

Moore matrices and automorphisms of plane cubics

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This note is based on Lars Fauskrud's masters thesis (cf. [1]).

Let

$$X = \mathbf{P}_x^2 \times \mathbf{P}_y^2 \times \mathbf{P}_z^2$$

with coordinates x_i, y_i, z_i $i = 0, 1, 2$ and the projections

$$p_x : X \rightarrow \mathbf{P}_x^2, \quad p_y : X \rightarrow \mathbf{P}_y^2, \quad p_z : X \rightarrow \mathbf{P}_z^2.$$

Let for each point $y \in \mathbf{P}_y^2$

$$X_y = p_y^{-1}(y).$$

Consider the matrices

$$M_{x,y} = (x_{i+j}y_{i-j}) = \begin{pmatrix} x_0y_0 & x_1y_2 & x_2y_1 \\ x_1y_1 & x_2y_0 & x_0y_2 \\ x_2y_2 & x_0y_1 & x_1y_0 \end{pmatrix}$$

and

$$M_{z,y} = \begin{pmatrix} z_0y_0 & z_1y_2 & z_2y_1 \\ z_1y_1 & z_2y_0 & z_0y_2 \\ z_2y_2 & z_0y_1 & z_1y_0 \end{pmatrix}.$$

Note that

$$M_{x,y} \cdot \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

iff

$$M_{z,y} \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The entries are the trilinear forms

$$\begin{aligned} & x_0y_0z_0 + x_1y_2z_1 + x_2y_1z_2 \\ & x_1y_1z_0 + x_2y_0z_1 + x_0y_2z_2 \\ & x_2y_2z_0 + x_0y_1z_1 + x_1y_0z_2 \end{aligned}$$

whose common zeros form a 3-fold $Z \subset X$. For a point $y \in \mathbf{P}_y^2$ let

$$Z_y = Z \cap X_y.$$

Z_y is a curve whose image $E_{xy} = p_x(Z_y)$ is defined by the determinant of $M_{x,y}$, ie. by

$$\det M_{x,y} = (y_0^3 + y_1^3 + y_2^3)x_0x_1x_2 - y_0y_1y_2(x_0^3 + x_1^3 + x_2^3),$$

the image $E_{zy} = p_z(Z_y)$ is similarly defined by

$$\det M_{z,y} = (y_0^3 + y_1^3 + y_2^3)z_0z_1z_2 - y_0y_1y_2(z_0^3 + z_1^3 + z_2^3).$$

Note that the restrictions of p_x and p_z to Z_y are isomorphisms, when Z_y is smooth, in fact we shall see below that Z_y form the graph of the restriction to E_{xy} of certain Cremona transformations. Identifying $\mathbf{P}_x^2$ and $\mathbf{P}_z^2$ we find that $E_{xy} = E_{zy}$. Therefore we denote this curve by E_y . Identifying also $\mathbf{P}_x^2$ and $\mathbf{P}_y^2$ we see that $y \in E_y$. In fact $E_y = E'_y$ for any point $y' \in E_y$. We proceed therefore with a fixed curve E in the pencil Γ generated by the cubics

$$\{x_0x_1x_2 = 0\} \quad \text{and} \quad \{x_0^3 + x_1^3 + x_2^3 = 0\}.$$

Consider the involution

$$\iota : x_i \rightarrow x_{-i} \quad i \pmod{3}.$$

It is easy to check that each curve E is invariant under ι , so it defines an involution on each curve E . Furthermore for each point $y \in E$

$$\varphi_y : E \rightarrow E \quad x \mapsto p_z(p_x^{-1}(x))$$

is an automorphism of E . The involution ι and the automorphisms φ_y define in fact an abelian group structure on E , when E is smooth: For $x, y \in E$ we define

$$x + y = \varphi_y(\iota(x)),$$

in coordinates, (with x, y general on E):

$$\varphi_y(\iota(x)) = (x_1x_2y_0^2 - x_0^2y_1y_2, x_0x_1y_2^2 - x_2^2y_0y_1, x_0x_2y_1^2 - x_1^2y_0y_2).$$

More precisely, it is defined by any nonzero row vector of the cofactor matrix of $M_{x,y}$. It is straight forward to check that this is a commutative operation with neutral element $(0, 1, -1)$ and with $\iota(x)$ inverse to x . What remains is:

Claim 1: *The operation $+$ on E is associative.*

There is the following characterization of collinearity (cf. another characterization below).

Claim 2: *Three points $x, y, z \in E$ are collinear iff $x + y + z = (0, 1, -1)$ on E .*

Notice that $\iota \circ \varphi_y : E \rightarrow E$ is the translation by y on E as a group, and that any base point of the pencil Γ has order 3 and every point invariant under ι has order 2.

On the four singular members of the pencil the situation is similar, for details see [1].

Next fix again a point $y \in \mathbf{P}_y^2$, and let $\mathbf{P}_c^2$ denote the plane of 2-dimensional subspaces of the row space of $M_{x,y}$. Let $CM_{x,y}$ be the cofactor matrix of $M_{x,y}$, ie.:

$$CM_{x,y} = \begin{pmatrix} x_1x_2y_0^2 - x_0^2y_1y_2 & x_0x_2y_2^2 - x_1^2y_0y_1 & x_0x_1y_1^2 - x_2^2y_0y_2 \\ x_1^2y_0y_2 - x_0x_2y_2^2 & x_2^2y_1y_2 - x_0x_1y_0^2 & x_0^2y_0y_1 - x_1x_2y_2^2 \\ x_0x_2y_0^2 - x_1^2y_1y_2 & x_1x_2y_1^2 - x_0^2y_0y_2 & x_0x_2y_0^2 - x_1^2y_1y_2 \end{pmatrix}.$$

For any point in $c \in \mathbf{P}_c^2$ the entries of

$$(c_0 \quad c_1 \quad c_2) \cdot CM_{x,y}$$

define a Cremona transformation

$$\psi_{y,c} : \mathbf{P}_x^2 \dashrightarrow \mathbf{P}_z^2.$$

The restriction of this Cremona transformation to E_y coincide with φ_y . It is independant of c because $CM_{x,y}$ have rank 1 on E_y .

If we identify $\mathbf{P}_z^2$ with the dual of $\mathbf{P}_c^2$ via the bilinear form

$$c_0z_0 + c_1z_1 + c_2z_2 = 0,$$

and restrict all maps to E , then $\varphi_y^{-1}(z)$ is a basepoint for $\psi_{y,c}$ if and only if $c_0z_0 + c_1z_1 + c_2z_2 = 0$. Thus the 3 basepoints of $\psi_{y,c}$ all lie on E_y and are precisely the preimage of the 3 points of intersection between $E = E_y \subset \mathbf{P}_z^2$ and the line $c_0z_0 + c_1z_1 + c_2z_2 = 0$. For details when E is singular see [1].

References:

[1] Lars Fauskrud: "Spesielle Cremona Transformasjoner." Cand. Scient. thesis, Oslo 1996.