ALMA MATER STUDIORUM – UNIVERSITA' DI BOLOGNA
FACOLTA' DI SCIENZE M.F.N.

DOTTORATO DI RICERCA IN MATEMATICA
XIV CICLO
(a.a. 2001-2002)

Tesi di Dottorato

Some problems of projective geometry

Relatore:
Ch.mo Prof. Ph.Ellia

Co-relatore:
Ch.mo Prof. K.Ranestad

Coordinatore:
Ch.mo Prof. B.Franchi

Candidato: Stefano Brangani

Acknowledgments:

It is a pleasure to thank

Prof. Philippe Ellía (University of Ferrara), for having accepted to be my advisor, for many useful discussions, suggestions and remarks;

Prof. Kristian Ranestad (Matematisk Institutt, Oslo), for having accepted to be my coadvisor, for the suggestions, the help he gave me with invaluable discussions and for introducing me to the real joy of discovering.

Phd. Alessandro Arsie, Dott. Enrico Carlini, Dott. Alessandra Dragotto, Dott. Johannes Kleppe for their friendship, the interesting discussions and the pleasant atmosphere they created.

January 12, 2003

Contents

1 Introduction **1**

2 Multilinear maps **3**
2.1 Introduction 3
2.2 The trilinear case 4
2.3 The general case 9

3 On non-rigid projective curves **13**
3.1 Introduction 13
3.2 First order deformations 15
3.3 Finite deformations 21
3.4 Some special classes of curves in $\mathbb{P}^3$ 26

4 Rational surfaces of degree 5 and 6 in $\mathbb{P}^3$ **33**
4.1 Introduction 33
4.2 Quintics with 4 triple points 38
4.2.1 Specifying the Cremona transformation 40
4.2.2 The adjunction map technique 47
4.3 A rational sextic surface with 10 triple points 52
4.3.1 Proof of Theorem 2. 53
4.3.2 Cremona transformations 68
4.3.3 Necessary conditions 70
4.4 Appendix 1: 82
4.4.1 The complete list of candidate linear systems 82
4.4.2 Cremona transformations 85

Chapter 1

Introduction

In this thesis I am presenting the works I have done (some with co-authors) during the four years of my dottorato di ricerca at the University of Bologna-Ferrara on some problems, of very different nature, in projective algebraic geometry.

Besides this introduction, the thesis is divided into three chapters and each one contains original results. Here I will give a brief description of the content of each chapter.

Chapter 2 deals with the very first problem I have been working on: the study of possible ranks of linear maps between finite dimensional vectorspaces. The problem was proposed to E.Carlini, J.Kleppe and to me by Anthony Geramita during the first section of the PRAGMATIC 2000 summer school held in Catania. The purely algebraic nature of the problem of finding the possible values for the ranks of linear maps induced by a tensor is connected with some geometric properties of the secant varieties of Segre varieties. The trilinear case is completely solved and we give some insights in the general case.

Chapter 3 is devoted to the study of some deformation properties of projective (embedded) curves: we study the rational functorial map Φ :

$Hilb\,(d,g,r) \dashrightarrow \mathcal{M}_g$, associating to each point $p\,(C)$, representing the smooth integral curve C, the corresponding isomorphism class $[C] \in \mathcal{M}_g$. The aim of the chapter is to study in which cases the image of Φ is positivedimensional. The key idea here is to study the problem for linearly normal curves (and it turns out we have better inequalities in this case), and then to study their projections in connection with the (first order) rigidity property (see section 3.1, definitions 1, 2, 3 and 4). We start giving a result for first order deformations, then, under strong assumptions on the point $p\,(C) \in Hilb\,(d,g,r)_{red}$ we find a result for finite deformations. At the end of the chapter we apply the results found to projective curves of maximal rank, moreover we give the resolution of their ideal sheaf. The results found in this chapter appear in a work, developed with Alessandro Arsie, accepted for pubblication in "Rendiconti del Seminario Matematico e Fisico dell'Università e del Politecnico di Torino".

Finally, the last chapter of the thesis deals with the study of rational surfaces of degree 5 and 6 in $\mathbb{P}^3$ with isolated ordinary triple points. The author developed this chapter at the Matematisk Institutt of Oslo under the supervision of Prof. Kristian Ranestad: the problem investigated is that of constructing rational surfaces with isolated singular locus. In particular we deal with low degree projective surfaces (degree $d = 5, 6$) having ordinary triple points. We start studying the case of a quintic surface with four ordinary triple points, and then to the much more complicated case of a sextic surface with ten ordinary triple points. We find numerical types for the hyperplane sections of such surfaces (we take as a smooth model the blow up of $\mathbb{P}^2$, $Bl\,(\mathbb{P}^2)$) and we study the geometry of such surfaces.

Chapter 2

Multilinear maps

2.1 Introduction

Let $V_1, \ldots, V_n$ be finite dimensional k-vectorspaces, where k is any field, and let T be a multilinear map $V_1 \times \ldots \times V_n \to k$, which we will look at as an element of $V_1^* \otimes \ldots \otimes V_n^* \cong (V_1 \otimes \ldots \otimes V_n)^*$ (that is a k-linear map from $V_1 \otimes \ldots \otimes V_n$ to k). Then for any subset $I \subseteq N = \{1, \ldots, n\}$, T induces a k-linear homomorphism $T_I : \otimes_{i \in I} V_i \to \otimes_{i \notin I} V_i^*$. Of course, $T_N = T$ and $T_\emptyset : k \to \otimes_{i \in N} V_i^*$ sends 1 to T, and these maps have rank 1 as long as T is nonzero.

What are the possibilities for the other ranks $(rank T_I)$ as T varies? The purpose of this first chapter is to solve this question in the trilinear case, and to give some inequalities in the general case.

In the following, let $s_i = \dim_k V_i$ be the dimension of V_i for $i = 1, \ldots, n$. Also, for any $I \subseteq N$, let $W_I = \otimes_{i \in I} V_i$ and $I^c = N \setminus I$. Often we will just write $T_{ijk\ldots}$ for $T_{\{i,j,k,\ldots\}}$, as long as there is no confusion.

2.2 The trilinear case

Before we discuss the trilinear case in detail, we will make precise the definition and some properties of T_I. For any k-vectorspace V, let tr be the canonical trace map $V \otimes V^* \to k$, and let $\delta(V)$ be its dual map $k \to V \otimes V^*$. Also, let id be the identity map $V \to V$. Note that all these commute with tensor products, i.e. $tr(U \otimes V) = tr\,(U) \otimes tr\,(V)$ and $id(U \otimes V) = id\,(U) \otimes id\,(V)$.

Definition 1: For any $T \in W_N^*$ and $I \subset \{1, \ldots, n\}$ define T_I by

$$T_I = (T \otimes id\,(W_{I^c}^*)) \circ (id\,(W_I) \otimes \delta\,(W_{I^c}))$$

Equivalently, the following diagram commutes:

$$\begin{array}{rcl} W_I \otimes W_{I^c} \otimes W_{I^c}^* & = & W_N \otimes W_{I^c}^* \\ id\,(W_I) \otimes \delta\,(W_{I^c}) \uparrow & & \downarrow T \otimes id\,(W_{I^c}^*) \\ W_I & \stackrel{T_I}{\to} & W_{I^c}^* \end{array}$$

Remark 1: We note that $T_I^* = T_{I^c}$, hence these maps have the same rank. Moreover, for any $J \subseteq I^c$, we have

$$\begin{array}{rcl} W_I \otimes W_J \otimes W_J^* & = & W_{I \cup J} \otimes W_J^* \\ {\scriptstyle id(W_I) \otimes \delta(W_J)} \uparrow & & \downarrow_{T \otimes id(W_J^*)} \\ W_I & \stackrel{T_I}{\to} & W_{I^c}^* = W_{(I \cup J)^c}^* \otimes W_J^* \end{array}$$

since the maps are canonically defined. Alternatively, we can compute it as follows, using definition 2.2 for $T_{I \cup J}$:

$T_{I \cup J} \otimes id\,(W_J^*) = \Big((T \otimes id(W_{(I \cup J)^c}^*)) \circ (id(W_{I \cup J}) \otimes \delta(W_{(I \cup J)^c}))\Big) \otimes id\,(W_J^*) =$
$(T \otimes id\left(W_{I^c \setminus J}^*\right) \otimes id\,(W_J^*))(id\,(W_I \otimes W_J) \otimes \delta(W_{I^c \setminus J}) \otimes id\,(W_J^*)) = (T \otimes$
$id\,(W_{I^c}^*)) \circ (id\,(W_I) \otimes id\,(W_J \otimes W_J^*) \otimes \delta(W_{I^c \setminus J}))$
and since
$(id\,(W_I) \otimes id\,(W_J \otimes W_J^*) \otimes \delta(W_{I^c \setminus J})) = id\,(W_I) \otimes \delta(W_J) \otimes \delta(W_{I^c \setminus J}) = id\,(W_I) \otimes$
$\delta(W_{I^c})$ we get

$$(T_{I \cup J} \otimes id\,(W_J^*)) \circ (id\,(W_I) \otimes \delta(W_J)) = (T \otimes id\,(W_{I^c}^*)) \circ (id\,(W_I) \otimes \delta(W_{I^c}))$$

which is the definition of T_I. Note that $(I \cup J)^c = I^c \setminus J$, and that putting $J = I^c$ would give definition 2.2 back. It follows that

$$im\,(T_I) \subseteq im(T_{I \cup J} \otimes id\,(W_J^*)) = W_J^* \otimes im\,(T_{I \cup J})\,.$$

Returning to the trilinear case, we notice that the equality $rank\,(T_I) = rank\,(T_{I^c})$ implies that it is enough to study those T_I where $|I| = 1$. As the proof of the following theorem shows, the trilinear case is combinatorial in its essence.

Theorem 1 *There exists $T \in V_1^* \otimes V_2^* \otimes V_3^*$ such that $rank\,(T_i) = r_i$ if and only if*

$$r_i \leq \min\left(s_i, \prod_{j \neq i} r_j\right) \text{ for all } i.$$

Proof. By the discussion preceeding this theorem, we have $im\,(T_1) \subseteq V_2^* \otimes im\,(T_{12})$ and $im\,(T_1) \subseteq im\,(T_{13}) \otimes V_3^*$. Hence $im\,(T_1) \subseteq (V_2^* \otimes im\,(T_{12})) \cap im\,(T_{13}) \otimes V_3^* = im\,(T_{12}) \otimes im\,(T_{13})$. It follows that

$$r_1 = \dim_k im\,(T_1) \leq \dim_k (im\,(T_{12}) \otimes im\,(T_{13})) = r_{12}r_{13} = r_2r_3$$

By symmetry we have $r_i \leq \min \prod_{j \neq i} r_j$ for all i. Since $r_i \leq s_i$ is obvious, since T_i is a map from V_i, we have proven the *only if* part.

To prove the *if* part, we construct T as follows: let $X_i = \{1, \ldots, r_i\}$. We may assume that r_1 is the biggest of the r_i's. Then the only inequality that is not trivial, is $r_1 \leq r_2r_3$. This inequality ensures us that it is possible to find an injective function $f : X_1 \hookrightarrow X_2 \times X_3$. Since r_1 is biggest, it is possible to choose this function such that the compositions with the projections $X_2 \times X_3 \to X_j$ are both surjective (just fill the "diagonal" first).

Let $H = \{(x, f(x))|x \in X_1\} \subseteq X_1 \times X_2 \times X_3$ and define T by

$$T = \sum_{(a,b,c)\in H} v_{1a}^* \otimes v_{2b}^* \otimes v_{3c}^* \in V_1^* \otimes V_2^* \otimes V_3^*$$

where $\{v_{ij}^*|j = 1, \ldots, s_i\}$ is any basis for V_i^*. Now for any i, the image under T_i of any basis element v_{ij}^* with $j \in X_i$ is nonzero (and zero if $j \notin X_i$), since the projection $H \to X_i$ is surjective. Moreover, the injectivity of f implies that two distinct elements of H have at least two different coordinates, hence their images are trivially linearly independent. It follows that $rank\,(T_i) = \dim_k imT_i = r_i$, as required. □

Remark 2: Although we proved theorem 1 by choosing a basis $\{v_{ij}^*\}$ for V_i^* for any i, we only used the first r_i basisvectors by our choice of the sets X_i. This is equivalent to choosing subvectorspaces $U_i \subseteq V_i$ such that $\dim_k U_i = r_i \leq s_i = \dim_k V_i$, then construct $T \in U_1^* \otimes U_2^* \otimes U_3^*$ such that $rankT_i = r_i$ as in the proof, and then extend T to $V_1^* \otimes V_2^* \otimes V_3^*$ by zero.

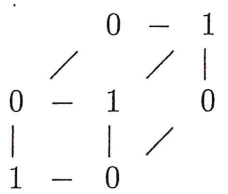
Remark 3: The "reduction" of constructing T to choosing a suitable $H \subseteq X_1 \times X_2 \times X_3$, simply means that we restrict our attention to those T's where all coefficients are either 0 or 1 (i.e. that can be represented as a $0-1$ box, see below). Since $|X_i| = r_i$, the restriction on H that all the projections $H \to X_j$ are surjective, is necessary to get $rankT_i = r_i$. The second restriction, that any two different element of H have at least two different coordinates (i.e. if $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are in H, then either $x = y$ or there exists at least two j such that $x_j \neq y_j$), is on the other hand not *necessary*. It is just a simple condition to ensure that the images of the basis elements are linearly independent, and it is a condition that is not too restrictive.

Remark 4: Theorem 1 and its proof generalize immediately to the gen-

eral case (any n) if we are only interested in those T_I where $|I| = 1$, i.e. there exists $T \in V_1^* \otimes \ldots \otimes V_n^*$ with $rank T_i = r_i$ if and only if $r_i \leq \min\left(s_i, \prod_{j \neq i} r_j\right)$ for all i.

Next we will explain how to find the matrices for the T_I's in a given basis. It was in this way we first found and proved theorem 1. Looking at a $T \in V_1^* \otimes V_2^* \otimes V_3^*$ as a box of dimension $s_1 \times s_2 \times s_3$ filled with the coefficients of T in given bases for V_1, V_2 and V_3, the matrix defining T_1 can be obtained by "collapsing" the box in the corresponding direction. This means arranging the entries of parallel planes as columnvectors in any chosen order. In other words we look at $V_2 \otimes V_3$ as a vectorspace of dimension $s_2 s_3$ and forget about T's internal structure with respect to this tensor product.

Example 1: Assume that all V_i's are two dimensional, and for $i = 1, 2, 3$ let $\{v_{ij}^* | j = 1, 2\}$ be a basis for V_i^*. Define $T = \varphi_{111} + \varphi_{121} + \varphi_{212} + \varphi_{222}$ where $\varphi_{ijk} = v_{1i}^* \otimes v_{2j}^* \otimes v_{3k}^*$. We can picture T as a $2 \times 2 \times 2$ box where the entries are the coefficients of T in the given basis:



Now the matrices representing the T's are as follows (arranging the entries of any face into a columnvector in a given order):

$$T_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \quad T_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \quad T_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Before moving on to the general case, we would like to explain the geometric interpretation of the inequalities of theorem 1. Here we will look at any nonzero T as a point in $\mathbb{P}(V_1 \otimes V_2 \otimes V_3) \cong \mathbb{P}_k^{s_1 s_2 s_3 - 1}$ (proportional T's give the same point). Note that $rank T_I = 0$ if and only if $T = 0$, hence all ranks are positive for a $T \in \mathbb{P}(V_1 \otimes V_2 \otimes V_3)$. Now using Segre embeddings we will identify the spaces $\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)$, $\mathbb{P}(V_1) \times \mathbb{P}(V_2 \otimes V_3)$, $\mathbb{P}(V_2) \times \mathbb{P}(V_1 \otimes V_3)$ and $\mathbb{P}(V_3) \times \mathbb{P}(V_1 \otimes V_2)$ with their images inside $\mathbb{P}(V_1 \otimes V_2 \otimes V_3)$. Of course, $\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)$ is contained within the intersection of the three others:

$$\begin{array}{ccccc} & & \mathbb{P}(V_1) \times \mathbb{P}(V_2 \otimes V_3) & & \\ & \nearrow & & \searrow & \\ \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3) & \rightarrow & \mathbb{P}(V_2) \times \mathbb{P}(V_1 \otimes V_3) & \rightarrow & \mathbb{P}(V_1 \otimes V_2 \otimes V_3) \\ & \searrow & & \nearrow & \\ & & \mathbb{P}(V_3) \times \mathbb{P}(V_1 \otimes V_2) & & \end{array}$$

The geometric interpretation relies on the fact that T_1 has rank 1 if and only if its matrix (in any chosen basis) can be written as a product of two nonzero matrices of sizes $s_1 \times 1$ and $1 \times s_2 s_3$ in that order. These matrices (vectors) correspond to points, one in $\mathbb{P}(V_1)$ and the other in $\mathbb{P}(V_2 \otimes V_3)$. Hence $rank T_1 = 1$ if and only if T as a point in $\mathbb{P}(V_1 \otimes V_2 \otimes V_3)$ actually lies in the subspace $\mathbb{P}(V_1) \times \mathbb{P}(V_2 \otimes V_3)$. From an idealtheoretical point of view, T_1 has rank 1 if and only if it is nonzero and all 2×2 minors of its matrix vanish, which by definition defines the image of the Segre embedding. Obviously a similar argument holds for any T_I.

Theorem 1 tells us that if any two of T_1, T_2 and T_3 have rank 1, then the third one has rank 1 too. This means that the intersection of any two of the spaces $\mathbb{P}(V_1) \times \mathbb{P}(V_2 \otimes V_3)$, $\mathbb{P}(V_2) \times \mathbb{P}(V_1 \otimes V_3)$ and $\mathbb{P}(V_3) \times \mathbb{P}(V_1 \otimes V_2)$ contains the third one. Actually the intersection of any two equals $\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)$, even idealtheoretically.

In general, T_I has rank $\leq k$ if and only if the matrix of T_I can be written as a sum of k matrices of rank 1. As we have seen, these matrices of rank one correspond to points on the image of the Segre embedding. Hence $rankT_i = k$ if and only if T is a point on the k-secant variety of $\mathbb{P}(V_1) \times \mathbb{P}\left(\bigotimes_{j\neq i} V_j\right)$.

In this setting theorem 1 tells us that if T is a point on the r-secant variety of $\mathbb{P}(V_1)\times\mathbb{P}(V_2 \otimes V_3)$ and on the s-secant variety of $\mathbb{P}(V_2)\times\mathbb{P}(V_1 \otimes V_3)$, then T sits on the t-secant variety of $\mathbb{P}(V_3) \times \mathbb{P}(V_1 \otimes V_2)$ where t is any number between $\max(\frac{r}{s}, \frac{s}{r})$ and rs. If $r = 1$, that is if $T \in \mathbb{P}(V_1) \times \mathbb{P}(V_2 \otimes V_3)$, then we get $t = s$.

Example: Continuing our example, we see that the ranks are $rankT_i = (2, 1, 2)$. Since $rankT_2 = 1$, we realize that $T \in \mathbb{P}(V_2) \times \mathbb{P}(V_1 \otimes V_3)$. Also, since the other two ranks are equal to 2, T sits on the 2-secant variety of the other two spaces. Actually, T sits on the 2-secant variety of the smallest space since T is obviously a sum of two tensors (boxes) of rank 1.

2.3 The general case

Theorem 2 *Let $T \in V_1^* \otimes \ldots \otimes V_n^* = W_N$ and $I \subseteq N$. If $J_1, \ldots, J_m \subseteq I^c$ such that $J_1 \cap \ldots \cap J_m = \emptyset$, then*

$$rankT_I \leq \prod_{j=1}^{m} rankT_{I\cup J_j}$$

Furthermore, these inequalities are generated by the ones where $m = 2$.

Proof. First we prove the inequality when $m = 2$. Letting $I' = I\backslash J_1\backslash J_2$, we note that N is the disjoint union of I, J_1, J_2 and I'. Redefining W_I, W_{J_1}, W_{J_2}, $W_{I'}$ to be V_1, V_2, V_3, V_4, we have reduced the problem to prove that $rankT_1 \leq rankT_{12}rankT_{13}$ in the tetralinear case.

Letting $r = rankT_{12}$, we can find $\varphi_1, \ldots, \varphi_r \in V_1^* \otimes V_2^*$ and $\psi_1, \ldots, \psi_r \in V_3^* \otimes V_4^*$ such that $T = \sum_{i=1}^{r} \varphi_i \otimes \psi_i$. Choosing bases $\{v_{ij}^*\}$ for all V_i's, we may write $\varphi_i = \sum_{a,b} \alpha_{ab}^i v_{1a}^* \otimes v_{2b}^*$ and $\psi_i = \sum_{c,d} \beta_{cd}^i v_{3c}^* \otimes v_{4d}^*$ and hence

$$T = \sum_{i=1}^{r} \sum_{a,b,c,d} \alpha_{ab}^i \beta_{cd}^i v_{1a}^* \otimes v_{2b}^* \otimes v_{3c}^* \otimes v_{4d}^* \tag{2.1}$$

Then for any b and d, we see that $T_{24}(v_{2b} \otimes v_{4d}) = \sum_i \sum_{a,c} \alpha_{ab}^i \beta_{cd}^i v_{1a}^* \otimes v_{3c}^*$. Let $s = rankT_{24} = rankT_{13}$ and choose $(b_1, d_1), \ldots, (b_s, d_s)$ such that imT_{24} is spanned by $\{T_{24}(v_{2b_j} \otimes v_{4d_j}) \,|\, j = 1, \ldots, s\}$. Then we can find λ_{bd}^j, such that $T_{24}(v_{2b} \otimes v_{4d}) = \sum_{j=1}^{s} \lambda_{bd}^j T_{24}(v_{2b_j} \otimes v_{4d_j})$ for all b and d. This is equivalent to

$$\sum_{i=1}^{r} \alpha_{ab}^i \beta_{cd}^i = \sum_{i=1}^{r} \sum_{j=1}^{s} \lambda_{bd}^j \alpha_{ab_j}^i \beta_{cd_j}^i \text{ for all } a,\ b,\ c \text{ and } d.$$

Putting this into 2.1 gives

$$\begin{aligned} T &= \sum_{a,b,c,d} \sum_{i=1}^{r} \sum_{j=1}^{s} \lambda_{bd}^j \alpha_{ab_j}^i \beta_{cd_j}^i v_{1a}^* \otimes v_{2b}^* \otimes v_{3c}^* \otimes v_{4d}^* = \\ &= \sum_{i,j} \left(\sum_a \alpha_{ab_j}^i v_{1a}^* \right) \left(\sum_c \beta_{cd_j}^i v_{3c}^* \right) \left(\sum_{b,d} \lambda_{bd}^j v_{2b}^* \otimes v_{4d}^* \right), \end{aligned}$$

from which it follows that $rankT_1 \leq rs = rankT_{12} rankT_{13}$.

To prove that the $m = 2$ inequalities generate all, we proceed as follows.

Let $J = J_2 \cap \ldots \cap J_m$. Then $J_1,\ J \subseteq I^c$ and $J_1 \cap J = \emptyset$, and hence $rankT_I \leq rankT_{I \cup J_1} rankT_{I \cup J}$.

Now let $J_i' = J_i \backslash J = J_i \cap J^c$ for $i = 2, \ldots, m$. Then $J_2' \cap \ldots \cap J_m' = J_2 \cap \ldots \cap J_m \cap J^c = \emptyset$ and $J_i' \subseteq I^c \cap J^c = (I \cup J)^c$. Hence by induction on m we have $rankT_{I \cup J} \leq \prod_{j=2}^{m} rankT_{I \cup J_j}$. □

Remark: Note that for $m \geq 3$ we can have $J_i \cap J_j \neq \emptyset$ even though all inequalities are generated by the subset with $m = 2$, where necessarily

$J_1 \cap J_2 = \emptyset$. Also note that the I' in the proof may in general be nonempty. If it is empty, we could also have reduced to the trilinear case, where things are very easy.

Remark: Putting $r = 1$ in the proof of theorem 2, yields a strong result when $rankT_{12} = 1$. Since in this case $T = \left(\sum_i v_{1i}^* \otimes \rho_i\right)\left(\sum_i v_{3j}^* \otimes \eta_j\right)$, we see that $T_{13}(v_{1i} \otimes v_{3j}) = \rho_i \otimes \eta_j$ and $T_{14}(v_{1i} \otimes v_{4j}) = \rho_i \otimes \left(\sum_k \eta_k (v_{4j}) v_{3k}^*\right)$. Also $T_1(v_{1i}) = \rho_i \otimes \left(\sum_k v_{3j}^* \otimes \eta_j\right)$.

It follows that $rankT_1 = rankT_2 = \dim_k \langle \rho_i \rangle$ and $rankT_3 = rankT_4 = \dim_k \langle \eta_j \rangle$, and furthermore that $rankT_{13} = \dim_k \langle \rho_i \otimes \eta_j \rangle = rankT_1 rankT_3 = rankT_{14}$.

As an example, consider the $2 \times 2 \times 2 \times 2$ case (i.e. all s_i equal to 2). If $rankT_{12} = 1$, then it follows that $rankT_1 = rankT_2 = x$, $rankT_3 = rankT_4 = y$ and $rankT_{13} = rankT_{14} = z$, and the only possibilities for (x, y, z) are $(1, 1, 1)$, $(1, 2, 2)$, $(2, 1, 2)$ and $(2, 2, 4)$.

It is an open problem if the inequalities provided by theorem 2 are all restrictions on the generic tuple $(rankT_I)$ that exist.

Chapter 3

On non-rigid projective curves

3.1 Introduction

Let k be any algebraically closed field of characteristic zero and, as usual, let $\mathbb{P}^r := Proj(k[x_0, \ldots, x_r])$ the associated projective space. Inside the Hilbert scheme $H(d, g, r)$ parametrizing closed subschemes of dimension 1, arithmetic genus g, degree d in $\mathbb{P}^r$ let us considere the so called *restricted* Hilbert scheme $Hilb(d, g, r)$, which is the subscheme of $H(d, g, r)$consisting of those points $p(C)$, such that every component K of $H(d, g, r)$ on which $p(C)$ lies has smooth, non degenerate and irreducible general element (see Definition 1.31 of [10]).

The aim of the present chapter is to get some insight in the behaviour of the rational functorial map $\phi : Hilb(d, g, r) \to \mathcal{M}_g$, which associates to each point $p(C)$ in $Hilb(d, g, r)$ representing a smooth non degenerate irreducible curve C the corresponding isomorphism class $[C] \in \mathcal{M}_g$. In particular, we study in which cases the image of ϕ has positive dimension.

Any non degenerate integral subscheme of dimension 1 C in $\mathbb{P}^r$ determines a point $p(C) \in Hilb(d, g, r)$. We then give the following:

Definition 1: *The projective curve $C \subset \mathbb{P}^r$ admits non-trivial first order*

deformations if the image of the map $D\phi : T_{p(C)}Hilb(d,g,r) \to T_{[C]}\mathcal{M}_g$ *has positive dimension (or equivalently if* $D\phi \neq 0$*). In this case we say that the corresponding curve is non-rigid at the first order, for the given embedding.*

Definition 2: *The projective curve* $C \subset \mathbb{P}^r$ *admits non-trivial deformations if it exists at least a curve* $\gamma \subset Hilb(d,g,r)$*, through* $p(C)$*, which is not contracted to a point via* ϕ*. Equivalently, if it exists an irreducible component of* $Hilb(d,g,r)$ *containing* $p(C)$*, such that its image in* $\mathcal{M}_g$ *through* ϕ *has positive dimension. In this case we say that the curve is non-rigid for the given embedding.*

We can somehow get rid of the fixed embedding in some projective space taking into account *all* possible nondegenerate embedding, as in the following:

Definition 3: *The (abstract) smooth curve* C *is non-rigid, at the first order as a smooth non degenerate projective curve if for* any *non degenerate projective embedding* $j : C\hat{E} \hookrightarrow \mathbb{P}^r$*, the corresponding map* $D\phi : T_{p(C)}Hilb(d,g,r) \to T_{[C]}\mathcal{M}_g$ *is non zero.*

Analogously, one has the following:

Definition 4: *The (abstract) smooth curve* C *is non-rigid as a smooth non degenerate projective curve if for* any *non degenerate projective embedding* $j : C \hookrightarrow \mathbb{P}^r$*, there exists an irreducible component of the associated* $Hilb(d,g,r)$ *containing* $p(C)$*, such that its image in* $\mathcal{M}_g$ *through* ϕ *has positive dimension.*

In this chapter we will prove that there exists a dense open subset $U \subset \mathcal{M}_g$ $(g \geq 1)$ such that any C, with $[C] \in U$, is non rigid at the first order as a smooth non degenerate projective curve in the sense of Definition 3; moreover, we prove that these curves are non-rigid (not only at the first order) under the additional assumption that $p(C)$ is a smooth point of $Hilb(d,g,r)_{red}$

(the restricted Hilbert scheme with reduced scheme structure) or at worst it is a singular point of $Hilb(d,g,r)_{red}$ (see Definition 5 in section 3).

3.2 First order deformations

First of all we deal with the case of smooth projective curves of genus $g \geq 2$ in $\mathbb{P}^r$. We will prove that there exists a dense open subset $U^0_{BN} \subset \mathcal{M}_g$ such that for any $[C] \in U^0_{BN}$ and for *any* non degenerate smooth embedding of C in $\mathbb{P}^r$ the corresponding projective curve is non-rigid at the first order (in the sense of Definition 3).

From the fundamental exact sequence:

$$0 \to TC \to T\mathbb{P}^r_{|C} \to N_{C/\mathbb{P}^r} \to 0, \tag{3.1}$$

taking the associated long exact cohomology sequence, since $H^0(TC) = H^0(K_C^{-1}) = 0$ (genus $g \geq 2$), we get:

$$0 \to H^0(T\mathbb{P}^r_{|C}) \to H^0(N_{C/\mathbb{P}^r}) \stackrel{D\phi}{\to} H^1(TC) \to H^1(T\mathbb{P}^r_{|C}) \to H^0(N_{C/\mathbb{P}^r}) \to 0 \tag{3.2}$$

In the sequence 3.2, as usual we identify $H^0(N_{C/\mathbb{P}^r})$ with the tangent space $T_{p(C)}Hilb(d,g,r)$ to the Hilbert scheme at the point represented by C and $H^1(TC)$ with $T_{[C]}\mathcal{M}_g$. Thus the coboundary map $D\phi$ represents the differential of the map $\phi : Hilb(d,g,r) \to \mathcal{M}_g$ we are interested in. If $D\phi = 0$ (i.e. the corresponding curve is rigid also at the first order) the sequence above splits and in particular $h^0(T\mathbb{P}^r_{|C}) = h^0(N_{C/\mathbb{P}^r})$; thus imposing $h^0(T\mathbb{P}^r_{|C}) < h^0(N_{C/\mathbb{P}^r})$ and estimating the dimension of the cohomology groups, we get a relation involving d, g, r, which, if it is fulfilled implies that the corresponding curve is not rigid (at least at the first order). This is the meaning of the following:

Proposition 1: *Let $C \subset \mathbb{P}^r$ a smooth non-degenerate curve of genus $g \geq 2$ and degree d. Then if $d > \frac{2}{r+1}[g(r-2)+3]$, or if $d > 2g-2$ ($\mathcal{O}_C(1)$ non special) then $D\phi \neq 0$. Furthemore, if $C \subset \mathbb{P}^r$ is linearly normal, then the $D\phi \neq 0$ if*

$$d > \frac{(r-2)g + r(r+1) + 3}{r+1}. \tag{3.3}$$

Proof. It is obvious from the exactness of 3.2 that if $h^0(N_{C/\mathbb{P}^r}) > h^0(T\mathbb{P}^R_{|C})$, then $D\phi \neq 0$. On the other hand, $h^0(N_{C/\mathbb{P}^r}) = dim(T_{p(C)}Hilb(d,g,r)) \geq dim(Hilb(d,g,r))$ and $dim(Hilb(d,g,r)) \geq (r+1)d - (r-3)(g-1)$, where the last inequality always holds at points of $Hilb(d,g,r)$ parametrizing locally complete intersection curves (in particular smooth curves), see for example [10]. Thus $h^0(N_{C/\mathbb{P}^r}) \geq (r+1)d - (r-3)(g-1)$. Now, applying Riemann-Roch for the vector bundle $T\mathbb{P}^r$ on C, we get $h^0(T\mathbb{P}^r_{|C}) = (r+1)d - r(g-1) + h^1(T\mathbb{P}^r_{|C})$. On the other hand, from the Euler sequence (twisted with $\mathcal{O}_C$):

$$0 \to \mathcal{O}_C \to (r+1)\mathcal{O}_C(1) \to T\mathbb{P}^r_{|C} \to 0, \tag{3.4}$$

we get immediately $h^1(T\mathbb{P}^r_{|C}) \leq (r+1)h^1(\mathcal{O}_C(1))$ and by Riemann-Roch the latter is equal to $(r+1)(h^0(\mathcal{O}_C(1)) - d + g - 1)$. Now, if $\mathcal{O}_C(1)$ is non special (i.e. if $d > 2g-2$), then $h^1(T\mathbb{P}^r_{|C}) = 0$, so that, imposing $h^0(N_{C/\mathbb{P}^r}) > h^0(T\mathbb{P}^r_{|C})$, we get $3(g-1) > 0$, which is always satisfied (if $g \geq 2$). This means that a smooth curve which is embedded via a non special linear system, is non-rigid at least at the first order.

If instead $\mathcal{O}_C(1)$ is special, by Clifford's theorem we have $h^0(\mathcal{O}_C(1)) \leq d/2 + 1$, so that $h^1(T\mathbb{P}^r_{|C}) \leq (r+1)(g - d/2)$. Imposing again $h^0(N_{C/\mathbb{P}^r}) > h^0(T\mathbb{P}^r_{|C})$, that is $(r+1)d-(r-3)(g-1) > (r+1)d-r(g-1)+(r+1)(g-d/2)$, we get the relation $d > \frac{2}{r+1}[g(r-2)+3]$.

Finally, if $C \subset \mathbb{P}^r$ is linearly normal and non degenerate, then $h^0(\mathcal{O}_C(1)) = r+1$. Substituting in $h^1(T\mathbb{P}^r_{|C}) \leq (r+1)h^1(\mathcal{O}_C(1)) = (r+1)(h^0(\mathcal{O}_C(1)) - d +$

$g-1$) and imposing the fundamental inequality, we get $d > \frac{(r-2)g+r(r+1)+3}{r+1}$.
□

Since the bound 3.3 is particularly good, but it holds only for linearly normal curves and since any curve can be obtained via a series of (generic) projections from a linearly normal curve, we are going to study what is the relation among first order deformations of a linearly normal curve and the first order deformations of its projections. This is the aim of the following:

Proposition 2: *Let $C \subset \mathbb{P}^r$ a smooth curve of genus $g \geq 2$ which is non-rigid at the first order. Then any of its smooth projections $C' := \pi_q(C) \subset \mathbb{P}^{r-1}$ from a point $q \in \mathbb{P}^r$ is non-rigid at the first order.*

Proof. First of all, let us remark that the proposition states that in the following diagram:

$$\begin{array}{ccc} T_{p(C)}Hilb(d,g,r) & \stackrel{D\phi}{\rightarrow} & T_{[C]}\mathcal{M}_g \\ \downarrow & \nearrow_{D\phi'} & \\ T_{p(C')}Hilb(d,g,r-1) & & \end{array}$$

if $Im(D\phi) \neq 0$, then $Im(D\phi') \neq 0$. Now consider the following commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & \rightarrow & ker(a) & \rightarrow & ker(b) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & TC & \rightarrow & T\mathbb{P}^r_{|C} & \rightarrow & N_{C/\mathbb{P}^r} & \rightarrow & 0 \\ & & \downarrow\cong & & \downarrow a & & \downarrow b & & \\ 0 & \rightarrow & TC' & \rightarrow & T\mathbb{P}^{r-1}_{|C'} & \rightarrow & N_{C'/\mathbb{P}^r} & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \rightarrow & 0 & \rightarrow & 0 & & \end{array}$$

where the morphisms a and b are induced by the projection of C to C'. Clearly $TC \cong TC'$, because C and C' are isomorphic curves ($q \notin Sec(C)$ and thus the projection is an isomorphism) and moreover a and b are surjective by construction. Applying the snake lemma to the previous diagram we see that $ker(a) \cong ker(b)$ and since a and b are surjective morphisms of vector

bundles, it turns out that $ker(a) = ker(b) = \mathcal{L}$, where $\mathcal{L}$ is a line bundle on C. Restricting the attention to the last column of the previous diagram, it is clear from a geometric reasoning that the line bundle $\mathcal{L}$ can be identified with the rulings of the projective cone, with vertex q through which we project. Indeed, it is sufficient to look at the projection induced map b at a point $x \in C$: $b : N_{C/\mathbb{P}^r}, x \to N_{C'/\mathbb{P}^r}, \pi(x)$; the kernel is always the ruling of the cone with vertex q going trough x and this is never a subspace of TC, because $q \notin Sec(C)$. Clearly, we can identify the projective cone with vertex q through which we project with the line bundle $\mathcal{L}$, since we can consider instead of just $\mathbb{P}^r$, the blowing-up $Bl_q(\mathbb{P}^r)$ in q in such a way to separate all the rulings in the cone (this however does not affect our reasoning since we are dealing with line bundles over C and $q \notin C$).

Applying the cohomology functor to the previous commutative diagram and recalling that $h^0(TC) = 0$ since $g \geq 2$ we get the following diagram:

$$\begin{array}{ccccccccc}
 & & H^0(\mathcal{L}) & \equiv & H^0(\mathcal{L}) & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(T\mathbb{P}^r_{|C}) & \to & H^0(N_{C/\mathbb{P}^r}) & \stackrel{D\phi}{\to} & H^1(TC) & \to & \dots \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \to & H^0(T\mathbb{P}^{r-1}_{|C'}) & \to & H^0(N_{C'/\mathbb{P}^{r-1}}) & \stackrel{D\phi'}{\to} & H^1(TC') & \to & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & coker(\alpha) & \to & coker(\beta) & \to & 0 & \to & 0
\end{array}$$

Now, $Im(D\phi) \subset H^1(TC)$ and via the isomorphism γ it is mapped inside the other $H^1(TC)$. On the other hand, by commutativity of the square having as edges the maps $\beta, \gamma, D\phi$ and $D\phi'$ it is clear that $Im(D\phi') \subseteq Im(D\phi)$ so that if $D\phi \neq 0$, then a fortiori $D\phi' \neq 0$. □

The following corollary gives two simple sufficient conditions for having $Im(D\phi) \cong Im(D\phi')$.

Corollary 1: *Let* C, C', $D\phi$and $D\phi'$as in Proposition 2. Then if $\mathcal{O}_C(1)$ is non special or if $h^1(T\mathbb{P}^r_{|C}) = 0$, then $Im(D\phi) \cong Im(D\phi')$.

Proof: First of all, rewrite the previous diagram as:

$$\begin{array}{ccccccccc}
 & & H^0(\mathcal{L}) & \cong & H^0(\mathcal{L}) & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(T\mathbb{P}^r_{|C}) & \to & H^0(N_{C/\mathbb{P}^r}) & \to & Im(D\phi) & \to & 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow & & \\
0 & \to & H^0(T\mathbb{P}^{r-1}_{|C'}) & \to & H^0(N_{C'/\mathbb{P}^{r-1}}) & \to & Im(D\phi') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & coker(\alpha) & \to & coker(\beta) & \to & coker(\alpha)/coker(\beta) & \to & 0
\end{array}$$

Observe that $coker(\alpha) \subset H^1(\mathcal{L})$ and the same is true for $coker(\beta)$. So if $H^1(\mathcal{L}) = 0$, then $Im(D\phi) = Im(D\phi')$. On the other hand, from the exact sequence $0 \to \mathcal{L} \to T\mathbb{P}^r_{|C} \to T\mathbb{P}^{r-1}_{|C'} \to 0$, taking Chern polynomials, we get that $\mathcal{L}$ is a line bundle of degree d (and one can identify $\mathcal{L}$ with $\mathcal{O}_C(1) \otimes \mathcal{L}'$ for some $\mathcal{L}' \in Pic^0(C)$). Thus, if $\mathcal{O}_C(1)$is non special we conclude. If instead $h^1(T\mathbb{P}^r_{|C}) = 0$, then $coker(\alpha) = H^1(\mathcal{L})$ and $coker(\alpha) \subseteq coker(\beta) \subseteq H^1(\mathcal{L})$ so that $coker(\alpha) = coker(\beta)$ and we conclude again. □

Now we deal with the much simpler case of curves of genus $g = 1$.

Proposition 3: *For any smooth curve* $[C] \in \mathcal{M}_1$ *and for any non degenerate projective embedding of* $C \hookrightarrow \mathbb{P}^r$, *the corresponding projective curve is non-rigid at the first order.*

Proof. From the fundamental exact sequence:

$$0 \to T_C \to T\mathbb{P}^r_{|C} \to N_{C/\mathbb{P}^r} \to 0,$$

since $T_C \cong \mathcal{O}_C$ $(g = 1)$, we obtain the long exact cohomology sequence:

$$0 \to H^0(\mathcal{O}_C) \to H^0(T\mathbb{P}^r_{|C}) \to H^0(N_{C/\mathbb{P}^r}) \stackrel{D\phi}{\to} H^1(\mathcal{O}_C) \to H^1(T\mathbb{P}^r_{|C}) \dots \quad (3.5)$$

Twisting the Euler sequence with $\mathcal{O}_C$ and taking cohomology, we have that $h^1(T\mathbb{P}^r_{|C}) \leq (r+1)h^1(\mathcal{O}_C(1))$, but $\mathcal{O}_C(1)$ is always non special for a curve of

genus $g = 1$ since $d > 2g-2 = 0$. Thus $H^1(T\mathbb{P}^r_{|C}) = 0$ and being $h^1(\mathcal{O}_C) \neq 0$, from (3.5) we have that $D\phi \neq 0$ and it is even always surjective. □

We conclude this section with the following theorem, which is the analogue of Proposition 3 for curves of genus $g \geq 2$ (in this case we can not work over all $\mathcal{M}_g$, but just on an open dense subset).

Theorem 1: *For any $g \geq 2$, there exists a dense open subset $U_{BN} \subset \mathcal{M}_g$ such that for any $[C] \in U_{BN}$ and for* any *non degenerate projective embedding of $C \subset \mathbb{P}^r$, the corresponding projective curve is non-rigid at the first order.*

Proof. According to theorem 1.8, page 216 of [1], there exists a dense open subset $U_{BN} \subset \mathcal{M}_g$ such that any $[C] \in U_{BN}$ can be embedded in $\mathbb{P}^r$ as a smooth non degenerate curve of degree d if and only if $\rho \geq 0$, where $\rho(d, g, r) := g - (r+1)(g-d+r)$ is the Brill-Noether number. Now we consider a curve $[C] \in U_{BN}$ and we embed it as a linerly normal curve $\bar{C}$ of degree d in some $\mathbb{P}^r$. Since $[C] \in U_{BN}$, we have that $\rho \geq 0$; on the other hand, $\bar{C}$ is linearly normal and the fundamental inequality (3.3) is satisfied since $\rho \geq 0$ (indeed, it is just a computation to see that (3.3) is equivalent to $\rho \geq -\epsilon$ for some $\epsilon > 0$). Thus, by Proposition 1 $\bar{C}$ is non-rigid at the first order, and moreover by Propositon 2 all of its smooth projections are non-rigid at the first order. To conclude, observe that any smooth non degenerate projective curve C such that $[C] \in U_{BN}$ can be obtained via a series of smooth projections from a linearly normal projective curve $\bar{C}$ with corresponding $\rho \geq 0$ (since for the curves in U_{BN} the Brill-Noether condition is necessary and sufficient). □

3.3 Finite deformations

Our problem is now to extend the first order deformations studied in the previous section to finite deformations. By Theorem 1, we know that, for the curves C such that $[C] \in U_{BN}$ ($g \geq 2$), the corresponding $Im(D\phi) \neq 0$ and an even stronger result holds for curves of genus $g = 1$. We need to prove that there exists a vector $v \in T_{p(C)}Hilb(d, g, r)$, corresponding to a smooth curve $\gamma \subset Hilb(d, g, r)$ through $p(C)$ such that the image of the curve via ϕ is positive dimensional. To this aim, observe that if $[C] \in U_{BN}$ is not a smooth point of $\mathcal{M}_g$, then there are $w \in T_{[C]}U_{BN}$ which are obstructed deformations, that is which do not correspond to any curve in U_{BN} through $[C]$. We can easily get rid of this problem, just by restricting further the open subset U_{BN}. Indeed, for $g \geq 1$, there is a dense open subset $U^0 \subset \mathcal{M}_g$ such that any $[C] \in U^0$ is a smooth point (see for example [10]). Thus, for curves of genus $g \geq 2$ we consider the dense open subset $U^0_{BN} := U_{BN} \cap U^0$ and for any $w \in T_{[C]}U^0_{BN}$, the corresponding first order deformations are unobstructed, while for curves of genus $g = 1$ we just restrict to the smooth part of $\mathcal{M}_1$, that we denote as U^0_1.

We can draw a first conclusion to this reasoning via the following:

Proposition 4: *Let $[C] \in U^0_{BN}$ or $[C] \in U^1_0$ and let $C \subset \mathbb{P}^r$ any projective embedding such that the corresponding point $p(C) \in Hilb(d, g, r)$ is a smooth point of the restricted Hilbert scheme. Then the projective curve $C \subset \mathbb{P}^r$ is non rigid.*

Proof. By Theorem 1 or Proposition 3, the associated map $D\phi \neq 0$, so that there exists a $w \in T_{p(C)}Hilb(d, g, r)$ such that $D\phi(w) \neq 0$. Since $p(C)$ is a smooth point of $Hilb(d, g, r)$, the tangent vector w corresponds to a smooth curve $\gamma \subset Hilb(d, g, r)$ through $p(C)$, such that $T_{p(C)}\gamma = w$. Now

consider the image Z of this curve in U^0_{BN} via ϕ. Since $\mathcal{M}_g$ exists as a quasi-projective variety, in particular we can represent a neighbourhood of $[C] \in \mathcal{M}_g$, as $Spec(B)$, for some finitely generated k-algebra B. This implies that the map ϕ can be viewed locally around $p(C)$ as a morphism of affine schemes. Thus the image of the curve γ (which is a reduced scheme) is the subscheme Z in $Spec(B)$. Then either Z is positive dimensional and in this case we are done, or it is a zero dimensional subscheme, supported at the point $[C]$; observe that this zero dimensional subscheme Z can not be the reduced point $[C]$, otherwise we would certainly have $D\phi(w) = 0$. So let us consider the case in which Z is a zero dimensional subscheme, supported at the point $[C]$, with non-reduced scheme structure: this case is clearly impossible since the image Z of a reduced subscheme (the curve γ) via the morphism of affine schemes ϕ can not be a non-reduced subscheme. Indeed, if it were the case, consider the restriction of ϕ to γ: $\phi\gamma$, $Z_{red} = [C]$; then $\phi\gamma^{-1}([C])$ is a reduced subscheme, which coincides with γ, since γ is reduced. But this would imply that $\phi(\gamma) = [C]$ and $D\phi(w) = 0$.

Thus, it turns out Z has necessarily positive dimension and we conclude. □

The hypothesis of Proposition 4, according to which $p(C)$ is a smooth point of $Hilb(d, g, r)$ is extremely strong. Ideally, one would like to extend the result of Proposition 4 to *any* non degenerate projective embeddings for curves $[C] \in U^0_{BN}$. Before giving a partial extension of Proposition 4 (Theorem 2) let us give the following:

Definition 5: *A point $p(C) \in Hilb(d, g, r)_{red}$ is called a reducible singularity if it is in the intersection of two or more irreducible components of $Hilb(d, g, r)_{red}$, each of which is smooth in $p(C)$.*

Theorem 2: *Let* $[C] \in U^0_{BN}$ *or* $[C] \in U^1_0$ *and let* $C \subset \mathbb{P}^r$ *any projective embedding such that the corresponding point* $p(C) \in Hilb(d,g,r)$ *is a smooth point of* $Hilb(d,g,r)_{red}$ *(restricted Hilbert scheme with reduced structure) or such that* $p(C)$ *is a reducible singularity of* $Hilb(d,g,r)_{red}$. *Then the projective curve* $C \subset \mathbb{P}^r$ *is non rigid.*

Proof. Let us consider the exact sequence:

$$0 \to H^0(T\mathbb{P}^r_{|C}) \to T_{p(C)}Hilb(d,g,r) \stackrel{D\phi}{\to} T_{[C]}\mathcal{M}_g \tag{3.6}$$

from which $ker(D\phi) \cong H^0(T\mathbb{P}^r_{|C})$. Take the reduced scheme $Hilb(d,g,r)_{red}$ and consider the induced morphism of schemes $r : Hilb(d,g,r)_{red} \to Hilb(d,g,r)$ (see for example [?] exercise 2.3, page 79).)

If $p(C)$ is a smooth point of $Hilb(d,g,r)_{red}$, then $dim(Hilb(d,g,r)) = dim(T_{p(C)}Hilb(d,g,r)_{red})$. On the other hand, to prove that there are first order deformations we have just imposed $h^0(T\mathbb{P}^r_{|C}) < dim(Hilb(d,g,r))$. Now, we want to prove that in the following diagram

$$\begin{array}{ccccc} 0 & \to & H^0(T\mathbb{P}^r_{|C}) & \to & T_{p(C)}Hilb(d,g,r) \stackrel{D\phi}{\to} T_{[C]}\mathcal{M}_g \\ & & & & \uparrow Dr \quad \nearrow \\ & & & & T_{p(C)}Hilb(g,d,r)_{red} \end{array}$$

the map Dr is injective, so that since $h^0(T\mathbb{P}^r_{|C}) < dim(Hilb(d,g,r)) = dim(T_{p(C)}Hilb(d,g,r)_{red})$, we can find a $w \in T_{p(C)}Hilb(d,g,r)_{red}$ the image of which in $T_{[C]}\mathcal{M}_g$ is non zero and then we can argue as in the proof of Proposition 4. Setting $Hilb(d,g,r)_{red} = X_{red}$, $p(C) = x$ and $Hilb(d,g,r) = X$, we have to prove that given $r : X_{red} \to X$, the associated morphism on tangent spaces is injective $Dr : T_xX_{red} \to T_xX$. Since X_{red} is a scheme, we can always find an open affine subscheme U_{red} of X_{red} containing x such that $U_{red} = Spec(A_{red})$, where A_{red} is a finitely generated k-algebra without nilpotent elements and the closed point x corresponds to a maximal

ideal m_x. Recall that, from the point of view of the functor of points, the closed point x corresponds to a morphism $\lambda : Spec(k) \to Spec(A_{red})$ (which is induced by $A_{red} \to A_{red,m_x} \to A_{red,m_x}/m_x A_{red,m_x} = k(x) = k$, where A_{red,m_x} is the localization of A_{red} at the maximal ideal m_x). Recall also that via the algebra map $k[\epsilon]/\epsilon^2 \to k$ and the corresponding inclusion of schemes $i : Spec(k) \to Spec(k[\epsilon]/\epsilon^2)$, $T_x X_{red}$ can be identified with $\{u \in Hom(Spec(k[\epsilon]/\epsilon^2, Spec(A_{red}))) \text{ such that } u \circ i = \lambda\}$. Clearly, an analogous description also holds for X and $T_x X$, (we denote the corresponding neighbourhood of x in X as $Spec\,(A)$). From the description of $T_x X_{red}$ just given, it turns out any $w \in T_x X_{red}$, $w \neq 0$, corresponds to a (non-zero) ring homomorphism $u^\natural : A_{red} \to k[\epsilon]/\epsilon^2$ such that the following diagram is commutative:

$$\begin{array}{ccc} A_{red} & \xrightarrow{u^\natural} & k[\epsilon]/\epsilon^2 \\ & {\scriptstyle \lambda^\natural}\searrow & \downarrow i^\natural \\ & & k\,(x) = k \end{array}$$

On the other hand, saying that $Dr\,(w) \neq 0$ is equivalent to say that we can lift the non zero ring homomorphism $u^\natural : A_{red} \to k[\epsilon]/\epsilon^2$ to a non zero ring homomorphism $\tilde{u}^\natural : A \to k[\epsilon]/\epsilon^2$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\tilde{\lambda}^\natural} & k\,(x) = k \\ {\scriptstyle r^\natural}\downarrow & {\scriptstyle \tilde{u}^\natural}\searrow & \uparrow i^\natural \\ A_{red} & \xrightarrow{u^\natural} & k[\epsilon]/\epsilon^2 \\ & {\scriptstyle \lambda^\natural}\searrow & \downarrow i^\natural \\ & & k\,(x) = k \end{array}$$

It is clear that we can always do such a lifting, since the homomorphisms $\tilde{u}^\natural$ and $\tilde{\lambda}^\natural$ are just given precomposing the corresponding homomorphisms from A_{red}, with $r^\natural$. Moreover, since $r^\natural$ is a non-zero ring homomorphism, it turns out that if $u^\natural \neq 0$, then also $\tilde{u}^\natural \neq 0$ and the previous diagram is commutative. This implies that $Dr\,(w) \neq 0$ and thus that $Dr : T_{p(C)} Hilb\,(d, g, r)_{red} \hookrightarrow$

$T_{p(C)}Hilb\,(d,g,r)$ is injective. Reasoning as in the proof of Proposition 4, we can find a curve $\gamma \subset Hilb\,(d,g,r)_{red}$ through $p\,(C)$ in such a way that $D\phi \circ Dr\left(T_{p(C)}\gamma\right) \neq 0$. Thus the image of this curve via $\phi \circ r$ contains the point $[C]$ in U^0_{BN} and a tangent direction. On the other hand the image via $\phi \circ r$ of a reduced scheme can not be a non reduced point (always because we can represent a neighbourhood of $[C]$ in $\mathcal{M}_g$ as an affine scheme and consider $\phi \circ r$ locally as a morphism of affine schemes). Thus the image of γ through $\phi \circ r$ must have positive dimension and in this way we conclude if $p\,(C)$ is a smooth point of $Hilb\,(d,g,r)_{red}$.

Finally, if $p(C)$ is a singular point of $Hilb(d,g,r)_{red}$, it will be sufficient to repeat the previous reasoning, substituting $T_{p(C)}Hilb(d,g,r)_{red}$, with $T_{p(C)}H$, where H is an irreducible component of $Hilb(d,g,r)_{red}$ through $p\,(C)$, smooth at $p\,(C)$ and of maximal dimension, so that $\dim_{p(C)} H = \dim_{p(C)} Hilb(d,g,r)_{red} = \dim_{p(C)} Hilb(d,g,r)$. In the same way, one can find a smooth curve $\gamma \subset H$, through $p\,(C)$, such that its image in $\mathcal{M}_g$ is positive dimensional, arguing again as in the proof of Proposition 4 (the image of γ has to be a reduced scheme, hence necessarily positive dimensional, in order to have $D\phi \neq 0$). □

Remark: If $p\,(C)$ is a reducible singularity of $Hilb\,(d,g,r)_{red}$, for the Theorem 2 to work, it is not necessary that *all* irreducible components of $Hilb\,(d,g,r)_{red}$ through $p\,(C)$ are smooth in a neighbourhood of $p\,(C)$. Indeed, from the proof of Theorem 2, it is clear that it is sufficient that there exists an irreducible component of maximal dimension H of $Hilb\,(d,g,r)_{red}$, which is smooth in a neighbourhood of $p\,(C)$.

In the light of the previous theorem, let us discuss Mumford's famous example of a component of the restricted Hilbert scheme which is non reduced (see [11]). He considered smooth curves C on smooth cubic surfaces S in $\mathbb{P}^3$,

belonging to the complete linear system $|4H + L|$, where H is the divisor class of an hyperplane section and L is the class of a line on S. It is immediate to see that the degree of such a curve is $d = 14$ and that its genus is $g = 24$. Therefore we are working with $Hilb\,(14, 24, 3)$. In [11], it is proved that the sublocus J_3 of $Hilb\,(14, 24, 3)$ cut out by curves C of this type, is dense in a component of the Hilbert scheme. Moreover, it turns out that this component is non-reduced. Indeed, Mumford showed that the dimension of $Hilb\,(14, 24, 3)$ at the point $p\,(C)$ representing a curve C of the type just described, is 56, while the dimension of the tangent space to $Hilb\,(14, 24, 3)$ at $p\,(C)$ is 57. On the other hand, it is proved in [3] that for the points of type $p\,(C)$ an infinitesimal deformation (i.e. a deformation over $Spec\,(k[\epsilon]/\epsilon^2)$) is either obstructed at the second order (i.e. you can not lift the deformation to $Spec\,(k[\epsilon]/\epsilon^3)$), or at no order at all. This implies that the corresponding component of $Hilb\,(14, 24, 3)_{red}$ is smooth. Since for curves of this type we have $d > \frac{g+3}{2}$, by Proposition 1 we know that $D\phi \neq 0$. If $[C] \in \mathcal{M}_g$ is a smooth point, then by Theorem 2, being $Hilb\,(14, 24, 3)_{red}$ smooth at $p\,(C) \in J_3$, we have that the curve $C \hookrightarrow \mathbb{P}^3$ is non rigid for the given embedding.

For other interesting examples of singularities of Hilbert schemes of curves and related constructions, see [5], [9], [2] and [13].

3.4 Some special classes of curves in $\mathbb{P}^3$

In this section, we take into account some special classes of curves and prove that they are non-rigid at the first order. As a first example, let us consider projectively normal curve C in $\mathbb{P}^3$, which does not lie on a quadric or on a cubic. We prove the curves of this class are non-rigid at the first order. Then their ideal sheaf has a resolution of the type (with $a_j \geq 4$ and consequently

$b_j \geq 5$):

$$0 \to \oplus_{j=1}^{s} \mathcal{O}_{\mathbb{P}^3}(-b_j) \to \oplus_{j=1}^{s+1} \mathcal{O}_{\mathbb{P}^3}(-a_j) \to \mathcal{I}_C \to 0,$$

from which, twisting with $T\mathbb{P}^3$, we get:

$$0 \to \oplus_{j=1}^{s} T\mathbb{P}^3(-b_j) \to \oplus_{j=1}^{s+1} T\mathbb{P}^3(-a_j) \to T_{\mathbb{P}^3} \otimes \mathcal{I}_C \to 0. \tag{3.7}$$

On the other hand, from the Euler sequence (suitably twisted) we have that $h^0(T\mathbb{P}^3(-k)) = 0$ and $h^1(T\mathbb{P}^3(-k)) = 0$ for $k \geq 4$. Thus, from (3.7) it follows that $h^0(T\mathbb{P}^3 \otimes \mathcal{I}_C) = 0$. Moreover, $H^2(T\mathbb{P}^3(-b_j))$ is equal by Serre duality to $H^1(\Omega^1_{\mathbb{P}^3}(b_j - 4))^*$ and this is zero by Bott formulas, since we assumed $b_j \geq 5$. Therefore, again from (3.7), it follows that $h^1(T\mathbb{P}^3 \otimes \mathcal{I}_C) = 0$. Finally, from the defining sequence of C, twisting by $T\mathbb{P}^3$, we get that $H^0(T\mathbb{P}^3) \cong H^0(T\mathbb{P}^3|_C)$. Now, $h^0(T\mathbb{P}^3) = 15$, so that $D\phi \neq 0$ as soon as $15 < 4d$ (recall that $h^0(N_{C/\mathbb{P}^3}) \geq 4d$), that is $D\phi \neq 0$ for $d \geq 4$.

Now we consider a projectively normal curve which sits on a smooth cubic surface S in $\mathbb{P}^3$ and prove that this curve is non-rigid at the first order. From the exact sequence:

$$0 \to N_{C/S} \to N_{C/\mathbb{P}^3} \to N_S|_C \to 0, \tag{3.8}$$

since $N_S|_C \cong \mathcal{O}_C(3)$ and $N_{C/S} \cong \omega_C \otimes \omega_S^{-1} \cong \omega_C(1) \cong \mathcal{O}_C(C)$, we get $\chi(N_{C/\mathbb{P}^3}) = \chi(\omega_C(1)) + \chi(\mathcal{O}_C(3))$. By Riemann-Roch $\chi(\mathcal{O}_C(3)) = 3d - g + 1$ and by Serre duality $h^1(\omega_C(1)) = h^0(\mathcal{O}_C(1)) = 0$, so that by Riemann-Roch $\chi(\omega_C(1)) = C^2 + 1 - g$ and $\chi(N_{C/\mathbb{P}^3}) = 3d - g + 1 + h^0(\omega_C(1)) = 3d - 2g + 2 + C^2$. Again from the sequence (3.8), taking cohomology, we have that $h^1(N_{C/\mathbb{P}^3}) = h^1(\mathcal{O}_C(3))$. On the other hand, from the exact sequence:

$$0 \to \mathcal{I}_C(3) \to \mathcal{O}_{\mathbb{P}^3}(3) \to \mathcal{O}_C(3) \to 0,$$

assuming that C is projectively normal and that it sits on a unique cubic, we have $1 - 20 + h^1(\mathcal{O}_C(3)) + 3d - g + 1 = 0$, so that $h^1(N_{C/\mathbb{P}^3}) = 18 - 3d + g$.

Thus $h^0(N_{C/\mathbb{P}^3}) = \chi(N_{C/\mathbb{P}^3}) + h^1(N_{C/\mathbb{P}^3}) = 20 - g + C^2$. As a remark, notice that since $h^0(N_{C/\mathbb{P}^3}) \geq 4d$, we obtain the inequality $4d \leq 20 - g + C^2$ for curves of this type. To give an estimate of $h^0(T\mathbb{P}^3_{|C})$, we use as before the Riemann-Roch Theorem and the Euler sequence, so that $h^0(T\mathbb{P}^3_{|C}) \leq 4d + 3(1-g) + 4h^1(\mathcal{O}_C(1))$. On the other hand, from the defining sequence of C twisted by $\mathcal{O}_{\mathbb{P}^3}(1)$, assuming C projectively normal and nondegenerate, we get $h^1(\mathcal{O}_C(1)) = g - d + 3$, so that $h^0(T\mathbb{P}^3_{|C}) \leq g + 15$. Thus $D\phi \neq 0$ as soon as $g + 15 < 20 - g + C^2$. Using adjunction formula, i.e. $C.(C + K_S) = 2g - 2$, we can rewrite this as $C.K_S < 3$. Now, since S is a smooth cubic $K_S \equiv -H$ where H is an effective divisor representing a hyperplane section. Moreover any C is linearly equivalent to $al - \sum b_i e_i$ and $h \equiv 3l - \sum e_i$ (we identify S with $\mathbb{P}^2$ blown-up in 6 different points), so that $D\phi \neq 0$ as soon as $3a - \sum b_i > 3$, but $3a - \sum b_i = d$, and so we get the condition $d \geq 4$.

Finally , as an example we consider the case of projectively normal curves on a smooth quadric Q, proving that these curves are non-rigid at the first order (indeed it is sufficient to assume that $h^1(\mathcal{I}_C(2)) = 0$). First of all, from the sequence:

$$0 \to N_{C/Q} \to N_{C/\mathbb{P}^3} \to N_Q|_C \to 0,$$

being $N_{C/Q} \cong \omega_C(2)$ and $N_Q|_C \equiv \mathcal{O}_C(2)$, we have that $h^1(N_{C/\mathbb{P}^3}) = h^1(\mathcal{O}_C(2))$; from the defining sequence $0 \to \mathcal{I}_C(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_C(2) \to 0$, since we assumed $h^1(\mathcal{I}_C(2)) = 0$, we have $1 - 10 + h^1(\mathcal{O}_C(2)) + 2d - g + 1 = 0$. Moreover, again by Serre duality and Kodaira vanishing $h^1(\omega_C(2)) = h^1(N_{C/Q}) = 0$ so that $h^0(N_{C/\mathbb{P}^3}) = \chi(\omega_C(2)) + \chi(\mathcal{O}_C(2)) + h^1(N_{C/\mathbb{P}^3})$ and this is equal to $10 - g + C^2$. The previous estimate for $h^0(T\mathbb{P}^3_{|C})$ works also in this case (we just used the fact that C is linearly normal and non degenerate), so that $D\phi \neq 0$ as soon as $g + 15 < 10 - g + C^2$. By adjunction $2g - 2 = C.(C + K_Q)$, and by the fact that $K_Q \equiv -2H$ this last inequality can be rewritten as

$2C.H > 7$ so that for $d \geq 4$ C is non rigid at the first order for the given embedding and so they are non-rigid (Theorem 2 and [6]).

Let us take into account the wider class of curves of maximal rank in $\mathbb{P}^3$. By definition a curve C is of maximal rank iff $h^0(\mathcal{I}_C(k))\, h^1(\mathcal{I}_C(k)) = 0$ for any $k \in \mathbb{Z}$. Since we have already dealt with projectively normal curves, from now on we assume that C is a smooth irreducible curve of maximal rank, which is not projectively normal. As usual, let $s := \min\{k/h^0(\mathcal{I}_C(k)) \neq 0\}$ be the postulation index of C. Observe that $h^1(\mathcal{I}_C(k)) = 0$ for any $k \geq s$, since C is of maximal rank. Thus, having set $c(C) := \max\{k/h^1(\mathcal{I}_C(k)) \neq 0\}$, we have that $c(C) \leq s-1$ ($c(C)$ is called the completeness index).

As a first case, let us consider $c = s-2$ and assume $h^1(\mathcal{O}_C(s-2)) = 0$ (which is certainly satisfied if $d(2-s)+2g-2 < 0$ or equivalently $d > \frac{2g-2}{s-2}$, $s \geq 3$). Observe that in this case C is s-regular, i.e. $h^i(\mathcal{I}_C(s-i)) = 0$ for any $i > 0$. Indeed, from the defining sequence of C, we have that $h^1(\mathcal{O}_C(k)) = h^2(\mathcal{I}_C(k))$ and since $h^1(\mathcal{O}_C(s-2)) = 0$, we are done. Set $u := h^0(\mathcal{I}_C(s))$. Then, if

$$0 \to \oplus\mathcal{O}_{\mathbb{P}^3}(-n_{3i}) \to \oplus\mathcal{O}_{\mathbb{P}^3}(-n_{2i}) \to \oplus\mathcal{O}_{\mathbb{P}^3}(-n_{1i}) \to \mathcal{I}_C \to 0$$

is the minimal free resolution of $\mathcal{I}_C$, setting $n_j^+ := \max\{n_{ji}\}$ and $n_j^- = \min\{n_{ji}\}$, it is easy to see that $n_3^+ = c+4 = s+2$. Moreover, we have $n_3^+ > n_2^+ > n_1^+$, $n_3^- > n_2^- > n_1^-$ and also $n_3^+ = s+2 > n_2^+ > n_1^- = s$. From these we get $n_2^+ = n_2^- = s+1$, that is $n_{2i} = s+1$ for any i. Thus, in this case, the minimal free resolution is

$$0 \to y\mathcal{O}_{\mathbb{P}^3}(-s-2) \to x\mathcal{O}_{\mathbb{P}^3}(-s-1) \to u\mathcal{O}_{\mathbb{P}^3}(-s) \to \mathcal{I}_C \to 0 \qquad (3.9)$$

(resolution of the first kind), where $y = h^1(\mathcal{I}_C(c)) = h^1(\mathcal{I}_C(s-2))$. If we have a resolution of the first kind we can split it as follows:

$$0 \to y\mathcal{O}_{\mathbb{P}^3}(-s-2) \to x\mathcal{O}_{\mathbb{P}^3}(-s-1) \to E \to 0 \qquad (3.10)$$

$$0 \to E \to u\mathcal{O}_{\mathbb{P}^3}(-s) \to \mathcal{I}_C \to 0 \tag{3.11}$$

where E is a locally free sheaf (indeed, if it were free, then C would be projectively normal by (3.11). Twisting (3.10) and (3.11) by $T\mathbb{P}^3$ and taking cohomology, we get:

$$0 \to uH^0\left(T\mathbb{P}^3(-s)\right) \to H^0\left(\mathcal{I}_C \otimes T\mathbb{P}^3\right) \to H^1\left(E \otimes T\mathbb{P}^3\right) \to \dots \tag{3.12}$$

$$\dots \to xH^1\left(T\mathbb{P}^3(-s-1)\right) \to H^1\left(E \otimes T\mathbb{P}^3\right) \to yH^2\left(T\mathbb{P}^3(-s-2)\right) \to \dots \tag{3.13}$$

On the other hand, in the sequence (3.13), $h^1\left(T\mathbb{P}^3(-s-1)\right) = h^2\left(\Omega^1_{\mathbb{P}^3}(s-3)\right) = 0$ by Serre duality and Bott formulas, while $h^2\left(T\mathbb{P}^3(-s-2)\right) = h^1\left(\Omega^1_{\mathbb{P}^3}(s-2)\right) = 0$, if $s \geq 3$. Thus, we get that if $s \geq 3$, then $H^1\left(E \otimes T\mathbb{P}^3\right) = 0$. Moreover, twisting the Euler sequence of C with $\mathcal{O}_{\mathbb{P}^3}(-s)$, we obtain that $h^0\left(T\mathbb{P}^3(-s)\right) = 0$ as soon as $s \geq 2$. Therefore, from the sequence (3.12), we have that $h^0\left(\mathcal{I}_C \otimes T\mathbb{P}^3\right) = 0$ as soon as $s \geq 3$.

Now, twisting the defining sequence of C by $T\mathbb{P}^3$ and taking cohomology, we get (assuming $s \geq 3$):

$$0 \to H^0\left(T\mathbb{P}^3\right) \to H^0\left(T\mathbb{P}^3_{|C}\right) \to H^1\left(\mathcal{I}_C \otimes T\mathbb{P}^3\right) \to 0 \tag{3.14}$$

We want to give an estimate to $h^1\left(\mathcal{I}_C \otimes T\mathbb{P}^3\right)$. Continuing the long exact cohomology sequence (3.12), using again Serre duality an Bott formulas and assuming $s \geq 5$, we get that $h^1\left(\mathcal{I}_C \otimes T\mathbb{P}^3\right) = h^2\left(E \otimes T\mathbb{P}^3\right)$. Moreover, going on with the sequence (3.13), applying Serre duality and Bott formulas ($s \geq 5$), we obtain $h^2\left(E \otimes T\mathbb{P}^3\right) = \frac{ys(s-1)(s-3)}{2}$ and from (3.14) we get $h^0\left(T\mathbb{P}^3\right) \leq 15 + \frac{ys(s-1)(s-3)}{2}$, $s \geq 5$. Thus, if $4d > 15 + \frac{ys(s-1)(s-3)}{2}$, or equivalently $d \geq 4 + \frac{ys(s-1)(s-3)}{8}$, $s \geq 5$, then a curve C of maximal rank, with a resolution of the first kind and with $h^1\left(\mathcal{O}_C(s-2)\right) = 0$, is non rigid at the first order for the given embedding.

As a final example, let us consider a curve C of maximal rank, such that $h^0(\mathcal{I}_C(s)) \leq 2$ and $h^1(\mathcal{O}_C(s-3)) = h^1(\mathcal{O}_C(s-2)) = h^1(\mathcal{O}_C(s-1)) = h^1(\mathcal{O}_C(s)) = 0$ (this happens for example if $d > \frac{2g-2}{s}$ and assuming $s \geq 4$). In this case, we have $c(C) = s-1$. Indeed, if it were $c < s-2$, then C would be $(s-1)$ −regular and this contradicts the fact that s is the postulation. Moreover, if it were $c = s-2$, then C would be s−regular and since $h^0(\mathcal{I}_C(s)) \leq 2$, C would be a complete intersection of type (s,s), and in particular it would be projectively normal.

Thus, $c(C) = s-1$ and from the given hypotheses, the fact that $h^2(\mathcal{I}_C(s-1)) = h^1(\mathcal{O}_C(s-1)) = 0$, and $h^1(\mathcal{I}_C(s)) = 0$ (since $c(C) = s-1$), it is easy to see that C is $(s+1)$ −regular. This implies that the homogeneous ideal $I(C)$ is generated in degree less or equal to $s+1$. With notations as above, we have $n_3^+ = c+4 = s+3 > n_2^+ > n_1^+ = s+1$, where the last equality holds since $I(C)$ is generated in degree less or equal to $s+1$. From this, we get $n_2^+ = s+2$ and moreover $n_2^- > n_1^- = s$, so that $n_2^- \geq s+1$. On the other hand, we can say more, because the map $H^0(\mathcal{I}_C(s)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathcal{I}_C(s+1))$ is injective; indeed, $h^0(\mathcal{I}_C(s)) \leq 2$ and from a relation of the form $H_1F_s = H_2F_s'$ between the two generators in degree s, we would have that $H_1|F_s'$ but this is clearly impossible. It turns out that we have no relations in degree $(s+1)$ between the generators of $I(C)$. Thus $n_2^- > s+1$, $n_3^- > n_2^- \geq s+2$, so that $n_{3i} = s+3$ for any i and also $n_{2i} = s+2$ for any i.

Hence, in this case, the minimal free resolution of $\mathcal{I}_C$ is the following:

$$0 \to v\mathcal{O}_{\mathbb{P}^3}(-s-3) \to x\mathcal{O}_{\mathbb{P}^3}(-s-2) \to w\mathcal{O}_{\mathbb{P}^3}(-s-1) \oplus u\mathcal{O}_{\mathbb{P}^3}(-s) \to \mathcal{I}_C \to 0 \tag{3.15}$$

(resolution of the second kind), where $v = h^1(\mathcal{I}_C(c)) = h^1(\mathcal{I}_C(s-1))$. In this case, that is under the hypotheses that $h^1(\mathcal{O}_C(k)) = 0$ fo $k = s, s-1, s-2, s-3$ (which is satisfied if for example $d > \frac{2g-2}{s}$, $s \geq 4$), $h^0(\mathcal{I}_C(s)) \leq 2$

$(c = s - 1)$, assuming $s \geq 5$ and arguing as in the previous case, starting from the sequence (3.15), we get that C is non rigid at the first order for the given embedding as soon as $d \geq 4 + \frac{vs(s-2)(s+1)}{8}$. We leave to the interested reader the details of this case, which is completely analogous to the previous one.

Chapter 4

Rational surfaces of degree 5 and 6 in $\mathbb{P}^3$

4.1 Introduction

In this chapter we are dealing with the classical problem of constructing rational surfaces in $\mathbb{P}^3$ with isolated singularities. More precisely, we deal with rational surfaces with isolated triple points: quintic surfaces with 4 ordinary triple points and sextic surfaces with 10 ordinary triple points.

Ordinary triple point means that blowing-up once the singularity we get a smooth surface and that the exceptional divisor over the triple point is a smooth elliptic curve of self intersection -3 (see [14], pages 636-646).

If $S \subseteq \mathbb{P}^3$ is any surface with ordinary isolated singularities and $\tilde{S}$ is the proper transform of S in $\mathbb{P}^3$ blown-up in $\mathrm{Sing}(S)$, then $\tilde{S}$ will be a smooth model of S but not minimal in general: blowing down the (-1) curves we find the **minimal model** of $\tilde{S}$. We can take $Bl\,(\mathbb{P}^2)$ to be this smooth model, hence we have a diagram

$$\begin{array}{ll} \tilde{S} & \xrightarrow{\psi} S \subseteq \mathbb{P}^3 \\ \varphi \searrow & \\ & \mathbb{P}^2 \end{array}$$

where φ contracts the (-1) curves and ψ is the minimal desingularization contracting the exceptional divisors to $\mathrm{Sing}(S)$.

The hyperplane serie of such surfaces can be given by specifying a linear system of plane curves, having the right dimension, passing (with multiplicity) through a set of points in the plane in special position.

More precisely, our task will be to exhibit a collection of points p_i, $i = 1, \ldots, N$ and a linear system $|H|$ of curves on $Bl(\mathbb{P}^2)$ in the explicit form $H = al - \sum_1^N b_i e_i$ where $Bl(\mathbb{P}^2)$ is the plane blown up in these points with exceptional divisors e_i and l denotes the pullback of a line. By this we need:
(i) to give the collection of numerical data a, b_i, $i = 1, \ldots, N$ representing the **numerical type** of $|H|$;
(ii) to specify the choice of points p_i's which is necessary for $|H|$ to define a birational morphism $\varphi_{|H|} : Bl(\mathbb{P}^2) \dashrightarrow \mathbb{P}^3$, with $\mathrm{Im}(\varphi_{|H|}) = S$.

One thing to point out is that rational surfaces having isolated singularities exist. As an example consider a surface $S \subseteq \mathbb{P}^3$ of degree d with a $(d-1)$-uple point and call H a presentation of its hyperplane section. S is clearly rational, a smooth model for S is the blow-up of $\mathbb{P}^2$ in the set of the $d(d-1)$ points of intersection of two smooth plane curves C and C' of degrees $d, d-1$ respectively. In the previous notations, H on the smooth model $Bl(\mathbb{P}^2)$ is given by $\left| dl - \sum_1^{d(d-1)} e_i \right|$ and it is immediate to observe that the curve C' is contracted by H to the singular point of S in $\mathbb{P}^3$.

This example reveals some of the difficulties we will find from the next section: once we have a guess for the **numerical type** of $|H|$, we need to find the special configuration of the points we are blowing-up ensuring $|H|$ is not empty and has projective dimension 3 (in this particular example the $d(d-1)$ points in $C \cap C'$ must be a complete intersection).

Following the program so far exposed, in section 2 we work out the case of quintics: we find the linear system $|H|$ using a Cremona transformation, we give an expression of the hyperplane serie $|H|$ over the smooth model $Bl(\mathbb{P}^2)$ for a generic quintic surface; moreover, once $|H|$ is given, we specify the conditions on the 16 points we are blowing-up in order to have a birational morphism $Bl(\mathbb{P}^2) \overset{|H|}{\dashrightarrow} \mathbb{P}^3$ with image a quintic with four ordinary triple points not on a plane. Then, after a brief introduction to the main properties of the adjunction map, we find the numerical type of $|H|$ in an alternative way applying the adjunction map technique.

From section 4.3 on we deal with sextic surfaces with 10 ordinary triple points not on a quadric: here the Cremona transformation does not work and we find the numerical type of $|H|$ using the adjunction map.

We will often refer to the work of Endrass, Persson and Stevens (see [15]): the authors studied surfaces in $\mathbb{P}^3$ having triple points, and give a classification for degree 5 and 6 surfaces. They proved that degree 6 surfaces can have at most 10 triple points; moreover, by specializing pencils of surfaces, they also showed the existence of sextic surfaces with $\upsilon = 1, \ldots, 10$ triple points, and that in case $\upsilon = 10$ the sextic is rational.

Our strategy to find $|H|$ will be the following:

i) apply the adjunction map and iterate the process of adjunction starting from the smooth model $\tilde{S}$ of the singular sextic until we get a smooth rational surface with low genus and degree: these surfaces are classified and we have a presentation of their hyperplane serie. We will find more than one candidate surface;

ii) reconstruct all the possible numerical types of the hyperplane series $|H|$ on $\tilde{S}$ we get from i): we will find a long list (see Appendix 1);

iii) extract, from the list found in ii), using some necessary conditions, a

smaller list of possible numerical types.

The necessary conditions we are talking about in iii) follow from the presence on $\tilde{S}$ of rigid configurations of (-1) curves: meaning they have precise degrees and fixed mutual intersections.

From the very beginning of our search for $|H|$ we assume that the triple points of the surface S come from the contraction of smooth elliptic curves E_i with selfintersection -3 on the smooth model $\tilde{S}$ of S.

The existence of a birational morphism $\varphi : \tilde{S} \to \mathbb{P}^3$ mapping $\tilde{S}$ to a sextic surface with ten triple points is clearly equivalent to the following necessary conditions involving the ten elliptic curves E_i contracted and the presentation of the hyperplane series H giving the map φ:

$$\begin{cases} \mathcal{O}_{E_j}(E_i) = \mathcal{O}_{E_j} & i \neq j \\ \mathcal{O}_{E_j}(H) = \mathcal{O}_{E_j} & j = 1, \ldots, 10 \end{cases} \tag{4.1}$$

moreover we prove that conditions (4.1) are also sufficient to have a birational morphism like φ.

The problem of finding the numerical types of $H = al - \sum_1^N b_i e_i$, is purely combinatoric: this fact is reflected in the several tables the reader will find in section 4.3.1. It is thanks to (4.1) that the problem of finding the intersection tables becomes solvable and we can finally find the only possible numerical types for $|H|$.

The main result is:

Theorem: *Let $S \subseteq \mathbb{P}^3$ be a sextic surface with 10 ordinary triple points p_i not on a quadric surface. If $\psi : \tilde{S} \to S \subseteq \mathbb{P}^3$ is the minimal desingularization of S, then there exists a blow-up map $Bl : \tilde{S} \to \mathbb{P}^2$, and we have the diagram*

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\psi} & \mathbb{P}^3 \\ & \overset{Bl}{\searrow} & \\ & & \mathbb{P}^2 \end{array}$$

Calling $e_1, ..., e_{15}$ *the exceptional divisors of* Bl, *and assuming that the* 10 *exceptional curves* $\psi^{-1}(p_i) =: E_i \subseteq \tilde{S}$, $i = 1, \ldots, 10$ *contracted to the triple points are smooth elliptic curves with self intersection* -3, *then the numerical type of the hyperplane section of* $\tilde{S}$ *given on* $Bl(\mathbb{P}^2)$ *is one of the following:*

i) $H(1) = 18l - 8e_1 - \sum_{2}^{7} 5e_i - \sum_{8}^{13} 4e_j - \sum_{14}^{15} 2e_k$,

ii) $H(2) = 18l - \sum_{1}^{3} 6e_i - \sum_{4}^{9} 5e_j - \sum_{10}^{12} 4e_k - \sum_{13}^{15} 2e_i$;

iii) $H(3) = 22l - 8e_1 - \sum_{2}^{7} 7e_i - \sum_{8}^{9} 6e_j - \sum_{10}^{11} 4e_k - \sum_{12}^{15} 2e_i$.

iv) $H(4) = 24l - \sum_{1}^{4} 8e_i - \sum_{5}^{10} 7e_j - \sum_{11}^{15} 2e_i$.

We have given in Lemma 5 and in Corollary 2 the flavour of the kind of conditions we need on the 15 points e_i for $|H|$ to define a birational morphism to $\mathbb{P}^3$ onto a sextic with 10 triple points: the task is particularly delicate because there are a lot of necessary conditions the 15 points must fulfil.

Looking for a while at surfaces S of degree $d \geq 7$, it is interesting to observe here that imposing ordinary triple points cannot force S to be rational: in [15] it is proved such surfaces are of general type. Hence if we want to construct rational septics with isolated singularities we must impose at least one point of multiplicity ≥ 4.

Following [15] we can compute the drop in the invariants imposed by a fourtuple point.

A quick computation of $\chi(\mathcal{O}_{\tilde{S}})$(where $\tilde{S}$ is the minimal desingularization of S): consider a quartic cone $Q \subseteq \mathbb{P}^3$ over a smooth plane curve B of degree 4. We have $\chi(\mathcal{O}_Q) = 1 - g(B) = -2$ while for a smooth quartic surface $\chi(\mathcal{O}_Q) = 2$. Hence a fourtuple point decreases χ by 4. Since in the same way one sees that a triple point decreases χ by 1, for a septic surface $\chi(\mathcal{O}_{\tilde{S}}) = 1$ if S has a fortuple point and 16 triple points (from now on we

will use the notation $4^1 3^{16}$).

Moreover, following the proof of Claim 2) in Section 2.2, we can see that a fortuple point decreases K^2 by 16.

Finally, since a smooth septic surface has $K^2 = 63$ we compute $K_{\tilde{S}}^2 = 63 - 3 * 16 - 16 = -1$. Hence if S exists, $\tilde{S}$ has the following invariants:

$$\begin{cases} \chi(\mathcal{O}_{\tilde{S}}) = 1 \\ q = p_g = 0 \\ K_{\tilde{S}}^2 = -1 \end{cases}$$

If such a surface is rational it could come from $\mathbb{P}^2$ blown-up in 10 points.
We can compile, for degree 7 and 8 surfaces with $\chi(\mathcal{O}_{\tilde{S}}) = 1$, the following table showing the invariants depending on the singularities:

deg(S)	Sing(S)	$\chi(\mathcal{O}_{\tilde{S}})$	$K_{\tilde{S}}^2$
7	$4^1 3^{16}$	1	-1
7	$4^2 3^{12}$	1	-5
7	$4^3 3^8$	1	-9
7	$4^4 3^4$	1	-13
7	4^5	1	-17
8	$4^8 3^1$	1	-3

Hence if we allow only singular points of multiplicity ≤ 4 in degree 8 there is only one possibility: if such a surface exists, its smooth model is $\mathbb{P}^2$ blown-up in 12 points.

4.2 Quintics with 4 triple points

In this section we construct a quintic surface S of $\mathbb{P}^3$ with four ordinary triple points in general position (that is not in a plane) and without other singularities. The model for such a surface will be $Bl_A(\mathbb{P}^2)$, $A = \{e_1, ..., e_{16}\}$ and we will use the linear system

$$|D| = \left| 9l - \sum_1^6 3e_i - \sum_7^{10} 2e_j - \sum_{11}^{16} e_k \right|$$

Remark 1:

(a) We are looking for a linear system $|D|$ of plane curves satisfying the following conditions:

(i) $\dim(D) = 4$, to have a birational map $\varphi_D : \tilde{S} \to S \subseteq \mathbb{P}^3$;

(ii) $\deg(D) = D^2 = \deg(S) = 5$;

(iii) $S = \varphi_D\left(\tilde{S}\right)$ has just isolated singularities.

(b) It is well known that the canonical divisor of a smooth surface $S \subseteq \mathbb{P}^3$ is $K_S \sim (d-4) H_S$ (H_S is the hyperplane section of S). If the surface is not smooth we consider the desingularization $\tilde{S}$ of S and $K_{\tilde{S}} \sim (d-4) H_{\tilde{S}} - \Gamma$: the curve Γ is called double curve (it is mapped by φ to the singular locus $\mathrm{Sing}(S)$).

(c) To achieve condition (iii) in (a) we must impose that the double curve Γ of S is contracted by φ_D or, equivalently, $\Gamma.D = 0$. In general $\Gamma = \Gamma_1 \cup ... \cup \Gamma_N$ where $\Gamma_i \xrightarrow{\varphi\delta} p_i \in S$ are contractions of the components Γ_i to the isolated singular points $p_i's$ (of multiplicity k) of S.

(d) Why do we ask S to have 4 triple points? Consider $\tilde{S}$ in $Bl(\mathbb{P}^3)$. In the notations of (c), since $\tilde{S} \sim dH - \sum k\Gamma_i$ and $K_{Bl(\mathbb{P}^3)} = -4\pi^* H + \sum_1^N 2E_i$ by adjunction we have $K_{\tilde{S}} = (d-4) H_{\tilde{S}} + \sum_1^N (2-k) E_i$. Since we ask S to be rational, the adjoint system must be empty, that is:

$$\left| (d-4) H_S - \sum_1^N (2-k) p_i \right| = \emptyset,$$

which is equivalent to $\frac{(d-1)(d-2)(d-3)}{6} \leq N\frac{k(k-1)(k-2)}{6}$, provided the N singular ($k-$uple) points impose independent conditions to $|(d-4) H_S|$. Substituting $d = 5$ and $k = 3$ we see that a necessary condition to have $h^0(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}})) = 0$ is that $N \geq 4$ (and the $p_i's$ in general position), so that $\left| H_S - \sum_1^4 p_i \right| = \emptyset$. Analogously, in case $d = 6$ and $k = 3$ we get $N \geq 10$.

(e) In this first construction of a degree 5 surface with four triple points, the difficult part is the computation of $h^0(\mathcal{O}_{\tilde{S}}(D))$; from the sequence

$$0 \to \mathcal{O}_{\tilde{S}} \to \mathcal{O}_{\tilde{S}}(D) \to \mathcal{O}_D(D) \to 0$$

recalling $q\left(\tilde{S}\right) = 0$, we get $h^1(\mathcal{O}_{\tilde{S}}(D)) = h^1(\mathcal{O}_D(D))$, $h^0(\mathcal{O}_{\tilde{S}}(D)) = 1 + h^0(\mathcal{O}_D(D)) = 2 + D^2 - \pi + h^1(\mathcal{O}_D(D)) = 1 + h^1(\mathcal{O}_D(D))$, hence $h^1(\mathcal{O}_{\tilde{S}}(D)) \geq 3$, and $\mathcal{O}_{\tilde{S}}(D)$ will be a special bundle. In other words, once the linear system $|D| = \left|9l - \sum_1^6 3e_i - \sum_7^{10} 2e_j - \sum_{11}^{16} e_k\right|$ is given, since the expected dimension (when $h^1(\mathcal{O}_{\mathbb{P}^2}(D)) = 0$) of $h^0(\mathcal{O}_{\mathbb{P}^2}(D))$ is 1, we see that the 16 points e_i, $i = 1, \ldots, 16$ we are blowing up in $\mathbb{P}^2$ must be in some special configuration. The problem will be to characterise such a configuration and to show it exists.

4.2.1 Specifying the Cremona transformation

There are essentially two ways to find the "good" linear system $|D|$ giving the map φ: one "ad-hoc" way is to apply a Cremona transformation, the other more general way is to use the adjunction map technique.

In this section we will describe the first way; now we start recalling some very basic properties of **standard Cremona transformations**: then we will specify one Cremona by choosing its fundamental points and then we will use it to find $|D|$.

Basic properties:

(i) Cr is given by the cubics passing twice through four fundamental points $p_1, \ldots, p_4 \in \mathbb{P}^3$ in general position;

(ii) Cr is not defined on the 6 edges of the tetrahedron Δ spanned in $\mathbb{P}^3$ by the p_i 's;

(iii) Cr contracts the four faces of Δ;

(iv) Cr:$\mathbb{P}^3$ $\dashrightarrow$ $\mathbb{P}^3$ is birational, more precisely $Cr : \mathbb{P}^3\backslash\Delta \xrightarrow{Cr} \mathbb{P}^3\backslash\Delta'$ is an isomorphism (Δ' is the tetrahedron spanned by the images of the four faces of Δ).

Now we specify the Cremona transformation we will use to find $|D|$. As usual, we let $Bl : \tilde{S} \to S$ be the map blowing up the triple points p_i of S. Call $E_i = Bl^{-1}(p_i)$ the exceptional divisor over the triple point $p_i \in S$, $i = 1, \ldots, 4$ and let $H_{\tilde{S}} = |Bl^*(H)|_{\tilde{S}}$ where H is an hyperplane of $\mathbb{P}^3$.

After a suitable change of coordinates, we can assume the four triple points (that we assume to be in general position) of the quintic S to be the four fundamental points $p_1, ..., p_4$ of the Cremona transformation Cr.

Remark 2: On $\tilde{S}$ the Cremona Cr is given by the linear system

$$\tilde{\delta} = \left|\tilde{D}\right| = \left|3H_{\tilde{S}} - 2E_1 - ... - 2E_4 - \sum_1^6 l_{ij}\right|$$

where the $l'_{ij}s$ are the strict transforms of the lines $L'_{ij}s$ of $\mathbb{P}^3$ through $p_i, p_j \in S$.

Claim 1: $Cr(S)$ is a smooth cubic surface.

Proof of Claim 1: a generic twisted cubic C through $q_1, \ldots, q_4$ is mapped on a line, in fact $C.\tilde{D} = \deg(C) \cdot \deg\left(\tilde{D}\right) - 4 \cdot 2 = 1$. Since the points in $Cr(C) \cap Cr(S) = \deg(Cr(S))$ are given by the residual intersection of C with S with respect to the p_i's, $Cr(C) \cap Cr(S)$ consists of $\deg(C) * \deg(S) - 4 * 3 = 3$ points, and $Cr(S)$ is a cubic irreducible surface T. Since Cr is an isomorphism outside the tetrahedron Δ, to prove $Cr(S)$ is smooth is equivalent to prove that the curves cut out on S by the faces of Δ are contracted to smooth points on $Cr(S)$. This is the same to ask $\tilde{C}_{ijk}\left(H - \tilde{C}_{ijk}\right) = 1$, where $\tilde{C}_{ijk}$ is the strict transform of the conic C_{ijk}

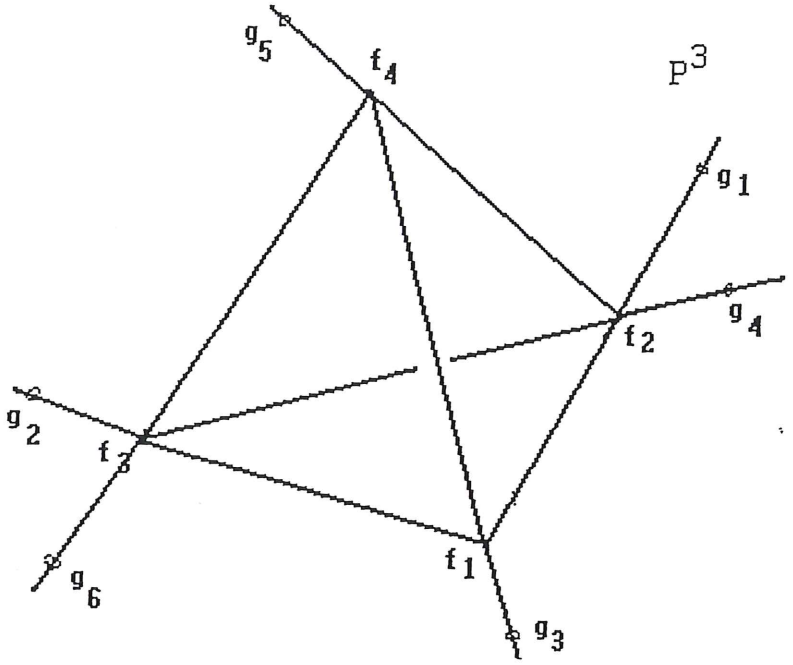
residual to the lines L_{ij}, L_{ik}, L_{jk} in the plane $\langle p_i, p_j, p_k \rangle$. So we have to prove $\tilde{C}^2_{ijk} = -1$. By adjunction $-2 = \tilde{C}^2_{ijk} + \tilde{C}_{ijk}.K_{\tilde{S}}$, and recalling $\tilde{C}_{ijk}.K_{\tilde{S}} = \tilde{C}_{ijk}.(H_{\tilde{S}} - \sum_1^4 E_i) = 2 - 3 = -1$ we have the claim. □

Conversely, we can start by taking four generic points $f_1, ..., f_4$ on a smooth cubic surface T of $\mathbb{P}^3$ and apply the Cremona transformation based on those points: the image $Cr(T)$ will be a quintic surface with four triple points coming from the contraction of the plane cubics cut out by the four faces of the tetrahedron spanned by the f_i's $(i = 1, \ldots, 4)$.

The linear system giving the Cremona on the blow up $\tilde{T}$ of the cubic surface is $D' = 3H_{\tilde{T}} - 2F_1 - ... - 2F_4 - \sum_1^6 G_{ij}$, where the F_i's are the exceptional divisors over the f_i's and the G_{ij}'s are the exceptional divisors over the points g_{ij}'s in the residual intersection of the line $\langle f_i, f_j \rangle$ with T. Since we obtain the smooth cubic of $\mathbb{P}^3$ as the embedding of $Bl_B(\mathbb{P}^2)$, $B = \{e_1, ..., e_6\}$, with hyperplane section $H_{\tilde{T}} = 3l - \sum_1^6 e_i$ (the e_i's are in general position), we find that the candidate linear system we must use to construct the quintic is $|D| = |9l - \sum_1^6 3e_i - \sum_7^{10} 2e_j - \sum_{11}^{16} e_k|$.

As we already mentioned the difficult work now is to specify the position of the 16 points we are blowing-up in order for $|D|$ to define a birational map to $\mathbb{P}^3$, this means we must ask the $e_i's$ to satisfy some "genericity conditions".

In this case the Cremona transformation gives us also the "special configuration" of the points and the "genericity" conditions we are looking for. I will illustrate this first with a picture:



Genericity conditions:

(a) We study the configuration of the 16 points e_i in the plane using the configuration of the above picture. In $\mathbb{P}^2$ the points $e_1, ..., e_6$ are generic: this means no 3 on a line and no 6 on a conic (recall the cubics through the e_i's give a smooth cubic surface). To get the configuration of the remaining points in $\mathbb{P}^2$, we use the Cremona. Let $\langle C_1, ..., C_4 \rangle$ be one basis of the vectorspace $|3l - \sum_1^6 e_i|$. We take three generic points f_2, f_3, f_4 on C_1 and we impose C_2 to pass through f_2 and f_3. Then we choose a generic point f_1 on C_2 and we impose C_3 to pass through f_1, f_2, f_4 and C_4 to pass through f_1, f_2, f_3. Any of the four independent cubics C_i will have 8 points of intersection with any other C_j, $j \neq i$, hence on every C_i are defined 3 more points g_{ij} (there are 6 such g_{ij}'s), depending only on the $e_1, ..., e_6, f_1, ..., f_4$. These last 6 points are completely determined by the choice of the previous 10 points.

(b) The genericity conditions we ask on the first ten points of the linear system $|D|$ are inspired by the Cremona transformation we defined:

$$dim\left|6l-\sum_{1}^{6}2e_k-2f_i-\sum_{\hat{i}}f_j\right|=4\ \forall i=1,...,4;(*)$$

$$dim\left|3l-\sum_{1}^{6}e_i-f_i-f_j\right|=2\ \forall i\neq j,\ i,j=1,...,4.(**).$$

Observe that the 16 points found in **(a)** satisfy the two conditions $(*)$ and $(**)$.

In the following Proposition 1 we are going to reformulate in a different setting some consequences of $(*)$ and $(**)$, and to do that we will need the next lemma:

Lemma 1: If H has a decomposition

$$H\equiv C+D$$

where C and D are curves on Φ, such that $\dim|C|\geq 1$, and if the restriction maps $H^0(\mathcal{O}_\Phi(H))\rightarrow H^0(\mathcal{O}_D(H))$ and $H^0(\mathcal{O}_\Phi(H))\rightarrow H^0(\mathcal{O}_C(H))$ are surjective, and $|H|$ restricts to very ample linear systems on D and on every C in $|C|$, then $|H|$ is very ample.

Proof. See [16]. □

Proposition 1:

(a) The linear system $|\bar{H}|=|9l-\sum_1^6 3e_i-\sum_7^{10}2e_j|$ defines an embedding $\varphi{:}Bl_A(\mathbb{P}^2)\rightarrow\mathbb{P}^6$, $A=\{e_1,...,e_6,f_1,...,f_4\}$, and $Im\varphi=\Phi$ is a surface of degree 11;

(b) the curves $C_i=3l-\sum_1^6 e_i-\sum_{\hat{i}}f_j$ are embedded on Φ as smooth plane cubics;

(c) on Φ we have: $\langle\varphi(C_i),\varphi(C_j)\rangle=\mathbb{P}^4$, $\langle\varphi(C_i),\varphi(C_j),\varphi(C_k)\rangle=\mathbb{P}^5$ and $\langle\varphi(C_1),\varphi(C_2),\varphi(C_3),\varphi(C_4)\rangle=\mathbb{P}^6$.

(d) Φ has a unique six secant plane π.

Proof.

(a) (b) To prove H gives an embedding, we look first at the four curves $\varphi(C_i)$, $i = 1, ..., 4$. Their images are plane smooth cubics. In fact from

$$0 \to \mathcal{O}_\Phi \to \mathcal{O}_\Phi(H) \to \mathcal{O}_H(H) \to 0$$

taking cohomology we get $h^0(\mathcal{O}_\Phi(H)) = 1 + h^0(\mathcal{O}_H(H)) = 1 + H^2 - \pi + 1 + h^1(\mathcal{O}_H(H)) = 7 + h^1(\mathcal{O}_H(H))$. By Serre's duality $h^1(\mathcal{O}_H(H)) = h^0(\mathcal{O}_H(K_H - H))$; moreover from $K_H.H = H^2 + K_\Phi.H$ and $2\pi - 2 = H^2 + K_\Phi.H$ we get $(K_H - H).H = -1$. Hence $h^0(\mathcal{O}_H(K_H - H)) = 0$ and $h^0(\mathcal{O}_\Phi(H)) = 7$, that is φ gives a map in $\mathbb{P}^6$.

From $dim\,|H - \varphi(C_i)| = \dim \left|6l - \sum_1^6 2e_k - 2f_i - \sum_{\hat{i}}^{\vdots} f_j\right| = 4$ (genericity condition $(*)$) and $\dim|H - \varphi(C_i)| = 7$ we deduce that $\langle C_i \rangle \simeq \mathbb{P}^2$. This implies that the restriction $\varphi_{|C_i}$ is an embedding: $\varphi_{|C_i}$ has degree 3 ($H.C_i = 3$) and it can not be a $3:1$ map to a line. Consider now the exact sequence

$$0 \to \mathcal{O}_\Phi(H - \varphi(C_i)) \to \mathcal{O}_\Phi(H) \to \mathcal{O}_{\varphi(C_i)}(H) \to 0$$

and observe that by Riemann-Roch $h^1(\mathcal{O}_\Phi(H - \varphi(E_i))) = 0$. Taking cohomology we have

$$H^0(\mathcal{O}_\Phi(H)) \twoheadrightarrow H^0\left(\mathcal{O}_{\varphi(C_i)}(H)\right)$$

so that φ can have base points only on $\varphi(C_i)$, but this is impossible, since we noticed that $\varphi_{|C_i}$ is an embedding. We conclude that φ has no base points.

To see φ is an embedding we use the four plane curves C_i and Lemma 2, in particular we let the curves C and D to be $H \backslash C_i$ and C_i respectively. Taking cohomology from the defining sequences

$$0 \to \mathcal{O}_\Phi(C_i) \to \mathcal{O}_\Phi(H) \to \mathcal{O}_{H-C_i}(H) \to 0$$

$$0 \to \mathcal{O}_\Phi(H - C_i) \to \mathcal{O}_\Phi(H) \to \mathcal{O}_{C_i}(H) \to 0$$

it follows immediately that the restriction maps $H^0(\mathcal{O}_\Phi(H)) \to H^0(\mathcal{O}_{C_i}(H))$, and $H^0(\mathcal{O}_\Phi(H)) \to H^0(\mathcal{O}_{H-C_i}(H))$ are surjective. We already noticed that $\varphi_{|C_i}$ is an embedding, that is $|H|$ restricts to very ample linear systems on D. The same holds for $H - C_i$. This is clear if $H - C_i$ is irreducible. If not, it can break in a finite number of ways, and the required condition is in each case easily verified. Hence we are in the assumptions of Lemma 2 and we conclude $|H|$ embeds $Bl(\mathbb{P}^2)$ in $\mathbb{P}^6$ as a surface of degree $H^2 = 11$ containing the smooth plane cubics $\varphi(C_i)$, $i = 1, \ldots, 4$.

(c) Follows immediately from the genericity assumptions $(*)$ and $(**)$.

(d) Existence of a six secant plane: we consider the 6 points g_i coming from the intersection of the 4 elliptic curves C_i and we show that $\langle g_1, ..., g_6 \rangle = \mathbb{P}^2$. We blow up Φ in $\{g_i\}_{1,...,6}$ and we consider the intersection $(C_1+C_2+C_3) \cap C_4$. Since $C_1+C_2+C_3 = H-F_4$ and $F_4.C_4 = 0$, we have $(C_1+C_2+C_3) \cap C_4 = (H-F_4) \cap C_4 = H \cap C_4$. Hence, since C_4 is a plane curve the three g_i coming from the last intersection are on a line. This holds for each combination of the four elliptic curves E_i, so that $\langle g_1, ..., g_6 \rangle = \mathbb{P}^2$.

Unicity: let $\pi' = \langle p_1, ..., p_6 \rangle$ be another 6-secant plane to Φ. We can assume π' intersect in at least one point each C_i, $i = 1, ..., 4$. On the contrary we would have $\langle \pi, \langle C_i \rangle \rangle = \mathbb{P}^5$, $i = 1, ..., 4$. Observe that $H \backslash C_i$ is a curve of degree 8 and genus 3 (it corresponds to plane curves in the class $6l - \sum_1^6 2e_i - 2f_i - \sum_{\hat{i}} f_j$). Moreover $\langle \bar{H} \backslash C_i \rangle = \mathbb{P}^5$, and the ∞^1 hyperplanes through $\langle \pi, p \rangle$, $p \in \bar{H} \backslash C_i$ a generic point, would cut a residual point on $\bar{H} \backslash C_i$, giving a rational parametrization; a contradiction. Since $\langle C_i \rangle \subset \langle \pi, \pi' \rangle$ $i = 1, ..., 4$ (remember that $\langle C_i \rangle \cap \pi$ is a line), if π and π' span a $\mathbb{P}^5$ we have $\langle C_1, ..., C_4 \rangle \subset \mathbb{P}^4$, a contradiction. Hence the two 6 secant plane must intersect, and the only way this is possible is they intersect in the g_i's. $\square$

As we already noticed, the 6 special points spanning the unique 6-secant plane to Φ from which we project Φ in $\mathbb{P}^3$ to get the quintic surface with the 4 triple points follow after the choice of the first 10 points. Observe also that using the Cremona transformation we showed at once the existence of a quintic surface with 4 triple points and the existence of a configuration of points in $\mathbb{P}^2$ satisfying conditions $(*)$ and $(**)$.

4.2.2 The adjunction map technique

In this paragraph we are going to find the linear system $|H|$ of the previous section using the adjunction map. We start with some remarks and a definition, then we recall the basic Theorem of Sommese-Van De Ven on the adjoint map.

Assume we start with a linear system $H = al - \sum_1^N b_i e_i$ over $\mathbb{P}^2$ defining a smooth rational surface $\tilde{S}$. We know $K_{\tilde{S}} = -3l + \sum_1^N e_i$. Set $a_1 \geq a_2 \geq \ldots \geq a_{15}$ and let $N \geq 2$, $a_N = 1$, $a_{N-1} \geq 2$.
We have then $H + K = (a-3)\, l - \sum_1^{N-1} (b_i - 1)e_i$ and this tells us:
Fact i) in this case the map defined by $H + K$ contracts the -1 curve e_{16} of $\tilde{S}$.

Fact ii) If we let K_1 to be the canonical divisor of the surface S_1 obtained by $H' = H + K_S$, we get $K_1 = -3l + \sum_1^{N-1} e_i$ and $K_1^2 = K^2 + \#\{i|a_i = 1\}$. In general, letting $K_i = K_{S_i}$ with $S_i = \mathrm{Im}\left(\varphi_{H+K_{S_{i-1}}}\right)$ given by $H_i = H_{i-1} + K_{i-1}$, with $H_0 = H$ $K_0 = K$, we have $K_i^2 = K^2 + \#\{i|a_i = 1\} + \#\{i|a_i = 2\} + \ldots + \#\{i|a_i = i\}$.

Fact iii) During this process, starting from the surface S, we construct some other rational surfaces S_i: however this process will eventually end, in fact since at every step we add $K_i = -3l + \sum_1^{N_i} e_i$ to H_{i-1}, at the i-th step we get

$H_i = (a-3i)\,l - \sum_1^{N-1}(b_i - i)e_i$, hence the process will end for some $i > 0$.

Definition 1: In the previous notations, the map given by $H + K$ is called **adjunction map.**

Sommese and Van de Ven proved the following:

Theorem 1: Let S be a smooth surface with a very ample divisor H and a canonical divisor K. Then

1) $|H + K| = \emptyset$ if and only if S is a scroll or a Veronese surface,

2) $|H + K| \neq \emptyset$ only if $|H + K|$ has no basepoints.

In case 2) we have furthermore that

A) $(H + K)^2 = 0$ if and only if S is ruled in conics,

B) $(H + K)^2 > 0$ only if the map φ_{H+K} defined by $|H + K|$ is the blowing-down of (-1) lines except for the following four cases:

i) S is $\mathbb{P}^2$ blown up in 7 points and $H \equiv 6l - \sum_1^7 2E_i$.

ii) S is $\mathbb{P}^2$ blown up in 8 points and $H \equiv 6l - \sum_1^7 2E_i - E_8$.

iii) S is $\mathbb{P}^2$ blown up in 8 points and $H \equiv 9l - \sum_1^7 3E_i$.

iv) $S \cong \mathbb{P}(E)$ where E is an indecomposable rank 2 bundle on an elliptic curve, and $H \equiv 3B$ where B is an effective divisor with $B^2 = 1$ on S.

Proof. See [17]. □

In our setting we iterate the adjunction map starting from the smooth model $\tilde{S} = S_0$ of the quintic surface S until we reduce the degree and the sectional genus π of the image surface S_i and we have a classified rational surface:

$$\tilde{S} = S_0 \overset{|H+K|}{\to} S_1 \overset{|H_1+K_1|}{\to} S_2 \to \dots$$

By fact iii) above we eventually end up on a smooth rational surface S_j, and

we reconstruct H by subtracting all the K_i's, $i < j$, from H_j.

Numerical data and iteration rules for the adjunction map:

To apply the adjunction map technique to the smooth resolution S_0 of the quintic surface S we start by filling up a table involving the invariants H^2, $H.K$, K^2 and the sectional genus π for the surfaces S_i, starting from S_0. Before we can do that we need the invariants for the starting surface $S = S_0$ as well as the recurrency rules to write them for every i.

Claim 2: Every triple point decreases K_S^2 by 3, hence $K_S^2 = -7$.

Proof: recall we are assuming that blowing up once the surface S at each triple point p_i we get a smooth surface (that is we are considering ordinary triple points) and that the exceptional divisors on the triple points are genus 1 curves E_i, with $E_i^2 = -3$. Following [15]: let $E = \sum_1^{\upsilon} E_i$ be the sum of all exceptional divisors on the smooth model S_0 of S and denote by S_{sm} a smooth quintic surface of $\mathbb{P}^3$. Since the E_i are elliptic curves by adjunction $E_i.(K_{S_0} + E_i) = 0$. Set $K_{S_0} \sim \pi^* K_{S_{sm}} - \alpha E$: then $E_i.(K_{S_0} + E_i) = E_i.(\pi^* K_{S_{sm}} - \alpha E + E_i) = 3(\alpha - 1)$. Hence $K_{S_0} \sim \pi^* K_{S_{sm}} - E$ and $c_1(S_0)^2 = K_{S_0}^2 = K_{S_{sm}}^2 - 3\upsilon$. □

By Claim 2, in our presentation $H = al - \sum_1^N b_i e_i$, we have $N = 16$. From $\pi = 10$ and the adjunction formula we get $H.K = 12$.

The next step is to compile the following table:

	H^2	H.K	K^2	π	n=dim$\lvert H_i \rvert$
$S=S_0$	5	5	-7	6	3
S_1	8	-2	$-7+n_1$	4	5
S_2	$-3+n_1$	$-9+n_1$	$-7+n_1+n_2$	$-5+n_1$	3

(*)

where we used the following **iteration Rules:**

A) $H_i^2 = (H_{i-1} + K_{i-1})^2$,

B) $K_i^2 = K_{i-1}^2 + n_i$

C) $H_i.K_i = H_{i-1}.K_{i-1} + K_{i-1}^2$.

Proof: A) is just the definition. B) follows from **Fact ii)** above. C) Since $H_i.K_i = (H_{i-1} + K_{i-1})(K_i)$, and the map $H_i + K_i$ is a birational map of smooth rational surfaces, we know $K_i = K_{i-1} + C_{i-1}$ with $C_{i-1}.H_i = 0$. C) is proved. □

Going back to table (∗), in the above notations, the parameters there are the numbers

$$n_i = \sharp\{-1 \text{ rational curves } C_i \text{ of } S, \text{ of degree } i\}$$

(where "C_i of degree i" means $C_i.H = i$).

Remark 3:

Since the adjunction map is birational, it moves from one smooth rational surface to another, hence we finally arrive on a smooth rational surface S_2 of $\mathbb{P}^3$: these are well known and there are two possibilities, either S_2 is a cubic Del Pezzo surface or a quadric smooth surface.

Performing the computations we find three possibilities:

(a) S_2 is a smooth quadric, then $H_2^2 = 2$ and $K_2^2 = 8$, and by the equalities in table (∗), $n_1 = 5$ and $n_2 = 10$. From $H_2 = (1,1)$ and $K_2 = -2(1,1)$ we get, subtracting $K_1 = K_2 + \sum_1^{10} e_i$ and $K = K_1 + \sum_{11}^{15} e_j$,

$$H = 5(1,1) - \sum_1^{10} 2e_i - \sum_{11}^{15} e_j$$

(b) If S_2 is a Del Pezzo cubic surface, then $H_2^2 = K_2^2 = 9$, giving $n_1 = 6$ and $n_2 = 4$. We get now

$$H = 9l - \sum_1^6 3e_i - \sum_7^{10} 2e_j - \sum_{11}^{16} e_k$$

(c) If $(H_2 + K_2)^2 = 0$, that is S_1 is ruled in conics, we deduce $n_1 = 3$ and we have

$$\begin{array}{ccc} S_1 & \overset{|H+K|}{\longrightarrow} & C \\ & \searrow & \uparrow |3F| \\ & & \mathbb{F}_1 \end{array}$$

where F is a fiber and $B^2 = -1$. Taking the model $Bl(\mathbb{P}^2)$, $F = l - e_1$, $B = e_1$ and $K_{\mathbb{F}_1} = -2B - 2F + e_1 + \ldots + e_{12}$ we find

$$H = 9l - 5e_1 - \sum_{2}^{13} 2e_i - \sum_{14}^{16} e_j$$

Now it is easily seen that the only linear system admitting a configuration of four compatible (-3) elliptic curves is **(b)**, in fact:

In case **(a)**: on a smooth quadric surface Q an elliptic curve E is of type $E = (a, b)$, with $a, b \geq 2$. Moreover we ask $\bigcup_{1}^{4} E_i = H - K = 7(1,1) - \sum_{1}^{10} 3e_i - \sum_{11}^{15} 2e_j$, giving a contradiction.

In case **(b)**: there is only one numerical possibility for a configuration of four disjoint (-3) elliptic curves, that is:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}
E_1 $(3l)$	1	1	1	1	1	1	1	1	1	0	1	1	1	0	0	0
E_2 $(3l)$	1	1	1	1	1	1	1	1	0	1	0	0	1	1	0	1
E_3 $(3l)$	1	1	1	1	1	1	1	0	1	1	1	0	0	0	1	1
E_4 $(3l)$	1	1	1	1	1	1	0	1	1	1	0	1	0	1	1	0

In case **(c)**: from $\sum_{1}^{4} E_i = H - K = 12l - \ldots$ it follows the four elliptic (-3) curves E_i are of type $3l - \ldots$moreover, the only curve of this kind is

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}
E_1 $(3l)$	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0

Remark 4:

The iteration of the adjunction map gives us just the numerical type of the

candidate linear system **(b)**: it does not give us the genericity conditions $(*)$ and $(**)$, found via the Cremona transformation, providing a birational morphism $\varphi : Bl(\mathbb{P}^2) \to \mathbb{P}^3$.

4.3 A rational sextic surface with 10 triple points

In this section we find necessary conditions in order to get a sextic surface of $\mathbb{P}^3$ with 10 triple points as the image of a morphism $Bl(\mathbb{P}^2) \stackrel{|H|}{\to} \mathbb{P}^3$.

The main result is:

Theorem 2: *Let $S \subseteq \mathbb{P}^3$ be a sextic surface with 10 ordinary triple points p_i not on a quadric surface. If $\tilde{S}$ is the minimal desingularization of S and $\psi : \tilde{S} \to S \subseteq \mathbb{P}^3$ is a morphism, then there exists a blow-up map $Bl : \tilde{S} \to \mathbb{P}^2$, and we have the diagram*

$$\begin{array}{ccc} \tilde{S} & \stackrel{\psi}{\to} & \mathbb{P}^3 \\ {\scriptstyle Bl}\searrow & & \\ & & \mathbb{P}^2 \end{array}$$

Calling $e_1, \ldots, e_{15}$ the exceptional divisors of Bl and assuming that the 10 exceptional curves $\psi^{-1}(p_i) =: E_i \subseteq \tilde{S}$, $i = 1, \ldots, 10$ contracted to the triple points are smooth elliptic curves with selfintersection -3, then the numerical type of the hyperplane section of $\tilde{S}$ is one of the following:

i) $H(1) = 18l - 8e_1 - \sum_{2}^{7} 5e_i - \sum_{8}^{13} 4e_j - \sum_{14}^{15} 2e_k,$

ii) $H(2) = 18l - \sum_{1}^{3} 6e_i - \sum_{4}^{9} 5e_j - \sum_{10}^{12} 4e_k - \sum_{13}^{15} 2e_i;$

iii) $H(3) = 22l - 8e_1 - \sum_{2}^{7} 7e_i - \sum_{8}^{9} 6e_j - \sum_{10}^{11} 4e_k - \sum_{12}^{15} 2e_i.$

iv) $H(4) = 24l - \sum_{1}^{4} 8e_i - \sum_{5}^{10} 7e_j - \sum_{11}^{15} 2e_i.$

Remark 5:

$\tilde{S}$ will not be, in general, a minimal surface. The presence of (-1) curves of

particular degree will be an important clue to find $|H|$ (see Lemma 3, page 57).

4.3.1 Proof of Theorem 2.

As in the case of the quintic surface (on page 49), we start by compiling a table showing the invariants H^2, $H.K$, K^2 and the sectional genus π of the surfaces S_i.

From Claim 2 in the previous section, we know $K^2 = -6$, hence in the presentation $H = al - \sum_1^N b_i e_i$ we have $N = 15$. From $\pi = 10$ and the adjunction formula $H.K = 12$.

The table with the invariants looks like:

	H_i^2	H_iK_i	K_i^2	$\pi\,(S_i)$	$m_i = \dim\|H_i\|$
S_0	6	12	-6	10	3
S_1	24	6	$-6+n_1$	16	9
S_2	$30+n_1$	n_1	$-6+\sum_1^2 n_i$	$16+n_1$	15
S_3	$24+4n_1+n_2$	$-6+2n_1+n_2$	$-6+\sum_1^3 n_i$	$10+3n_1+n_2$	n_1+15
S_4	$6+\sum_1^3 (4-i)^2 n_i$	$-12+\sum_1^3 (4-i) n_i$	$-6+\sum_1^4 n_i$	$-2+6n_1+3n_2+n_3$	$3n_1+n_2+9$
S_5	$-24+\sum_1^4 (5-i)^2 n_i$	$-18+\sum_1^4 (5-i) n_i$	$-6+\sum_1^5 n_i$	$-20+\sum_1^4 \frac{i}{2}(i+1)n_{5-i}$	$6n_1+3n_2+$ $+n_3-3$
S_6	$-66+\sum_1^5 (6-i)^2 n_i$	$-24+\sum_1^5 (5-i) n_i$	$-6+\sum_1^6 n_i$	$-44+\sum_1^5 \frac{i}{2}(i+1)n_{6-i}$	$10n_1+6n_2+$ $3n_3+n_4-21$

where we used, as in the case of the quintic, the rules $H_i^2 = (H_{i-1} + K_{i-1})^2$, $K_i^2 = K_{i-1}^2 + n_i$ and $H_i.K_i = H_{i-1}.K_{i-1} + K_{i-1}^2$.

Remark 6: There are some obvious inequalities we can use to discuss the parameters appearing in the table:

i) $\sum_i n_i \leq 15$;

ii) if S_i is a surface then $\pi(S_i) \geq 0$ and $H_i^2 > 0$;
iii) if $S_i \subset \mathbb{P}^{m_i}$ is non degenerate, then $H_i^2 \geq m_i - 1$.

To iterate the adjunction on the rational surfaces S_i we need also the following:

Lemma 2: Let S be a rational surface and H be a very ample divisor on S. If $K^2 \geq 0$, then $H.K \leq -3$.

Proof. Following [16]: Riemann-Roch says $h^0(\mathcal{O}_S(-K)) = h^1(\mathcal{O}_S(-K)) + K^2 + 1 \geq 1$. Take $C \in H^0(\mathcal{O}_S(-K))$: C has arithmetic genus 1, hence $C.H \geq 3$. □

To make it easy one can write down a program to use a software like CoCoA (for example) and finally gets 61 numerical possibilities compatible with conditions i),...,iii) of Remark 6 (see Appendix 1 for the complete list). The work is then to find "the right numerical types" among those in the list.

Rewiew:

In [15] Endrass, Persson and Stevens study the existence of sextic surfaces in $\mathbb{P}^3$ with many triple points and give a detailed classification. After proving the existence of sextic surfaces with ≤ 9 triple points, the authors prove the existence of a sextic surface with 10 triple points. For completeness we give here the idea of their argument.

They start observing that if a sextic surface S with 10 triple points exists, then it is always contained in a pencil of the kind $S(\mu, \lambda) = \mu F + \lambda Q^3$, where $\{F = 0\}$,$\{Q = 0\}$ are the equations of the sextic S and of a quadric surface Q through nine of the ten triple points of S.

Observe that the generic surface in the pencil so constructed has exactly 9 triple points, since by Bertini's theorem the generic element of $S(\mu, \lambda)$ is smooth outside the base locus of $S(\mu, \lambda)$ (that consists of the nine triple

points we choose). Since in [15] the authors provide the complete classification in the case of sextic surfaces with 9 triple points, they are able to conclude that the generic surface in $S(\mu, \lambda)$ is a $K3$ surface or an elliptic surface. Finally they work out all the computations with a specific pencil and they prove the existence of the sextic surface S.

Now we look for necessary conditions to reduce the long list reported in Appendix 1 to a shorter one.

We start looking at a particular class of curves on the blown-up $\tilde{S}$ of S that will be of great interest for our purpose: (-1) **curves.**

Remark 7:
In the previous notations, every (-1) curve A on the smooth model $\tilde{S}_{\lambda,\mu}$ of a generic surface in the pencil $S(\lambda, \mu)$ (that is a sextic with 9 triple points) is contained in the adjoint system $K_{\tilde{S}_{\lambda,\mu}}$. In fact by adjunction on $\tilde{S}_{\lambda,\mu}$:

$$A.K_{\tilde{S}_{\lambda,\mu}} = 2g(A) - 2 - A^2 = -1$$

hence $A \subseteq K_{\tilde{S}_{\lambda,\mu}}$. Moreover recall the adjoint system is cut out by the unique quadric surface through the nine triple points of $S_{\lambda,\mu}$. This shows that the (-1) curves of the surface $\{F = 0\}$ are fixed in the pencil $S(\mu, \lambda)$, as they are fixed in the intersection $F \cap Q. = K_{\tilde{S}_{\lambda,\mu}}$.

To complete the picture we refer once more to [15]: in their classification of sextic surfaces with 9 triple points they studied in detail (-1) curves and their possible configurations and dinstinguish two cases:
(i) if the canonical divisor $K_{\tilde{S}_{\lambda,\mu}}$ is the union of all the (-1) curves C_i, then $\tilde{S}_{\lambda,\mu}$ is a $K3$ surface blown-up in 3 points; moreover on $\tilde{S}_{\lambda,\mu}$ there are exactly 3 (-1) curves of degrees (d_1, d_2, d_3) and three possibilities occurs:

$$(d_1, d_2, d_3) \in \{(2, 2, 8), (2, 4, 6), (4, 4, 4)\}$$

(ii) if $K_{\tilde{S}_{\lambda,\mu}}$ is not the union of the (-1) curves, then $\tilde{S}_{\lambda,\mu}$ is an elliptic minimal surface blown up in three points; moreover, on $\tilde{S}_{\lambda,\mu}$ there are exactly three (-1) curves, and their degrees d_1, d_2 and d_3 satisfy:

$$(d_1, d_2, d_3) \in \{(2,2,2), (2,2,4)\}$$

From (i) and (ii) it follows that **a sextic surface with nine triple points does not have (-1) curves of degree 1 or 3.**

The key remark is: assume $\tilde{S}$ has a (-1) curve C coming from a curve $\Gamma \subset S$ that does not pass through a triple point, say p_{10}: then when we look at the pencil $S(\mu, \lambda)$ of sextics through p_i, $i = 1, \ldots, 9$, specializing to $\tilde{S}$, we will find C_i as a (-1) rational curve on the generic surface in $S(\mu, \lambda)$. Moreover, the configuration of the (-1) curves on $\tilde{S}$ must be as described in [15] (page 23), that is:

in case i):

type		p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
(2, 2, 8)	C_1	1	0	0	0	0	1	1	1	1
	C_2	0	1	1	1	0	0	0	1	1
	C_3	2	2	2	2	3	2	2	1	1
(2, 4, 6)	C_1	1	0	0	0	0	1	1	1	1
	C_2	0	2	1	1	1	1	1	1	1
	C_3	2	1	2	2	2	1	1	1	1
(4, 4, 4)	C_1	2	1	0	1	1	1	1	1	1
	C_2	0	2	1	1	1	1	1	1	1
	C_3	1	0	2	1	1	1	1	1	1

where the rows of the table show the rational curves C_i, the p_i's are the triple points of S, and the numbers in the j-th column are the multiplicities p_j has on C_i (So, for example, looking at the first row: C_1 is a conic passing through p_1, p_6, p_7, p_8, p_9 and C_3 is a degree 8 curve through p_8, p_9, with double points in p_i, $i = 1, 2, 3, 4, 6, 7$ and a triple point in p_5).

In case ii):

type		p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
$(2,2,2)$	C_1	0	0	1	1	1	1	1	0	0
	C_2	1	1	0	0	1	1	0	1	0
	C_3	1	1	1	1	0	0	0	0	1
$(2,2,4)$	C_1	0	0	1	1	1	1	1	0	0
	C_2	1	1	0	0	1	1	0	1	0
	C_3	1	1	1	1	0	1	1	1	2

With this in mind, we have the following:

Lemma 3: In the previous notations, on the blow-up $\tilde{S}$ of a sextic surface S with 10 triple(points not on a quadric), we have $n_1 = n_3 = 0$.

Proof. Assume $\tilde{S}$ has a (-1) line L. Since the singular locus $\{p_1, \ldots, p_{10}\}$ of S is not on L (the surface S would have L as a double curve) we can assume $p_{10} \notin L$. Then we would find L as a (-1) curve of degree 1 on a sextic with the nine triple points $\{p_1, \ldots, p_9\}$, in the pencil $\lambda S + \mu Q$ (Q is the equation of the quadric through $p_1, \ldots, p_9$). Hence $n_1 = \#\left\{-1 \text{ lines on } \tilde{S}\right\} = 0$.
To prove $n_3 = 0$, we observe that, since $\tilde{S}$ is rational, its adjoint system $\left|2H - \sum_{i=1}^{10} E_i\right|$ is empty, hence the singular locus $\{p_1, \ldots, p_{10}\}$ of S is not on a quadric and hence not on a twisted cubic C. Say $p_{10} \notin C$ and consider the pencil $P_{\lambda,\mu} = \lambda S + \mu Q$, where $Q \in \left|2H - \sum_{i=1}^{9} E_i\right|$. Since the (-1) curves are fixed in $P_{\lambda,\mu}$, we would find sextic surfaces with a (-1) twisted cubic, which is impossible. The Lemma is proved. □

This last necessary condition, $n_1 = n_3 = 0$, reduces the number of numerical possibilities for the linear systems in our list, and we are left with just 11 candidates. We look now at (-1) conics on $\tilde{S}$.

Maximal configuration of -1 conics

Another necessary condition we can derive is that $\tilde{S}$ has no more than 6 (-1) conics: this follows from the particular configuration these curves must have on a sextic surface with 9 triple points S' and consequently on $\tilde{S}$. To be more precise, (-1) conics on S' must have two triple points in common (see [15]), and following the above notations, their configuration is:

in case i)

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
C_1	1	0	0	0	0	1	1	1	1
C_2	0	1	1	1	0	0	0	1	1

in case ii)

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
C_1	0	0	1	1	1	1	1	0	0
C_2	1	1	0	0	1	1	0	1	0
C_3	1	1	1	1	0	0	0	0	1

(where the p_i's are the triple points of S', the rows show the (-1) conic C_i and in the i-th j-th place we find the multiplicity of p_i on C_j: so in case i), the conic C_1 pass through the five triple points p_1, p_6, p_7, p_8 and p_9).

On $\tilde{S}$ we have two possibilities: either two (-1) conics are disjoint or they appear in the above configurations.

In the first case we have

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}
C_1	1	1	1	1	1	0	0	0	0	0
C_2	0	0	0	0	0	1	1	1	1	1

and there is no room for another (-1) conic (any two of the C_i must have two "1"'s in common).

In the second case we can have at most 6 (-1) conics and the maximal

configuration, up to permutation of the p_i's, looks like

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}
C_1	1	0	1	0	1	1	0	0	0	1
C_2	1	1	0	1	0	1	0	1	0	0
C_3	0	1	1	1	0	0	0	0	1	1
C_4	0	1	1	0	1	0	1	1	0	0
C_5	1	0	0	1	1	0	1	0	1	0
C_6	0	0	0	0	0	1	1	1	1	1

In our list of numerically compatible linear systems, after imposing these necessary conditions, we are left with the 11 following candidates:

(1) $H = 11\,(1,1) - \sum_{1}^{4} 5e_i - \sum_{5}^{12} 4e_j - \sum_{13}^{14} 2e_j$ (on a quadric Q);

(2) $H = 18l - \sum_{1}^{3} 6e_i - \sum_{4}^{9} 5e_j - \sum_{10}^{12} 4e_k - \sum_{13}^{15} 2e_i$;

(3) $H = 19l - 7e_1 - \sum_{2}^{6} 6e_i - \sum_{7}^{8} 5e_j - \sum_{9}^{12} 4e_k - \sum_{13}^{15} 2e_i$;

(4) $H = 18l - 8e_1 - \sum_{2}^{7} 5e_i - \sum_{8}^{13} 4e_j - \sum_{14}^{15} 2e_k$;

(5) $H = 22l - 8e_1 - \sum_{2}^{7} 7e_i - \sum_{8}^{9} 6e_j - \sum_{10}^{11} 4e_k - \sum_{12}^{15} 2e_i$;

(6) $H = 21l - \sum_{1}^{5} 7e_i - \sum_{6}^{8} 6e_j - \sum_{9}^{10} 5e_k - 4e_{11} - \sum_{12}^{15} 2e_i$;

(7) $H = 20l - 7e_1 - \sum_{2}^{9} 6e_i - 5e_{10} - 4e_{11} - \sum_{12}^{15} 2e_i$;

(8) $H = 24l - \sum_{1}^{4} 8e_i - \sum_{5}^{10} 7e_j - \sum_{11}^{15} 2e_i$;

(9) $H = 27l - \sum_{1}^{7} 9e_i - 8e_8 - \sum_{9}^{10} 6e_j - \sum_{11}^{15} 2e_i$;

(10) $H = 25l - 9e_1 - \sum_{2}^{7} 8e_i - \sum_{8}^{9} 7e_j - 6e_{10} - \sum_{11}^{15} 2e_i$;

(11) $H = 27l - \sum_{1}^{6} 9e_i - \sum_{7}^{9} 7e_j - 5e_{10} - \sum_{11}^{15} 2e_i$.

A case by case computation shows the candidate linear systems are (2), (4), (5) and (8) (respectively $H\,(2)$, $H\,(1)$, $H\,(3)$ and $H\,(4)$ of Theorem 2).

As we will see, the elimination of the other systems is possible since the configuration of exceptional curves $E_1, ..., E_{10}$ we want on $\tilde{S}$ is very special, we recall in fact we are assuming the following:

Conditions:

(a) E_i is an elliptic curve $\forall i = 1, ..., 10$;

(b) $E_i.H = 0$ $\forall i = 1, ..., 10$ (the E_i's are contracted by $|H|$);

(c) $E_i^2 = -3$ (the E_i's are contracted to an ordinary triple point);

(d) $E_i.E_j = 0$ $i \neq j$, $i, j = 1, ..., 10$.

To see that the situation is rigid, here follows a detailed computation in cases (3) , (2), (4), (5) and (8).

Computations in case (3)

The presentation here is $H = 19l - 7e_1 - \sum_{2}^{6} 6e_i - \sum_{7}^{8} 5e_j - \sum_{9}^{12} 4e_k - \sum_{13}^{15} 2e_i$. We are going to show we have to exclude this H.

In order to find a configuration of 10 elliptic curves E_i satisfying conditions (a)...(d) above, we fill-up tables showing, for the given linear system H, elliptic curves in different degrees. Hence E_i could be the strict transform of a smooth plane cubic, of a quartic with two double points, of a quintic with 5 double points or with a triple point and two double points, etc...

In the next table we report **all** -3 elliptic curves of degree 3:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1
E_2	1	1	1	1	1	1	0	0	1	1	1	1	0	1	1
E_3	1	1	1	1	1	0	1	1	0	1	1	1	1	1	0
E_4	1	1	1	1	0	1	1	1	1	0	1	1	0	1	1
E_5	1	1	1	0	1	1	1	1	1	1	0	1	1	0	1

The notation in the table is the one used before, just now we are looking at the curves E_i's in the plane blown-up in the 15 points $e_1, \ldots, e_{15}$. Hence in the rows of the tables we read the elliptic curves E_i and in the i-th j-th place the multiplicities of intersection of the divisor e_i with E_i; so, for example, $E_1 = 3l - \sum_1^9 e_i - \sum_{13}^{15} e_j$, where $l = \pi^* L$, $\pi : Bl\,(\mathbb{P}^2) \dashrightarrow \mathbb{P}^2$ the blow-up map, L the class of a line of $\mathbb{P}^2$.

To compile the table above we used the following facts:

i) E_i is elliptic for every i, hence it does not have singular points, that is the multiplicity $E_j.e_i$ is 0 or 1;
ii) since $E_i^2 = -3$ and $E_i = 3l - \sum_{j=1}^{15} \beta_{ij} e_j$, we deduce $\beta_{ij} := 1$ for 12 β_{ij}'s;
iii) since we want $H.E_i = 0$, we have a finite choice of places where we can put the "0"'s.

Using the same arguments we find **all** the -3 elliptic curves coming from quartics:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1	2	2	1	1	1	1	1	1	1	1	1	0	1	1	0
E_2	2	1	2	1	1	1	1	1	1	1	0	1	1	0	1
E_3	2	1	1	2	1	1	1	1	0	1	1	1	1	0	1
E_4	2	1	1	1	1	1	1	1	2	1	1	1	1	0	0
E_5	1	1	1	1	2	1	2	1	1	1	1	1	0	0	1
E_6	1	1	1	1	1	2	1	2	1	1	1	1	0	1	0

the E_i's coming from quintics come from quintics with 5 double points, and they are:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1	2	2	2	2	1	1	1	2	1	1	1	1	1	0	0
E_2	2	1	1	2	2	2	2	1	1	1	1	1	1	0	0
E_3	2	1	2	1	2	2	1	2	1	1	1	1	0	1	0
E_4	2	2	2	1	2	1	2	1	1	1	1	1	0	0	1

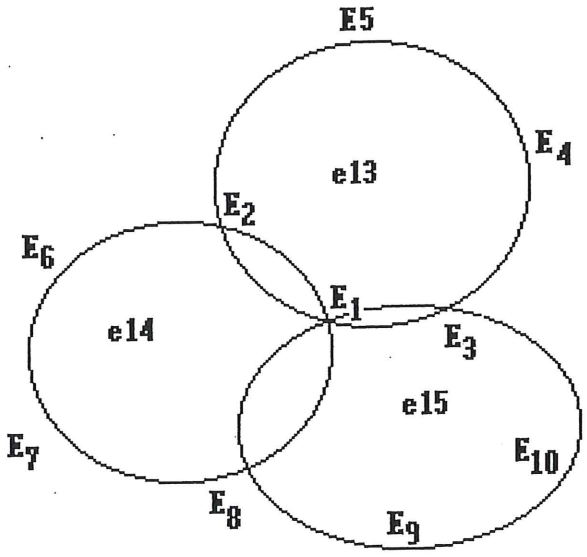
only two E_i's come from a sextic: one comes from a sextic with 9 double points, one from a sextic with a triple point and 6 double points; they are:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1 (6)	2	2	2	2	2	2	2	2	2	1	1	1	0	0	0
E_2 (6)	3	2	2	2	2	2	2	1	1	1	1	1	1	0	0

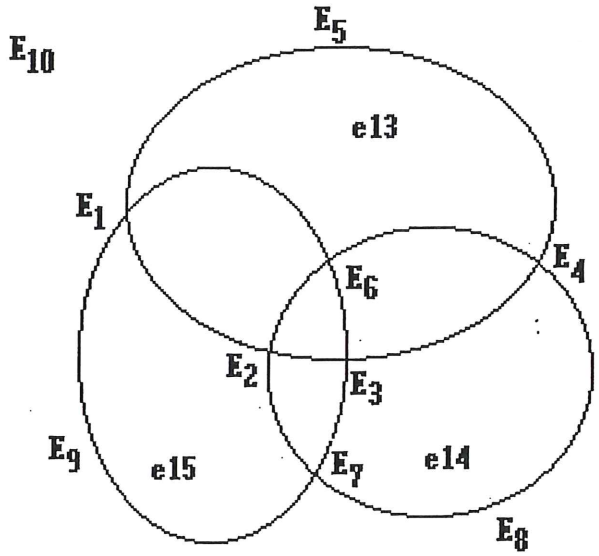
Remark 8:

In the presentation of H the three (-1) conics e_{13},e_{14} and e_{15} appear. These (-1) curves come from conics C_i, $i = 13, 14, 15$ on $S \subseteq \mathbb{P}^3$ passing through 5 triple points. It is easily seen that the planes $\langle C_i \rangle$ intersect S in $3C_i$. With this in mind it is not surprising to find the following special expressions for the hyperplane section H:

i) $\begin{cases} H = 3e_{13} + E_1 + E_2 + E_3 + E_4 + E_5 \\ H = 3e_{14} + E_1 + E_2 + E_6 + E_7 + E_8 \\ H = 3e_{15} + E_1 + E_3 + E_8 + E_9 + E_{10} \end{cases}$ in case the configuration of the (-1) conics looks like



ii) $\begin{cases} H = 3e_{13} + E_1 + E_2 + E_3 + E_4 + E_5 \\ H = 3e_{14} + E_2 + E_4 + E_6 + E_7 + E_8 \\ H = 3e_{15} + E_1 + E_3 + E_6 + E_7 + E_9 \end{cases}$ in case the configuration of the (-1) conics is



Since there are no more than 3 compatible elliptic curves (meaning they are disjoint) coming from cubics in case i) it is easily seen, using the equality $\alpha(H) = 19 = \sum_{i=1}^{5} \alpha(E_i)$, that $\alpha^+ := \max\{\alpha(E_i)\} = 6$, where we write $E_i = \alpha(E_i)\, l - \sum_{j=1}^{15} \beta_{ij} e_j \; \beta_{ij} \geq 0$. The same result holds in case ii). This is the reason why we stopped the computations of the tables for the elliptic curves E_i at $\alpha(E_i) = 6$.

The next step is to fit 10 of these curves in a configuration satisfying the compatibility condition (d) on page 60, that is $E_i.E_j = 0$, $\forall i \neq j$.

If we do not allow to use the first elliptic sextic curve $E_1(6)$, it is easy to see that we can get a configuration of just 9 compatible curves. Also using $E_1(6)$ we arrive at the same conclusion. We need 10 elliptic curves, hence we have to exclude the linear system (3) from our list.

Computations in case (2) (H (2) in Theorem 2)

As in the previous case we start giving the complete list of candidate curves we have to use in the construction of our table of the E_i's, $i = 1, \ldots, 10$.

Elliptic curves coming from cubics:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1
E_2	1	1	1	1	1	1	1	0	0	1	1	1	1	1	0
E_3	1	1	1	1	1	0	0	1	1	1	1	1	1	0	1
E_4	1	1	1	0	0	1	1	1	1	1	1	1	0	1	1
E_5	1	1	0	1	1	1	1	1	1	1	1	0	1	1	0
E_6	1	0	1	1	1	1	1	1	1	1	0	1	1	0	1
E_7	0	1	1	1	1	1	1	1	1	0	1	1	0	1	1

Coming from quartics:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1	2	2	1	1	1	1	1	1	1	1	1	0	1	1	0
E_2	2	1	2	1	1	1	1	1	1	1	0	1	1	0	1
E_3	1	2	2	1	1	1	1	1	1	0	1	1	0	1	1
E_4	1	1	1	1	1	2	1	2	1	1	1	1	1	0	0
E_5	1	1	1	2	2	1	1	1	2	1	1	1	0	1	0
E_6	1	1	1	2	1	1	2	1	1	1	1	1	0	0	1
E_7	2	1	1	1	1	1	1	1	1	2	1	1	1	0	0
E_8	1	2	1	1	1	1	1	1	1	1	2	1	0	1	0
E_9	1	1	2	1	1	1	1	1	1	1	1	2	0	0	1

Coming from quintics:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1	2	2	2	1	1	1	2	1	2	1	1	1	1	0	0
E_2	2	2	2	2	1	1	1	2	1	1	1	1	0	1	0
E_3	2	2	2	1	2	2	1	1	1	1	1	1	0	0	1

and just one coming from a sextic curve:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1	2	2	2	2	2	2	2	2	2	1	1	1	0	0	0

Remark 9:

In the notations of Remark 8, also in this case we can stop at $\alpha(E_i) = 6$.

The only possible configuration is the one in the following table:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1 (3)	1	1	0	1	1	1	1	1	1	1	1	0	1	1	0
E_2 (3)	1	0	1	1	1	1	1	1	1	1	0	1	1	0	1
E_3 (3)	0	1	1	1	1	1	1	1	1	0	1	1	0	1	1
E_4 (4)	1	1	2	1	1	1	1	1	1	1	2	1	1	0	0
E_5 (4)	2	1	1	1	1	1	1	1	1	1	1	2	0	1	0
E_6 (4)	1	2	1	1	1	1	1	1	1	2	1	1	0	0	1
E_7 (4)	2	2	1	1	1	1	1	1	1	0	1	1	1	0	1
E_8 (4)	2	1	2	1	1	1	1	1	1	1	1	0	0	1	1
E_9 (4)	1	2	2	1	1	1	1	1	1	1	0	1	1	1	0
E_{10} (6)	2	2	2	2	2	2	2	2	2	1	1	1	0	0	0

we observe that this surface could come as a degeneration from a surface with 9 triple points having configurations of -1 curves of degrees $(d_1, d_2, d_3) \in \{(2,2,2), (2,2,8), (2,4,6)\}$.

Computations in case (4) ($H(1)$ in Theorem 2)

The complete list of -3 elliptic curves, up to degree 6 is given in the following tables.

Coming from cubics:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0
E_2	1	0	0	1	1	1	1	1	1	1	1	1	1	0	1
E_3	1	1	1	0	0	1	1	1	1	1	1	1	1	1	0
E_4	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1
E_5	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1

Coming from quartics:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1	2	1	1	1	1	1	1	2	0	1	1	1	1	1	0
E_2	2	1	1	1	1	1	1	1	2	0	1	1	1	1	0
E_3	2	1	1	1	1	1	1	0	1	2	1	1	1	1	0
E_4	2	1	1	1	1	1	1	1	1	1	2	0	1	0	1
E_5	2	1	1	1	1	1	1	1	1	1	1	2	0	0	1
E_6	2	1	1	1	1	1	1	1	1	1	0	1	2	0	1

and

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_7	2	2	1	1	1	1	0	1	1	1	1	1	1	0	1
E_8	2	1	2	1	1	0	1	1	1	1	1	1	1	1	0
E_9	2	1	1	1	0	2	1	1	1	1	1	1	1	1	0
E_{10}	2	0	1	2	1	1	1	1	1	1	1	1	1	0	1
E_{11}	2	1	0	1	2	1	1	1	1	1	1	1	1	1	0
E_{12}	2	1	1	0	1	1	2	1	1	1	1	1	1	0	1
E_{13}	1	2	2	1	1	1	1	1	1	1	1	1	1	0	0

Coming from quintics:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1	2	1	2	1	2	2	2	1	1	1	1	1	1	0	0
E_2	3	2	2	1	1	1	1	1	1	1	1	1	1	1	0
E_3	3	1	1	2	2	1	1	1	1	1	1	1	1	0	1

Coming from sextics:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1	1	1	1	2	2	2	2	2	0	0	0	2	2	2	2
E_2	1	2	2	2	2	2	0	2	1	0	2	1	0	2	2
E_3	1	2	2	2	2	2	0	2	2	1	0	1	0	2	2
E_4	1	2	2	2	2	2	0	2	0	2	1	1	0	2	2
E_5	3	2	2	2	2	2	2	1	1	1	1	1	1	0	0

Using these -3 elliptic curves we find exactly two possible configurations compatible with condition (d) on page 60. They are:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1 (3)	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0
E_2 (3)	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1
E_3 (3)	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1
E_4 (4)	2	1	1	1	1	1	1	2	0	1	1	1	1	1	0
E_5 (4)	2	1	1	1	1	1	1	1	2	0	1	1	1	1	0
E_6 (4)	2	1	1	1	1	1	1	0	1	2	1	1	1	1	0
E_7 (4)	2	1	1	1	1	1	1	1	1	1	2	0	1	0	1
E_8 (4)	2	1	1	1	1	1	1	1	1	1	1	2	0	0	1
E_9 (4)	2	1	1	1	1	1	1	1	1	1	0	1	2	0	1
E_{10} (6)	3	2	2	2	2	2	2	1	1	1	1	1	1	0	0

and

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1 (3)	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1
E_2 (3)	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1
E_3 (4)	1	2	2	1	1	1	1	1	1	1	1	1	1	0	0
E_4 (4)	2	1	1	1	1	1	1	2	0	1	1	1	1	1	0
E_5 (4)	2	1	1	1	1	1	1	1	2	0	1	1	1	1	0
E_6 (4)	2	1	1	1	1	1	1	0	1	2	1	1	1	1	0
E_7 (4)	2	1	1	1	1	1	1	1	1	1	2	0	1	0	1
E_8 (4)	2	1	1	1	1	1	1	1	1	1	1	2	0	0	1
E_9 (4)	2	1	1	1	1	1	1	1	1	1	0	1	2	0	1
E_{10} (5)	2	1	1	2	2	2	2	1	1	1	1	1	1	0	0

These two tables correspond to a unique case, meaning we can go from the first to the second one using the transformation

$$\begin{aligned} e_1 &\rightarrow l - e_1 - e_2 \\ e_2 &\rightarrow l - e_1 - e_3 \\ e_3 &\rightarrow l - e_2 - e_3 \end{aligned}$$

Computations in case (8) (H (4) **in Theorem 2)**

We can perform the computations as in the previous cases, listing all the -3 elliptic curves, up to degree 7, contracted by H (4) and find finally the

following intersection table:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1 (3)	1	1	1	0	1	1	1	1	1	1	0	0	1	1	1
E_2 (3)	1	1	0	1	1	1	1	1	1	1	1	1	1	0	0
E_3 (4)	1	1	2	2	1	1	1	1	1	1	0	1	1	1	0
E_4 (4)	1	2	2	1	1	1	1	1	1	1	1	1	0	0	1
E_5 (4)	2	1	1	2	1	1	1	1	1	1	1	0	0	1	1
E_6 (6)	2	1	2	2	2	2	2	2	2	2	0	1	0	0	1
E_7 (6)	1	2	2	2	2	2	2	2	2	2	1	0	0	1	0
E_8 (7)	3	3	2	2	2	2	2	2	2	2	0	1	0	1	0
E_9 (7)	3	2	3	2	2	2	2	2	2	2	1	0	1	0	0
E_{10} (7)	2	3	2	3	2	2	2	2	2	2	0	0	1	0	1

This proves the Theorem. □

4.3.2 Cremona transformations

If $S \subseteq \mathbb{P}^3$ is a surface of degree 6 with ten triple points and we perform a Cremona transformation with fundamental points four out of the ten triple points, we get another sextic surface $S' \subseteq \mathbb{P}^3$ with ten triple points. This is the reason why in this case the Cremona transformation does not solve the problem like in the case of quintics.

Definition 2: If H,H' are the hyperplane sections of S, S', we say H, H' are linked by a Cremona transformation.

From the computations of the previous section it follows:

Corollary 1: The linear systems $H(1)$, $H(2)$, $H(3)$ and $H(4)$ in Theorem 2 are linked by Cremona transformations.

Proof.

Starting from the presentation $H(1)$ and applying a Cremona transformation based on the triple points p_4, p_5, p_7, p_8, we get $H' = 3H - 2(E_4 + E_5 + E_7 + E_8)$,

that is $H' = 22l - 8e_1 - \sum_{2}^{7} 7e_i - 2e_8 - 4e_9 - 6e_{10} - 2e_{11} - 4e_{12} - 6e_{13} - 2e_{14} - 2e_{15}$, which is linear system $H\,(3)$, and the configuration of elliptic curves is given by

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1 (3)	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0
E_2 (3)	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1
E_3 (3)	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1
E_6 (4)	2	1	1	1	1	1	1	0	1	2	1	1	1	1	0
E_9 (4)	2	1	1	1	1	1	1	1	1	1	0	1	2	0	1
E_{10} (6)	3	2	2	2	2	2	2	1	1	1	1	1	1	0	0
Γ_1 (6)	2	2	2	2	2	2	2	0	1	2	0	2	1	0	1
Γ_2 (6)	2	2	2	2	2	2	2	0	1	2	1	0	2	0	1
Γ_3 (6)	2	2	2	2	2	2	2	0	2	1	0	1	2	1	0
Γ_4 (6)	2	2	2	2	2	2	2	1	0	2	0	1	2	1	0

Moreover, applying to $H\,(1)$ a Cremona transformation with fundamental points p_4, p_5, p_8, p_{10}, we get linear system $H'' = 18l - \sum_{i=1,10,13} 6e_i - \sum_{2}^{7} 5e_i - \sum_{j=9,11,15} 4e_j - \sum_{k=8,12,14} 2e_k$, that is system $H\,(2)$ in Theorem 2. The configuration of the corresponding elliptic curves for this presentation reads like:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
E_1 (3)	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0
E_2 (3)	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1
E_3 (3)	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1
Γ_4 (4)	1	1	1	1	1	1	1	1	0	2	1	0	2	1	1
Γ_5 (4)	1	1	1	1	1	1	1	0	2	1	1	0	2	1	1
E_6 (4)	2	1	1	1	1	1	1	0	1	2	1	1	1	1	0
E_7 (4)	2	1	1	1	1	1	1	1	1	1	2	0	1	0	1
Γ_8 (4)	1	1	1	1	1	1	1	0	1	2	1	1	1	0	2
E_9 (4)	2	1	1	1	1	1	1	1	1	1	0	1	2	0	1
Γ_{10} (6)	2	2	2	2	2	2	2	0	1	2	1	0	2	0	1

Finally: starting from the presentation $H\,(4)$ and performing a Cremona with base points p_6, p_8, p_9, p_{10} we find linear system $H\,(2)$. The Corollary is proved □

In Appendix 2 we work out more in detail all possible Cremona trasformations starting from each presentation $H(i)$, $i = 1, 2, 3, 4$.

4.3.3 Necessary conditions

With the last section we have finished the computations of the "data tables" giving numerical data that the ten elliptic curves E_i, whenever they exist, must fulfil.

We did not analize the undelying geometry: however it is clear, looking through the computations so far, that the combinatorics involved is very rigid (and this is why we were able to find only four possible presentations H) and this should be reflected somehow in the geometry of the surface.

From now on we will focus our attention on the geometry of the surface, and we mean we will prove the existence on it of particular curves or pencils of curves.

The final aim of this work is, of course, to construct such a sextic surface in $\mathbb{P}^3$. To have such a construction is equivalent to construct ten smooth elliptic curves E_i and an hyperplane serie H of dimension 4 such that

$$(*) \begin{cases} \mathcal{O}_{E_j}(E_i) = \mathcal{O}_{E_j} & i \neq j \\ \mathcal{O}_{E_j}(H) = \mathcal{O}_{E_j} & \forall j \end{cases}$$

holds.

In this section we will derive from $(*)$ a lot of necessary conditions of different kind:

i) conditions the points e_i should satisfy on the curves E_i (these are direct consequences of $(*)$, see Lemma 5);

ii) existence of particular curves or pencil of curves on the surface $\tilde{S}$ (more difficult to prove and in general implied by $(*)$ and conditions in i), see Corollary 2).

A strategy to find the construction is try to reduce the conditions implied by (∗) and find an "intermediate statement" of necessary (and may be sufficient) conditions on which we have a better control.

In what follows we will use the notion of triangle of tangencies on an elliptic curve: we first give the definition, then the entire section is devoted to derive-reduce the necessary conditions. At the end of the section we state some of the necessary condition that should appear in the "intermediate statement".

Triangles of tangencies on an elliptic curve

Definition 3: Given an elliptic cubic curve $E \subseteq \mathbb{P}^2$ three distinct points $p, q, r \in E$ are a triangle of tangencies if the three tangents to E at these points form a triangle with vertexes at p, q, r.

Remark 9:

Clearly a generic choice of the points p, q, r is not a triangle of tangencies: starting from say p we draw the tangent line which intersects the cubic in a unique third point q, then we follow the tangent line to E at q and we call r the unique third point of intersection with E. Now there is no reason for the tangent line to E at r to pass through p.

What is true is the following:

Lemma 4:

a) Once a flex o is fixed as the origin for the group structure on E, p, q, r is a triangle of tangencies if and only if

i) $2p + q = o$;

ii) $2q + r = o$;

iii) $2r + p = o$.

b) In a triangle of tangencies (p, q, r), the three vertexes are 9-torsion points

with respect with the fixed group structure on the curve E.

c) If p is a 9– torsion point and i), ii) hold, then iii) holds.

Proof. a) is clear. b) From a) we have immediately that $9p = 9q = 9r = o$.

c) From a), part ii), $2r = -4q = 8p$, hence $2r + p = 9p = 0$. This shows there is a map

$$\begin{array}{ccc} \{9 \text{ torsion points}\} \setminus \{3 \text{ torsion points}\} & \to & \{3 \text{ torsion points}\} \setminus \{o\} \\ p & \mapsto & p+q+r \end{array}$$

hence for each of the eight flexes (starting from alligned points p, q, r gives the origin o) the fiber consists of 9 points, grouped in three triangles (since the role played by p, q and r is symmetric). □

Now we state some necessary conditions coming directly from $(*)$ and we remark these are **local necessary conditions** on the specified curves:

Lemma 5:

From conditions $(*)$ it follows that:

On E_1:

N1) $e_8 + e_9 + e_{10} \sim e_{11} + e_{12} + e_{13}$

N2) $2e_8 + e_{10} \sim e_8 + 2e_9 \sim e_9 + 2e_{10}$ and $2e_{11} + e_{13} \sim e_{11} + 2e_{12} \sim e_{12} + 2e_{13}$

N3) $e_{10} + e_{12} = e_8 + e_{11}$;

On E_2:

N4) $3e_{14} \sim e_{11} + e_{12} + e_{13} \sim 3e_{15}$ and $3e_1 \sim e_{11} + e_{12} + e_{13}$

N5) $2e_1 \sim e_{14} + e_{15}$

N6) $2e_{11} + e_{13} \sim 2e_{12} + e_{11} \sim 2e_{13} + e_{12}$

N7) $e_{11} + e_{15} \sim e_{12} + e_{14}$;

On E_3:

N8) $e_8 + e_9 + e_{10} \sim e_1 + e_{14} + e_{15}$

N9) $2e_8 + e_{10} \sim e_8 + 2e_9 \sim e_9 + 2e_{10}$

N10) $2e_8 + e_{10} + e_{14} \sim e_8 + e_9 + e_{10} + e_{15}$

N11) $e_8 + e_9 + e_{10} \sim 3e_1$;

On E_4:

N12) $e_{11} + e_{12} + e_{13} \sim e_8 + e_{10} \sim e_1 + e_{14}$

N13) $2e_{11} + e_{13} \sim e_{11} + 2e_{12} \sim e_{12} + 2e_{13}$

N14) $e_{10} + e_{12} + e_{14} \sim e_8 + e_{11}$

N15) $2e_{10} \sim e_8$

N16) $e_8 + e_{10} + e_{11} + e_{12} + e_{13} \sim 3e_1$

N17) $e_1 \sim 2e_{14}$.

On E_5:

N18) $e_{11} + e_{12} + e_{13} \sim e_8 + e_9 \sim e_1 + e_{14}$

N19) $2e_{11} + e_{13} \sim e_{11} + 2e_{12} \sim e_{12} + 2e_{13}$

N20) $e_{12} + e_{14} \sim e_8 + e_{11}$

N21) $2e_8 \sim e_9$

N22) $e_8 + e_{10} + e_{11} + e_{12} + e_{13} \sim 3e_1$

N23) $e_1 \sim 2e_{14}$.

On E_6:

N24) $e_9 + e_{10} \sim e_1 + e_{14} \sim e_{11} + e_{12} + e_{13}$

N25) $e_{10} \sim 2e_9$

N26) $e_{12} + e_{14} \sim e_9 + e_{11}$

N27) $2e_{11} + e_{13} \sim e_{11} + 2e_{12} \sim e_{12} + 2e_{13}$

N28) $e_1 \sim 2e_{14}$

On E_7:

N29) $e_{11} + e_{13} \sim e_1 + e_{15} \sim e_8 + e_9 + e_{10}$

N30) $e_{11} \sim 2e_{13}$

N31) $e_{10} + e_{13} \sim e_8 + e_{15}$

N32) $2e_8 + e_{10} \sim e_8 + 2e_9 \sim e_9 + 2e_{10}$

N33) $e_1 \sim 2e_{15}$

On E_8:

N34) $e_{11} + e_{12} \sim e_1 + e_{15} \sim e_8 + e_9 + e_{10}$

N35) $e_{12} \sim 2e_{11}$

N36) $e_{10} + e_{11} \sim e_8 + e_{15}$

N37) $2e_8 + e_{10} \sim e_8 + 2e_9 \sim e_9 + 2e_{10}$

N38) $e_1 \sim 2e_{15}$

On E_9:

N39) $e_{12} + e_{13} \sim e_1 + e_{15} \sim e_8 + e_9 + e_{10}$

N40) $e_{13} \sim 2e_{12}$

N41) $e_{10} + e_{12} \sim e_8 + e_{15}$

N42) $2e_8 + e_{10} \sim e_8 + 2e_9 \sim e_9 + 2e_{10}$

N43) $e_1 \sim 2e_{15}$

On E_{10}:

N44) $e_1 \sim e_8 + e_9 + e_{10} \sim e_{11} + e_{12} + e_{13}$

N45) $2e_8 + e_{10} \sim e_8 + 2e_9 \sim e_9 + 2e_{10}$ and $2e_{11} + e_{13} \sim e_{11} + 2e_{12} \sim e_{12} + 2e_{13}$

N46) $3(e_8 + e_9 + e_{10}) = 3(2e_{11} + e_{13}) = 3(2e_8 + e_{10})$

N47) $e_8 = e_{12}$

N48) $2e_{11} \sim e_{12} + e_{13}$, $2e_{12} \sim e_{11} + e_{13}$, $2e_{13} \sim e_{11} + e_{12}$;

Proof.

We work explicitly the equivalences involving E_{10} and E_2 (equivalences for the other curves are found in exactly the same way).

On E_{10}: looking at $E_1, E_2, E_3 \in \left|3l - \sum_2^7 e_i\right|$ we find, recalling that $E_1|_{E_{10}} \sim E_2|_{E_{10}} \sim E_3|_{E_{10}}$ as divisors, that

$$e_8 + e_9 + e_{10} + e_{11} + e_{12} + e_{13} \sim e_1 + e_{11} + e_{12} + e_{13} \sim e_1 + e_8 + e_9 + e_{10}$$

that is

$$e_1 \sim e_{11} + e_{12} + e_{13} \sim e_8 + e_9 + e_{10}.$$

Looking at $E_4, E_5, \ldots, E_9 \in \left|4l - 2e_1 - \sum_2^7 e_i\right|$ we find

$$\begin{aligned}
e_1 + 2e_8 + e_{10} + \sum_{11}^{13} e_i &\sim e_1 + e_8 + 2e_9 + \sum_{11}^{13} e_i \sim \\
e_1 + e_9 + 2e_{10} + \sum_{11}^{13} e_i &\sim e_1 + \sum_{8}^{10} e_i + 2e_{11} + e_{13} \sim \\
e_1 + \sum_{8}^{10} e_i + e_{11} + 2e_{12} &\sim e_1 + \sum_{8}^{10} e_i + e_{12} + 2e_{13} \\
&\vdots
\end{aligned}$$

that we can rewrite as

$$\begin{aligned}
2e_8 + e_{10} &\sim e_8 + 2e_9 \sim e_9 + 2e_{10} \\
2e_{11} + e_{13} &\sim e_{11} + 2e_{12} \sim e_{12} + 2e_{13} \\
e_{12} &\sim e_8 \\
e_{10} + e_{12} &\sim e_8 + e_{11} \\
e_{10} + e_{13} &\sim e_8 + e_{12} \\
e_{10} + e_{11} &\sim e_8 + e_{13}
\end{aligned}$$

Finally we can rewrite the last three equivalences as

$$\begin{aligned}
2e_{12} &\sim e_{11} + e_{13} \\
2e_{13} &\sim e_{11} + e_{12}
\end{aligned}$$

this completes the proof of N44),...,N48).

On E_2: looking at $E_1, E_3 \in \left|3l - \sum_2^7 e_i\right|$ we find:

$$e_{11} + e_{12} + e_{13} \sim e_1 + e_{14} + e_{15}$$

Restricting to E_2 the curves $E_4, \ldots, E_9 \in \left|4l - 2e_1 - \sum_2^7 e_i\right|$ we get:

$$e_{11} + e_{12} + e_{13} + e_{14} \sim 2e_{11} + e_{13} + e_{15} \sim e_{11} + 2e_{12} + e_{15} \sim e_{12} + 2e_{13} + e_{15}$$

that is

$$\begin{aligned} 2e_{11} + e_{13} &\sim 2e_{12} + e_{11} \sim 2e_{13} + e_{12} \\ e_{11} + e_{15} &\sim e_{12} + e_{14} \end{aligned}$$

Furthermore from $E_{10} \sim 2E_1$ when we consider them in $\left|3l - \sum_2^7 e_i\right|$, we find

$$2\left(e_{11} + e_{12} + e_{13}\right) \sim 3e_1 + e_{11} + e_{12} + e_{13}$$

that is

$$3e_1 \sim e_{11} + e_{12} + e_{13}.$$

□

Now we want to find **global necessary conditions**, in the sense we will derive form Lemma 5 the existence of curves on $\tilde{S}$ (or even of pencils of curves).

Corollary 2:

From the equivalences of Lemma 5 it follows that:

GN1) The following linear systems on $\tilde{S}$ are pencils of curves:

i) $\left|6l - \sum_1^7 2e_i - e_9 - 2e_{10} - 2e_{12} - e_{13}\right|$, where one of the tangents of the sextics in e_{10} and e_{12} is the tangent direction of E_{10};

ii) $\left|6l - \sum_1^7 2e_i - e_9 - 2e_{10} - e_{11} - 2e_{13}\right|$, where one of the tangents of the sextics in e_{10} and e_{13} is the tangent direction of E_{10};

iii) $\left|6l - \sum_1^7 2e_i - 2e_9 - e_{10} - e_{12} - 2e_{13}\right|$, where one of the tangents of the sextics in e_9 and e_{13} is the tangent direction of E_{10}.

GN2) On $\tilde{S}$ there is a pencil of curves $\left|6l - \sum_{2}^{7} 2e_i\right|$ having the same tangent directions as E_{10} in e_i, $i = 2, ..., 7$;
GN3) On $\tilde{S}$ there is a curve in $\left|3l - \sum_{2}^{7} e_i\right|$ with a triple contact in e_1 with E_2 and a curve with a triple contact in e_1 with E_3.

Proof.

Remember we are assuming we have the ten elliptic smooth curves E_i, $i = 1, \ldots, 10$.
GN1) Let $D \in \left|6l - \sum_{1}^{7} 2e_i\right|$. We give the precise argument to prove that the linear system $|D - e_9 - 2e_{10} - 2e_{12} - e_{13}|$ of sextics with the same tangent direction of E_{10} in e_{10} and e_{12} is a pencil of curves on $\tilde{S}$: From the decomposition $(E_2 + E_3 + E_6 - E_7)\,|_{E_{10}} = (D - e_9 - 2e_{10} - 2e_{12} - e_{13})\,|_{E_{10}}$ and $(*)$ we deduce $h^0\left(\mathcal{O}_{E_{10}}\left(D - e_9 - 2e_{10} - 2e_{12} - e_{13}\right)\right) = 1$.

Blowing-up e_{10} and e_{12} on $\tilde{S}$ we get divisors that by abuse of notation we call e_{10} and e_{12}: we denote by f_{10} and f_{12} the two points on e_{10}, e_{12} giving the tangent directions of E_{10} at e_{10}, e_{12}. After blowing-up f_{10} and f_{12} we consider divisors of the form $\bar{D} = D - e_9 - 2e_{10} - 2e_{12} - e_{13} - \sum_{1}^{2}(g_{10,i} + h_{12,i})$.

From the exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}\left(-f_{10} - f_{12} + \sum_{1}^{2}(g_{10,i} + h_{12,i})\right) \to \mathcal{O}_{\tilde{S}}\left(\bar{D}\right) \to \mathcal{O}_{E_{10}}\left(\bar{D}\right) \to 0$$

we see that if $\bar{D}$ has the same tangent directions of E_{10} at e_{10}, e_{12} then $h^0\left(\mathcal{O}_{\tilde{S}}\left(-f_{10} - f_{12} + \sum_{1}^{2}(g_{10,i} + h_{12,i})\right)\right) = 1$ and
$h^1\left(\mathcal{O}_{\tilde{S}}\left(-f_{10} - f_{12} + \sum_{1}^{2}(g_{10,i} + h_{12,i})\right)\right) = 0$. Hence, since $h^0\left(\mathcal{O}_{E_{10}}\left(\bar{D}\right)\right) = 1$, we get $h^0\left(\mathcal{O}_{\tilde{S}}\left(\bar{D}\right)\right) = 2$.
To prove the result for the other linear systems in the statement one can use

the following relations:

$$(E_2 + E_3 + E_6 - E_8)\,|_{E_{10}} = (D - e_9 - 2e_{10} - e_{11} - 2e_{13})\,|_{E_{10}}$$
$$(E_2 + E_3 + E_9 - E_4)\,|_{E_{10}} = (D - 2e_9 - e_{10} - e_{12} - 2e_{13})\,|_{E_{10}}$$

GN2) From $\left(4l - 2e_1 - \sum_2^7 e_i\right)|_{E_{10}} = e_{11}+e_{12}+e_{13}+2e_8+e_9$ and $e_8+e_9+e_{10} = \left(3l - \sum_1^7 e_i\right)|_{E_{10}}$ it follows $(2e_8 + e_9) = (l - e_1)\,|_{E_{10}}$. Using N46) of Lemma 5, we get $\left(6l - \sum_2^7 3e_i\right)|_{E_{10}} = \mathcal{O}_{E_{10}}$.

Now we blow-up the six points $e_2, ..., e_7$ in $\mathbb{P}^2$ and we consider the two points f_{i1}, f_{i2} on the exceptional divisors e_i parametrizing the tangent directions at e_i on $\tilde{S}$, corresponding to the tangent directions of E_{10} at the e_i's. After blowing-up the f_{ij}'s we write $\tilde{E}_{10} \sim 6l - 3e_1 - \sum_2^7 2e_i - \sum_2^7 (f_{i1} + f_{i2})$, where $e_i.\tilde{E}_{10} = 2$ and $f_{ij}.\tilde{E}_{10} = 1$.

From the chain of equivalences

$$\begin{array}{ccc}
\left(6l - \sum_2^7 3e_i\right)|_{E_{10}} & \sim & \left(6l - \sum_2^7 2e_i - \sum_2^7 (f_{i1} + f_{i2})\right)|_{E_{10}} \\
\Updownarrow & & \\
\left(\sum_2^7 3e_i\right)|_{E_{10}} & \sim & \left(\sum_2^7 2e_i + \sum_2^7 (f_{i1} + f_{i2})\right)|_{E_{10}} \\
\Updownarrow & & \\
\left(\sum_2^7 e_i\right)|_{E_{10}} & \sim & \left(\sum_2^7 (f_{i1} + f_{i2})\right)|_{E_{10}}
\end{array}$$

we deduce that the condition "have triple contact at $e_2, \ldots, e_7$ with E_{10}" is fullfilled by a curve having double points at the e_i's and having the same tangent directions as E_{10} at the e_i's. Finally, from the exact sequence

$$0 \to \mathcal{O}_{\tilde{S}} \to \mathcal{O}_{\tilde{S}}\left(6l - \sum_2^7 2e_i - \sum_2^7 (f_{i1} + f_{i2})\right) \to$$
$$\to \mathcal{O}_{E_{10}}\left(6l - \sum_2^7 2e_i - \sum_2^7 (f_{i1} + f_{i2})\right) \to 0$$

we deduce, since $q(S) = 0$ and $h^0\left(\mathcal{O}_{E_{10}}\left(6l - \sum_2^7 2e_i - \sum_2^7 (f_{i1} + f_{i2})\right)\right) = 1$, that $h^0\left(\mathcal{O}_S\left(6l - \sum_2^7 2e_i - \sum_2^7 (f_{i1} + f_{i2})\right)\right) = 2$.

GN3) On E_2 we have $3e_{14} \sim 3e_{15} \sim e_{11} + e_{12} + e_{13}$, hence also $3(e_{14} + e_{15}) \sim 2(e_{11} + e_{12} + e_{13})$ holds. Since $e_{14} + e_{15} = E_3|_{E_2}$ and $e_{11} + e_{12} + e_{13} = E_1|_{E_2}$, we have

$$3\left(3l - \sum_1^7 e_i\right) \sim_{E_2} 2\left(3l - \sum_1^7 e_i\right) \text{ that is } \mathcal{O}_{E_2}\left(3l - 3e_1 - \sum_2^7 e_i\right) \sim \mathcal{O}_{E_2}$$

This means that in the linear system $\left|3l - \sum_2^7 e_i\right|$ there is a curve with a triple contact in e_1 with E_2.
Since the relation $3(e_{14} + e_{15}) \sim 2(e_{11} + e_{12} + e_{13})$ holds also on the curve E_3 we are done. □

The next step is to prove that conditions $(*)$ are sufficient to define a map $\tilde{S} \dashrightarrow \mathbb{P}^3$: this is the meaning of the following

Proposition 2: Conditions $(*)$ are sufficient to define a map $\tilde{S} \dashrightarrow \mathbb{P}^3$.

Proof.
Existence of four sections: Assuming conditions $(*)$ hold for $H(3)$, we have the special decompositions

$$\begin{aligned} H(3) &= E_1 + E_3 + E_9 + E_{10} + \Gamma_4 + 3e_8 = \qquad ((\#)) \\ &= E_1 + E_2 + E_6 + E_{10} + \Gamma_2 + 3e_{11} = \\ &= E_2 + E_3 + E_6 + \Gamma_3 + \Gamma_4 + 3e_{14} = \\ &= E_2 + E_3 + E_9 + \Gamma_1 + \Gamma_2 + 3e_{15} \end{aligned}$$

giving exact sequences

$$\begin{array}{l} 0 \to \mathcal{O}_S(3e_8) \to \mathcal{O}_S(H(3)) \to \mathcal{O}_{E_1+E_3+E_9+E_{10}+\Gamma_4}(H(3)) \to 0 \\ 0 \to \mathcal{O}_S(3e_{11}) \to \mathcal{O}_S(H(3)) \to \mathcal{O}_{E_1+E_2+E_6+E_{10}+\Gamma_2}(H(3)) \to 0 \\ 0 \to \mathcal{O}_S(3e_{14}) \to \mathcal{O}_S(H(3)) \to \mathcal{O}_{E_2+E_3+E_6+\Gamma_3+\Gamma_4}(H(3)) \to 0 \\ 0 \to \mathcal{O}_S(3e_{15}) \to \mathcal{O}_S(H(3)) \to \mathcal{O}_{E_2+E_3+E_9+\Gamma_1+\Gamma_2}(H(3)) \to 0 \end{array}$$

Hence from the injections $H^0(\mathcal{O}_S(3e_8)) \simeq \mathbb{C} \hookrightarrow H^0(\mathcal{O}_S(H(3)))$ we get four sections $H_{e_i} \in H^0(\mathcal{O}_S(H(3)))$. Now we want to prove that the H_{e_i} $(i = 8, 11, 14, 15)$ are independent. To this aim it is enough to observe that $H_{e_{14}}$ and $H_{e_{15}}$ are independent since the last vanish along Γ_1 while the first does not. Analogously $H_{e_{15}}, H_{e_{14}}$ and $H_{e_{11}}$ are independent since the first two vanish on E_3 while the last does not. Finally H_{e_8} is independent from the other three sections since it does not vanish on E_2. □

Proposition 3: The map given by $|H(3)|$ is birational.

Proof.

$|H(3)|$ has no base points: such a point can not sit on any of the ten elliptic curves E_i's (a base point p would be in the zero section of every element in $H^0(\mathcal{O}_S(H(3))^*)$, moreover the sections are constant and nonzero on the E_i's, since they are contracted by $H(3)$). Hence, recalling the special decompositions $(\#)$ in the proof of Proposition 2, such a base point should sit in the intersection $e_8 \cap e_{11} \cap e_{14} \cap e_{15}$ which is empty.

The map $\varphi_{|H(3)|}$ is birational: since $(H(3))^2 = 6$, $\varphi_{|H'|}$ could be $3:1$ on a quadric or $2:1$ on a cubic surface of $\mathbb{P}^3$. We will show this is impossible, so that the map is birational.

i) $\varphi_{|H'|}$ is not $3:1$ on a quadric. Start from the sequence

$$0 \to \mathcal{O}_S(2e_{15}) \to \mathcal{O}_S(H(3) - e_{15}) \to \mathcal{O}_{E_2+E_3+E_9+\Gamma_1+\Gamma_2}(H(3) - e_{15}) \to 0$$

we deduce $h^0(\mathcal{O}_S(H(3) - e_{15})) = 1$, equivalently $\varphi_{|H(3)|}(e_{15})$ is a smooth conic, that is $\varphi_{|H(3)||e_{15}}$ is $1:1$. Hence $\varphi^{-1}_{|H(3)|}(\varphi_{|H(3)|})(e_{15}) = e_{15} + C$, where

$\varphi_{|H(3)|} : C \overset{2:1}{\to} \varphi_{|H(3)|}(e_{15})$. Pulling back the two families $\{R_s\}$ and $\{L_t\}$ of lines sitting on the quadric we get two basepoint free pencils of curves $\{\Gamma_i\}$ and $\{\Phi_j\}$ on S such that $\Gamma_i \overset{2:1}{\to} R_s$ and $\Phi_j \overset{2:1}{\to} L_t$. If we look at the curves $\{\Gamma_j\}$ in $\mathbb{P}^2$ we see they can come from conics or cubics through the 4 points e_i, $i = 8, 11, 14, 15$. It is easy to see the latter is impossible. Hence there is just one pencil $\{\Gamma_j\}$, a contradiction.

ii) The proof is as in i) (just fix two lines on the cubis surface and consider, instead of pencil of lines, pencil of conics.

We conclude the map given by $H(3)$ is birational and its image is a sextic surface with ten triple points. □

Remark 10:

The ten triple points coming from the contraction of the elliptic curves E_i and Γ_j (by (**)), are not on a quadric: this is equivalent to say, since the map $\varphi_{|H(3)|}$ is birational, we can not have an expression like

$$2H(3) = E_1 + E_2 + E_3 + E_6 + E_9 + E_{10} + \sum_1^4 \Gamma_i + C$$

where C is an effective divisor on S). Recalling the expressions of the ten curves involved we get an expression of the kind $47l-..$on the right hand side and $36l$ on the left hand side, giving the desired contradiction.

4.4 Appendix 1:

4.4.1 The complete list of candidate linear systems

Here follows the complete list of the 62 possible numerical types for the hyperplane section H of $\tilde{S}$:

$$H = 11\,(1,1) - \sum_{1}^{6} 5e_i - \sum_{7}^{8} 4e_j - \sum_{9}^{14} 3e_k$$

$$H = 18l - \sum_{1}^{6} 6e_i - \sum_{7}^{9} 4e_j - \sum_{10}^{15} 3e_k$$

$$H = 18l - \sum_{1}^{5} 6e_i - \sum_{6}^{8} 5e_j - \sum_{9}^{15} 3e_k$$

$$H = 11\,(1,1) - \sum_{1}^{5} 5e_i - \sum_{6}^{10} 4e_j - \sum_{11}^{13} e_k - 2e_{14}$$

$$H = 17l - 6e_1 - \sum_{2}^{8} 5e_i - \sum_{9}^{10} 4e_j - \sum_{11}^{14} 3e_k - 2e_{15}$$

$$H = 18l - \sum_{1}^{5} 6e_i - \sum_{6}^{7} 5e_j - \sum_{8}^{10} 4e_k - \sum_{11}^{14} 3e_r - 2e_{15}$$

$$H = 18l - \sum_{1}^{4} 6e_i - \sum_{5}^{9} 5e_j - \sum_{10}^{14} 3e_k - 2e_{15}$$

$$H = 19l - 7e_1 - \sum_{2}^{7} 6e_i - 5e_8 - 4e_9 - \sum_{10}^{14} 3e_j - 2e_{15}$$

$$H = 11\,(1,1) - \sum_{1}^{4} 5e_i - \sum_{5}^{12} 4e_j - \sum_{13}^{14} 2e_k$$

$$H = 17l - 6e_1 - \sum_{2}^{7} 5e_i - \sum_{8}^{12} 4e_j - 3e_{13} - \sum_{14}^{15} 2e_k$$

$$H = 18l - \sum_{1}^{5} 6e_i - 5e_6 - \sum_{7}^{12} 4e_j - 3e_{13} - \sum_{14}^{15} 2e_k$$

$$H = 19l - 7e_1 - \sum_{2}^{7} 6e_i - \sum_{8}^{11} 4e_j - \sum_{12}^{13} 3e_k - \sum_{14}^{15} 2e_r$$

$$H = 19l - 7e_1 - \sum_{2}^{6} 6e_i - \sum_{7}^{9} 5e_j - 4e_{10} - \sum_{11}^{13} 3e_k - \sum_{14}^{15} 2e_r$$

$$H = 21l - \sum_{1}^{6} 7e_i - \sum_{7}^{8} 6e_j - 5e_9 - \sum_{10}^{15} 3e_k$$

$$H = 18l - \sum_{1}^{3} 6e_i - \sum_{4}^{9} 5e_j - \sum_{10}^{12} 4e_k - \sum_{13}^{15} 2e_r$$

$$H = 19l - 7e_1 - \sum_{2}^{6} 6e_i - \sum_{7}^{8} 5e_j - \sum_{9}^{12} 4e_k - \sum_{13}^{15} 2e_r$$

$$H = 19l - \sum_{1}^{7} 6e_i - \sum_{8}^{9} 5e_j - \sum_{10}^{11} 4e_k - 3e_{12} - \sum_{13}^{15} 2e_r$$

$$H = 19l - 7e_1 - \sum_{2}^{7} 6e_i - \sum_{8}^{11} 4e_j - \sum_{12}^{13} 3e_k - \sum_{14}^{15} 2e_r$$

$$H = 21l - \sum_{1}^{6} 7e_i - \sum_{7}^{8} 6e_j - \sum_{9}^{11} 4e_k - 3e_{12} - \sum_{13}^{15} 2e_r$$

$$H = 21l - \sum_{1}^{5} 7e_i - \sum_{6}^{9} 6e_j - 4e_{10} - \sum_{11}^{12} 3e_k - \sum_{13}^{15} 2e_r$$

$$H = 22l - 8e_1 - \sum_{2}^{8} 7e_i - 5e_9 - 4e_{10} - \sum_{11}^{12} 3e_j - \sum_{13}^{15} 2e_k$$

$$H = 24l - \sum_{1}^{6} 8e_i - \sum_{7}^{9} 7e_j - \sum_{10}^{12} 3e_k - \sum_{13}^{15} 2e_r$$

$$H = 22l - 8e_1 - \sum_{2}^{7} 7e_i - \sum_{8}^{9} 6e_j - \sum_{10}^{11} 4e_k - \sum_{12}^{15} 2e_r$$

$$H = 25l - 9e_1 - \sum_{2}^{8} 8e_i - 7e_9 - 4e_{10} - 3e_{11} - \sum_{12}^{15} 2e_r$$

$$H = 11\,(1,1) - \sum_{1}^{3} 5e_i - \sum_{4}^{13} 4e_j - e_{14}$$

$$H = 17l - 6e_1 - \sum_{2}^{6} 5e_i - \sum_{7}^{13} 4e_j - 3e_{14} - e_{15}$$

$$H = 18l - \sum_{1}^{5} 6e_i - \sum_{6}^{13} 4e_j - 3e_{14} - e_{15}$$

$$H = 18l - \sum_{1}^{4} 6e_i - \sum_{5}^{7} 5e_j - \sum_{8}^{12} 4e_k - \sum_{13}^{14} 3e_r - e_{15}$$

$$H = 18l - \sum_{1}^{3} 6e_i - \sum_{4}^{9} 5e_j - \sum_{10}^{11} 4e_k - \sum_{12}^{14} 3e_r - e_{15}$$

$$H = 19l - 7e_1 - \sum_{2}^{6} 6e_i - \sum_{7}^{8} 5e_j - \sum_{9}^{11} 4e_k - \sum_{12}^{14} 3e_r - e_{15}$$

$$H = 19l - \sum_{1}^{7} 6e_i - \sum_{8}^{9} 5e_j - 4e_{10} - \sum_{11}^{14} 3e_k - e_{15}$$

$$H = 21l - \sum_{1}^{6} 7e_i - \sum_{7}^{8} 6e_j - \sum_{9}^{10} 4e_k - \sum_{11}^{14} 3e_r - e_{15}$$

$$H = 22l - 8e_1 - \sum_{2}^{8} 7e_i - 5e_9 - \sum_{10}^{14} 3e_j - e_{15}$$

$$H = 21l - \sum_{1}^{5} 7e_i - \sum_{6}^{9} 6e_j - \sum_{10}^{14} 3e_k - e_{15}$$

$$H = 18l - \sum_{1}^{3} 6e_i - \sum_{4}^{8} 5e_j - \sum_{9}^{13} 4e_k - 2e_{14} - e_{15}$$

$$H = 19l - 7e_1 - \sum_{2}^{6} 6e_i - 5e_7 - \sum_{8}^{13} 4e_j - 2e_{14} - e_{15}$$

$$H = 19l - 7e_1 - \sum_{2}^{5} 6e_i - \sum_{6}^{9} 5e_j - \sum_{10}^{12} 4e_k - 3e_{13} - 2e_{14} - e_{15}$$

$$H = 19l - \sum_{1}^{7} 6e_i - 5e_8 - \sum_{9}^{12} 4e_j - 3e_{13} - 2e_{14} - e_{15}$$

$$H = 21l - \sum_{1}^{6} 7e_i - 6e_7 - \sum_{8}^{9} 5e_j - \sum_{10}^{11} 4e_k - \sum_{12}^{13} 3e_r - 2e_{14} - e_{15}$$

$$H = 13\,(1,1) - \sum_{1}^{7} 6e_i - 5e_8 - \sum_{9}^{10} 4e_j - \sum_{11}^{12} 3e_k - 2e_{13} - e_{14}$$

$$H = 22l - 8e_1 - \sum_{2}^{8} 7e_i - \sum_{9}^{11} 4e_j - \sum_{12}^{13} 3e_k - 2e_{14} - e_{15}$$

$$H = 22l - 8e_1 - \sum_{2}^{7} 7e_i - \sum_{8}^{9} 6e_j - 4e_{10} - \sum_{11}^{13} 3e_k - 2e_{14} - e_{15}$$

$$H = 25l - 9e_1 - \sum_{2}^{8} 8e_i - 7e_9 - \sum_{10}^{13} 3e_j - 2e_{14} - e_{15}$$

$$H = 22l - \sum_{1}^{8} 7e_i - 6e_9 - \sum_{10}^{11} 4e_j - 3e_{12} - \sum_{13}^{14} 2e_k - e_{15}$$

$$H = 25l - \sum_{1}^{9} 8e_i - 4e_{10} - \sum_{11}^{12} 3e_j - \sum_{13}^{14} 2e_k - e_{15}$$

$$H = 27l - \sum_{1}^{6} 9e_i - \sum_{7}^{9} 8e_j - \sum_{10}^{11} 4e_k - \sum_{12}^{14} 2e_r - e_{15}$$

$$H = 22l - 8e_1 - \sum_{2}^{7} 7e_i - 6e_8 - 5e_9 - \sum_{10}^{12} 4e_j - 3e_{13} - \sum_{14}^{15} 2e_k$$

$$H = 21l - \sum_{1}^{4} 7e_i - \sum_{5}^{9} 6e_j - \sum_{10}^{12} 4e_k - 3e_{13} - \sum_{14}^{15} 2e_r$$

$$H = 24l - \sum_{1}^{6} 8e_i - \sum_{7}^{8} 7e_j - 6e_9 - \sum_{10}^{11} 4e_k - \sum_{12}^{13} 3e_r - \sum_{14}^{15} e_s$$

$$H = 15\,(1,1) - \sum_{1}^{8} 7e_i - \sum_{9}^{10} 4e_j - \sum_{11}^{12} 3e_k - \sum_{13}^{14} e_r$$

$$H = 27l - \sum_{1}^{6} 9e_i - \sum_{7}^{9} 8e_j - 4e_{10} - \sum_{11}^{13} 3e_k - \sum_{14}^{15} e_r$$

$$H = 24l - \sum_{1}^{5} 8e_i - \sum_{6}^{9} 7e_j - \sum_{10}^{12} 4e_k - 2e_{13} - \sum_{14}^{15} e_r$$

$$H = 25l - 9e_1 - \sum_{2}^{8} 8e_i - 6e_9 - \sum_{10}^{12} 4e_j - 2e_{13} - \sum_{14}^{15} e_k$$

$$H = 28l - 10e_1 - \sum_{2}^{8} 9e_i - 8e_9 - \sum_{10}^{11} 4e_j - 3e_{12} - 2e_{13} - \sum_{14}^{15} e_k$$

$$H = 21l - \sum_{1}^{5} 7e_i - \sum_{6}^{8} 6e_j - \sum_{9}^{13} 4e_k - \sum_{14}^{15} e_r$$

$$H = 30l - \sum_{1}^{6} 10e_i - \sum_{7}^{9} 9e_j - \sum_{10}^{12} 4e_k - \sum_{13}^{15} e_r$$

$$H = 21l - \sum_{1}^{6} 7e_i - \sum_{7}^{9} 5e_j - \sum_{10}^{13} 4e_k - \sum_{14}^{15} e_r$$

$$H = 13\,(1,1) - \sum_{1}^{6} 6e_i - \sum_{7}^{8} 5e_j - \sum_{9}^{12} 4e_k - \sum_{13}^{14} e_r$$

$$H = 25l - 9e_1 - \sum_{2}^{7} 8e_i - \sum_{8}^{9} 7e_j - 6e_{10} - \sum_{11}^{15} 2e_k$$

$$H = 24l - \sum_{1}^{4} 8e_i - \sum_{5}^{10} 7e_j - \sum_{11}^{15} 2e_r$$

$$H = 27l - \sum_{1}^{6} 9e_i - \sum_{7}^{9} 7e_j - 5e_{10} - \sum_{11}^{15} 2e_i.$$

$$H = 27l - \sum_{1}^{7} 9e_i - 8e_8 - \sum_{9}^{10} 6e_j - \sum_{11}^{15} 2e_r$$

4.4.2 Cremona transformations

Here we look more carefully at the Cremona transformations applied to the three presentations of H given in Theorem 2.

Notations: **the type of a (-3) elliptic curve** E_i is the unique triplet (d_1, d_2, d_3) where the d_i's are the degrees of the configuration of (-1) curves disjoint from E_i. **The type of a Cremona transformation** with fundamental points $p_1, \ldots, p_4$, where the p_i come from the exceptional divisor E_i, is the set consisting of the four types of $E_1, \ldots, E_4$. When we write $(d_1, d_2, d_3)^n$ in the type of a Cremona transformation we mean we have used n curves

having the same type (d_1, d_2, d_3).

(A) Starting from $H(1) = 18l - 8e_1 - \sum_{2}^{7} 5e_i - \sum_{8}^{13} 4e_j - \sum_{14}^{15} 2e_k$ in Proposition 6, we get the following linear systems:

1) $H' = 24l - 14e_1 - \sum_{2}^{7} 5e_i - \sum_{8}^{13} 6e_j - \sum_{14}^{15} 2e_k$ coming from a Cremona of type $\left\{(228)^2, (444)^2\right\}$, that reduces, with the base change $l' = 4l - 3e_1 - \sum_{8}^{13} e_i$, to $H'' = 18l - 8e'_1 - \sum_{2}^{7} 5e_i - \sum_{8}^{13} 4e'_j - \sum_{14}^{15} 2e_k$, that is again linear system $H(1)$;

2) $H' = 26l - 14e_1 - 8e_9 - \sum_{2}^{7} 7e_i - \sum_{10,12} 6e_j - \sum_{8,13} 4e_k - \sum_{11,14,15} 2e_i$ coming from a Cremona of type $\left\{(228), (444), (246)^2\right\}$. After a base change it reduces to linear system $H(2)$ of Theorem 2;

3) $H' = 22l - 14e_1 - 8e_9 - \sum_{2}^{7} 5e_i - 6e_{10} - \sum_{8,11,12,13,15} 4e_j - 2e_{14}$ from a Cremona of type $\left\{(228)^2, (444), (246)\right\}$ reduces to $H(1)$;

4) $H' = 20l - \sum_{1,9} 8e_i - \sum_{2}^{7} 5e_i - \sum_{10,12} 6e_j - \sum_{8,13} 4e_k - \sum_{11,14,15} 2e_i$ from a Cremona of type $\left\{(228), (444), (246)^2\right\}$ reduces to $H(2)$ of Theorem 2;

5) $H' = 24l - 12e_1 - \sum_{2}^{7} 7e_i - \sum_{10,12} 6e_j - \sum_{9,13,15} 4e_k - \sum_{8,11,14} 2e_i$ from a Cremona of type $\left\{(228), (246)^3\right\}$ reduces to $H(2)$;

6) $H' = 18l - \sum_{2}^{7} 5e_i - \sum_{1,10,12} 6e_j - \sum_{9,13,15} 4e_k - \sum_{8,11,14} 2e_i$, which is linear system $H(2)$ in Theorem 2, comes from a Cremona of type $\left\{(228), (246)^3\right\}$;

7) $H' = 22l - 8e_1 - \sum_{2}^{7} 7e_i - \sum_{10,13} 6e_j - \sum_{9,12} 4e_k - \sum_{8,11,14,15} 2e_i$ from a Cremona of type $\left\{(246)^4\right\}$;

8) $H' = 20l - 10e_1 - \sum_{2}^{7} 5e_i - \sum_{9,15} 6e_j - \sum_{8,11,12,13} 4e_k - \sum_{10,14} 2e_i$ from a Cremona of type $\left\{(228)^2, (246)^2\right\}$ reduces to $H(1)$ with a base change.

Hence the linear system H' depends only on the type of the Cremona we have applied starting from H, the only types giving a new presentation are:

i) types $\{(228),(444),(246)^2\},\{(228),(246)^3\}$ giving linear system $H(2)$ with types $\{(228),(222),(246)^2\}$, $\{(228)^2,(222),(246)\}$ (respectively);
ii) type $\{(246)^4\}$ giving linear system $H(3)$ with type $\{(224)^2,(222)^2\}$.

Performing all the computations for the presentations $H(2)$ and $H(3)$, and summing up only the types giving new linear systems:

(B) Starting from $H(2)$

i) types $\{(228),(222),(246)^2\},\{(228)^2,(222),(246)^3\}$ give linear system $H(1)$ with types $\{(228),(444),(246)^2\},\{(228),(246)^3\}$ respectively;
ii) type $\{(246),(228)^3\}$ gives linear system $H(3)$ with type $\{(224),(222)^2,(228)\}$;
iii) type $\{(246)^2,(228)^2\}$ gives $H(4)$ with type $\{(222)^4\}$.

(C) Starting from $H(3)$

i) types $\{(222)^2,(224),(228)\},\{(228),(222),(246)^2\}$ give linear system $H(2)$ with types $\{(228)^3,(246)\},\{(228)^2,(246)^2\}$ respectively;
ii) type $\{(246)^2,(228)^2\}$ give linear system $H(1)$ with type $\{(246)^4\}$;
iii) type $\{(224)^2,(222),(228)\}$ reduces to $H(3)$ with the same type, after a base change.
iv) type $\{(224)(246)(228)^2\}$ gives linear system $H(4)$ with type $\{(222)^3(228)\}$

(D) Starting from $H(4)$

i) type $\{(222)^4\}$ gives linear system $H(2)$ with type $\{(246)^2(228)^2\}$
ii) type $\{(222)^3(228)\}$ gives linear system $H(3)$ with type $\{(224)(246)(228)^2\}$
iii) type $\{(222)^2(228)^2\}$ reduces to $H(4)$ with the same type, after a base change.

Bibliography

[1] Arbarello E., Cornalba M., Griffiths P., Harris J., *Geometry of Algebraic Curves, I* Number 267 in Grundlehren der mathematischen Wissenschaften (New York, Springer Verlag, 1985).

[2] Bolondi G., Kleppe J.O., Miró-Roig R.M., *Maximal rank curves and singular points of the Hilbert scheme*, Compositio Math., 77, (1991), no.3, 269-291.

[3] Curtin D., *Obstructions to deforming a space curve*, Trans. Amer. Math. Soc., 267, 83-94 (1981).

[4] Eisenbud D., Harris J., *Geometry of Schemes*, Springer Verlag, 1999.

[5] Ellia Ph., *D'autres composantes non réduites de* $Hilb\,(\mathbb{P}^3)$, Math. Ann., 277 (1987), no. 3, 433-446.

[6] Ellingsrud G., *Sur le schéma de Hilbert des variétés de codimension* 2 *dans* $\mathbb{P}^e$, *à cône de Cohen-Macaulay*, Ann. Sci. Ecole Nornal Sup. (4), 8, 1975, 423-431.

[7] Grothendieck A., *Les Schémas de Hilbert*, Sem. Bourbaki, 221 (1960-61).

[8] Hartshorne R., *Algbraic Geometry*, Springer Verlag, 1977.

[9] Kleppe J.O., *Nonreduced components of the Hilbert scheme of smooth space curves*, Space Curves (Rocca di Papa, 1985), LNM 1266, 181-207, Springer, Berlin, 1987.

[10] Morrison I, Harris J., *Moduli of curves*, Springer Verlag, 1998.

[11] Mumford D., *Further pathologies in algebraic geometry*, Amer. J. Math., 84, 642-648, 1962.

[12] Okonek Ch., Schneider M., Spindler H., *Vector bundles on complex projective spaces*, Progress in Mathematics, Vol 3, Birkäuser, 1980.

[13] Walter Ch., Some examples of obstructed curves in $\mathbb{P}^3$, in Complex Projective Geometry, London Math. Society, LNS 179, 324-340, 1992, Cambridge University Press.

[14] Griffiths, P., Harris, J.: *Principles of Algebraic Geometry.* John Wiley & Sons.

[15] Endrass, S., Persson, U., Stevens, J.: *Surfaces with triple points.* Preprint, Dep. of Math., Gøteborg University, 2000.

[16] Ranestad, K.: *On Smooth Surfaces of Degree 10 in the Projective Fourspace*, thesis, University of Oslo (1988). Ps file from *http* : *//w.w.w.math.uio.no/~ranestad/papers.html.*

[17] Sommese, A.J., Van de Ven, A.: *On the adjunction mapping*, Math. Ann. **278** (1987).