $V(F, x_i) = S_i$, $i=0,1$.

Suppose $s=3$. We have the exact sequence

$$0 \rightarrow I_S(1) \xrightarrow{x_0 x_1} I_S(3) \rightarrow I_{C_0 \cup C_1 | H_0 \cup H_1}(3) \rightarrow 0$$

By Severi's Theorem we know that $H^1(I_S(1))=0$ when S is not a Veronese surface. Since $0 \neq F \in H^0(I_{C_0 \cup C_1}(3))$ and $H^0(I_S(1))=0$, we then get from the sequence that $H^0(I_S(3)) \neq 0$, and we are done.

Recall, a closed subscheme $X \subseteq \mathbb{P}^N$ is said to be projectively Cohen-Macaulay if the graded ring of the embedding is a Cohen-Macaulay ring. If $\dim X > 1$, then this is equivalent to the statement

$$H^i(I_X(n)) = 0, \quad n \in \mathbb{Z}, \quad 0 < i < \dim X$$

Lemma 1.2.2:

Let $X \subseteq \mathbb{P}^N$ be a closed subscheme of dimension 2 or more. If a hyperplane section of X is projectively Cohen-Macaulay then so is X .

Proof:

Let Y denote the hyperplane section of X and consider the induced exact sequences

$$0 \rightarrow I_X(n) \rightarrow I_X(n+1) \rightarrow I_Y(n+1) \rightarrow 0, \quad n \in \mathbb{Z}.$$

By assumption we have $H^1(I_Y(n))=0$, $n \in \mathbb{Z}$. If we use that $H^1(I_X(-m))=0$ when $m \gg 0$, we then find by induction that $H^1(I_X(n))=0$, $n \in \mathbb{Z}$. Furthermore, for $2 \leq j \leq \dim X$ we have injections $H^j(I_X(n)) \rightarrow H^j(I_X(n+1))$. Since $H^j(I_X(n))=0$ when $n \gg 0$, the Lemma is proved.

The result concerning curves in $\mathbb{P}^3$ that will be most helpful is the following ([G.P.1])

Theorem 1.2.3 (Gruson-Peskine):

Let C be a smooth curve of degree d and genus g in $\mathbb{P}^3$. If C is not lying on a surface of degree $< s$, where $s(s-1) < d$, then

$$(*) \quad g \leq 1 + \frac{d}{2} \left(\frac{d}{s} + s - 4 \right) - r(s-r)(s-1)/2s,$$

where $d+r \equiv 0 \pmod{s}$, $0 \leq r < s$. Furthermore, equality in $(*)$ occurs if and only if C is linked to a plane curve of degree r by surfaces of degree s and $\frac{d+r}{s}$ respectively.

We are now able to combine the results given in this paragraph in a form that will be most suitable for us. This Proposition will, in the cases $s \leq 4$, be the essential tool in Chapter 3.

Proposition 1.2.4:

If a smooth surface S of degree d and sectional genus π in $\mathbb{P}^4$ is not contained in a hypersurface of degree $< s$, where $s(s-1) < d$, then

$$(*) \quad \pi < 1 + \frac{d}{2} \left(\frac{d}{s} + s - 4 \right) - r(s-r)(s-1)/2s,$$

where $d+r \equiv 0 \pmod{s}$, $0 < r < s$. Furthermore, equality in $(*)$ occurs if and only if S is projectively Cohen-Macaulay and linked to a degenerate surface of degree r by hypersurfaces of degree s and $\frac{d+r}{s}$ respectively.

Proof:

Since $(s-1)^2 < s(s-1) < d$, we get $(*)$ from Proposition 1.2.1 and Theorem 1.2.3. If we have equality in $(*)$, then Theorem 1.2.3 tells that a hyperplane section of S , say C , is linked to a plane curve of degree r by surfaces of degree s and $\frac{d+r}{s}$ respectively. From Lemma 1.2.2 we get S is projectively Cohen-Macaulay. The surjection

$$\bigoplus_n H^0(I_S(n)) \rightarrow \bigoplus_n H^0(I_C(n))$$

shows that there exist hypersurfaces of degree s and $\frac{d+r}{s}$ respectively, both containing S , whose hyperplane sections are the surfaces considered containing C . The residual intersection of these hypersurfaces is a scheme whose hyperplane section is a plane curve of degree r , hence degenerate.

The converse implication follows immediately from Theorem 1.2.3.

Remark 1.2.5:

We have chosen to mention the projectively Cohen-Macaulay property in Proposition 1.2.4. The reason is that we will be interested in the irregularity of surfaces, q , in the next chapters. Clearly a projectively Cohen-Macaulay surface is regular; $q=0$ and $\chi>0$.

§3.

We end this chapter by giving some of the known classification of surfaces in $\mathbb{P}^4$.

Proposition 1.3.1:

Let $S \subseteq \mathbb{P}^4$ be a smooth non-degenerate surface with characters π and d , then the following 2 conditions are equivalent:

1) S is contained in a quadric hypersurface.

$$2) \quad \pi = 1 + \left\lfloor \frac{d}{4}(d-4) \right\rfloor$$

When this is so, then S is projectively Cohen-Macaulay and either

a) d is even, and S is a complete intersection of the quadric and another hypersurface

or

b) d is odd, and S is linked to a plane by the quadric and another hypersurface.

This result is well known (see for instance [Ro, p.151]): If the hyperquadric has only isolated singularities, then it can

be easily deduced from the Jacobi formula and Proposition 1.2.4 (use $\mu=0$ or 1). If the quadric has a double line, then a general hyperplane section of S is a smooth curve on a quadric cone, and must have maximal genus.

The following table gives the smooth non-degenerate surfaces of degree 6 or less in P^4 not on a quadric hypersurface, together with their characters (see [S.R, p.218]).

Table 1.3.2:

	d	π	χ	q	K^2	H.K
Veronese	4	0	1	0	9	-6
Elliptic scroll	5	1	0	1	0	-5
Bordiga	6	3	1	0	-1	-2

Remark 1.3.3:

The Table 1.3.2 still gives all possibilities if we replace the condition "smooth" with "normal and not a cone". By lack of references, we will give some partial results in this direction when needed.

The surfaces in P^4 of degree 7 and 8 are classified by Okonek ([O1],[O2]).

CHAPTER 2

Smooth surfaces on cubic hypersurfaces in $\mathbb{P}^4$

§0

For the surfaces on quadric hypersurfaces in $\mathbb{P}^4$ we have a complete classification in Prop. 1.3.1, and a natural question is if a similar result is true in the case of cubic hypersurfaces.

In §1 we will see that this is so when the cubic hypersurface has isolated singularities; a surface on such a hypersurface is linked to a surface of degree 5 or less (Theorem 2.1.1). The method of the proof will be to utilize the genus of a general hyperplane section of the surface; this curve lies on a smooth cubic Del Pezzo surface in $\mathbb{P}^3$ and because of the "nearly maximal" genus we find, we can link the curve to a curve of degree 5 or less on the Del Pezzo surface. Finally we will be able to lift this linkage to a linkage between the surface and another surface. In the end of §1 we give a brief discussion of the result.

We have not succeeded in the classification of surfaces on cubic hypersurfaces with non-isolated singularities. However, from the special cases we do know, we believe that the list in Theorem 2.1.1 essentially is complete for any cubic hypersurface. Rather than to present a number of partial results, not too interesting in themselves, we have chosen only to discuss two approaches to this problem in §2. The first one is a prolongation of the method used in §1, the second is a geometric method that can be used for any non-conical cubic hypersurface.

§1

The surfaces on a cubic hypersurface with isolated singularities.

Theorem 2.1.1:

Let S be a smooth surface with characters π and d , contained in a cubic hypersurface in $\mathbb{P}^4$ with only isolated singularities. Then we have

$$\pi = 1 + \frac{1}{6}d(d-3) - \frac{\mu}{6},$$

with the following possibilities:

- 0) $\mu = 0$:
 S complete intersection,
- 1) $\mu = 4, d \equiv 2 \pmod{3}$:
 S linked to a plane,
- 2) $\mu = 4, d \equiv 1 \pmod{3}$:
 S linked to a quadric surface (being a cone if the cubic hypersurface is a cone)
- 3) $\mu = 6$:
 S linked to a rational cubic scroll,
- 4) $\mu = 10, d \equiv 2 \pmod{3}$:
 S linked to a Veronese surface,
- 5) $\mu = 10, d \equiv 1 \pmod{3}$:
 S linked to an elliptic quintic scroll.

In all cases, the cubic hypersurface is used in the linkage.

Remark 2.1.2:

The surfaces involved in the linkage in 1)-5) are precisely the surfaces in $\mathbb{P}^4$ having $|H+K|=\emptyset$ (see the comments after Theorem 1.1.2 and use $K=(s-4)H$ for a surface of degree s in $\mathbb{P}^3$). We have not succeeded in finding a proof of the Theorem, utilizing this property.

Remark 2.1.3:

A Segre cubic primal is a cubic hypersurface in $\mathbb{P}^4$ with 10 isolated singular points (see [S.R, p.169]). We can find examples of all of the surfaces listed in the Theorem 2.1.1 on such a hypersurface.

Proof of the Theorem:

Let V denote the cubic hypersurface, and let $F=0$ be the equation defining it. By the Jacobi formula, we have

$$(2.1.4) \quad \pi = 1 + \frac{1}{6}d(d-3) - \frac{\mu}{6},$$

where μ is the degree of the finite scheme defined by the restriction of the Jacobi ideal of V to S .

Since V has only isolated singularities, Bezout's Theorem implies $\mu \leq 2^4 = 16$. If we consider (2.1.4), then we easily find the following possibilities:

$$(2.1.5) \quad \begin{aligned} d \equiv 0 \pmod{3} \quad \text{and} \quad \mu \in \{0, 6, 12\} \\ d \not\equiv 0 \pmod{3} \quad \text{and} \quad \mu \in \{4, 10, 16\} \end{aligned}$$

First of all we will exclude the trivial case $\mu=16$. This can be done in several ways, but we choose one that allows us to make use of the definition of μ .

The case $\mu=16$:

Let $L \subseteq H^0(\mathcal{O}_{\mathbb{P}^4}(2))$ be the linear system generated by $\frac{\partial F}{\partial x_i}$, $i=0,1,\dots,4$. Since $\mu=16$, it follows from Bezout's Theorem that 4 general elements of L define the Jacobi scheme of V . Let I denote the ideal sheaf of this complete intersection scheme.

Clearly $h^0(I(2))=4$, and since $L \subseteq H^0(I(2))$, the set

$\{\frac{\partial F}{\partial x_i}\}_{i=0,\dots,4}$ is linearly dependent.

We can then assume $\frac{\partial F}{\partial x_0} = 0$ (by a suitable change of coordinates in P^4), hence V is a cone with vertex $P=(1,0,\dots,0)$ and smooth outside P . Let $(R,\underline{m})$ denote the local ring of P^4 in p . The $\frac{\partial F}{\partial x_i}$'s are homogeneous of degree 2 in the variables $x_1,\dots,x_4$, so they can be considered as elements in R . Let $J \subseteq R$ be the ideal they generate. Furthermore, let $f=g=0$ be local equations for S in p . We find

$$16 = \mu = \dim_k [R/J+(f,g)] < \dim_k R/J = 16.$$

Hence $(f,g) \subseteq J$, but this implies S is singular in p because $J \subseteq \underline{m}^2$.

It might seem natural to try a similar approach for the remaining μ 's, for instance we also want to exclude the case $\mu=12$. But to do so we would have to find all cubic forms with a Jacobi scheme of degree 12 or more (one example is $x_0x_1x_2+x_3^3+x_4^3$); a task we have not found too attractive. Instead, we will choose a method where we avoid to have to treat all different types of singular loci separately.

Let us first give an outline of this method:

A general hyperplane section of V is a smooth cubic Del Pezzo surface. Since π is "nearly maximal", we are able to link the induced hyperplane section of S to a curve of "expected" (according to the Theorem if $\mu \leq 10$) type on the Del Pezzo surface.

This is proved separately in Appendix A. We will show that this linkage is a restriction to $\mathbb{P}^3$ of a linkage between S and a surface S' . After some effort we also show that we can choose S' smooth. It should, however, be pointed out that this is really a minor problem; the surfaces of low degree in $\mathbb{P}^4$, including singular ones, are easily classified (see Remark 1.3.3). The case $\mu=12$ will be excluded because the smooth S' we find does not exist.

We need two Lemmas before we go on (In the first of these Lemmas we also consider $\dim(\text{Sing}V)=1$ because it is needed in Chapter 3).

Lemma 2.1.6:

Let S be a smooth surface on a cubic conic hypersurface V in $\mathbb{P}^4$ and suppose $\dim(\text{Sing}V) < 1$. Then either

A) $d \equiv 0 \pmod{3}$; $\pi = 1 + \frac{1}{6}d(d-3)$ and S is a complete intersection on V ,

or

B) $d \equiv 1 \pmod{3}$; $\pi = 1 + \frac{1}{6}d(d-3) - \frac{2}{3}$ and S is linked to a quadric cone on V .

Proof:

V has as vertex either a line or a point. In both cases a general hyperplane section of V intersecting the vertex, is a cone over a smooth plane cubic curve. The induced hyperplane section of S is a smooth curve (choose the hyperplane transversal to S in the intersection with the vertex and use Bertini's Theorem), lying on this cone. Let C denote the curve. It is an easy exercise to show that there are two possibilities depending on whether C

passes through the vertex or not (see [H, Example 2.11.4, p. 374]):

A') C is a complete intersection on the cone

B') C is linked to two generatrices on the cone.

Furthermore, the genus equals the π written in A) and B) respectively. The Lemma follows from Prop. 1.2.4 (the quadric surface is necessarily a cone).

Remark 2.1.7:

If we consider the Theorem, then A) and B) belong to the cases 0) and 2) respectively. Henceforth we can assume that our cubic hypersurface with isolated singularities is not conical.

Lemma 2.1.8:

Let V be a cubic hypersurface in $\mathbb{P}^4$ with isolated singularities and suppose V is not a cone. Then V contains only a finite number of quadric cones or cones over space curves.

Proof:

First of all we will show that the cones we consider, with the exception of a finite number of quadric cones, have vertex in a singular point on V :

A cone over a space curve has degree 3 or more, thus a general hyperplane section of V through the vertex contains 3 or more lines through the vertex, not lying in a plane. Hence such a hyperplane section of V is singular in the vertex and consequently V is also singular in that point.

A quadric cone is linked to a plane. Let P denote the plane and say $P=V(x_0, x_1)$. We can then express V as the zero locus of $x_0 Q_0 + x_1 Q_1$, where Q_1 and Q_2 are quadratic forms. There is a $\mathbb{P}^1$ of quadric surfaces linked to P and a generic one is smooth outside $V(x_0, x_1, Q_0, Q_1)$ by Bertini's Theorem. One easily checks that this locus is contained in the singular locus of V . Furthermore, there is only a finite number of planes in V (for instance less than the number of lines on a cubic Del Pezzo surface; 27), so there is only a finite number of quadric cones with vertex outside the singular points of V .

Now we can assume that the vertex of the cone is $p=(0,0,0,0,1)$ and V is singular in p . Since V is not a cone, we have

$$V = V(x_4 Q + G) \quad \text{where} \quad Q, G \in k[x_0, x_1, x_2, x_3].$$

By parametrizing a line in V through p , we find that the line must be contained in the sextic cone $V(Q, G)$. Hence a cone with vertex p must be a component of this sextic cone.

Since we have assumed isolated singularities we find a finite number of these cones.

Notation:

Let L be a general hyperplane in $\mathbb{P}^4$. $S_3 = L \cap V$ is a smooth cubic Del Pezzo surface. In each of the following cases we have chosen a basis $\{\lambda, E_1, \dots, E_6\}$ for $\text{Pic } S_3$, where the linear system $|\lambda|$ defines a morphism $S_3 \rightarrow \mathbb{P}^2$ and $E_1, \dots, E_6$ are the six lines contracted (see Appendix A). Let H_3 denote a hyperplane section of S_3 . Furthermore, let $C = L \cap S$, $\mathcal{I}_C = \mathcal{I}_C|_{S_3}$, and $\mathcal{I}_S = \mathcal{I}_S|_V$.

Multiplication with the linear form defining L induces the exact sequence

$$(2.1.9) \quad 0 \rightarrow \mathcal{I}_S(n-1) \rightarrow \mathcal{I}_S(n) \rightarrow \mathcal{I}_C(n) \rightarrow 0, \quad n \in \mathbb{Z}.$$

The case $\mu=10, d \equiv 2 \pmod{3}$:

From Appendix A we have that C is linked to a smooth rational quartic curve, say C_4 , on S_3 . In $\text{Pic}S_3$ we have that $C+C_4=tH_3$, where $d+4=3t$. Furthermore,

$$\begin{aligned} h^1(\mathcal{I}_C(n)) &= \delta_{n,t-2} \quad (\delta = \text{The Kronecker-symbol}), \\ H^0(\mathcal{I}_C(t)) &\simeq H^0(\mathcal{O}_{S_3}(C_4)), \quad \text{and the rank is 4,} \\ H^0(\mathcal{I}_C(t-1)) &\simeq H^0(\mathcal{O}_{S_3}(C_4-H_3)) = 0. \end{aligned}$$

By induction, using $h^1(\mathcal{I}_S(m))=0$ when $m < 0$ and $h^1(\mathcal{I}_C(n))=\delta_{n,t-2}$, we get from (2.1.9) that $h^1(\mathcal{I}_S(n))=0$ when $n < t-2$. Furthermore, for $n=t-2, t-1$, and t in (2.1.9) we get the exact sequences

$$\begin{aligned} 0 &\rightarrow H^1(\mathcal{I}_S(t-2)) \rightarrow H^1(\mathcal{I}_C(t-2)) \simeq k, \\ 0 &\rightarrow H^1(\mathcal{I}_S(t-2)) \rightarrow H^1(\mathcal{I}_S(t-1)) \rightarrow 0, \\ H^0(\mathcal{I}_S(t)) &\rightarrow H^0(\mathcal{I}_C(t)) \rightarrow H^1(\mathcal{I}_S(t-1)). \end{aligned}$$

We will show that $H^1(\mathcal{I}_S(t-1))=0$. Assume the contrary, we then get

$$H^1(\mathcal{I}_S(t-1)) \simeq H^1(\mathcal{I}_S(t-2)) \simeq k.$$

This implies that there is a linear form L such that the multiplication

$$H^1(\mathcal{I}_S(t-2)) \xrightarrow{\cdot L} H^1(\mathcal{I}_S(t-1))$$

is not injective. For a corresponding hyperplane section C_L , of S , we have $H^0(\mathcal{I}_{C_L}(t-1)) \neq 0$. But this gives C_L is linked to a line on $V \cap (V(L))$; by the "Théorème de liaison globale" ([P.S]), we find that C_L is projectively Cohen-Macaulay and we have a contradiction by Lemma 1.2.2 since C_4 and then S is not projectively Cohen-Macaulay.

Consequently $H^1(\mathcal{I}_S(t-1))=0$, and we have a surjection

$$H^0(\mathcal{I}_S(t)) \rightarrow H^0(\mathcal{I}_C(t)) \simeq H^0(\mathcal{O}_{S_3}(C_4)) .$$

We can therefore find forms $F_t \in H^0(\mathcal{O}_{\mathbb{P}^4}(t))$ such that $V(F_t) \cap V = S \cup S_4$, where S_4 (being dependent on F_t) is a quartic surface having a smooth rational quartic curve as general hyperplane section. If S_4 is smooth, then we are done, because a surface with this property is a Veronese surface.

Suppose a general such S_4 is singular: Since $h^0(\mathcal{I}_S(t)) > h^0(\mathcal{I}_C(t)) = 4$, we can avoid the finite number of quartic cones (Lemma 2.1.8). Furthermore, a general hyperplane section is irreducible and smooth so S_4 is irreducible and has only isolated singularities. By Serre's criterion (" R_1 and S_2 "), S_4 is normal (the S_2 property comes from S by linkage, [P.S, Prop. 1.3]), and by Severi's Theorem S_4 is not a projection from $\mathbb{P}^5$.

Let $\tilde{S}$ be a desingularization of S_4 , and let $|L|$ be the linear system giving $\tilde{S} \rightarrow S \subset \mathbb{P}^4$. A general member of $|L|$ is a rational curve, and we have $L^2=4$. By the adjunction formula on $\tilde{S}$, we find $L.K=-6<0$ so $\tilde{S}$ is ruled ([B, p.112]). $\tilde{S}$ is a rational surface since L is rational, so

$$\chi(\mathcal{O}_{\tilde{S}}) = 1 \quad \text{and} \quad h^2(\mathcal{O}_{\tilde{S}}(L)) = h^0(\omega_{\tilde{S}}(-L)) = 0 .$$

By the Riemann-Roch Theorem we find

$$h^0(\mathcal{O}_{\tilde{S}}(L)) - h^1(\mathcal{O}_{\tilde{S}}(L)) = \chi + \frac{1}{2}L(L-K) = 6.$$

But this contradicts $h^0(\mathcal{O}_{\tilde{S}}(L))=5$, so a general such S_4 is smooth.

The case $\mu=10, d \equiv 1 \pmod{3}$:

From Appendix A we have that C is linked to a smooth elliptic quintic curve, say C_5 , on S_3 . In $\text{Pic}S_3$ we have that $C+C_5=tH_3$, where $d+5=3t$ and $C_5=3\ell-E_1-E_2-E_3-E_4$. Furthermore,

$$h^1(\mathcal{I}_C(n)) = \delta_{n,t-2},$$

$$H^0(\mathcal{I}_C(t)) \simeq H^0(\mathcal{O}_{S_3}(C_5)), \text{ and the rank is } 6,$$

$$H^0(\mathcal{I}_C(t-1)) \simeq H^0(\mathcal{O}_{S_3}(C_5-H_3)), \text{ and the rank is } 1,$$

where E_5+E_6 is the global section.

As in the previous case, we consider the exact sequence (2.1.9).

We find $h^1(\mathcal{I}_S(t-2)) < 1$ in a similar way, and for $n=t-1$ we get

$$0 \rightarrow H^0(\mathcal{I}_S(t-1)) \rightarrow H^0(\mathcal{I}_C(t-1)) \rightarrow H^1(\mathcal{I}_S(t-2)) \rightarrow H^1(\mathcal{I}_S(t-1)) \rightarrow 0$$

If $H^0(\mathcal{I}_S(t-1)) \neq 0$, then we can link S to the union of two planes having the skew lines E_5 and E_6 as hyperplane section. But the union of two planes intersecting in a point is not locally Cohen-Macaulay. This contradicts the fact that S is locally Cohen-Macaulay since this property is preserved by linkage ([P.S, Prop. 1.3]).

From the exact sequence above, we then find $H^0(\mathcal{I}_C(t-1)) = H^1(\mathcal{I}_S(t-2))$ and $H^1(\mathcal{I}_S(t-1)) = 0$, leading to a surjection

$$H^0(\mathcal{I}_S(t)) \rightarrow H^0(\mathcal{I}_C(t)) \simeq H^0(\mathcal{O}_{S_3}(C_5)) .$$

As in the previous case, we can link S to an irreducible normal surface, say S_5 , this time of degree 5, not being a cone or a projection from $\mathbb{P}^5$. If S_5 is smooth, then we are done because $\pi(S_5) = 1$ implies S_5 is an elliptic quintic scroll.

Suppose S_5 is singular and let $\tilde{S}$ be a minimal desingularization (i.e. no -1 curves are contracted by $\tilde{S} \rightarrow S_5$). Let $|L|$ denote the linear system on $\tilde{S}$ defining this morphism.

We have $L^2 = 5$, $L.K = -5$, and $\tilde{S}$ is ruled, say of genus g .

We will show that S_5 is a cone: Since a general section of $|L|$ is an elliptic curve, we have $g = 0$ or 1 , hence $\chi = 1 - g = 1$ or 0 . Let $\mathcal{O} = \mathcal{O}_{\tilde{S}}$. We have $h^0(\mathcal{O}(L)) = 5$ and $h^2(\mathcal{O}(L)) = h^0(\omega_{\tilde{S}}(-L)) = 0$. By the Riemann-Roch Theorem we find

$$-h^1(\mathcal{O}(L)) = \chi ,$$

so we must have $\chi = 0$ and $\tilde{S}$ is ruled over an elliptic curve.

This implies S_5 is ruled in lines and L is a section of $\tilde{S}$: Otherwise, by Hurwitz' Theorem, we have an unramified covering between a general element of $|L|$ and the elliptic curve that defines the ruling. But this is impossible, because if S_5 is not ruled in lines we can find a smooth connected elliptic section of $|L|$ that touches a curve in the ruling (consider a general hyperplane in $\mathbb{P}^4$ containing a tangent), hence we must have some ramification.

A singular fiber in the ruling of a ruled surface contains at least two -1 curves (as one easily proves by induction on the number of components). Since L intersects a fiber in the ruling of $\tilde{S}$ once, and the morphism defined by $|L|$ does not contract any -1 curve, we find that $\tilde{S}$ is a minimal ruled surface. But this leaves us with the possibility that only a section of $\tilde{S}$ is contracted by $\tilde{S} \rightarrow S_5$, hence S_5 is a cone.

The case $\mu=12$, $d \equiv 0 \pmod{3}$:

We want to show that this case is impossible. The method will be to link S to a normal surface of degree 6, not a cone, with $\pi=2$, and then show that such a surface does not exist.

From Appendix A we have that C is linked to a smooth sextic curve of genus 2, say C_6 , on S_3 . In $\text{Pic}S_3$ we have $C+C_6=tH_3$, where $d+6=3t$ and $C_6=4\lambda-2E_1-E_2-E_3-E_4-E_5$. Furthermore,

$$h^1(\mathcal{I}_C(n)) = \begin{cases} 1, & n=t-3, t-2 \\ 0, & \text{otherwise} \end{cases}$$

$$H^0(\mathcal{I}_C(t)) \simeq H^0(\mathcal{O}_{S_3}(C_6)), \text{ and the rank is } 8,$$

$$H^0(\mathcal{I}_C(t-1)) \simeq H^0(\mathcal{O}_{S_3}(C_6-H_3)), \text{ and the rank is } 2,$$

and all sections have the form C_2+E_6 , where C_2 is a conic section in $|\lambda-E_1|$.

Using the sequence (2.1.9) as in the previous cases, we find

$$h^1(\mathcal{I}_S(t-3)) < h^1(\mathcal{I}_C(t-3)) = 1$$

and

$$H^1(\mathcal{I}_S(t-3)) \rightarrow H^1(\mathcal{I}_S(t-2)) \rightarrow H^1(\mathcal{I}_C(t-2))$$

Since $h^1(\mathcal{I}_C(t-2))=1$, we get $h^1(\mathcal{I}_S(t-2))<2$.

Let us show as before that $H^0(\mathcal{I}_S(t-1))=0$. If not, the corresponding hypersurface links S to a scheme with C_2+E_6 as hyperplane section. Since $C_2.E_6=0$, this scheme is the union of a plane and a quadric surface intersecting in 2 points. Such a scheme is not locally Cohen-Macaulay, and once more this is impossible because S is locally Cohen-Macaulay; a property which is preserved by linkage.

We then get the exact sequence

$$0 \rightarrow H^0(\mathcal{I}_C(t-1)) \rightarrow H^1(\mathcal{I}_S(t-2)) \rightarrow H^1(\mathcal{I}_S(t-1)) \rightarrow 0,$$

which proves $h^1(\mathcal{I}_S(t-1))=0$ because $h^0(\mathcal{I}_C(t-1))=2 > h^1(\mathcal{I}_S(t-2))$.

The surjection $H^0(\mathcal{I}_S(t)) \rightarrow H^0(\mathcal{I}_C(t))$ shows as in the previous cases, that we can link S to a sextic surface S_6 that is irreducible, normal, not a cone, and with sectional genus 2.

Let $\tilde{S} \rightarrow S_6$ be a minimal desingularization and L the induced linebundle on $\tilde{S}$. Let $\mathcal{O} = \mathcal{O}_{\tilde{S}}$. We have $h^0(\mathcal{O}(L))=5$, $L^2=6$, $L.K=-4$, and $\tilde{S}$ is ruled say of genus g ; $\chi=1-g$.

Remark: Theorem 1.1.2 does not apply directly on the linear system $|L+K|$, since L is not ample. However, one should bear the idea of that Theorem in mind in the sequel.

Lemma 2.1.10:

Let X be a smooth surface and $\phi: X \rightarrow \mathbb{P}^n$ a morphism defined by a linear system $|L|$. Suppose that ϕ does not contract any -1 curve and that $|L+K| \neq \emptyset$. Then we have $(L+K)^2 > 0$.

Proof:

Suppose $(L+K)^2 < 0$, and let $C \in |L+K|$. Since $C^2 < 0$, there exists an effective irreducible divisor $C_1 \subseteq C$ with $C_1.C < 0$ and $C_1^2 < 0$. Since $C_1.L > 0$ (L comes from $\mathbb{P}^n$) and $C_1.(L+K) < 0$, we find $C_1.K < 0$. By the adjunction formula we have $C_1^2 + C_1.K > -2$. If we combine these inequalities, we get $C_1^2 = -1$, $C_1.K = -1$, and $C_1.L = 0$ as the only possibility. But this implies C_1 is a -1 curve contracted by ϕ .

By the Riemann-Roch Theorem, we find $\chi(\mathcal{O}(L)) = 5 - h^1(\mathcal{O}(L)) = 1 - g + 5$ and $\chi(\mathcal{O}(L+K)) = h^0(\mathcal{O}(L+K)) - h^1(\mathcal{O}(L+K)) = 2 - g$.

The first equality shows $g > 1$. If $g = 1$, then the second equality shows $|L+K| \neq \emptyset$, but this contradicts Lemma 2.1.10 because $(L+K)^2 = -2 + K^2 < 0$. Furthermore, since $p(L) = 2$, the only possibility left is $g = 2$ and L is a section of $\tilde{S}$ by Hurwitz' Theorem. Hence S_6 is ruled in lines. Finally, by the same argument as in the previous case, we find that a section of $\tilde{S}$ has to be contracted, but this is impossible since S_6 is not a cone.

The cases $\mu \leq 6$:

From Appendix A we know that if $\mu = 6$, then C is linked to a twisted cubic. By linkage and Lemma 1.2.2, C and S are projectively Cohen-Macaulay. By Lemma 2.1.8 we can link S to a normal cubic surface that is not a cone; obviously, such a surface is smooth.

The cases $\mu \in \{0, 4\}$ can also be dealt with in the same way, but since π is maximal Prop. 1.2.4 applies. When the linked surface is a quadric, we can choose it smooth by Lemma 2.1.8, and the Theorem is finally proved.

Corollary 2.1.11:

The only irregular smooth surfaces on a cubic hypersurface in $\mathbb{P}^4$ with isolated singularities are the elliptic quintic scrolls.

Proof:

Let S denote such a surface of degree d . The surfaces in 0)-3) in the Theorem are all projectively Cohen-Macaulay by linkage, hence regular.

If S is linked to a Veronese surface V , $d+4=3t$, then $\omega_S \simeq \mathcal{I}_V(t-2)$ by the exact sequence of linkage. For the Veronese surface it is well known that the only incomplete series is $H^0(\mathcal{O}_V(1))$, hence $h^1(\mathcal{O}_S) = h^1(\omega_S) \neq 0$ if and only if $t=3$; that is, S is an elliptic quintic scroll.

If S is linked to an elliptic quintic scroll E , then we see that S is regular because all the series $H^0(\mathcal{O}_E(n))$ are complete.

Remark: In [Ro, p.153] there is a result saying that an irregular normal surface on a cubic hypersurface verifying certain conditions (for instance with isolated singularities) has $p_g=0$.

Definition 2.1.12:

When one can obtain a surface from another by a sequence of linkages, we say that the surfaces are in the same linkage class. If we use a fixed hypersurface V in all of these linkages we say we have a linkage class on V .

Our Theorem can then be reformulated in the following way

Theorem 2.1.13:

Let $V \subset \mathbb{P}^4$ be a cubic hypersurface with isolated singularities. Then a smooth surface on V is in the linkage class on V of one of the following types: The linkage class of

- a) the empty surface, b) a plane,
- c) a cubic scroll or d) a Veronese surface.

The linkage class a), consists of all complete intersections (of V and other hypersurfaces). A quadric surface on V is linked to a plane, and an elliptic quintic scroll is linked to a Veronese surface.

Remark 2.1.14:

If we merely demand "linkage" and not "linkage on V ", then the linkage classes of type a), b), and c) coincide; they all represent the linkage class of projectively Cohen-Macaulay surfaces.

Before we end this section we should also remark that the following Theorem is easy to prove. It is the subtle subject of irregular surfaces in $\mathbb{P}^4$ that is the barrier.

Theorem 2.1.15:

If a linkage class contains a smooth surface on a cubic hypersurface in $\mathbb{P}^4$ (no restrictions on the singularities), then it contains a smooth surface on a quadric hypersurface or an irregular surface.

Proof:

Let S be a smooth surface on a cubic hypersurface $V \subset \mathbb{P}^4$, and let J be the ideal of S in V . Clearly we can assume S is not a Veronese surface. The dualizing sheaf of V is $\mathcal{O}_V(-2)$. By duality we find

$$h^1(J^\vee(-2)) = h^2(J) = h^1(\mathcal{O}_S)$$

$$h^2(J^\vee(-3)) = h^1(J(1)) = 0, \text{ by Severi's Theorem}$$

$$h^3(J^\vee(-4)) = h^0(J(2))$$

If S is regular and not on a quadric hypersurface, this proves $J^\vee$ is -1 regular, therefore $J^\vee(-1)$ is generated by its global sections. Hence it has a smooth section since $J^\vee$ had a smooth section (defining S). The vanishing variety of this section is a smooth surface of degree $\deg S - 3$ lying on the cubic hypersurface and in the same linkage class as S (it is bilinked to S). We are done by induction on the degree of S .

§2

Remarks on smooth surfaces on cubic hypersurfaces in $\mathbb{P}^4$ with non-isolated singularities.

We have not managed to solve the classification question about surfaces on a cubic hypersurface with non-isolated singularities. However, we conjecture that Theorem 2.1.13 is valid for any cubic hypersurface. We refer to this Theorem rather than Theorem 2.1.1 to avoid difficulties like the fact that a Veronese surface is

contained in a cubic hypersurface with a double plane (the cone over a cubic scroll with a double line obtained by projection of the Veronese surface from a point on itself). On this cubic hypersurface the Veronese surface is not linked to a smooth quintic scroll but only to a degenerate such scroll, but it is in the same linkage class as itself (1).

In this paragraph we will sketch two approaches to this problem to show the difficulties that arise.

In the proof of Theorem 2.1.1, we could ignore the specific nature of the singular loci (we only used $\mu \leq 12$); this is clearly no longer possible.

List 2.2.1:

If the singular locus of an irreducible cubic hypersurface has dimension one or more, then the possibilities for the reduced structure of the non-isolated part are

- 1) Line.
- 2) Conic section (possibly reducible).
- 3) Three lines through a point, spanning a $\mathbb{P}^3$.
- 4) Normal rational quartic curve.
- 5) Two irreducible conic sections, intersecting in one point, spanning a $\mathbb{P}^4$.
- 6) Four lines through a point, spanning a $\mathbb{P}^4$.
- 7) Plane.

This list is readily verified, and we omit the details:

Let V denote the cubic hypersurface. A general hyperplane section of V is a cubic surface in $\mathbb{P}^3$, and we know that such a

surface has a double line or at most 4 nodes. This explains why 7) is the only possibility with dimension 2 and why the degree is 4 or less in 1)-6). By Bezout's Theorem, a line joining 2 singular points on V is contained in V , hence the chord variety of the singular curve is contained in V . Since V is irreducible, most curves of degree 4 or less are then excluded. Finally, one shows that a plane curve in $\text{Sing}(V)$ has degree 2 or less by considering an equation for V .

Remark 2.2.2:

In the cases 6) and 7) one can easily classify the smooth surfaces on the cubic hypersurface.

In case 6) one can show that the cubic hypersurface is a cone over a Cayley surface in $\mathbb{P}^3$ (4 nodes), and for cones over a normal cubic surface Lemma 2.1.6 applies. In case 7) the cubic hypersurface is a projection of a cubic threefold in $\mathbb{P}^5$, and by Severi's Theorem one shows that a smooth surface on this hypersurface has degree 4 or less (see Lemma 3.1.19 for details).

In the following let S denote a smooth surface on the cubic hypersurface V .

The first method we would like to illustrate is in the spirit of §1:

Method A:

If the singular locus of V contains a curve on S , then we define D as the largest positive divisor on S contained in the scheme defined by the Jacobi ideal of V . We can then easily adjust the Jacobi formula to this case; we get a section of

$$I_S/I_S^2(3) \otimes \mathcal{O}_S(-D)$$

vanishing in codimension 2, say of degree μ' . As in Chapter 1 we find

$$(2.2.3) \quad \pi = 1 + \frac{1}{6}d(d-3) - \frac{\mu}{6},$$

where $\mu = \mu' - (2p(D) - 2) + D.H.$. The essential tool in proving Theorem 2.1.1 was the fact $\mu \leq 16$; obviously we do not get it for free anymore.

Let D' denote the one-dimensional part of $\text{Sing}(V)$ and assume V is not of the type 7):

In some cases we can bound the μ ($\mu \leq 12$), such that we get a π in the same range as we had in §1. For instance this is the case if $D=D'$ (one has to examine each kind of singular locus carefully, the divisors may be nonreduced). A hyperplane section of V is a cubic surface in $\mathbb{P}^3$ having a Del Pezzo surface as desingularization, and the calculation in Appendix A goes through as before. Hence the method of §1 can be copied (there are some problems in getting linkage with a smooth surface, but these difficulties seem possible to overcome).

However if $D \neq D'$, then two severe problems may occur:

- i) A component of D' is not contained in S ; then μ' includes the number of intersections between this component and S , and we cannot give a bound.
- ii) A common component of D and D' occurs with different multiplicities; then μ' may include an unknown number of embedded points with support on this component.

Besides some partial results we have not resolved these problems.

Another approach is to use the monoidal structure of a singular cubic hypersurface (not necessarily with non-isolated singularities):

Method B:

Suppose V is singular in the point $p=(0,0,0,0,1)$ and that V is not a cone. The equation defining V is then

$$x_4 Q = G ,$$

where Q and G are homogeneous polynomials in $k[x_0, x_1, x_2, x_3]$ of degree 2 and 3 respectively. The projection from p defines a birational map $V \dashrightarrow \mathbb{P}^3$. The inverse map, say ϕ , is given by

$$(x_0, x_1, x_2, x_3) \mapsto (x_0 Q, x_1 Q, x_2 Q, x_3 Q, G) ,$$

and it is defined outside the sextic complete intersection curve

$$C_6 = V(Q, G) \subseteq \mathbb{P}^3 .$$

Alternatively, ϕ is given by the linear system of cubic surfaces that contain C_6 .

If $\mathbb{P}^2 \subseteq \mathbb{P}^3$, then $\phi(\mathbb{P}^2)$ is a hyperplane section of V through p , and for a general $\mathbb{P}^2$ this cubic surface is obtained from the blowing up of $\mathbb{P}^2$ in the six points $\mathbb{P}^2 \cap C_6$.

If S is a surface on V , then S can be represented by a surface $\bar{S}$ in $\mathbb{P}^3$ via ϕ .

Let S_2 denote the quadric surface $V(Q) \subseteq \mathbb{P}^3$. If $\bar{S} \cap S_2$ contains a curve C outside C_6 , then S contains p because $S_2 \setminus C_6$ is contracted to this point. If S is smooth in p , then a hyperplane through p transversal to S in this point, inter-

sects S in a curve that is smooth in p . This curve corresponds to a plane curve in $\mathbb{P}^3$ and since the intersection of the plane and S_2 is contracted, we get that C is a line (in fact, C is the projection of the tangent plane of S in p).

Example 2.2.4:

Let S_2 be a smooth quadric surface, and let C_6 be the union of six distinct lines; 3 from each of the 2 line-systems on S_2 . Let $\bar{S}$ be a cubic scroll in $\mathbb{P}^3$ that contains 3 of these lines in the ruling, and such that the double line of $\bar{S}$ is a fourth of these lines. If the exceptional line on $\bar{S}$ is not one of the 2 remaining lines of C_6 , then S is a Veronese surface, otherwise S is a cubic scroll in $\mathbb{P}^4$.

When C_6 is a "complicated" curve (e.g. reducible and non-reduced), then it is hard to examine which surfaces $\bar{S}$ represent smooth surfaces $S \subseteq V$. However, to illustrate this method we can solve the classification question in the case 4) of List 2.2.1:

Proposition 2.2.5:

A smooth surface on a cubic hypersurface in $\mathbb{P}^4$ with a normal rational quartic curve in the singular locus, is linked to a cubic scroll that contains the quartic curve.

Proof:

As usual, let S denote the surface and V the cubic hypersurface. Furthermore, let C denote the normal rational quartic curve.

By Bezout's Theorem, V is the chord variety of C . For a suitable choice of coordinates in P^4 , there is a matrix

$$M = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{bmatrix}$$

such that C is defined by the 2×2 minors of M , and V is given by $\det(M)=0$.

We will stick to the notation of Method B in the following.

S_2 is then the cone $V(x_0x_2 - x_1^2) \subseteq P^3$, and we get that C_6 is a double twisted cubic curve $2C_3 \subseteq S_2$; C_3 is the projection of C .

If S is represented by $\bar{S}$, there are two possibilities

- a) $\bar{S} \cap S_2 = rC_3$ and r is even
- b) $\bar{S} \cap S_2 = rC_3 \cup L$ and r is odd,

where L is a line on S_2 .

Recall that the map $\phi: P^3 \dashrightarrow P^4$ is given by the cubic surfaces that contain C_6 . In case a) we have $\bar{S} \cap S_2 = \frac{r}{2}C_6$; i.e. S is a complete intersection of V , clearly singular.

In case b), we have that a cubic scroll X that contains C is represented by a quadric surface $\bar{X}$, such that $\bar{X} \cap S_2 = C_3 \cup L'$, where L' is a line. Since $L \cup L'$ is a complete intersection of S_2 and a plane, P^2 , and $S_2 \cup P^2$ is a cubic surface that contains C_6 , we get that $\bar{X} \cup \bar{S}$ represents a complete intersection of V , and the Proposition follows. (One can also easily show that any (singular) surface on V is either a complete intersection or linked to a cubic scroll.)

Remark 2.2.6:

A surface of the kind we have in Prop. 2.2.5 is of the type 3) in Theorem 2.1.1 with $\mu=6$ (use for instance formula (2.2.3) for π). We get examples of surfaces with $\mu=6$ when we consider surfaces on "determinantal cubic hypersurfaces" (i.e. defined by a 3×3 determinant); a general such hypersurface has 6 isolated singular points by Porteous' formula. The hypersurface of Prop. 2.2.5 is defined by the determinant of a 3×3 symmetric matrix. This may indicate that a study of deformations of surfaces can resolve our classification problem. The representation of Method B seems suitable for this purpose, because it makes it possible to study families of pairs (S, V) , where S is a surface contained in a cubic hypersurface $V \subset \mathbb{P}^4$.

CHAPTER 3

The surfaces in $\mathbb{P}^4$ without apparent triple points.

§0.

Definition 3.0.1:

A surface in $\mathbb{P}^4$ is said to be without apparent triple points if the union of its trisecant lines does not fill $\mathbb{P}^4$. (Equivalently: A general projection of the surface to $\mathbb{P}^3$ has no triple points.)

We will use this chapter to prove the following

Theorem 3.0.2:

The only smooth surfaces in $\mathbb{P}^4$ without apparent triple points are elliptic quintic scrolls and surfaces contained in a hyperquadric.

Severi has shown ([Se, p.45]) that a surface (not necessarily smooth) in $\mathbb{P}^4$ without apparent triple points and not on a hyperquadric, has a hypersurface ruled in planes as trisecant variety. Furthermore, such a plane intersects the surface in a curve of degree 3 or more.

We will not use Severi's method or result in this paper. In the case of elliptic quintic scrolls we can describe the family of planes in the following way. Two lines in the ruling generate a $\mathbb{P}^3$. The residual intersection of this $\mathbb{P}^3$ and the surface is an elliptic plane curve of degree 3. Any line in the plane is a trisecant and conversely we get every trisecant this way.

Z. Ran has shown ([Ra]) that a codimension 2 variety in $\mathbb{P}^n$ defined by a section of a rank 2 vector bundle, is contained in a hypersurface of degree $n-2$ if and only if it is without apparent

(n-1)-uple points. Moreover when this is so, then the variety is a complete intersection.

The method we will use to show the Theorem is to utilize how the lack of apparent triple points affects the sectional genus. At the end of the proof, the use of a formula concerning foursecants is crucial.

§1

The number of apparent triple points for a surface in $\mathbb{P}^4$ is ([S.R, p.203])

$$(3.1.1) \quad t = \binom{\mu_0}{3} - \frac{1}{2}\mu_0\mu_1 + \frac{1}{3}(2\mu_1 + 2v_2 + \mu_2),$$

where v_2 and the μ 's are the so called elementary projective characters of a surface ([S.R, p.193]).

We will stick to the notation and the characters introduced in Chapter 1, §0. For a smooth surface in $\mathbb{P}^4$ with Chern classes c_1 and c_2 , we have (see [LB])

$$\mu_0 = d$$

$$\mu_1 = 3d - H \cdot c_1$$

$$2\mu_2 = c_1^2 + c_2 - 12d + 8\mu_1 - v_2$$

$$v_2 = 6d - 4H \cdot c_1 + c_1^2 - c_2$$

Finally, if we use $c_1 = -K$,

$$H \cdot K = 2\pi - 2 - d \quad (\text{adjunction formula}),$$

$$\chi = \frac{1}{12}(c_1^2 + c_2) \quad (\text{Noether's formula}),$$

and the c_2 -formula of Chapter 1, then we get after some calculation

$$(3.1.2) \quad t = \binom{d-1}{3} - \pi(d-3) + 2\chi - 2.$$

From Table 1.3.2 we find $t=1$ for the Veronese and the Bordiga surfaces, and $t=0$ for the elliptic quintic scrolls as stated in the Theorem.

By Bezout's Theorem it is clear that the trisecant variety of a surface contained in a hyperquadric is the quadric itself, so the Theorem is trivial in one direction.

Example 3.1.3:

Consider a surface defined by a section of a rank 2 bundle E ; $K=(v-5)H$ where $v=c_1(E)$. If we use

$$2\pi - 2 = H \cdot (H+K) = d(v-4)$$

and eliminate χ by means of the c_2 -formula, then (3.1.2) reads

$$(3.1.4) \quad t = \frac{1}{3}d(v-(d+1))(v-(\frac{d}{2} + 2)) \dots$$

Hence $t=0$ if and only if $v=d+1$ or $v = \frac{d}{2} + 2$. Applying Castelnuovo's bound to the genus of a hyperplane section of the surface, we find $v=d+1$ if and only if the surface is contained in a $\mathbb{P}^3$, and $v = \frac{d}{2} + 2$ if and only if the surface is a complete intersection of a quadric and a hypersurface of degree $\frac{d}{2}$.

Example 3.1.5:

For a complete intersection surface of type (m,n) we have $v=m+n$ and $d=mn$. In this case (3.1.4) reads

$$t = \frac{mn}{6}(m-1)(m-2)(n-1)(n-2)$$

Now we can start the proof of the Theorem. First of all we have the following basic identity.

Lemma 3.1.6:

The characters of a surface in $\mathbb{P}^4$ without apparent triple points satisfy

$$\pi = 1 + \frac{d}{6}(d-3) - \frac{2}{3} + \frac{2(\chi-1)}{d-3}.$$

Proof:

Let $t=0$ in (3.1.2) and solve the equation with respect to π .

Lemma 3.1.7:

If a surface $S \subset \mathbb{P}^4$ is without apparent triple points and not on a quadric hypersurface, then S is an elliptic quintic scroll or a birationally ruled surface with $\chi < 0$.

Proof:

Table 1.3.2 shows that we can assume $d > 6$. Furthermore, since S is not on a hyperquadric Prop. 1.2.4 yields

$$(3.1.8) \quad \pi < \pi_3 = 1 + \frac{d}{6}(d-3) - \frac{r(3-r)}{3}$$

where $d+r \equiv 0 \pmod{3}$, $0 < r < 3$.

We will use the following trivial observations to compare (3.1.8) and Lemma 3.1.6:

$$\frac{d}{6}(d-3) \in \mathbb{Z} \iff d \equiv 0 \pmod{3},$$

$$\frac{r(3-r)}{3} = \frac{2}{3} \iff d \not\equiv 0 \pmod{3}.$$

We separate in three cases

$\chi > 1$:

If $d \not\equiv 0 \pmod{3}$, then $\pi > \pi_3$.

If $d \equiv 0 \pmod{3}$, then $\pi > \pi_3$ or

$$\frac{2(\chi-1)}{d-3} = \frac{2}{3} \quad \text{and} \quad \pi = \pi_3.$$

But in the latter case Prop. 1.2.4 tells that S is a complete intersection of a cubic hypersurface and a hypersurface of degree $\frac{d}{3}$, hence $t \neq 0$ by Example 3.1.5, when $d > 6$.

$\chi = 1$:

Lemma 3.1.6 gives $\pi = \pi_3 = 1 + \frac{d}{6}(d-3) - \frac{2}{3}$, hence by Prop. 1.2.4

we get S is projectively Cohen-Macaulay and consequently

$h^1(\mathcal{O}_S(n)) = 0$ for all $n \in \mathbb{Z}$. Since $\chi = 1$ and $h^1(\mathcal{O}_S) = 0$, we have

$h^2(\mathcal{O}_S) = h^0(\omega_S) = 0$. By Serre duality, we then get $h^2(\mathcal{O}_S(2)) =$

$h^0(\omega_S(-2)) = 0$, and by the Riemann-Roch Theorem

$$h^0(\mathcal{O}_S(2)) = \chi(\mathcal{O}_S(2)) = \chi + \frac{1}{2}2H(2H-K).$$

If we substitute $\pi = \pi_3$ in H.K, we find

$$h^0(\mathcal{O}_S(2)) = \frac{1}{3}(-d^2 + 12d + 7) < 15 = h^0(\mathcal{O}_{P^4}(2)).$$

Hence S is on a quadric hypersurface.

$\chi = 0$:

If $d \not\equiv 0 \pmod{3}$, then $\frac{-2}{d-3} \in \mathbb{Z}$ implies $d < 5$.

If $d \equiv 0 \pmod{3}$, then $\frac{2}{3} + \frac{2}{d-3} \in \mathbb{Z}$ gives $d = 9$ as the only possibility, and then we get $\pi = 9$, $H.K = 7$. From the c_2 -formula we find $K^2 = -22$, hence

$$(H+K)^2 = 1 .$$

But Theorem 1.1.2 applies, leading to S is birationally equivalent to $\mathbb{P}^2$ via the morphism defined by $|H+K|$. This is clearly impossible since $\chi=0$.

This leaves us with surfaces having $\chi < 0$. Such surfaces are ruled according to the Enriques' classification, so the Lemma is proved.

Remark 3.1.8:

It is an open question whether there exists any smooth surface in $\mathbb{P}^4$ with $\chi < 0$. Such a surface belongs to the set of irregular surfaces ($q > 0$) in $\mathbb{P}^4$, and this set is not well understood; the only known examples are the elliptic quintic scrolls and surfaces constructed from the Horrocks-Mumford bundle.

To proceed in the proof of the Theorem we even need more information

Lemma 3.1.9:

Let $S \subset \mathbb{P}^4$ be a birationally ruled surface with $\chi < 0$. If S is without apparent triple points, then S is contained in a cubic hypersurface.

Proof:

S is not a rational surface since $\chi < 0$. We refer to [B, Ch.III] for the basic properties of (nonrational) ruled surfaces.

If we blow down the exceptional curves on S we reach a geometrically ruled surface, say S_0 ([B, Thm.III.10]). That is, there exists a morphism

$$\phi : S_0 \rightarrow C$$

where C is a smooth curve of genus

$$(3.1.10) \quad g = 1 - \chi ,$$

such that every fiber of ϕ is isomorphic to $\mathbb{P}^1$.

Let n denote the number of blowing downs in the morphism $S \rightarrow S_0$, then

$$(3.1.11) \quad K^2 = 8\chi - n .$$

If we substitute for χ and K^2 in the c_2 -formula, then we find

$$(3.1.12) \quad \pi = \frac{1}{10}(d+2)(d-3) + \frac{1}{5}(n+2g)$$

Furthermore, Lemma 3.1.6 now reads

$$(3.1.13) \quad \pi = \frac{1}{6}(d-1)(d-2) - \frac{2g}{d-3}$$

The scrolls in $\mathbb{P}^4$ have $\chi > 0$, hence Theorem 1.1.2 yields by adjunction and (3.1.11)

$$(3.1.14) \quad 0 < (H+K)^2 = 4\pi - 8g + 4 - n - d .$$

We can clearly assume $d > 6$, hence by (3.1.14) we get

$$(3.1.15) \quad g < \frac{\pi-1}{2} .$$

If we combine (3.1.13) and (3.1.15), then we have

$$(3.1.16) \quad \pi > \frac{1}{6}(d-1)(d-3) + \frac{1}{d-2}.$$

The maximal sectional genus π_4 , for a surface not contained in a cubic hypersurface is, by Prop. 1.2.4 when $d > 12$

$$(3.1.17) \quad \pi_4 = \begin{cases} 1 + \left\lfloor \frac{d^2}{8} \right\rfloor, & \text{when } d \equiv 0 \pmod{4} \\ \left\lfloor \frac{d^2}{8} \right\rfloor, & \text{otherwise} \end{cases}$$

When one compares (3.1.16) and (3.1.17) one easily finds $\pi > \pi_4$ for $d > 12$. Since $\pi = \pi_4$ implies S is projectively Cohen-Macaulay and $\chi > 0$, it suffices to show that there are no surfaces with $6 < d < 12$ and $g > 1$ that satisfy 3.1.12-15, in order to prove the Lemma; this is indeed straightforward:

$d = 12$: $\pi = 18 - \frac{2g-3}{9}$ by (3.1.13), hence $g=6$ and $\pi=17$ (only possibility by (3.1.15)) leading to $n=28$ by (3.1.12), contradicting (3.1.14).

For $d=11$ one reaches a similar contradiction (the numbers are $\pi=14$, $g=4$, $n=26$).

For $d < 10$ we get a contradiction from (3.1.13) and (3.1.15); we just set up the table we get from (3.1.13)

d	π
10	$12 - \frac{2g}{7}$
9	$9 - \frac{g-1}{3}$
8	$7 - \frac{2g}{5}$
7	$5 - \frac{g}{2}$

By Lemma 3.1.7 and Lemma 3.1.9, we have reduced the problem to studying surfaces without apparent triple points on cubic hypersurfaces.

Since we know all surfaces of degree 6 or less, the following Proposition finishes the proof of the Theorem.

Proposition 3.1.18:

Let S be a smooth surface of degree $d > 6$ on an irreducible cubic hypersurface $V \subset \mathbb{P}^4$. Then S has apparent triple points.

Proof:

The idea will be to consider the foursecant lines of S ; by Bezout's Theorem, they are contained in V . The lines in a cubic hypersurface (with isolated singularities) are well studied, cf. the Fano surface in the Grassmannian of lines in $\mathbb{P}^4$ ([A.K]). We only need to be sure that the dimension of the set of foursecant lines does not exceed the "expected" dimension; 2.

First of all we consider "an exception", namely the nonnormal cubic hypersurface.

Lemma 3.1.19:

A smooth surface on a cubic hypersurface in $\mathbb{P}^4$ with a double plane is a Veronese surface or a surface of degree 3 or less.

Proof:

Let the plane be $V(x_0, x_1)$, then the cubic hypersurface is given by

$$x_0^2 L_0 - x_0 x_1 L_1 + x_1^2 L_2 = 0,$$

where L_i , $i=0,1,2$ are linear forms.

One easily checks that this cubic is the projection from $(0,0,0,0,0,1)$ of the cubic threefold in $\mathbb{P}^5$ given by the 2×2 minors of the matrix

$$\begin{bmatrix} L_1 - x_5 & x_0 & L_2 \\ L_0 & x_1 & x_5 \end{bmatrix}$$

(This is typical for cubic hypersurfaces in any $\mathbb{P}^n$ with a singular locus of codimension one.)

Our surface is then a projection from $\mathbb{P}^5$, and by Severi's Theorem the only possibility is a Veronese surface or a surface contained in a hyperplane in $\mathbb{P}^5$. If the latter is the case, then the surface is contained in a hyperplane section of the cubic threefold, hence of degree 3 or less.

Henceforth, we assume that V has not a double plane. Furthermore, we can assume that V is not a cone: Lemma 2.1.6 implies that a surface on a cubic cone is projectively Cohen-Macaulay and $\chi > 0$. By Lemma 3.1.7 such a surface without apparent triple points is also contained in a quadric hypersurface, so the degree is 6 or less.

Lemma 3.1.20:

There is only a finite number of lines in V through a general point of V .

Proof:

If there were a one dimensional family of lines through a general point, then a general hyperplane section of V contains an infi-

nite number of lines. A normal cubic surface in $\mathbb{P}^3$ has 27 or less lines, so the hyperplane section must be a cone or a cubic ruled surface with a double line. But this implies V has a double plane, or V is a cone.

Remark: One can show that this number is 6 (If V is smooth it is precisely 6, otherwise it may be with multiplicity, see [A.K]).

Since the foursecant lines of S are contained in V , Lemma 3.1.20 implies there is a finite number of foursecants intersecting a general line in $\mathbb{P}^4$. Furthermore, we know that S is not a $\mathbb{P}^2$ or a scroll, hence S contains only a finite number of lines.

Under these conditions, the number of foursecants intersecting a general line in $\mathbb{P}^4$ is (counted with multiplicity) by [LB]

$$(3.1.21) \quad \underline{f} = 2\binom{d}{4} + t(d-3) + h - s\binom{d-2}{2},$$

where

t = the number of apparent triple points,
and if Γ is the double curve of a general projection of S to $\mathbb{P}^3$, then

$$s = \deg \Gamma = \frac{1}{2}(d-1)(d-2) - \pi,$$

h = the number of apparent double points on Γ
 $= \frac{1}{2}s(s-d+2) - \frac{3}{2}t$ (S is smooth).

Assume $t=0$; we will show this is impossible when $d>6$. We get

$$\begin{aligned} h &= \frac{1}{2}s(s-d+2) \\ &= \frac{1}{8}[(d-1)(d-2)^2(d-3)-4\pi(d-2)^2+4\pi^2] , \\ s\binom{d-2}{2} &= \frac{1}{4}[(d-1)(d-2)^2(d-3)-2\pi(d-2)(d-3)] , \end{aligned}$$

and by some calculation

$$\begin{aligned} f &= 2\binom{d}{4} + h - s\binom{d-2}{2} \\ &= \frac{1}{24}[12\pi^2-12\pi(d-2)-(d-1)(d-2)(d-3)(d-6)] \end{aligned}$$

The discriminant of this polynomial, of degree 2 in π , is

$$\Delta = 12^2(d-2)^2 + 4 \cdot 12(d-1)(d-2)(d-3)(d-6) ,$$

and the roots are

$$\pi_{\pm} = \frac{12(d-2) \pm \sqrt{\Delta}}{24} .$$

Since $d > 6$, we immediately find $\sqrt{\Delta} > 12(d-2)$. Thus $\pi > 0$ and $f > 0$ implies

$$\pi > \pi_+ .$$

By a rough estimate

$$\pi_+ > \frac{12(d-2) + \sqrt{4 \cdot 12(d-3)^2}}{24} .$$

But then π exceeds the maximal genus of a surface not on a quadric hypersurface:

$$\pi_3 = 1 + \left\lfloor \frac{d}{6}(d-3) \right\rfloor$$

Hence S is contained both in a quadric hypersurface and in the irreducible cubic V , clearly contradicting $d > 6$.

APPENDIX A

Curves of "nearly maximal" genus on a smooth cubic surface in $\mathbb{P}^3$.

§0

The maximal genus of a curve of degree d on a cubic surface is, by Theorem 1.2.3

$$\pi_3 = 1 + \left\lfloor \frac{d}{6}(d-3) \right\rfloor$$

Furthermore, the Theorem says that depending on $d \pmod{3}$, maximum is achieved only for complete intersections or curves linked to a line or a conic section.

For a smooth (or normal) cubic surface S this can be shown directly, using the well known properties of $\text{Pic}S$. In this Appendix we will give a similar result for the curves of genus $\pi_3 - 1$ and also $\pi_3 - 2$ in the case $d \equiv 0 \pmod{3}$.

The result seems to be known, but no references exist. In Lemma A.2 we have calculated some cohomology groups to be used in Chapter 2.

§1

A smooth cubic surface $S \subset \mathbb{P}^3$ is a Del Pezzo surface achieved by blowing up $\mathbb{P}^2$ in 6 points in "general position". (For details see [H. pp.395-406].)

We have $\text{Pic}S \cong \mathbb{Z}^7$ and we will say a curve $C \subset S$ has the type $(\delta, m_1, \dots, m_6)$ if $C = \delta L - \sum_{i=1}^6 m_i E_i$ in $\text{Pic}S$, where $|L|$ defines a morphism to $\mathbb{P}^2$ by blowing down six skew lines E_i , $i=1, \dots, 6$.

Notation: Let a smooth space curve of degree d and genus g be denoted by C_d^g .

Proposition A.1: Let S be a smooth cubic surface in $\mathbb{P}^3$ and $C = C_d^g \subset S$. Let $\pi_3 = 1 + \left\lfloor \frac{d}{6}(d-3) \right\rfloor$, then we have

If $g = \pi_3 - 1$, then C is linked on S to a smooth curve D according to

- i) $d \equiv 0 \pmod{3}$, $D = C_3^0$ of type $(1, 0, 0, 0, 0, 0, 0)$
- ii) $d \equiv 2 \pmod{3}$, $D = C_4^0$ of type $(2, 1, 1, 0, 0, 0, 0)$
- iii) $d \equiv 1 \pmod{3}$, $D = C_5^1$ of type $(3, 1, 1, 1, 1, 0, 0)$

If $g = \pi_3 - 2$ and $d \equiv 0 \pmod{3}$, then C is linked to a C_6^2 of type $(4, 2, 1, 1, 1, 1, 0)$.

Proof: Given a smooth curve C on S , there exists a so called adequate basis $\{L, E_1, \dots, E_6\}$ for $\text{Pic}S$ (see [G.P.2]), That is, $|L|$ defines a morphism to $\mathbb{P}^2$ contracting the exceptional lines E_i , $i=1, \dots, 6$, and when we express the divisor class of C in $\text{Pic}S$

$$1) \quad C = \delta L - \sum_{i=1}^6 m_i E_i$$

we have

$$2) \quad \delta > m_1 + m_2 + m_3,$$

$$3) \quad m_1 > m_2 > \dots > m_6 > 0$$

In $\text{Pic}S$ we have $H = -K = 3L - \sum_{i=1}^6 E_i$, and the intersection pairing is given by

$$L^2 = 1, \quad E_i \cdot E_j = -\delta_{ij}, \quad L \cdot E_i = 0.$$

Let $r = \delta - m_1$; 2) implies

$$4) \quad r > m_2 + m_3$$

Furthermore,

$$5) \quad d = H.C = 3\delta - \sum m_i$$

and by the adjunction formula

$$\begin{aligned} 6) \quad g = p(c) &= \frac{1}{2}(\delta-1)(\delta-2) - \frac{1}{2} \sum_{i=1}^6 m_i(m_i-1) \\ &= 1 + \frac{1}{2}[(r-1)d - \frac{3}{4}r^2 - \sum_{i=2}^6 (\frac{r}{2} - m_i)^2] \end{aligned}$$

(The latter expression is taken from [G.P.2].) Using 5) we get

$$7) \quad m_1 = \frac{1}{2}(d-3r + \sum_{i=2}^6 m_i)$$

Another property of the adequate basis is that $r < \frac{2d}{3}$, hence for suitable integers $\alpha > 0$ and $\beta \in \{0, 1, 2\}$, $\beta + d \equiv 0 \pmod{3}$ we have

$$r = \frac{2d-\beta}{3} - \alpha$$

If we write $g = 1 + \frac{1}{6}d(d-3) - \frac{\mu}{6}$ and compare with 6), we find

$$8) \quad \frac{\mu}{6} = \frac{1}{24}(3\alpha+\beta)^2 + \frac{1}{2} \sum_{i=2}^6 (\frac{r}{2} - m_i)^2$$

Before we start examining this equation for our possible μ 's, we will define another basis for $\text{Pic}S$, the "coadequate" basis:

Let $G_j = 2L - \sum_{i \neq j} E_i$, $j=1, \dots, 6$. Then $\{G_j\}$ is an exceptional system, i.e. a set of pairwise skew exceptional lines. If we define $L' = 5L - 2 \sum E_i$ then $H = 3L' - \sum G_i$ and $\{L', G_1, \dots, G_6\}$ is a basis for $\text{Pic}S$. Again, $|L'|$ defines a morphism to $\mathbb{P}^2$ contracting the G_j 's.

Case 1: $\mu=6$, $d \equiv 0 \pmod{3}$

This yields $\beta=0$, $r = \frac{2d}{3} - \alpha$, and 8) reads

$$1 = \frac{3}{8}\alpha^2 + \frac{1}{2}\sum_{i=2}^6\left(\frac{r}{2} - m_i\right)^2.$$

$\alpha=0$: $r = \frac{2d}{3}$ implies r is even, and we find

$$2 = \sum_{i=2}^6\left(\frac{r}{2} - m_i\right)^2.$$

3) and 4) give

$$m_2 = m_3 = m_4 \text{ and } m_5 = m_6 = \frac{r}{2} - 1.$$

Substituted in 7) this gives

$$m_1 = \frac{1}{2}\left(\frac{3}{2}r - 3r + 3 \cdot \frac{r}{2} + 2\left(\frac{r}{2} - 1\right)\right) = \frac{r}{2} - 1 < m_2$$

which is impossible by 3).

$\alpha=1$: We find r is odd and by 8)

$$5 = \sum_{i=2}^6(r-2m_i)^2.$$

Hence $|r-2m_i|=1$, $i=2, \dots, 6$. Following the recipe of the case $\alpha=0$, we find two possibilities

a) $m_2 = \frac{r+1}{2}$, $m_3 = \dots = m_6 = \frac{r-1}{2}$
leading to $m_1 = \frac{r}{2} < m_2$; impossible.

b) $m_2 = \dots = m_6 = \frac{r-1}{2}$
giving $m_1 = \frac{r-1}{2}$ and $\delta = r + m_1 = \frac{3r-1}{2}$.

Let $t = \frac{r+3}{2}$, using 5) we find $d+3=3t$, and furthermore,
 $C=(3t-5)L-\sum_{i=1}^6(t-2)E_i$ by 1).

Hence $C + (5L - 2 \sum_{i=1}^6 E_i) = tH$, and in terms of the "coadequate" basis

$$C + L' = tH.$$

But that means C is linked to curves of type $(1, 0, \dots, 0)$; that is (smooth) twisted cubics.

$\alpha > 2$ is clearly impossible, and the case i) in the Proposition is proved.

Case 2: $\mu=10$, $d \equiv 2 \pmod{3}$

We have $\beta=1$, $r = \frac{2d-1}{3} - \alpha$

$\alpha=0$: r odd and 8) reads

$$13 = \sum_{i=2}^6 (r - 2m_i)^2.$$

In some way we must have $13=9+1+1+1+1$, giving

a) $m_2 = \frac{r+3}{2}$, $m_3 = \frac{r+1}{2}$; contradicting 4).

b) $m_2 = \dots = m_5 = \frac{r-1}{2}$, $m_6 = \frac{r-3}{2}$
leading to $m_1 = \frac{r-3}{2} < m_2$; impossible.

$\alpha=1$: r even, and $2 = \sum_{i=2}^6 (\frac{r}{2} - m_i)^2$

As in Case 1, $\alpha=0$ we find

$$m_2 = m_3 = m_4 = \frac{r}{2} \text{ and } m_5 = m_6 = \frac{r}{2} - 1.$$

This gives $m_1 = \frac{r}{2}$ and $\delta = \frac{3r}{2}$. Let $t = \frac{r}{2} + 2$, we then find $d+4=3t$ and

$$C = 3(t-2)L - (t-2) \sum_{i=1}^4 E_i - (t-3)(E_5 + E_6).$$

Hence $C + (6L - 2\sum_{i=1}^4 E_i - 3(E_5 + E_6)) = tH$, and in terms of the "coade-quate" basis

$$C + (2L' - G_5 - G_6) = tH.$$

Renumbering the G_j 's, we find C is linked to curves of the type $(2, 1, 1, 0, 0, 0, 0)$. We can clearly find a smooth curve of this type; giving a rational quartic in P^3 .

$\alpha > 2$ is impossible by 8).

Case 3: $\mu=10, d \equiv 1 \pmod{3}$

We have $r = \frac{2d-2}{3} - \alpha$

$\alpha=0$: r even. $3 = \sum_{i=2}^6 (\frac{r}{2} - m_i)^2$

$$m_2 = m_3 = \frac{r}{2}, \quad m_4 = m_5 = m_6 = \frac{r}{2} - 1$$

giving $m_1 \equiv \frac{r}{2} - 1 < m_2$; impossible.

$\alpha=1$: r odd. $5 = \sum_{i=2}^6 (r - 2m_i)^2$

a) $m_2 = \dots = m_6 = \frac{r-1}{2}$ implies $m_1 = \frac{r}{2} \notin \mathbb{Z}$.

b) $m_2 = \frac{r+1}{2}, m_3 = \dots = m_6 = \frac{r-1}{2}$
giving $m_1 = \frac{r+1}{2}$ and $\delta = \frac{3r+1}{2}$.

Let $t = \frac{r+5}{2}$, we find $d+5=3t$ and

$$C = (3t-7)L - (t-2)(E_1 + E_2) - (t-3)\sum_{i=3}^6 E_i.$$

Hence $C + (7L - 2(E_1 + E_2) - 3\sum_{i=3}^6 E_i) = tH$, and using the "coadequate" basis we find

$$C + (3L' - G_3 - G_4 - G_5 - G_6) = tH.$$

Since $\alpha > 2$ is impossible, we have found that C is linked to a smooth elliptic quintic curve of type $(3,1,1,1,1,0,0)$ as stated.

Case 4: $\mu=12, d \equiv 0 \pmod{3}$

We have $r = \frac{2d}{3} - \alpha$

$\alpha=0$: r even, $4 = \sum_{i=2}^6 \left(\frac{r}{2} - m_i\right)^2$

a) $m_2 = \frac{r}{2}, m_3 = \dots = m_6 = \frac{r}{2} - 1$
giving $m_1 = \frac{r}{2} - 2 < m_2$; impossible,

b) $m_2 = \dots = m_5 = \frac{r}{2}, m_6 = \frac{r}{2} - 2$
giving $m_1 = \frac{r}{2} - 1 < m_2$; impossible.

$\alpha=1$: r odd, and $13 = \sum_{i=2}^6 (r - 2m_i)^2$

As in Case 2, $\alpha=0$ we find

$$m_2 = \dots = m_5 = \frac{r-1}{2}, m_6 = \frac{r-3}{2}.$$

But then $m_1 = \frac{1}{2}(r-2) \notin \mathbb{Z}$.

$\alpha=2$: r even, $1 = \sum_{i=2}^6 \left(\frac{r}{2} - m_i\right)^2$

$$m_2 = \dots = m_5 = \frac{r}{2}, m_6 = \frac{r}{2} - 1.$$

We find $m_1 = \frac{r}{2} + 1$ and $\delta = 3\frac{r}{2} + 1$. Let $t = \frac{r}{2} + 3$, we then find $d+6=3t$ and

$$C = (3t-8)L - (t-2)E_1 - (t-3)\sum_{i=2}^5 E_i - (t-4)E_6.$$

Hence $C + (8L - 2E_1 - 3\sum_{i=2}^5 E_i - 4E_6) = tH$, and using the "coadequate" basis we find

$$C + (4L' - \sum_{j=2}^5 G_j - 2G_6) = tH.$$

Renumbering the G_j 's, we find C is linked to curves of the type $(4, 2, 1, 1, 1, 1, 0)$. By [H, p.405], there exist smooth curves of this type.

$\alpha > 3$ is impossible and the Lemma is finally proved.

If we stick to the notation introduced, we have

Lemma A2: Suppose a curve C is linked on S to a curve D by a hypersurface of degree t . If D is as indicated in the leftmost column, then we have

D	$h^0(\mathcal{O}_S(D))$	$h^0(\mathcal{O}_S(D-H))$	$h^1(I_{C S}(n))$
C_4^0	4	0	$\delta_{n,t-2}$
C_5^1	6	1	$\delta_{n,t-2}$
C_6^2	8	2	$\delta_{n,t-2} + \delta_{n,t-3}$

(δ denotes the Kronecker-symbol)

The calculation is straightforward and we omit the details: We have $H^1(\mathcal{O}_S(n)) = 0$, $n \in \mathbb{Z}$, and $K = -H$. Considering the exact sequence

$$(*) \quad 0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S(D) \rightarrow \mathcal{O}_D(D) \rightarrow 0,$$

we find $h^0(\mathcal{O}_S(D)) = D \cdot H + p(D)$ by Riemann-Roch on D .

Tensoring $(*)$ with $\mathcal{O}_S(-H)$, we find by adjunction

$$h^0(\mathcal{O}_S(D-H)) = h^0(\mathcal{O}_D(D-H)) = h^0(\omega_D) = p(D) .$$

To fill in the rightmost column, we have $C+D=tH$ in $\text{Pic}S$. Hence

$$h^1(I_{C|S}(n)) = h^1(\mathcal{O}_S(D+(n-t)H)) = h^1(I_{D|S}(t-1-n)) \text{ by Serre duality.}$$

Calculating $h^1(I_{D|S}(r))$, $r \in \mathbb{Z}$, is easily done considering the sequence

$$0 \rightarrow I_{D|S}(r) \rightarrow \mathcal{O}_S(r) \rightarrow \mathcal{O}_D(r) \rightarrow 0 ,$$

and using induction on r by means of

$$0 \rightarrow \mathcal{O}_S(-D+rH) \rightarrow \mathcal{O}_S(-D+(r+1)H) \rightarrow \mathcal{O}_H(-D+(r+1)H) \rightarrow 0 .$$

We skip this calculation. (Alternatively, we could have used known results on the postulation of space curves.)

APPENDIX B

The smooth scrolls in $\mathbb{P}^4$.

A scroll S in a projective space is a surface ruled in lines;
i.e. there exists a morphism

$$\phi: S \rightarrow C,$$

where C is a smooth curve, such that every fibre of ϕ is a line. In this Appendix we will give a proof of the following

Theorem B.1:

The only non-degenerate smooth scrolls in $\mathbb{P}^4$ are the rational cubic and the elliptic quintic scrolls.

Remark: It is easily shown that the only smooth scrolls in $\mathbb{P}^3$ are the quadric surfaces.

This Theorem "should be" classical, but the only proofs we have found are somewhat unsatisfactory.

One approach is to use an exercise in [B, p.115];
unfortunately it is not clear under what circumstances the claim of the exercise is correct. We will follow [L] and fill a gap in the end of that proof.

Proof of the Theorem:

Let g denote the genus of the base curve C , and for the scroll S let

- K = a canonical divisor
- H = a hyperplane section
- π = the sectional genus

$d = H^2 =$ the degree
 $\chi =$ the Euler-Poincaré characteristic.

Since S is ruled in lines, ϕ induces an isomorphism between a general member of $|H|$ and C , hence $\pi=g$. Furthermore, $K^2=8\chi$, $\chi=1-g$ ($[H, V_2]$), and by the adjunction formula; $H.K=2\pi-2-d$. If we use the c_2 -formula for a smooth surface in $\mathbb{P}^4$

$$d^2 - 10d - 5H.K - 2K^2 + 12\chi = 0 ,$$

we find

$$(B.2) \quad \pi = \frac{1}{6}(d-2)(d-3) .$$

Let G denote the Grassmannian of lines in $\mathbb{P}^4$, considered as a subscheme of $\mathbb{P}^9$ by the Plücker embedding. Let $C_S \subset G \subset \mathbb{P}^9$ denote the curve corresponding to S .

A Schubert cycle of codimension one in G is the set of lines in $\mathbb{P}^4$ intersecting a plane. Such a cycle is a special hyperplane section of $G \subset \mathbb{P}^9$ (in fact, the section of a tangent hyperplane). Therefore, the degree of C_S equals the number of lines in the ruling of S that meets a general plane in $\mathbb{P}^4$, but that is the degree of S , d .

Castelnuovo's bound on the genus p of a curve of degree d in $\mathbb{P}^5$ yields ([G.H, p.527])

$$(B.3) \quad p < \frac{1}{8}(d-1-\varepsilon)(d-5+\varepsilon) ,$$

where $0 < \varepsilon < 3$ and $d-1 \equiv \varepsilon \pmod{4}$.

A simple comparison of (B.2) and (B.3) gives that C_S does not span a $\mathbb{P}^5$; let $\mathbb{P}_S^4$ denote a $\mathbb{P}^4$ containing C_S .

So far this is a shortcut of [L]. The proof in that paper ends at this point because it is assumed that the intersection $\mathbb{P}_S^4 \cap G$ is proper, giving a curve of degree 5 ($\dim G = 6$ and the degree is 5), hence $d \leq 5$. It is not clear why the intersection should be proper, so we go on.

Let $\mathbb{P}^{v9}$ denote the hyperplanes in $\mathbb{P}^9$, and let $V \subset \mathbb{P}^{v9}$ be the set of hyperplanes that intersect G in a Schubert cycle of codimension one. Then V is the Grassmannian of planes in $\mathbb{P}^4$ via duality.

The correspondence is natural; a plane P corresponds to the Schubert cycle: {lines intersecting P }. There is a $\mathbb{P}^4 \subset \mathbb{P}^{v9}$ of hyperplanes containing $\mathbb{P}_S^4$, and since $\dim V = 6$, we find $\dim(V \cap \mathbb{P}^4) > 1$. But this implies there is a curve in V of hyperplanes containing C_S ; that is, we have found a one dimensional family of planes in our original $\mathbb{P}^4$ intersecting all the lines in the ruling of S .

Let $C_i \subset \mathbb{P}_i^2$, $i=1,2$ be two distinct plane curves we get on S from (an irreducible component of) this family. Then C_1 and C_2 are algebraically equivalent, and furthermore, they are both smooth, being sections of S .

If $\dim \mathbb{P}_1^2 \cap \mathbb{P}_2^2 = 1$, then S is contained in the $\mathbb{P}^3$ spanned by the two planes, because the C_i 's are sections. Hence, $\dim \mathbb{P}_1^2 \cap \mathbb{P}_2^2 = 0$, and we find

$$C_1^2 = C_1 \cdot C_2 = 0 \text{ or } 1.$$

Let f denote a line in the ruling, then $f^2=0$ and $C_1.f=H.f=1$.
Furthermore, for an integer r we get (numerically) $H=C_1+rf$ and
 $d=H^2=C_1^2+2r$.

Let d' denote the degree of C_1 , then

$$d' = C_1.H = C_1^2 + r = \begin{cases} \frac{d}{2} & , \text{ if } C_1^2=0 \\ \frac{d+1}{2} & , \text{ if } C_1^2=1 \end{cases}$$

By the formula for the genus of a plane curve, we get from (B.2)

$$\frac{1}{6}(d-2)(d-3) = \frac{1}{2}(d'-1)(d'-2) ,$$

leading to

$$d \in \{0, 2\} \text{ if } C_1^2=0, \text{ and } d \in \{3, 5\} \text{ if } C_1^2=1.$$

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