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THE EXTENDED POWER CONSTRUCTION AND COHOMOTOPY

by

CLIFFORD R. COOLEY

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of the requirements for the degree of

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INTRODUCTION

The extended power construction is a functor from spaces to spaces. It was first used in published work by Toda, to find a differential in the Adams spectral sequence for the homotopy of spheres. Kahn showed how to relate this construction to squaring operations Sq^i in the cohomology $H^*(\underline{A})$ of the Steenrod algebra $\underline{A}$ and used this to give some systematic formulas for differentials in the spectral sequence at the prime 2. Later, Milgram generalized Kahn's work to other primes. Singer has described these operations in the cohomology of the unstable Steenrod algebra and has shown how they are related to maps in an algebraic EHP sequence. Nishida has studied the homology of the extended powers of a space as a module over the Steenrod algebra and has used these results to show the homotopy of spheres is nilpotent. In recent theses, Steinberger and Bruner have developed the theory for H_∞ ring spectra.

The present goal is to study the extended powers from the point of view of the cohomotopy of complexes. This construction is a source of maps from a complex to a sphere. Given a map $\alpha: K \rightarrow L$, the extended power construction gives another map

$$ep^i(\alpha): ep^i(K) \rightarrow ep^i(L)$$

To use this map, the new complexes $ep^i(K)$ and $ep^i(L)$ must be identified. In general the homology of the new complexes $ep^i(K)$ as a module over the Steenrod algebra is

understood by work of Nishida. In the special case L is a sphere, $ep^i(S^n)$ is a piece of real projective space

$$ep^i(S^n) = S^n RP_n^{n+i}.$$

When $n = 0 \pmod{2^N}$ for N large enough, there is a retraction

$$S^n RP_n^{n+i} \rightarrow S^{2n}.$$

By composing with the retraction, the extended power can be regarded as a map

$$\overline{ep}^i(\alpha): ep^i(K) \rightarrow S^{2n}.$$

So, beginning with an element in the cohomotopy of K , the extended power construction gives a new element in the cohomotopy of $ep^i(K)$. For example, when K is a sphere S^k , this construction is a source of nontrivial elements in the homotopy groups $[S^k RP_k^{k+i}, S^{2n}]$.

These maps contain information about homotopy. In order to relate this information to the homotopy of spheres, we use two invariants: the Hopf invariant and detectors in the Adams spectral sequence. As a result, elements of certain invariants can be constructed, and relations in homotopy and differentials in the Adams spectral sequence can be determined.

Section 1 describes a version of the Adams spectral sequence for the homotopy groups $[S^t K, X]$ from Bousfield and Kan. We give a differential complex using the Λ

algebra for computing the E_2 term of this spectral sequence. The Puppe sequence for the cohomotopy of a cofibration can be used to transfer information about the cohomotopy of a complex to information about the cohomotopy (homotopy) of a sphere. We develop a spectral sequence analogue of the Puppe sequence, using filtration shifts, where the filtration shifts can be calculated using differentials. Since the extended powers provide many examples of filtration shifts, this theory provides the basis for the calculation of differentials in section 4 which extend the results of Kahn and Milgram.

In Section 2, we describe the general facts about the extended power construction. Also, using the representation of RP_k^{k+1} as a Thom space, we study the attaching maps of this complex using results of Adams. This gives a precise answer to the question of when a retraction

$S^{k+N}_{RP_k^{k+1}} \rightarrow S^{2k+N}$ exists and also gives some information about $\pi_*^S(RP_k^{k+1})$.

The third section studies the extended power construction from the point of view of the EHP sequence. We show if $\alpha: K \rightarrow S^n$, and the dimension of $K < 3n-3$, the following formula gives the Hopf invariant H of $\overline{ep}^1(\alpha)$ in terms of $H(\alpha)$

$$EH\overline{ep}^1(\alpha) = \overline{ep}^{1-1}EH(\alpha).$$

Also in this section, we describe the relationship between

a construction of Toda called the star construction and the construction ep^1 . We show construction ep^1 satisfies the same theorems.

In Section 4, we extend the squaring operations in $H^*(\underline{A})$ to operations which can be used to detect maps $\overline{ep}^i(K) \rightarrow S^n$. Then using the results about the spectral sequence in section 1, we find differential formulas for the spectral sequence. This gives a proof of the formulas already known and also gives new formulas. The E_1 -term of section 1 plays an important role in the statement and proof of these formulas. Also in section 4 are examples of constructions of elements and homotopy relations using the results of sections 2 and 3.

SECTION 1. THE ADAMS SPECTRAL SEQUENCE

FOR THE COHOMOTOPY OF A COMPLEX

The purpose of the first part of this section is to give an E_1 term for the Adams spectral sequence for the group of homotopy classes of maps from $S^q W$ into X where W and X are suitably nice spaces. Let $[S^q W, X]$ denote this group. Bousfield and Kan have given a spectral sequence which works even unstably. Some main properties of their spectral sequence are summarized below. See [7] for details.

Throughout let H_* and H^* denote reduced homology with Z_2 coefficients. Assume X is a simply connected topological space, $\pi_t(X)$ is finitely generated for all t , and W has the weak homotopy type of a finite complex.

Then

Theorem 1.1 [7] There is a spectral sequence $E_r^{s,t}(W,X)$ converging (modulo torsion prime to 2) to $[S^{t-s} W, X]$ when $t-s \geq 1$ such that there is a natural isomorphism

$$E_2^{s,t}(W,X) \simeq \text{Ext}_{\underline{CA}}^s(H_*(S^t W), H_*(X)).$$

Here the category $\underline{CA}$ is the category of connected unstable coalgebras over the Steenrod algebra $\underline{A}$.

In case $H_*(X)$ is very nice (see below), the E_2 term can be calculated as an Ext in the category $\underline{MA}$ of connected unstable right $\underline{A}$ modules. As in [7] let

$$J': \underline{CA} \rightarrow \underline{MA}$$

be the forgetful functor and let

$$U: \underline{MA} \rightarrow \underline{CA}$$

be the right adjoint of J . Then a space X is called very nice provided

$$H_*(X) \simeq U_* M \in \underline{CA} \text{ for some } M \in \underline{MA}.$$

The sphere S^n is very nice for all $n \geq 1$ (since we are working at the prime 2). Let X be very nice and M be as above. Then

Theorem 1.2 [7] For $t > s \geq 0$, there is a natural isomorphism

$$E_2^{s,t}(W, X) \simeq \text{Ext}_{\underline{MA}}^s(H_* S^t W, M)$$

To give an E_1 term for the spectral sequence we mean to give a differential complex for computing the Ext groups above. This task is simplified in $\underline{MA}$ because $\underline{MA}$ is an abelian category with enough injectives. As in [5], [6] and [7] the algebra Λ plays an important role.

The Λ algebra

The Λ algebra is a differential graded algebra over $\mathbb{Z}_2$ with a generator λ_i for each $i \geq 0$ subject to the relations

$$\lambda_i \lambda_{2i+1+m} = \sum_{j \geq 0} (m-1-j, j) \lambda_{i+m-j} \lambda_{2i+1+j}$$

and with differential d

$$d\lambda_m = \sum_{j > 0} (m-j, j) \lambda_{m-j} \lambda_{j-1}$$

where (i, j) is the binomial coefficient $i!/((i-j)!j!)$.

Using the relations it is easy to show the Λ algebra has

a vector space basis consisting of allowable monomials

$$\lambda_I = \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_n} \quad \text{where } i_{j+1} \leq 2i_j.$$

As in [5], for $M \in \underline{MA}$, let $M \hat{\otimes} \Lambda \subseteq M \otimes \Lambda$ be the subgroup generated by $m \otimes \lambda_I$ where λ_I is allowable and $i_1 < \deg(m)$. Then $M \hat{\otimes} \Lambda$ can be made into a cochain complex by letting the coboundary δ act as follows:

$$\delta m \otimes \lambda_I = \sum_{i>0} m Sq^i \otimes \lambda_{i-1} \lambda_I + m \otimes d\lambda_I.$$

Let $m \otimes \lambda_I$ have grading $s = \text{length of } I$ and $t = \deg(m) + \deg(\lambda_I) + s$. Then

Theorem 1.3 [5] There is a natural isomorphism

$$\text{Ext}_{\underline{MA}}^s(H_*(S^t), M) \simeq H^{s,t}(M \hat{\otimes} \Lambda)$$

The goal is to generalize this technique to the computation of $\text{Ext}_{\underline{MA}}^s(S^t L, M)$ where $L \in \underline{MA}$ and $S^t L$ stands for the t -fold suspension of L in the category $\underline{MA}$. For $L \in \underline{MA}$, let

$$L^* = \text{Hom}(L, Z_2).$$

There is the pairing

$$\langle \cdot, \cdot \rangle: L \otimes L^* \rightarrow Z_2$$

given by $\langle \ell, f \rangle = f(\ell)$ where $f \in L^*$. Since L is a right $\underline{A}$ module, L^* is a left $\underline{A}$ module by setting

$$\langle \ell, Sq^i f \rangle = \langle \ell Sq^i, f \rangle$$

for all $\ell \in L$.

The complex $V(L, M)$:

For L and M in $\underline{MA}$, define a cochain complex

$V(L, M)$ by

$$V(L, M) = M \hat{\otimes} \Lambda \otimes L^*$$

with differential given by

$$\delta(m \otimes \lambda_I \otimes f) = \delta(m \otimes \lambda_I) \otimes f + \sum_{i>0} m \otimes \lambda_I \lambda_{i-1} \otimes Sq^i f.$$

Let $m \otimes \lambda_I \otimes f$ be graded by (s, t) where s = length of I and $t = s + \deg(m) + \deg(\lambda_I) - \deg(f)$.

Since $\delta^2 = 0$ on $L \otimes \Lambda$ by [5], in order to check that $\delta^2 = 0$ in $V(L, M)$ it suffices to check that $\delta^2(1 \otimes f) = 0$ in $\Lambda \otimes L^*$. This follows from the relations in Λ and in $\underline{A}$. It can also be derived as follows.

Using the formula for the differential,

$$\delta^2(1 \otimes f) = \sum \lambda_{i-1} \lambda_{j-1} \otimes Sq^j Sq^i f + \sum d\lambda_{i-1} \otimes Sq^i f.$$

Pairing the right hand side with an element $\ell \in L$ we get

$$\begin{aligned} & \sum \lambda_{i-1} \lambda_{j-1} \langle \ell, Sq^j Sq^i f \rangle + \sum d\lambda_{i-1} \langle \ell, Sq^i f \rangle \\ &= \sum \lambda_{i-1} \lambda_{j-1} \langle \ell Sq^j Sq^i, f \rangle + \sum d\lambda_{i-1} \langle \ell Sq^i, f \rangle. \end{aligned}$$

The right hand side of this equality is zero because it is just the element f in L^* paired with $\delta^2(\ell \otimes 1) = 0$ in $L \otimes \Lambda$. Since this is true for all $\ell \in L$, $\delta^2(1 \otimes f) = 0$ in $\Lambda \otimes L^*$. Hence $V(L, M)$ is a cochain complex.

Let a module $L \in \underline{MA}$ be called finite if L is finite dimensional as an ungraded vector space.

Proposition 1.4 When L is finite, there is a natural isomorphism

$$H^{0,0}(V(L,M)) \approx \text{Hom}_{\underline{MA}}(L,M)$$

Proof. Consider the restriction of the usual isomorphism

$$M \otimes L^* \xrightarrow{\sim} \text{Hom}_{Z_2}(L,M)$$

to the subgroup $H^{0,0}(V(L,M)) = \ker(\delta)$. (The map is an isomorphism because L is finite). We must verify that the image of $\ker(\delta)$ actually lies in $\text{Hom}_{\underline{MA}}(L,M)$. The map is given by

$$(m \otimes f)\ell \rightarrow \langle \ell, f \rangle m.$$

Now if $\sum m_\alpha \otimes 1 \otimes f_\alpha$ is in $\ker(\delta)$, then for fixed i

$$\sum m_\alpha \otimes \text{Sq}^i f_\alpha = \sum m_\alpha \text{Sq}^i \otimes f_\alpha$$

Hence

$$(\sum m_\alpha \otimes f_\alpha)\ell \text{Sq}^i \rightarrow \sum \langle \ell, \text{Sq}^i f_\alpha \rangle m_\alpha = \sum \langle \ell, f_\alpha \rangle m_\alpha \text{Sq}^i$$

So the image of $\sum m_\alpha \otimes f_\alpha$ in $\text{Hom}(L,M)$ respects the action of $\underline{A}$ and so lies in $\text{Hom}_{\underline{MA}}(L,M)$ as desired.

Conversely, using a similar argument it is easy to check that any element of $M \otimes L^*$ whose image lies in $\text{Hom}_{\underline{MA}}(L,M)$ must be in $\ker(\delta)$. This completes the proof.

Now suppose

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

is a short exact sequence in $\underline{MA}$. Write $H^{s,t}(L,M)$ for $H^{s,t}(V(L,M))$.

Proposition 1.5 There is a natural long exact sequence

$$\rightarrow H^{s,t}(L,M') \rightarrow H^{s,t}(L,M) \rightarrow H^{s,t}(L,M'') \rightarrow H^{s+1,t}(L,M')$$

Proof: The short exact sequence of modules yields a short exact sequence of complexes $V(L, -)$. This gives a long exact sequence in cohomology as desired.

Similarly, assume

$$0 \rightarrow L' \rightarrow L \rightarrow L'' \rightarrow 0$$

is a short exact sequence in $\underline{MA}$.

Proposition 1.6 There is a natural long exact sequence

$$\rightarrow H^{s,t}(L'', M) \rightarrow H^{s,t}(L, M) \rightarrow H^{s,t}(L', M) \xrightarrow{\delta} H^{s+1,t}(L'', M)$$

Proposition 1.7 Suppose I is an injective in $\underline{MA}$ and L is finite. Then

$$H^{s,t}(L, I) = 0 \quad \text{for } s > 0.$$

Proof: Reference [5] shows this when $L = H_*(S^t)$ where S^t is the t dimensional sphere. It follows for arbitrary finite L by induction on the dimension of L as a vector space using 1.6.

Theorem 1.8 There is a natural isomorphism for $s, t \geq 0$

$$\text{Ext}_{\underline{MA}}^{s,t}(L, M) \simeq H^{s,t}(V(L, M)).$$

Proof: Propositions 1.4, 1.5, and 1.7 show that $H^{s,t}(L, M)$ is a functor which satisfies the axioms for Ext . See [14] for details.

This theorem together with Theorem 1.2 shows that $V(L, M)$ is an E_1 term for the spectral sequence.

The proof of Theorem 1.8 constructs a natural isomorphism

$$\text{Ext}_{\underline{\text{MA}}}^{s,t}(L,M) \rightarrow H^{s,t}(V(L,M))$$

which respects the long exact sequence of Proposition 1.5. (Call this the long exact sequence on the right). This natural isomorphism also respects the long exact sequence "on the left". To see this suppose $C(L,M)$ is any natural cochain complex of $\mathbb{Z}_2$ vector spaces which carries short exact sequences on the left and on the right to short exact sequences of cochain complexes.

Proposition 1.9 Suppose there is a natural isomorphism

$$H^s(C(L,M)) \simeq \text{Ext}_{\underline{\text{MA}}}^s(L,M)$$

which respects the long exact sequence on the right. Then the natural isomorphism also respects the long exact sequence on the left.

Proof: Recall [14] that the connecting homomorphism $\delta: \text{Ext}^s(L'',M) \rightarrow \text{Ext}^{s+1}(L',M)$ is given by Yoneda composition with the extension $0 \rightarrow L' \rightarrow L \rightarrow L'' \rightarrow 0$. (We do not need to worry about signs with $\mathbb{Z}_2$ coefficients.) Alternatively, choose a short exact sequence $0 \rightarrow M \rightarrow I \rightarrow M'' \rightarrow 0$ where I is injective. Then when $s = 0$, δ can be defined from the following commutative diagram.

Diagram 1.10

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \uparrow & & & & \\
 0 \rightarrow & \text{Ext}^0(L'',M) & \rightarrow & \text{Ext}^0(L'',I) & \rightarrow & \text{Ext}^0(L'',M'') & \\
 & & & \uparrow & & \uparrow & \\
 & & & \text{Ext}^0(L,I) & \rightarrow & \text{Ext}^0(L,M'') & \\
 & & & & & \uparrow & \\
 & & & & & \text{Ext}^0(L',M'') & \rightarrow \text{Ext}^1(L',M) \\
 & & & & & \uparrow & \\
 & & & & & 0 &
 \end{array}$$

Define $\delta: \text{Ext}^0(L'', M) \rightarrow \text{Ext}^1(L', M)$ by "switchback" along the left-hand jagged edge. It is easy to check that this definition agrees with the definition of δ using Yoneda composition. Replacing $\text{Ext}^i(-, -)$ with $H^i C(-, -)$ in diagram 1.10, it is easy to verify that δ defined by the "switchback" also agrees with δ defined from the short exact sequence of cochain complexes

$$0 \rightarrow C(L'', M) \rightarrow C(L, M) \rightarrow C(L', M) \rightarrow 0.$$

Thus, we have proved that the following diagram commutes when $s = 0$.

$$\begin{array}{ccc} H^s C(L', M) & \xrightarrow{\delta} & H^{s+1} C(L'', M) \\ \uparrow & & \uparrow \\ \text{Ext}^s(L', M) & \xrightarrow{\delta} & \text{Ext}^{s+1}(L'', M) \end{array}$$

The vertical maps are the isomorphisms in the hypothesis of Proposition 1.9.

Proposition 1.9 now follows by induction on s because the following diagram commutes, the vertical maps are onto, and the same diagram commutes with $\text{Ext}^*(-, -)$ replaced by $H^* C(-, -)$.

$$\begin{array}{ccc} \text{Ext}^s(L'', M'') & \xrightarrow{\delta} & \text{Ext}^{s+1}(L', M'') \\ \downarrow \partial & & \downarrow \partial \\ \text{Ext}^{s+1}(L'', M) & \xrightarrow{\delta} & \text{Ext}^{s+2}(L', M) \end{array}$$

Pairings

There is a pairing

$$\text{Ext}_{\underline{MA}}^s(M, N) \otimes \text{Ext}_{\underline{MA}}^{s'}(L, M) \rightarrow \text{Ext}_{\underline{MA}}^{s+s'}(L, N)$$

given by the Yoneda product. There is also a pairing

$$V(M,N) \otimes V(L,M) \rightarrow V(L,N)$$

given by

$$(n \otimes \lambda_I \otimes f) \otimes (m \otimes \lambda_J \otimes g) \rightarrow \langle m, f \rangle (n \otimes \lambda_I \lambda_J \otimes g)$$

where $n \in N$, $f \in M^*$, $m \in M$, $g \in L^*$. Letting the differential act as a derivation on the tensor product, it is easy to show that this is a map of cochain complexes. Using the Kunneth formula we get the following.

Proposition 1.11 There is a natural pairing

$$H^S(M,N) \otimes H^{S'}(L,M) \rightarrow H^{S+S'}(L,N)$$

The goal is to show that this pairing corresponds to the Yoneda product under the natural isomorphism of Theorem 1.8.

Recall that $\text{Ext}_{\underline{MA}}^0(M,N) \simeq \text{Hom}_{\underline{MA}}(M,N)$. The Yoneda product $\text{Ext}^0 \otimes \text{Ext}^S \rightarrow \text{Ext}^S$ is given by the map

$$\underline{1.12} \quad f \otimes x \rightarrow f_*(x).$$

Recall also that if

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

is a short exact sequence in $\underline{MA}$, then the boundary map ∂

$$\partial: \text{Ext}^S(M, N'') \rightarrow \text{Ext}^{S+1}(M, N')$$

is given by Yoneda composition with the element of $\text{Ext}^1(N'', N)$ represented by the short exact sequence. This fact, together with the associativity of the Yoneda product shows that the Yoneda product respects the boundary map ∂ . That is, the following diagram commutes

Diagram 1.13

$$\begin{array}{ccc}
 \text{Ext}^s(M, N'') \otimes \text{Ext}^{s'}(L, M) & \xrightarrow{\partial \otimes 1} & \text{Ext}^{s+1}(M, N') \otimes \text{Ext}^{s'}(L, M) \\
 \downarrow & & \downarrow \\
 \text{Ext}^{s+s'}(1, N'') & \xrightarrow{\partial} & \text{Ext}^{s+s'+1}(L, N')
 \end{array}$$

Proposition 1.14 These two properties (1.12 and 1.13) characterize the Yoneda product

$$\text{Ext}^s(M, N) \otimes \text{Ext}^{s'}(L, M) \rightarrow \text{Ext}^{s+s'}(L, N)$$

Proof: Assume inductively the proposition is true for $s \leq j$ and all L, M, N and arbitrary $s' \geq 0$. When $j = 0$, the inductive hypothesis holds by 1.12. Now let I be an injective so that

$$0 \rightarrow N' \rightarrow I \rightarrow N'' \rightarrow 0$$

is short exact. Then in the commutative diagram 1.13 the horizontal maps are onto because $\text{Ext}^s(-, I) = 0$ for $s > 0$ since I is injective. Thus the product on the right (vertical arrow) is determined by the left hand product. Thus the inductive hypothesis is true for $s = j + 1$.

Proposition 1.15 The pairing of Proposition 1.11 corresponds to the Yoneda product under the natural isomorphism of Theorem 1.8.

Proof: In view of Proposition 1.14, it suffices to show the pairing satisfies 1.12 and 1.13. The pairing evidently satisfies 1.12 using the natural isomorphism of Proposition 1.4. It is a straightforward calculation to show that the pairing satisfies 1.13.

Proposition 1.16 In the stable range $H_*(X)$ is very nice

for any connected space X . In fact $H_*(X) = U_*H_*(X)$ as coalgebras in the stable range.

Proof: U_*M is the dual to the free unstable $\underline{A}$ algebra generated by M . If $M_j = 0$ for all $j < n$, then $U_*M_k = M_k$ for all $k < 2n$.

Proposition 1.17 When W is a finite complex, then in the stable range there is a natural isomorphism

$$E_2^{s,t}(W, X) \simeq \text{Ext}_{\underline{MA}}^s(H_*(S^t W), H_*(X)).$$

In the stable range means $t < 2n$ where $n-1$ is the connectivity of X . Note: this is not the same stable range as the range in which $[S^t W, X]$ is stable.

Proof:

$$\text{Ext}_{\underline{MA}}^s(H_*(S^t W), H_*X) \simeq \text{Ext}_{\underline{CA}}^s(H_*(S^t W), U_*H_*(X))$$

by the change of rings lemma [7;13.4]. Using Proposition 1.16 we can replace the right hand side with $\text{Ext}_{\underline{CA}}^s(H_*S^t W, H_*X)$ in the range specified. Applying Theorem 1.1 gives the desired result.

In order to study various constructions, we need an alternative but equivalent definition for the spectral sequence $E_r(W, X)$. In [7;7.2] the function complex $\text{hom}(W, X)$ is used. As for any space, there is constructed a sequence of fibre maps

$$D_S \text{hom}(W, X) \rightarrow D_{S-1} \text{hom}(W, X) \rightarrow \dots \rightarrow \text{Hom}(W, X)$$

The homotopy spectral sequence of this sequence of fibre maps is the spectral sequence $E_r(W, X)$.

Proposition 1.18 There is a natural morphism of the sequence of fibre maps above and the sequence of fibre maps

$$\text{hom}(W, D_s X) \rightarrow \text{hom}(W, D_{s-1} X) \rightarrow \dots \rightarrow \text{hom}(W, X).$$

This morphism induces isomorphisms in homotopy for each s .

Proof: Recall [7;7.2] that $D_s \text{hom}(W, X)$ is the fibration over $D_{s-1} \text{hom}(W, X)$ which is the pull back of the diagram

$$\begin{array}{ccc} & \Lambda D_{s-1} \text{hom}(W, RX) & \\ & \downarrow & \\ D_{s-1} \text{hom}(W, X) & \longrightarrow & D_{s-1} \text{hom}(W, RX) \end{array}$$

where Λ is the path fibration functor [7;2.2] and R is a ring (in our case $R = \mathbb{Z}_2$).

Lemma 1.19 There is a natural map

$$\Lambda \text{hom}(W, X) \rightarrow \text{hom}(W, \Lambda X)$$

which commutes with projection onto $\text{hom}(W, X)$.

Proof: An n -simplex in $\Lambda \text{hom}(W, X)$ is a simplicial map $\phi: \Delta(n+1) \times W \rightarrow X$ with the property that $\phi: [0] \times W \rightarrow *$ where $[0]$ denotes the subcomplex of $\Delta(n+1)$ generated by the 0^{th} vertex. Also $\phi: \Delta(n+1) \times \{*\} \rightarrow *$. Now let $g(n)$ denote the simplex which generates the complex $\Delta(n)$. Given $\phi \in \Lambda \text{hom}(W, X)$, define $\bar{\phi} \in \text{hom}(W, \Lambda X)$ by

$$s_I d_J g(n) \times w \mapsto (s_{I+1} d_{J+1} g(n+1) \times s_0 w) \in \Lambda X$$

where $I = (i_1, i_2, \dots, i_s)$ and

$I+1 = (i_1+1, i_2+1, \dots, i_s+1)$. It is easy to show that the

image of this map really lies in $\Delta X \subset X$ (this is not a simplicial inclusion!). The projection of ϕ into $\text{hom}(W, X)$ is given by the composition

$$\Delta(n) \wedge W \xrightarrow{\delta_0} \Delta(n+1) \wedge W \xrightarrow{\phi} X$$

or $s_I d_J g(n) \wedge W \rightarrow \phi(s_I d_J d_0 g(n+1) \wedge W)$.

This is evidently the same as the projection of ϕ using the simplicial identities $d_0 s_{I+1} = s_I$, $d_0 d_{J+1} = d_J$, and $d_0 s_0 w = w$. This completes the proof of the lemma,

Lemma 1.20 $\text{hom}(W, \Delta X)$ is a fibration over $\text{hom}(W, X)$.

Proof: See [17; 7.16]

Lemma 1.21 $\text{hom}(W, D_S X)$ is the fibration induced as the pullback in the following diagram

$$\begin{array}{ccc} & \text{hom}(W, \Delta D_{S-1} RX) & \\ & \downarrow & \\ \text{hom}(W, D_{S-1} X) & \rightarrow & \text{hom}(W, D_{S-1} RX) \end{array}$$

Proof of Proposition 1.18

Inductively, the composition

$$\Delta D_{S-1} \text{hom}(W, RX) \rightarrow \Delta \text{hom}(W, D_{S-1} RX) \rightarrow \text{hom}(W, \Delta D_{S-1} RX)$$

induces a map from $D_S \text{hom}(W, X)$ to the pullback

$\text{hom}(W, D_S X)$. Using the long exact sequences of the fibrations $\text{hom}(W, D_S X) \rightarrow \text{hom}(W, D_{S-1} X)$ and

$D_S \text{hom}(W, X) \rightarrow D_{S-1} \text{hom}(W, X)$ it is easy to show inductively that the maps

$$D_S \text{hom}(W, X) \rightarrow \text{hom}(W, D_S X)$$

are isomorphisms in homotopy using the fact that $\text{hom}(W, \text{AD}_S RX)$ is contractible. This completes the proof of the proposition.

Filtration shifts

Suppose G and H are abelian groups.

Definition 1.22 A construction on G with values in H is a relation on $G \times H$ which is additive. For each $g \in G$, there is an $h \in H$ so that (g, h) is in the relation, and if (g_1, h_1) and (g_2, h_2) is in the relation, then so is $(g_1 + g_2, h_1 + h_2)$.

If α is a construction on G with values in H we write $\alpha: G \rightarrow H$. Note that the image and kernel of α are well defined subgroups of H and G respectively. (The kernel of α is the subgroup of elements g so that (g, e) is in the relation α .)

Now let $f: Y \rightarrow W$ be a map between finite complexes. Then there is an induced map $f^*: E_r^{s,t}(W, X) \rightarrow E_r^{s,t}(Y, X)$. This map is related to the map on the homotopy level. However, if $f^*(b) = 0$, the corresponding element $f^\#(\beta) \in [S^{t-s}Y, X]$ may not be zero (where b detects $\beta \in [S^{t-s}W, X]$). The purpose of this part is to define constructions

$$f_r^{*,1}: \ker(f^*) \rightarrow E_{r-1}^{s+1, t+1}(Y, X)$$

$$f_r^{*,k}: \ker(f_r^{*,k-1}) \rightarrow E_{r-k}^{s+k, t+k}(Y, X).$$

Taken together, these constructions will define a

construction

$$f_r^{*,*}: E_r^{s,t}(W,X) \rightarrow \bigoplus_{0 \leq k < r} E_{r-k}^{s+k,t+k}(Y,X)$$

We need some basic facts about the spectral sequence. In view of Proposition 1.18 an element $b \in E_1^{s,t}(W,X)$ can be represented by a map $S^{t-s}W \rightarrow D_sRX$ or equivalently by a map of pairs

$$b: (CS^{t-s-1}W, S^{t-s-1}W) \rightarrow (D_sX, D_{s+1}X).$$

The element survives to E_r provided there is a map of pairs into the pair $(D_sX, D_{s+r}X)$ which has the same projection as b in the group of maps into the pair $(D_{s-r+1}X, D_{s+1}X)$.

$$\begin{array}{ccc} (D_sX, D_{s+r}X) & \xrightarrow{\quad} & (D_sX, D_{s+1}X) \\ & \searrow \quad \swarrow & \\ & (D_{s-r+1}X, D_{s+1}X) & \end{array}$$

Letting $F_{s,s+r}$ denote the fibre of the map $D_{s+r}X \rightarrow D_sX$, the spectral sequence may be discussed in terms of maps of spaces instead of maps of pairs. For instance, a map of pairs

$$(CW, W) \rightarrow (D_sX, D_{s+r}X)$$

corresponds under a natural isomorphism to a map

$$W \rightarrow F_{s,s+r}X.$$

Definition of $f_r^{*,1}$

Consider the following commutative diagram of groups with exact rows (long exact sequences of fibrations)

$$\begin{array}{ccccccc}
 [S^2Y, F_{s-r+1,s+1}] & & & & & & \\
 \partial \rightarrow [SY, F_{s+1,s+r}] & \rightarrow & [SY, F_{s-r+1,s+r}] & \rightarrow & [SY, F_{s-r+1,s+1}] & & \\
 \uparrow & & \uparrow & & \uparrow & & \\
 [SW, F_{s+1,s+r}] & \rightarrow & [SW, F_{s-r+1,s+r}] & \rightarrow & [SW, F_{s-r+1,s+1}] & & \\
 & & & & \uparrow & & \\
 & & & & [SW, F_{s,s+1}] & &
 \end{array}$$

Then if $b \in E_r^S(W, X)$, choose a representative

$b' \in [SW, F_{s-r+1,s+r}]$. Then $f^\#(b') \in [SY, F_{s-r+1,s+r}]$

can be lifted into $[SY, F_{s+1,s+r}]$ by the assumption

$b' \in \text{Ker}(f^*)$. Then let $f_r^{*,1}(b)$ be the projection of this lift into $[SY, F_{s-r+1,s+2}]$. The indeterminacy due to the choice of b is just $f^*[SW, F_{s+1,s+r}]$ and due to the choice of the lift of $f^\#(b)$ is just the image of the boundary map ∂ . In terms of the spectral sequence

$$\text{indeterminacy} = f^*E_{r-1}^{s+1,t+1}(W, X) + \sum_{q \leq r} \text{image}(d_q).$$

Definition of $f_r^{*,k}$

The hypothesis $f^{*,k-1}(\beta) = 0$ implies there is a choice of b so that $f^\#(b) \in [SY, F_{s-r+1,s+r}]$ projects to zero in $[SY, F_{s-r+1,s+k}]$. Then $f_r^{*,k}(\beta)$ is a lift of $f^\#(b)$ into $[F_{s+k,s+r}]$.

The indeterminacy is

$$\bigoplus_{0 < \underline{l} \leq k} (f^{*,k-1}_{E_{r-1}^{s+1,t+1}}(W,X)) + \sum_{q \leq r+k-1} \text{image } d_q$$

Note that the notation $f^{*,k}_{E_r^{s,t}}$ is being abused to mean $f^{*,k}(\ker f^{*,k-1})$.

Now let $g: W \rightarrow Z$ be the cofibre of the map $f: Y \rightarrow W$ so that $Y \rightarrow W \rightarrow Z$ is a cofibration.

Theorem 1.23 The following sequence is exact ($t-s \geq 2$)

$$\bigoplus_{0 \leq k < r} E_{r+k}^{s-k,t-k}(Z,X) \xrightarrow{g^{*,*}} E_r^{s,t}(W,X) \xrightarrow{f^{*,*}} \bigoplus_{0 \leq k < r} E_{r-k}^{s+k,t+k}(Y,Z)$$

Proof: Consider the finite tower of fibrations

$$D_{s+r}X \rightarrow D_{s+r-1}X \rightarrow \dots \rightarrow D_{s-r+1}X$$

Then the homotopy spectral sequences associated with this tower converge (with a shift in dimension) to the groups

$[S^{t-s-1}_K, F_{s-r+1,s+r}]$ where $K = Z, W$, or Y . The terms

$E_r^{s,t}(W,X)$ and $E_{r+k}^{s-k,t-k}(S,X)$ are the E_∞ terms of these

spectral sequences when $K = W$ or Z , while a quotient of

$E_{r-k}^{s+k,t+k}(Y,X)$ is the E_∞ term when $K = Y$.

The theorem follows because the following sequence of homotopy groups is exact because $X \rightarrow Y \rightarrow Z$ is a cofibration. (Let $F = F_{s-r+1,s+r}$).

$$[S^{t-s-1}_Z, F] \xrightarrow{g^\#} [S^{t-s-1}_W, F] \xrightarrow{f^\#} [S^{t-s-1}_Y, F].$$

For if $b \in E_r^{s,t}(W,X)$ is in the kernel of the construction $f^{*,*}$, then b detects an element $b' \in [S^{t-s-1}W, F]$ which is in the kernel of the map $f^\#$ in homotopy. Hence there is an $a' \in [S^{t-s-1}Z, F]$ so that $g^\#a' = b'$ and if a is any element which detects a' , it is easy to show that (a,b) is in the construction $g^{*,*}$. Hence $\ker(f^{*,*}) \subset \text{image}(g^{*,*})$. The opposite inclusion follows since if b is in the image of $g^{*,*}$, then b detects an element b' so that $f^\#(b') = 0$. This easily implies that $(b,0)$ is in the construction $f^{*,*}$.

Remark. Theorem 1.23 involves E_1 terms of the spectral sequence as part of the image of $f^{*,*}$. However, if $b \in E_r^{s,t}$ is actually a d_r cycle, then $f^{*,*}(b)$ can be regarded as in $\oplus E_k$ where $k \geq 2$.

The constructions $f^{*,*}$ are defined in a way that makes the proof of Theorem 1.23 easy. However, they can be calculated in a different way, by means of differentials in the spectral sequence for the cofibre $E(Z,X)$. To do this, first extend the cofibre sequence to

$$Y \xrightarrow{f} W \xrightarrow{g} Z \xrightarrow{h} SY \xrightarrow{Sf} SW \xrightarrow{Sg} SZ \xrightarrow{Sh} S^2Y.$$

The following theorem shows how to calculate the constructions $f_r^{s,k}$ in terms of differentials in the spectral sequence for the cofibre Z . Suppose

$\alpha \in E_r^{s,t}(W,X)$ and $t-s \geq 3$.

Theorem 1.24 $f_r^{s,k}(\alpha) \in E(Y,X)$ is the set of all $\gamma \in E(Y,X)$ such that $d_{k+l+m}(\beta) = h^{s+k,m}(\gamma)$ for some $\beta \in E(Z,X)$ where β satisfies $g_{r+1}^{s-1,1}(\beta) = \alpha$ (for some non-negative integers $l < r$ and $m < r-k$).

Proof: Let $F' \rightarrow F \rightarrow F''$ denote the fibration

$$F_{s+k,s+r} \rightarrow F_{s-r+1,s+r} \rightarrow F_{s-r+1,s+k}.$$

Then we get the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccc} [S^a Y, F'] & \rightarrow & [S^a Y, F] & \rightarrow & [S^a Y, F''] \\ \downarrow & & \downarrow f^\# & & \downarrow \\ [S^a W, F'] & \rightarrow & [S^a W, F] & \rightarrow & [S^a W, F''] \\ \downarrow & & \downarrow g^\# & & \downarrow \\ [S^a Z, F'] & \rightarrow & [S^a Z, F] & \rightarrow & [S^a Z, F''] \end{array}$$

To show $f_r^{s,k}(\alpha) \subset \{\gamma \text{ as above}\}$, suppose a representative of α has been chosen in $[S^a W, F]$. Choose a representative $\tilde{\alpha}$ of $f_r^{s,k}(\alpha) \in [S^a Y, F']$ so that $h^\#(\tilde{\alpha}) \in [S^{a-1} Z, F']$ can be lifted to an element of $[S^{a-1} Z, F_{s+k+m,s+r}]$ which then by definition represents $h^{s+k,l+m} f_r^{s,k}(\alpha)$. We must define $\tilde{\beta} \in [S^a Z, F_{s-r+1,s+k+m}]$ so that $g^{*,*}(\tilde{\beta}) = \alpha$ where β detects $\tilde{\beta}$ and so that $d_{k+l+m}(\beta)$ is equal to the representative of $h^{s+k,l+m} f_r^{s,k}(\alpha)$.

First fix a homotopy equivalence $Z \simeq W \bigcup_f CY$. We define $\tilde{\beta}$ by defining a map of pairs b

$$b: (CS^{a-1}(W \bigcup CY), S^{a-1}(W \bigcup CY)) \rightarrow (F, F_{s+k+m, s+r}).$$

On $S^{a-1}(W \bigcup CY)$ let b be a representative of $h^{\# \sim}_\alpha$. Use the representative of α to extend over $CS^{a-1}W$. To extend over $CS^{a-1}CY$, use the representative $\alpha \in [S^a Y, F']$ composed with the map from $CS^{a-1}CY$ to $S^{a-1}CY$ which adds the coning coordinates. (Write $CY = (0, \infty) \times Y / ([1, \infty) \times Y)$). This completes the definition of b .

Now ∂b by construction is $h^{\# \sim}_\alpha$ and so is detected by $h^{*, l+m} f^{*, k}(\alpha)$. The map $\tilde{\beta}$ defined by b must be detected by some $\beta \in E_{r+l}^{s-l, t}(Z, X)$ for some l and has the properties desired.

To show $\{\gamma \text{ as above}\} \subseteq f^{*, k}_\alpha$, suppose a representative b of β has been chosen as a map of pairs as above. Then by the hypothesis $f^{*, l}(\beta) = \alpha$, we may choose b so that, as a map into the pair (F, F') , $S^{a-1}W$ is carried to the basepoint. We need only show that the map b restricted to $S^{a-1}CY$ when regarded as a map $\phi_0: S^a Y \rightarrow F$ is homotopic to $\phi_1 = b \circ g \circ f$. Now write $CS^{a-1}CY$ with coordinates (t_1, t_2, y) where $y \in S^{a-1}Y$ and $t_1, t_2 \in [0, \infty)$ modulo suitable identifications. Now b restricts to a map $\bar{b}$ on $CS^{a-1}CY$. Now

$$\phi_0: CS^{a-1}Y \rightarrow F$$

is given by

$$\phi_0(t, y) = \bar{b}(t, 0, y).$$

Also

$$\phi_1(t, y) = \bar{b}(0, t, y).$$

These two maps are plainly homotopic via the parameter τ by

$$\phi_\tau(t, y) = \bar{b}((1-\tau)t, \tau t, y).$$

This completes the proof.

The geometric coboundary lemma

Suppose $Y \xrightarrow{f} W \xrightarrow{g} Z$ is a cofibration of finite complexes inducing a short exact sequence in Z_2 homology. Then if $M \in \underline{MA}$, there is a coboundary map

$$\delta: \text{Ext}_{\underline{MA}}^{s,t}(H_*Y, M) \rightarrow \text{Ext}_{\underline{MA}}^{s+1,t}(H_*Z, M)$$

which can be computed using the Λ algebra (Prop. 1.6, 1.9, Theorem 1.8). In case X is a space so that $H_*X = U_*M$ we have

$$\text{Ext}_{\underline{MA}}^{s,t}(H_*Y, M) \simeq E_2(K, X)$$

by Theorem 1.2. The purpose of this part is to show that δ is well related to the topological coboundary.

As before, choose $h: Z \rightarrow SY$ to extend the cofibre sequence.

Lemma 1.25 Suppose $t-s \geq 3$ and $a \in \text{Ext}_{\underline{MA}}^{s,t}(H_*Y, M)$

detects $\alpha \in [S^{t-s}Y, X]$. Then $\delta a \in \text{Ext}_{\underline{MA}}^{s+1,t}(H_*Z, M)$

detects $h^\#(\alpha) \in [S^{t-s-1}Z, X]$.

Proof: The first part of the proof is to note that the isomorphism $\text{Ext}_{\underline{\text{MA}}} \simeq E_2$ is induced by a natural chain equivalence. This involves examining the proof in [7] of Theorem 1.2. A sketch follows. [7: Remark 13.6] shows that there is a natural isomorphism $\text{Ext}_{\underline{\text{CA}}} \simeq \text{Ext}_{\underline{\text{MA}}}$ which is induced by a chain map. There is a natural isomorphism $E_2 \simeq \text{Ext}_{\underline{\text{CA}}}$ [7: Theorem 11.5, Corollary 11.7] induced by a natural chain map where the chains for E_2 are the groups $\pi_t \text{hom}(W, \underline{\text{RX}})$. A suitable double complex argument [7; p. 95] shows that $\pi_t(W, \underline{\text{RX}})$ is naturally chain equivalent to $\pi_{t-s} \text{hom}(W, D_s \text{RX})$. Hence $E_2 \simeq \text{Ext}_{\underline{\text{MA}}}$ by a map induced by a natural chain equivalence.

To complete the proof of the proposition, note that $\pi_{t-s}(\text{hom}(W, D_s \text{RX})) \simeq [S^{t-s} W, D_s \text{RX}] \simeq [S^{t-s} W, F_{s,s+1} X]$. Then the construction $h^{*,1}: E_2^{s,t}(Y,X) \rightarrow E_1^{s+1,t}(Z,X)$ can be computed using d_1 in $E_1^{s,t}(W,X)$ by Theorem 1.24. Because natural chain maps induce the isomorphism $\text{Ext}_{\underline{\text{MA}}} \simeq E_2$, $h^{*,1}$ plainly corresponds to δ on $\text{Ext}_{\underline{\text{MA}}}$. Since the Proposition is true when $\text{Ext}_{\underline{\text{MA}}}$ is replaced by E_2 and δ is replaced by $h^{*,1}$ we are done.

SECTION 2. THE EXTENDED POWER CONSTRUCTION

References for this section are [26], [11].

Let X be a topological space with base point. The group Z_2 of integers reduced mod 2 acts on the i -sphere S^i by the antipodal action and on the product $X \wedge X$ by interchanging factors. If Y is another space, then the half smash is defined by

$$Y \rtimes X = (Y \times X)/(Y \times *)$$

Now Z_2 acts on the product spaces $S^i \rtimes (X \times X)$ and $S^i \rtimes (X \wedge X)$ by the product action.

Definition. The extended powers of X are the spaces

$$ep^i X = S^i \rtimes_{Z_2} X \wedge X.$$

In case X is a finite C.W. complex, the extended powers of X are C.W. complexes. In fact, since the spheres S^i have a C.W. structure so that the antipodal action is cellular, Z_2 acts freely on the cells of $S^i \times X \times X$ and the map

$$S^i \times (X \times X) \rightarrow S^i \times_{Z_2} (X \times X)$$

is a cellular covering map. The right action of the Steenrod algebra on the homology of $ep^i(X)$ has been determined by Nishida [20].

More generally, the extended powers can be constructed with the group Z_2 replaced by other finite groups [26], [21], [19]. Let π be any finite group and $\rho: \pi \rightarrow S_n$ be a representation of π in the symmetric group on n letters. Let W_π be the universal cover of the classifying

space $B\pi$ with π acting on W . Let $X^{(n)}$ denote the n -fold smash product of X . Then

$$W \rtimes_{\pi} X^{(n)}$$

is an extended power of X . When $\pi = Z_p$ and $\rho: Z_p \rightarrow S_p$ is the regular representation, the extended power is known [26] as the extended p^{th} power of X .

The extended power construction is invariant under certain choices. Let W be any contractible C.W. complex with a free Z_2 action (i.e. no fixed point) which carries cells to cells.

Proposition 2.1 There is a Z_2 equivariant map

$$S^{\infty} \xrightarrow{f} W$$

and any two such maps are equivariantly homotopic.

Proof: On $S^0 \subset S^{\infty}$, define the map on one vertex arbitrarily and extend to the other vertex by equivariance. Inductively, assume the map is defined on $S^{n-1} \subset S^{\infty}$. Then extend over the upper hemisphere of S^n using the fact that W is contractible and extend over the bottom hemisphere by equivariance. Thus, there is a map $S^{\infty} \rightarrow W$. Equivariant homotopies between any two maps can be constructed in a similar manner.

Corollary 2.2 W/Z_2 is homotopy equivalent to $S^{\infty}/Z_2 = \mathbb{R}P^{\infty}$.

Proof: $W \rightarrow W/Z_2$ and $S^{\infty} \rightarrow S^{\infty}/Z_2$ are two coverings with Z_2 as group. The map of Proposition 2.1 is a map of coverings. Comparing the long exact sequences in homotopy for the coverings, we see that f induces an isomorphism

in homotopy. Since W/Z_2 and S^∞/Z_2 are C.W. complexes, the map f is a homotopy equivalence.

Proposition 2.3 Suppose X is a C.W. complex and $g: S \times X \wedge X \rightarrow W \times X \wedge X$ is any Z_2 equivariant map which is a homotopy equivalence. Then $g_{Z_2}: S \times_{Z_2} X \wedge X \rightarrow W \times_{Z_2} X \wedge X$ is a homotopy equivalence.

Note: $S^\infty \times X \wedge X$ is homotopy equivalent to $X \wedge X$.

Proof: The maps $- \times X \wedge X \rightarrow - \times_{Z_2} X \wedge X$ are coverings with group Z_2 . Then since g is a homotopy equivalence and a covering map, g_{Z_2} induces an isomorphism in all homotopy groups.

Corollary 2.4 When X is a connected C.W. complex

$$\bar{g}: S^\infty \rtimes_{Z_2} X \wedge X \rightarrow W \rtimes_{Z_2} X \wedge X$$

is a homotopy equivalence.

Proof: When X is connected, $X \wedge X$ is simply connected and so the spaces $- \times_{Z_2} X \wedge X$ have fundamental group Z_2 . Taking half smashes identifies the subspaces S^∞/Z_2 and W/Z_2 to the basepoint, and so by the Van Kampen theorem, the spaces $- \rtimes_{Z_2} X \wedge X$ are simply connected. Now

2.3 implies that g and $\bar{g}$ induce isomorphisms in homology. Hence $\bar{g}$ is a homotopy equivalence.

Remark: The above results allow one to recognize a nonstandard construction as the ep construction. Suppose $f: S^\infty \times X \wedge X \rightarrow W \times Y \wedge Y$ is any Z_2 equivariant map which

ignoring first factors is equal to $\alpha \wedge \alpha: X \wedge X \rightarrow Y \wedge Y$.

Proposition 2.5 $\bar{f}: S^\infty \rtimes_{Z_2} X \wedge X \rightarrow W \rtimes_{Z_2} Y \wedge Y$ is the

extended power construction on α up to a homotopy equivalence of the domain and range.

Proof: Let $\pi: S \times X \wedge Y \rightarrow W$ be the projection on the first factor of f . Then the composition of the map

$$\pi \times 1_{X \wedge X}: S^\infty \times X \wedge X \rightarrow W \times X \wedge X$$

and the map

$$1_W \times \alpha \wedge \alpha: W \times X \wedge X \rightarrow W \times Y \wedge Y$$

is equal to f . The first map induces a homotopy equivalence, and the second map is the extended power construction with S^∞ replaced by W . Now using the map $S^\infty \rightarrow W$ of proposition 2.1 gives the desired result.

In order to study the behavior of the extended powers under suspension, we use the following theorem.

Theorem 2.7 There is a homeomorphism

$$\sigma: S(\text{ep}^{i+1}X/\text{ep}^0X) \rightarrow \text{ep}^i(SX)$$

Proof: In order to conveniently give a formula for this homeomorphism, regard S^1 as the one point compactification of R^1 . Also regard S^{i+1} as the two point compactification of the cylinder $S^i \times R^1 \cup \{-\infty, +\infty\}$ with $Z_2 = \langle e, \tau \rangle$ action given by the product of the antipodal action on S^i and the action $\tau(\theta) = -\theta$ for $\theta \in R^1$. Also $S^0 \subset S^{i+1}$ is the set $\{-\infty, \infty\}$. Then the map

$S^1 \wedge (S^{m+1} \rtimes_{Z_2} X \wedge X) \rightarrow S^m \rtimes_{Z_2} SX \wedge SX$ is given by

$$\theta_1 \wedge (s \times \theta_2 \rtimes x \wedge y) \rightarrow s \rtimes (\phi_1 \wedge x) \wedge (\phi_2 \wedge y)$$

where $\phi_1 = \theta_1 + \theta_2$ and $\phi_2 = \theta_1 - \theta_2$

for $\theta_1, \theta_2 \in R^1$ and if either θ_1 or θ_2 is equal to $\pm\infty$ set the image to the base point. It is readily verified that the formula respects the Z_2 action. Hence σ is defined. Further, it is easy to show that σ is continuous and has an inverse.

Corollary 2.8 The extended power of a sphere is a stunted projective space ($n \geq 1$).

$$ep^i(S^n) = S^n RP_n^{i+n} = S^n(RP^{i+n}/RP^{n-1})$$

Proof:

$$W \rtimes S^0 \wedge S^0 = W \cup \{*\}.$$

Hence $ep^i S^0 = RP^i \cup \{*\}$. Now

$$ep^i(S^1) = S^1 ep^{i+1}(S^0)/S^1 ep^0(S^0) = S^1 RP_1^{i+1}.$$

The corollary now follows by induction on n .

Notation: Let ep_j^i denote ep^i/ep^{j-1} .

Later we will need to know how the extended power construction behaves with respect to loop adjoint. Let $\alpha: SK \rightarrow X$. Then the adjoint $\alpha_0: K \rightarrow \Omega X$ is the composition of the inclusion $i: K \rightarrow \Omega SK$ and the loops on α , i.e. $K \rightarrow \Omega SK \rightarrow \Omega X$.

Let τ be a map $\tau: S\Omega X \rightarrow X$ given by $\tau(t, f) = f(t)$

where $f \in \Omega X$ and t is the suspension coordinate. Then the following facts are easily verified.

Proposition 2.9

(a) The following composition is the identity

$$SK \xrightarrow{Si} S SK \xrightarrow{\tau} SK$$

(b) The following composition is .

$$SK \xrightarrow{S\alpha_0} S\Omega X \xrightarrow{\tau} X.$$

(c) Any map α'_0 satisfying (b) must be α_0 .

Lemma 2.10 The following diagram commutes

$$\begin{array}{ccc} \text{Sep}^{i+1}(\Omega SK) & \xrightarrow{\text{Sep}^{i+1}(\Omega \alpha)} & \text{Sep}^{i+1}\Omega X \\ \downarrow \text{ep}^i(\tau) \circ \sigma & & \downarrow \text{ep}^i(\tau) \circ \sigma \\ \text{ep}^i(SK) & \xrightarrow{\text{ep}^i(\alpha)} & \text{ep}^i X \end{array}$$

Proof: The lemma follows immediately by the naturality of ep and the easily checked commutativity of the following diagram

$$\begin{array}{ccc} & S\Omega(\alpha) & \\ S\Omega(SK) & \xrightarrow{\quad} & S\Omega(X) \\ \downarrow \tau & & \downarrow \tau \\ SK & \xrightarrow{\alpha} & X \end{array}$$

Now let $\alpha: SK \rightarrow X$ be given and let α_0 denote its adjoint. Then $\text{Sep}_1^{i+1}(\alpha_0)$ is a map

$$\text{Sep}_1^{i+1}(\alpha_0): \text{Sep}_1^{i+1}K \rightarrow \text{Sep}_1^{i+1}\Omega X.$$

Using the map σ we get

$$\text{ep}^i \text{SK} \xrightarrow{\sigma^{-1}} \text{Sep}_1^{i+1} \xrightarrow{\text{Sep}^{i+1}(\alpha_0)} \text{Sep}_1^{i+1} \Omega X \xrightarrow{\sigma} \text{ep}^i S \Omega X.$$

Let f denote this composition. Then the following proposition shows $\text{Sep}^{i+1}(\alpha_0)$ and $\text{ep}^i(\alpha_0)$ are closely related.

Proposition 2.11 $\text{ep}^i(\tau) \circ f = \text{ep}^i(\alpha).$

Proof: Consider the commutative square of Lemma 2.10. Then the left hand composition above can be obtained by composing the top of the square with the inclusion

$\text{ep}^i \text{SK} \xrightarrow{\sigma^{-1}} \text{Sep}_1^{i+1} K \xrightarrow{i} \text{Sep}_1^{i+1} \Omega \text{SK}.$ Since the bottom map in the square is $\text{ep}^i(\alpha)$, to finish the proof we need only show that $\text{ep}^i(\tau) \circ \sigma \circ i \circ \sigma^{-1} = \text{identity on } \text{ep}^i \text{SK}.$ This is contained in the following lemma.

Lemma 2.13 $\text{ep}^i \text{SK}$ is a retract of $\text{ep}^i S \Omega \text{SK}$ and hence a retract of $\text{Sep}_1^{i+1} \Omega \text{SK}.$

Proof: Applying the functor ep to (a) Proposition 2.9 gives $\text{ep}^i \text{SK}$ as a retract of $\text{ep}^i S \Omega \text{SK}.$ To get the second part observe that the following diagram commutes because the map σ is natural.

$$\begin{array}{ccc} \text{Sep}_1^{i+1} K & \xrightarrow{i} & \text{Sep}_1^{i+1} \Omega \text{SK} \\ \downarrow \sigma & & \downarrow \sigma \\ \text{ep}^i \text{SK} & \xrightarrow{i} & \text{ep}^i S \Omega \text{SK} \xrightarrow{\text{ep}^i \tau} \text{ep}^i \text{SK} \end{array}$$

Thus $\text{ep}^i(\tau) \circ \sigma \circ i \circ \sigma^{-1}$ is the identity on $\text{ep}^i \text{SK}.$

Attaching maps in stable $\mathbb{R}P^\infty$

This next part summarizes results of Adams, Mahowald, and Milgram on the attaching maps in stable $\mathbb{R}P^\infty$. To do this we recall some known facts from the KO theory of $\mathbb{R}P^n$. The reference for this part is [11].

If $S^n \rightarrow \mathbb{R}P^n$ is the universal cover of $\mathbb{R}P^n$, then S^n can be regarded as a principal Z_2 bundle over $\mathbb{R}P^n$. The associated line bundle $S^n \times_{Z_2} \mathbb{R}^1 \rightarrow \mathbb{R}P^n$ where Z_2 acts on $\mathbb{R}^1$ by $\tau(x) = -x$ is called the canonical line bundle and is denoted by ξ_n .

As usual, let $KO(X)$ denote the real K theory of finite dimensional real vector bundles over X . Then the reduced group $\widetilde{KO}(\mathbb{R}P^k)$ is a cyclic group of order c_k where c_k is given by the following table [11; 15.12.7].

k	0	1	2	3	4	5	6	7	8
c_k	1	2	4	4	8	8	8	8	16

and $c_{k+8} = 16c_k$. The generator of $\widetilde{KO}(\mathbb{R}P^k)$ is ξ_k^{-1} .

If $\eta: E \rightarrow B$ is a vector bundle with a Riemannian metric, let $D(\eta)$ denote the disk bundle associated with η . $D(\eta)$ is the bundle of vectors of length ≤ 1 . Let $S(\eta)$ denote the sphere bundle of vectors of length $= 1$. Then the Thom space of η is the space

$$T(\eta) = D(\eta)/S(\eta).$$

Let η and ξ denote bundles over a compact space X

and let θ denote the trivial bundle $X \times \mathbb{R}^1$. Then the following are standard facts about Thom spaces [11].

$$T(\theta) = X \rtimes S^1 = (X \cup \{*\}) \wedge S^1$$

$$T(\eta \times \xi) \text{ is homeomorphic to } T(\eta) \wedge T(\xi)$$

$$T(\eta \oplus \theta) \text{ is homeomorphic to } S^1(T(\eta))$$

The Thom spaces of the various multiples of the canonical line bundle ξ_k over $\mathbb{R}P^k$ are given by the following

Proposition 2.14 $T(m\xi_k) = \mathbb{R}P_{m-1}^{m+k}$

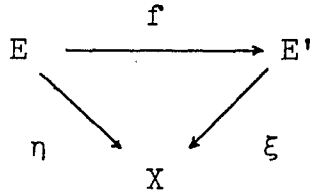
Let X be a finite dimensional C.W. complex and suppose a map $\alpha: S^n \rightarrow X$ is given. Then a map $f: X \rightarrow S^n$ is called a coreduction (over α) provided $f \circ \alpha$ is homotopic to the identity on S^n . Similarly, if $\beta: X \rightarrow S^n$ is given, then a map $g: S^n \rightarrow X$ is a reduction (over β) provided $\beta \circ g$ is homotopic to the identity.

Example: Suppose η is an n -dimensional real vector bundle over Y . Then there is a map of the trivial bundle $\theta_n: \mathbb{R}^n \rightarrow *$ to $\eta: E \rightarrow Y$ which is just the inclusion of the fibre over the base point. Applying the Thom space functor produces a map $T(\theta_n) \rightarrow T(\eta)$ which can be regarded as a map $\alpha: S^n \rightarrow T(\eta)$. When a coreduction exists (over α) the Thom space $T(\eta)$ is said to be coreducible. When a coreduction exists over $S^k(\alpha): S^{n+k} \rightarrow S^k T(\eta)$ for some k the Thom space is said to be stably coreducible or S-coreducible. S-coreducibility has been characterized.

In order to state this characterization we need the following.

Fibre Homotopy type

Let η and ξ be fibre bundles over a space X .



Then f is a fibre map provided $f(\eta^{-1}(x)) \subset \xi^{-1}(x)$ for all $x \in X$. Then f is a fibre-homotopy equivalence between η and ξ provided f is a homotopy equivalence in the category of bundles over X and fibre maps between them.

Remark: Because $\mathbb{R}^n$ is contractible, any two vector bundles over a space X are fibre homotopy equivalent. In the context of vector bundles, the relation of fibre homotopy equivalence is usually applied to the associated sphere bundles.

Consider the quotient of $\widetilde{KO}(X)$ by the relation of stable fibre homotopy equivalence of the associated sphere bundles. Then it can be shown [11] that this quotient denoted $J(X)$ is a group and that the map

$$J: KO(X) \rightarrow J(X)$$

is a group epimorphism.

Now we can state the characterization of coreducible Thom spaces.

Theorem 2.15 [11; 15.7.7] Let ξ be a vector bundle over a connected finite C.W. complex X . Then the following are equivalent.

- (1) $J(\xi) = 0 \in J(X)$
- (2) The space $T(\xi)$ is S -coreducible
- (3) The spaces $T(\xi)$ and $X \cup \{+\}$ are stably equivalent.

Generally, the computation of $J(X)$ is difficult [3]. For $\mathbb{R}P^n$ we have the following pleasant state of affairs.

Theorem 2.16 [11; 15.13.9]

$$J(\mathbb{R}P^n) = KO(\mathbb{R}P^n).$$

Combining the facts in this section so far, one can give a complete solution to the problem of finding stable coreductions of the Thom spaces of bundles over $\mathbb{R}P^n$. These coreductions are retractions of $\mathbb{R}P_i^{k+i}$ onto the bottom cell S^1 .

Proposition 2.17 Suppose $M \geq k-i$. Then there is a retraction

$$S^M \mathbb{R}P_i^{i+k} \longrightarrow S^{M+i}$$

if and only if i is a multiple of c_k where c_k is the order of $\xi_k - 1$ in $KO(\mathbb{R}P^k)$.

Proof: (if) When M is large enough the theorem is just the statement that $c_k \xi_k$ plus an M dimensional trivial bundle is a trivial bundle over $\mathbb{R}P^k$ and so the stable Thom space $S^M \mathbb{R}P_i^{k+i}$ is coreducible. Then given a coreduction for some large M , we may lower M by using

the fact that the homomorphism of homotopy groups

$$[S^M_{\mathbb{RP}^i_1}, S^{M+i}] \longrightarrow [S^{M+1}_{\mathbb{RP}^i_1}, S^{M+i+1}]$$

is onto for $M+k+i \leq 2(M+i)$ when $M+i > 1$ by the Blakers-Massey result.

(only if): the existence of a coreduction implies that $J(i\xi_k) = 0$ in $J(\mathbb{RP}^k)$. Since $KO(\mathbb{RP}^k) = J(\mathbb{RP}^k)$ this implies that i is a multiple of the order of the generator $\xi_k - 1$ by Proposition 2.15.

In order to translate information about stable coreductions into information about stable reductions we need the concept of S-duality. Suppose X and Y are C.W. complexes with base point. Let $[SX, Y]$ denote the group of homotopy classes of continuous (base point preserving) maps from SX to Y . Let $\{SX, Y\}$ denote the direct limit of groups over n of $[S^{n+1}X, S^nY]$.

Definition An n -pairing $X \wedge X' \rightarrow S^n$ is called an n -duality provided the induced maps of stable homotopy groups

$$\{S^k, X'\} \rightarrow \{X \wedge S^k, S^n\} \quad \text{and} \quad \{S^k, X\} \rightarrow \{X' \wedge S^k, S^n\}$$

are both isomorphisms. The spaces X and Y are called S-dual provided there is an m -duality

$$S^kX \wedge S^\ell Y \rightarrow S^m \quad \text{for some } k, \ell \text{ and } m.$$

Remark: Suppose X has an n -dual X' . Then if

$f \in [X, S^\ell]$ is an S-coreduction over a map $\alpha \in [S^\ell, X]$ then by the definitions $f' \in \{S^{n-\ell}, X\}$ is an S-reduction

of X' over $\alpha' \in \{X, S^{n-l}\}$.

Theorem 2.18 (Atiyah) Let η and ξ be two vector bundles over a closed differentiable manifold (compact and no boundary). Then if $\xi \oplus \eta \oplus \tau$ is trivial where τ is the tangent bundle, then the Thom spaces $T(\xi)$ and $T(\eta)$ are S-dual.

In order to apply this theorem to the problem of finding reductions of RP_k^n , recall the tangent bundle for the manifold RP^k satisfies

$$(RP^k) \oplus \theta = (k+1)\xi_k$$

where θ is a trivial line bundle. Applying Atiyah's theorem we immediately get

Proposition 2.19 The Thom space $T(m\xi_k)$ is S-dual to the Thom space $T((nc_k - m - k - 1)\xi_k)$ for each n . In other words,

$$RP_m^{k+m} \text{ is S-dual to } RP_{nc_k - m - k - 1}^{nc_k - m - 1}.$$

Corollary 2.20 If $N \geq k-m$, there is a reduction on the top cell

$$S^{N+k+m} \rightarrow S^N RP_m^{k+m}$$

if and only if $(m+k+1)\xi_k$ is stably trivial.

Proof: A coreduction of the Thom space is a coreduction over the inclusion map

$$S^{nc_k - m - k - 1} \rightarrow RP_{nc_k - m - k - 1}^{nc_k - m - 1}. \text{ The S-dual is then a reduction}$$

over the pinch map

$$S^M RP_m^{k+m} \rightarrow S^{M+k+m}.$$

Stably the coreduction exists if and only if $(nc_k - m - k - 1)\xi_k$ is trivial which implies $m+k+1$ is a multiple of c_k . The integer M can be lowered to $M = k - m$ as in the proof of Proposition 2.17.

Corollary 2.21 If $N \geq k - m$, the reductions exist if and only if

$$k \leq \phi(k+m) - 1$$

where ϕ is defined as follows.

$$\text{Write } L+1 = (2j+1)2^{4c+d} \quad (0 \leq d \leq 3).$$

$$\text{Then } \phi(L) = 8c + 2^d.$$

Proof: Given $L = k+m$, we wish to find the maximum of k so that $L+1 \equiv 0 \pmod{c_k}$. By looking at a table of values for c_k , we see that the maximum for k is congruent to one of the values 0, 1, 3 or 7 mod 8. The formula plainly is correct for $c = 0$ and $0 \leq d \leq 3$. The corollary follows by induction on c using the relation $c_{k+8} = 16c_k$.

When $k = \phi(k+m)$, there is no reduction. In this case we get the following [19].

Theorem 2.22 If $k = \phi(k+m)$ and $N \geq m$, then there is a map

$$S^{N+m} \bigcup_{\beta} e^{N+k+m} \rightarrow S^N_{RP_m^{k+m}}$$

extending the inclusion of the $N+m$ sphere, where β is an odd multiple of the generator of the 2-component of the image of J in the $k-1 = \phi(k+m) - 1$ stem of the stable homotopy of spheres.

In order to prove this we recall some facts.

Lemma 2.23 [3] Let η be a k vector bundle over a sphere S^n ($n \geq 2$). Then

$$T(\eta) = S^k \bigcup_{\alpha} e^{n+k}$$

where $\alpha \in \pi_{n+k-1}(S^k)$ is some element in the image of J .

Recall that the k dimensional vector bundles over X are classified by the set of maps $X \rightarrow BO(k)$, where $BO(k)$ is the classifying space for the orthogonal group $O(k)$ and X is a finite C.W.-complex. Let BO be the classifying space for the infinite orthogonal group, i.e.

$BO = \lim_k BO(k)$. Then we have $KO(X)$ is naturally isomorphic as a group to the homotopy group $[X, BO]$ addition is given by a map $BO \wedge BO \rightarrow BO$.

The connection between vector bundles and the image of J is given as follows. Let $S^n \rightarrow BO(k)$ classify η as in the lemma above. Then the adjoint $S^{n-1} \rightarrow \Omega BO(k) = O(k)$ is an element $\alpha' \in \pi_{n-1}(O(k))$. Then $J(\alpha') = \alpha$ as in the lemma.

In order to prove Theorem 2.22 we first prove the S-dual statement.

Lemma 2.24 Suppose $M > k-i$ and there is a coreduction $S^M_{RP_i^{k+i}} \rightarrow S^{M+i}$. Then there is a map extending a coreduction

$$S^M_{RP_i^{k+i+1}} \rightarrow S^{M+i} \bigcup_{\alpha} e^{M+k+i+1}$$

which is onto in Z_2 -homology where α is in the image of J .

Proof: RP_i^{k+i} is the Thom space $T(i\xi_k)$. To say there is a coreduction implies that the classifying map $i\xi_k: RP^k \rightarrow BO$ is trivial. Hence $i\xi_{k+1}$ defines a map from the quotient

$$RP^{k+1}/RP^k \rightarrow S^{k+1} \rightarrow BO.$$

By the naturality of the Thom construction and Lemma 2.23 we get

$$S^N T(i\xi_{k+1}) \rightarrow S^{N+i} \bigcup_{\alpha} e^{N+k+i+1}$$

for some suitably large N . Then N can be reduced to $M > k-i$ by Blakers-Massey as before (Proposition 2.17). The map is onto in Z_2 homology because the map of bases $RP^{k+1} \rightarrow S^{k+1}$ is onto in Z_2 homology and because the Thom isomorphism between the cohomology of the base and the reduced cohomology of the Thom space is natural in maps of the base. On S^{M+i} , the map is just the identity on the Thom space over the base point and so the map extends a coreduction as desired.

Lemma 2.25 If α is in the image of J , then $S^n \bigcup_{\alpha} e^{n+k}$ is its own S -dual.

Proof: Let η be the bundle determined by the map $\alpha': S^{k-1} \rightarrow SO$ where $J(\alpha') = \alpha$. That is, η is classified by the adjoint map $\bar{\alpha}: S^k \rightarrow BSO$. Since the tangent bundle to a sphere is stably trivial ($= 0$ in $\widetilde{KO}(S^k)$), applying Theorem 2.18 we get that $T(\eta)$ is S -dual to $T(-\eta)$ where

$-\eta$ is a bundle representing $-\eta \in KO(S^k)$. Because S^k is a suspension, $-\eta$ is classified by $-\bar{\alpha}$ in $\pi_k(BO)$ because the π_* group structure is the same as that induced by the Whitney sum map $BO \wedge BO \rightarrow BO$. The lemma follows because $S^n \cup_{-\alpha} e^{n+k}$ is homeomorphic to $S^n \cup_{\alpha} e^{n+k}$.

Proof of Theorem 2.22 Lemmas 2.24 and 2.25 supply maps

$$S^{N+m} \cup_{\alpha} e^{N+k+m} \rightarrow S_{RP_m}^{N+k+m}$$

where α is in the image of J which induce isomorphisms in homology in dimension $N+k+m$. Hence α can be regarded as the attaching map of the top cell. Precisely, let α_1 be the composition

$$\alpha_1: S^{N+k+m-1} \rightarrow S^{N+m} \rightarrow S_{RP_m}^{N+k+m-1}.$$

Then there is a map

$$S_{RP_m}^{N+k+m-1} \cup_{\alpha_1} e^{N+k+m} \rightarrow S_{RP_m}^{N+k+m}$$

which is an isomorphism in homology. Now α_1 must be nontrivial when $k = \phi(k+m)$ for otherwise there would exist a reduction on the top cell of $S_{RP_m}^{N+k+m}$ which is impossible by Corollary 2.21. When m is odd, α_1 cannot be divisible by 2 because $\pi_m(RP_m) = \mathbb{Z}_2$. When m is even, the element in the image of J in stem $\phi(m+k) - 1 = k - 1$ has order 2. Hence in both cases α_1 must be a generator of the 2-component of the image of J .

SECTION 3. THE EHP SEQUENCE

There is a sequence of maps [12], [25]

$$S^n \longrightarrow \Omega S^{n+1} \xrightarrow{h} \Omega S^{2n+1}$$

which is a fibration when localized at the prime 2. In homotopy this gives the EHP sequence (exact on the 2-component)

$$\pi_t(S^n) \xrightarrow{E} \pi_{t+1}(S^{n+1}) \xrightarrow{H} \pi_{t+1}(S^{2n+1}) \xrightarrow{P} \pi_{t-1}(S^n)$$

More generally, if K is a finite C.W. complex and $t > 1$, the fibration yields a long exact sequence of homotopy groups (on the 2-component)

$$[S^t K, S^n] \rightarrow [S^{t+1} K, S^{n+1}] \rightarrow [S^{t+1} K, S^{2n+1}] \rightarrow [S^{t-1} K, S^n]$$

where $[SX, Y]$ denotes the group of homotopy classes of maps from SX to Y .

Now suppose $\alpha: SK \rightarrow S^n$. Then applying the extended power construction gives a map

$$ep^i(\alpha): ep^i(SK) \rightarrow ep^i(S^n).$$

Suppose further there is a retraction

$$r_1: ep_1^{i+1}(S^{n-1}) \rightarrow S^{2n-1}.$$

Using the map σ of Theorem 2.7, we can then also give a retraction

$$r: ep^i(S^n) \rightarrow S^{2n}.$$

The composition $r \circ ep^i(\alpha)$ is a map

$$\overline{ep}^i(\alpha) \in [ep^i SK, S^{2n}].$$

Remark 3.0 To avoid cumbersome statements later, we denote by $\overline{ep}^i(\alpha)$ any element so obtained where r is any retraction. Thus $\overline{ep}^i(\alpha)$ is used to denote any one element in a class of elements; it will not be used to denote the whole class.

Using σ we can regard $ep^i SK$ as a suspension (Theorem 2.7). Hence the Hopf invariant $H(\overline{ep}^i(\alpha))$ is defined.

Theorem 3.1 If $\dim(K) < 3n-3$, as elements of $[ep^{i-1}S^{2n}K, S^{4n}]$ we have

$$EH(\overline{ep}^i(\alpha)) = \overline{ep}^{i-1}(EH(\alpha)).$$

Remark: In view of 3.0, this theorem is to be interpreted as follows: Given an element $\overline{ep}^i(\alpha)$, there is an element $\overline{ep}^{i-1}(EH\alpha)$ which makes the equality hold. (That is, there is an appropriate retraction $ep^{i-1}S^{2n} \rightarrow S^{4n}$.)

The goal of the first part of this section is to prove 3.1. Before proceeding with the proof, recall the reduced product construction of James. If X is a finite C.W. complex with basepoint, let X_∞ denote the free monoid on the points of X . That is, X_∞ may be considered as the set of monomials $x_1x_2\dots x_n$ where $x_i \in X$ and subject to the relations $xe = ex = x$ for all $x \in X$ where e denotes the basepoint. X_∞ is given a topology by regarding it as the quotient of the weak infinite product of X , where weak means that only

finitely many coordinates are not at the basepoint and also that the topology is the weakest topology which makes the inclusions of the finite products continuous. The main result on X_∞ is the following.

Theorem 3.2 (James) The space X_∞ is homotopy equivalent to the space ΩSX .

In what follows, we need a formula for this equivalence. To do this conveniently we use Moore loops. A Moore loop on X with basepoint e is a map $f: \mathbb{R} \rightarrow X$ so that $f(t) = e$ if $t \leq 0$ or if $t \geq t_f$ where $t_f \in \mathbb{R}$ depends on f . The space of Moore loops will be denoted ΩX .

It will be convenient to represent elements of the reduced product by elements of the ordinary products. Let $X_{(k)}$ stand for the k fold product of X . We use the notation $\bar{x} = (x_1, x_2, \dots, x_k)$ for an element of $X_{(k)}$.

Let ρ be a bounded real valued function defined on X which is zero on the basepoint and positive elsewhere. Let the starting point of x_a be denoted by

$$\rho_a(\bar{x}) = \sum_{j < a} \rho(x_j) \text{ and set } \rho_1(\bar{x}) = 0. \text{ The map } X_\infty \rightarrow \Omega SX$$

of Theorem 3.2 is given by

$$\text{where } \begin{array}{l} \bar{x} \rightarrow f_{\bar{x}} \\ f_{\bar{x}}(t) = \begin{cases} \frac{(t - \rho_a(\bar{x}), x_a)}{(x_a)} & \text{if } \rho_a(\bar{x}) < t < \rho_{a+1}(\bar{x}) \\ \text{basepoint} & \text{otherwise} \end{cases} \end{array}$$

Here as usual, SX is the reduced suspension represented

by $[0,1] \times X$ where the following set is identified to the basepoint $(0 \times X) \cup (1 \times X) \cup (I \times e)$.

A formula for the map $h: \Omega S^{n+1} \rightarrow \Omega S^{2n+1}$ can be given in terms of the James construction as follows. Regarding h as a map $S_\infty^n \rightarrow (S^n \wedge S^n)_\infty$ we may describe h in general as a map $X_\infty \rightarrow (X \wedge X)_\infty$ by

$$h(x_1, x_2, \dots, x_n) = \prod_{(a,b)} x_a \wedge x_b$$

where the order of the product over all pairs (a,b) on the right is lexicographical.

The main idea for the proof of Theorem 3.1 is to construct a homotopy commutative diagram.

Diagram 3.3

$$\begin{array}{ccc} (ep_1^{i+1}(S^{n-1}))_\infty & \xrightarrow{h} & (ep_1^{i+1}S^{n-1} \wedge ep_1^{i+1}S^{n-1}) \\ \bar{\Delta} \uparrow & & \Delta \uparrow \\ ep_1^i(S_2^{n-1}) & \xrightarrow{ep(h)} & ep_1^i(S^{n-1} \wedge S^{n-1}) \end{array}$$

The maps $\bar{\Delta}, \Delta$

To simplify the formulas, let $\tilde{I}$ and $\tilde{R}$ denote two spaces homeomorphic to S^1 . $\tilde{I}$ is a quotient of $[-1,1]$ so that both 1 and -1 denote the basepoint, and $\tilde{R}$ is the one point compactification of the real numbers. The group Z_2 acts on both $\tilde{I}$ and $\tilde{R}$ by multiplication by -1 . It is easy to see (writing $W^{i+1} = S^i \times R/\sim$ as in Theorem 2.7) that

$$ep_1^{i+1}(X) = S^i \rtimes_{Z_2} \tilde{R} \wedge (X \wedge X).$$

This obviously implies that there is a homeomorphism

$$ep_1^{i+1}(X) \approx S^i \rtimes_{Z_2} \tilde{I} \wedge X \wedge X.$$

Now define Δ_0 as a map

$$W \rtimes \tilde{R} \wedge (S_{(2)}^{n-1} \wedge S_{(2)}^{n-1}) \rightarrow (W \rtimes \tilde{I} \wedge (S^{n-1} \wedge S^{n-1}))_{(\infty)}$$

by the formula

$$w \rtimes \theta \wedge \bar{x} \wedge \bar{y} \rightarrow \prod_{a,b} w \rtimes \theta_{a,b} \wedge x_a \wedge y_b$$

where the product on the right is taken in lexicographic order and $\theta_{a,b}$ depends on $\theta, \bar{x}, \bar{y}$ and is given by

$$\theta_{a,b} = \frac{\rho_b(\bar{y}) - \rho_a(\bar{x}) + 2\theta}{\rho(z)}$$

where $z = x_a$ if the numerator is ≥ 0 and
 $z = y_b$ if the numerator is ≤ 0 .

Interpret $\theta_{a,b}$ as in $\tilde{I}$ by setting $\theta_{a,b}$ to the basepoint in $\tilde{I}$ whenever $\theta_{a,b}$ is not in the interval $(-1, 1)$.

Lemma 3.4 Δ_0 is continuous.

Proof: We need to show only that each coordinate function $w \rtimes \theta_{ab} \wedge x_a \wedge y_b$ is continuous, for then Δ_0 is the product of continuous maps. Now θ_{ab} is continuous except when either x_a or y_b is the basepoint and the

$(a,b)^{\text{th}}$ image coordinate is then at the basepoint. Because $W \rtimes \tilde{I}$ is compact, any neighborhood of $W \rtimes \tilde{I} \wedge e$ must contain a set of the form $W \rtimes \tilde{I} \wedge U$ where U is a neighborhood of e in $S^{n-1} \wedge S^{n-1}$. So any neighborhood of the basepoint must contain a set of the form $W \rtimes \tilde{I} \wedge U$. Then for any $\bar{x}$ and $\bar{y}$ so that $x_a \wedge y_b \in U$, the $(a,b)^{\text{th}}$ image coordinate of $w \quad \theta \wedge \bar{x} \wedge \bar{y}$ is in $W \rtimes \tilde{I} \wedge U$ and so Δ_0 is continuous.

It is easy to check that Δ_0 respects the identifications in the reduced product and so defines a continuous map

$$\Delta_1: W \rtimes \tilde{R} \wedge S_2^{n-1} \wedge S_2^{n-1} \rightarrow (W \rtimes \tilde{I} \wedge S^{n-1} \wedge S^{n-1})_\infty$$

Lemma 3.5 Δ_1 is equivariant.

It is easy to see that $\theta_{ab}(\theta, \bar{x}, \bar{y}) = -\theta_{ba}(-\theta, \bar{y}, \bar{x})$. But lexicographic order in (a,b) is not the same as lexicographic order in (b,a) . However, not all the terms contribute to the product.

Lemma 3.6 Either $\theta_{1,2}(\theta, \bar{x}, \bar{y})$ or $\theta_{2,1}(\theta, \bar{x}, \bar{y})$ is at the basepoint.

Proof: Case 1. $\theta_{1,2} \geq 0$.

Then $\theta_{1,2} < 1$ implies $\rho_2(\bar{y}) - \rho_1(\bar{x}) + 2\theta = \rho(y_1) + 2\theta < \rho(x_1)$, and so $\theta_{2,1} < -1$ as desired.

Case 2. $\theta_{1,2} \leq 0$.

Then $\theta_{1,2} > -1$ implies $0 > \rho(y_1) + 2\theta > -\rho(y_1)$ so $-\rho(x_1) > -\rho(x_1) + 2\theta$ and $\theta_{2,1} < -1$ as desired.

Since $\bar{\Delta}_1$ is equivariant, $\bar{\Delta}$ can be defined as desired

$$\bar{\Delta}: W \rtimes_{\mathbb{Z}_2} \tilde{R} \wedge S_2^{n-1} \wedge S_2^{n-1} \rightarrow (W \rtimes_{\mathbb{Z}_2} \tilde{I} \wedge S^{n-1} \wedge S^{n-1})_{\infty}.$$

Remark $\bar{\Delta}$ can actually be extended to $ep_1^{i+1}(S_{\infty}^{n-1})$ using the same formula. The proof is not much harder.

Similarly, define $\tilde{\Delta}$ as a map

$$\begin{aligned} W \rtimes \tilde{R} \wedge (S^{n-1} \wedge S^{n-1}) \wedge (S^{n-1} \wedge S^{n-1}) \\ \rightarrow (W \rtimes \tilde{I} S^{n-1} \wedge S^{n-1}) \wedge (W \rtimes \tilde{I} S^{n-1} \wedge S^{n-1}) \end{aligned}$$

by $w \rtimes \theta \wedge x_1 \wedge x_2 \wedge y_1 \wedge y_2$

$$\rightarrow (w \rtimes \theta_{11} x_1 \wedge y_1) \wedge (w \rtimes \theta_{22} x_2 \wedge y_2)$$

where $\theta_{1,1}$ and $\theta_{2,2}$ are defined letting $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$. This map $\tilde{\Delta}$ is continuous by the same argument which shows $\bar{\Delta}$ is continuous.

Lemma 3.7 The diagram 3.3 is homotopy commutative.

Proof: Plainly $\tilde{\Delta} \circ h$ is given by

$$w \rtimes \theta \wedge \bar{x} \wedge \bar{y} \rightarrow (w \rtimes \theta_{11} x_1 \wedge y_1) \wedge (w \rtimes \theta_{12} x_2 \wedge y_2).$$

On the other hand, $h \circ \bar{\Delta}$ sends $w \rtimes \theta \wedge \bar{x} \wedge \bar{y}$ in general to a nontrivial product. One of the extra factors is

$$(w \rtimes \theta_{12} x_1 \wedge y_2) \wedge (w \rtimes \theta_{21} x_2 \wedge y_1)$$

which is the basepoint by Lemma 3.6. All the other extra factors have the property that x_a or y_b occurs twice for some a or b . For instance x_1 occurs twice in the

$(1,1,1,2)^{\text{th}}$ factor

$$(w \rtimes \theta_{11} \wedge x_1 \wedge y_1) \wedge (w \rtimes \theta_{12} \wedge x_1 \wedge y_2).$$

There are four of these factors: (1112) , (1121) , (1222) , (2122) . The idea is to use the fact that $S^{n-1} \rightarrow S^{n-1} \wedge S^{n-1}$ given by $x \rightarrow x \wedge x$ is homotopic to the trivial map. This would complete the proof except that each factor is not equivariant and so does not define a function on $\text{ep}_1^{i+1}(S_2^{n-1})$. However, each pair $(1112, 1121)$ and $(1222, 2122)$ does define a map

$$\text{ep}_1^{i+1}(S_2^{n-1}) \rightarrow (\text{ep}_1^{i+1}S^{n-1} \wedge \text{ep}_1^{i+1}S^{n-1})_2.$$

Letting (1122) denote the map given by the 1122 factor (which is equivariant), we can view $h \circ \bar{\Delta}$ as the reduced product of the three maps $(1112, 1121)$, (1122) , and $(1222, 2122)$.

We show the map $(1112, 1121)$ is homotopy trivial. Then $(1222, 2122)$ is trivial by a similar argument and the proof will then be complete.

Let A denote the closure of the set of points

$$w \rtimes \theta \wedge \bar{x} \wedge \bar{y} \in W \rtimes \tilde{R} \wedge S_2^{n-1} \wedge S_2^{n-1}$$

such that $\theta_{12}(\theta, \bar{x}, \bar{y}) \neq * \in \tilde{I}$. Define a homotopy on A of (1112) to the trivial map using the fact that (1112) has the diagonal map as a factor (after an obvious change of coordinates). It is easy to see that this homotopy can be chosen so that at each time the points where $\theta_{12} = *$ are

carried to the basepoint.

Now let A' denote the image of the points of A under the Z_2 action. If $\theta_{21}(\theta, \bar{x}, \bar{y})$ is not at the basepoint, then $w \rtimes \theta \wedge \bar{x} \wedge \bar{y} \in A'$ because $\theta_{21}(\theta, \bar{x}, \bar{y}) = -\theta_{12}(-\theta, \bar{y}, \bar{x})$. The homotopy can be extended using equivariance to A' because $A \cap A'$ contains only points where both θ_{21} and θ_{12} are at the basepoint. On the closure of the complement of $(A \cup A')$, let the homotopy be the trivial map at the basepoint for all times. These definitions on the three closed sets A , A' and the closure of the complement of $(A \cup A')$ agree on the various overlaps (all give the basepoint) and so a homotopy to the trivial map of $(1112, 1121)$ is defined. This completes the proof of the Lemma.

Lemma 3.8 On the subspace $ep_1^{i+1}(S^{n-1}) \subset ep_1^{i+1}(S_2^{n-1})$ the map $\bar{\Delta}$ is homotopic to the inclusion

$$ep_1^{i+1}S^{n-1} \subset (ep_1^{i+1}S^{n-1})_\infty.$$

Proof: As a map $W \rtimes \tilde{R} \wedge S^{n-1} \wedge S^{n-1} \rightarrow W \rtimes \tilde{I} \wedge S^{n-1} \wedge S^{n-1}$

$\bar{\Delta}$ is the product of the identity map with the map $\tilde{R} \rightarrow \tilde{I}$ given by

$$\begin{aligned} t &\longmapsto t & \text{if } t \in (-1, 1) \\ t &\longmapsto * & \text{otherwise} \end{aligned}$$

Since this map $S^1 = \tilde{R} \rightarrow \tilde{I} = S^1$ is homotopic to the identity, we are done.

Proof of Theorem 3.1

Recall the assumption there is a retraction

$$r_1: \text{ep}_1^{i+1}(S^{n-1}) \rightarrow S^{2n-1}.$$

Using Diagram 3.3 we get the following commutative diagram.

Diagram 3.9

$$\begin{array}{ccccc}
 & (S^{2n-1})_\infty & \xrightarrow{h} & (S^{2n-1} S^{2n-1})_\infty & \\
 & \uparrow (r_1)_\infty & & \uparrow (r_1 \wedge r_1)_\infty & \\
 & (\text{ep}_1^{i+1} S^{n-1}) & \xrightarrow{h} & (\text{ep}_1^{i+1} S^{n-1} \wedge \text{ep}_1^{i+1} S^{n-1})_\infty & \\
 & \uparrow \bar{\Delta} & & \uparrow \tilde{\Delta} & \\
 \text{ep}_1^{i+1}(K) & \xrightarrow{\text{ep}(\alpha_0)} & \text{ep}_1^{i+1}(S_2^{n-1}) & \xrightarrow{\text{ep}(h)} & \text{ep}_1^{i+1}(S^{n-1} \wedge S^{n-1})
 \end{array}$$

g (diagonal arrow from $\text{ep}_1^{i+1}(K)$ to $(S^{2n-1})_\infty$)

Here $\alpha_0: K \rightarrow S_2^{n-1}$ is the adjoint of $\alpha: SK \rightarrow S^n$. The adjoint lies in S_2^{n-1} instead of S^{n-1} by the hypothesis on the dimension of K and because $S_2^{n-1} \rightarrow S^{n-1}$ is an equivalence through dimension $3n-4$.

Theorem 3.1 then follows from the following two Propositions (whose proofs are postponed).

Proposition 3.10 The map g in Diagram 3.9 is the adjoint of $\overline{\text{ep}}^i(\alpha)$.

Proposition 3.11 The map $\tilde{\Delta}$ is homotopic to a map $\tilde{\Delta}'$ which can be regarded as a map on $\text{ep}_2^{i+1}(S^{n-1} \wedge S^{n-1})$ and so that $\tilde{\Delta}'$ is an isomorphism in integral homology in dimension $4n - 2$.

Proposition 3.10 shows that the composition along the

top of Diagram 3.9 is the adjoint of $H(\overline{ep}^1(\alpha))$. Taking adjoint and suspending we get

$$EH\overline{ep}^1(\alpha): Sep^i SK \rightarrow S^{4n}$$

using the homeomorphism α of Theorem 2.7.

The map along the bottom of diagram 3.9 when regarded as a map

$$ep_2^{i+1}(K) \rightarrow ep_2^{i+1}(S^{n-1} \wedge S^{n-1})$$

gives by suspending twice and using S twice a map

$$ep^{i-1}(S^2 K) \rightarrow ep^{i-1}(S^{2n})$$

which is plainly $ep^{i-1}(EH\alpha)$. Now Proposition 3.11 shows the adjoint and suspension of $(r_1 \wedge r_1)_\infty \circ \tilde{\Delta}$ is a retraction $ep^{i-1}(S^{2n}) \rightarrow S^{4n}$ and so the composition is $\overline{ep}^{i-1}(EH\alpha)$. This completes the proof of Theorem 3.1.

Proof of Proposition 3.11

We define a homotopy $\tilde{\Delta}_t$ of maps

$$\begin{aligned} W^i \rtimes \tilde{R} \wedge (S^{n-1} \wedge S^{n-1}) \wedge (S^{n-1} \wedge S^{n-1}) \\ \rightarrow (W^i \rtimes \tilde{R} \wedge S^{n-1} \wedge S^{n-1}) \wedge (W^i \rtimes \tilde{R} \wedge S^{n-1} \wedge S^{n-1}) \end{aligned}$$

First write W^i as $W^{i-1} \times R^1 / \sim$ (as in Theorem 2.7). Then let Δ_t be given by

$$\begin{aligned} w \times \theta_1 \rtimes \theta \wedge (x_1 \wedge x_2) \wedge (y_1 \wedge y_2) \rightarrow \\ (w \times \theta_1 \rtimes (2 + 2t\theta_1) \wedge x_1 \wedge y_1) \wedge (w \times \theta_1 \rtimes \\ ((1-t)(\rho y_1 - \rho x_1) + 2\theta - 2t\theta_1) \wedge x_2 \wedge y_2). \end{aligned}$$

When $t = 1$, $\tilde{\Delta}_1$ is a homeomorphism of the bottom cell. Precisely, let w_0 be a point in W^{i-1} . Then points with w coordinate equal to w_0 make up the bottom cell of $ep_2^{i+1}(S^{n-1} \wedge S^{n-1})$ in dimension $4n-2$. This cell is carried homeomorphically onto the bottom cell of $ep_1^{i+1}S^{n-1} \wedge ep_1^{i+1}S^{n-1}$.

Now there is a map f from $ep_1^{i+1}S^{n-1}$ to itself so that $(f \wedge f)\tilde{\Delta}_0 = \tilde{\Delta}$ and f is a homotopy equivalence. The map f is given by

$$w \rtimes \theta \wedge x \wedge y \rightarrow w \rtimes \frac{\theta}{\rho z} \wedge x \wedge y$$

Here $\theta \in \tilde{R}$ and $\theta/\rho(z) \in \tilde{I}$ by letting $\theta/\rho(z)$ be the basepoint when $\theta/\rho(z)$ is not in the interval $(-1,1)$.

Also

$$z = \begin{cases} x & \text{if } \theta \geq 0 \\ y & \text{if } \theta \leq 0. \end{cases}$$

To see that f is a homotopy equivalence, define a homotopy f_t by

$$w \rtimes \theta \wedge x \wedge y \rightarrow w \rtimes \frac{\theta}{t+(1-t)\rho z} \wedge x \wedge y.$$

When $t = 1$, f_1 is a homotopy equivalence as in the proof of Lemma 3.8.

This completes the proof of Proposition 3.11.

Proof of Proposition 3.10

Lemma 3.12 The following diagram is homotopy commutative.

$$\begin{array}{ccc}
\text{Sep}^{i+1}(S_2^{n-1}) & \xrightarrow{S\bar{\Delta}} & S(\text{ep}_1^{i+1}S^{n-1})_\infty \\
\downarrow \text{ep}^i(\tau) \circ \sigma & & \downarrow \sigma \circ \tau \\
\text{ep}^i(S^n) & = & \text{ep}^i(S^n)
\end{array}$$

Here σ is the map of Theorem 2.7 and τ is the map of section 2

$$\tau : S\Omega X \rightarrow X.$$

Proof postponed.

To see how this implies Proposition 3.10 use Proposition 2.11 to see that

$$\text{ep}^i(\tau) \circ \sigma \circ \text{Sep}(\alpha_0) = \text{ep}^i(\alpha).$$

Then we also have by 3.11

$$\tau \circ \sigma \circ S\tilde{\Delta} \circ \text{Sep}(\alpha_0) = \text{ep}^i(\alpha).$$

Proposition 3.10 then follows because τ is natural and $(r_1)_\infty$ is a loop map.

Proof of Lemma 3.12

By Lemma 2.13 and Lemma 3.8, it suffices to give any map $\text{ep}^i S^n \rightarrow \text{ep}^i S^n$ which makes the diagram commute (in place of $=$), for any such map will be homotopic to the identity.

Recall that ρ is a non-negative real valued function on S^{n-1} which is 0 only at the basepoint. To discuss τ on $S(\text{ep}_1^{i+1} S^{n-1})_\infty$, we define a similar function ρ^* on $\text{ep}_1^{i+1}(S^{n-1})$.

Represent $ep_1^{i+1}S^{n-1}$ as $W \rtimes \tilde{I} \wedge S^{n-1} \wedge S^{n-1}$. Then define $\rho^*(\theta, x, y)$ by

$$\rho^* = \begin{cases} \min((1-\theta)\rho(x), \rho(y), 0) & \text{when } \theta \geq 0 \\ \min(\rho(x), (1+\theta)\rho(y), 0) & \text{when } \theta \leq 0 \end{cases}$$

Then ρ^* is equivariant and so defines a function of $ep_1^{i+1}(S^{n-1})$ which is 0 only at the basepoint. Using ρ^* , one can define a map (as in 3.2)

$$(ep_1^{i+1}S^{n-1})_\infty \rightarrow \Omega \text{Sep}_2^{i+1}S^{n-1} \quad \text{and so } \tau \text{ is defined on } S(ep_1^{i+1}S^{n-1}).$$

We need to give a homotopy of the map $S\bar{\Delta}$. Let $(S\Delta)_t$ as a map

$$\tilde{R} \wedge (W \rtimes \tilde{R} \wedge S_2^{n-1} \wedge S_2^{n-1}) \rightarrow \tilde{R} \wedge (W \rtimes \tilde{I} \wedge S^{n-1} \wedge S^{n-1})$$

be given by

$$\theta_1 \wedge (w \rtimes \theta_2 \wedge \bar{x} \wedge \bar{y}) \rightarrow \prod_{a,b} (\theta_1 - t|\theta_2|) \wedge w \rtimes \theta_{ab} \wedge x_a \wedge y_b.$$

Then Lemma 3.12 follows easily from the following.

Lemma 3.13 The map $ep_1^i(\tau) \circ \sigma$ in Lemma 3.12 is onto.

Furthermore if $z \in \text{Sep}_1^{i+1}(S_2^{n-1})$, the image of z under the map $\sigma \circ \tau \circ (S\Delta)_1$ is continuously determined by its image under $ep_1^i(\tau) \circ \sigma$.

Proof: Let $z = \theta_1 \wedge (w \rtimes \theta_2 \wedge \bar{x} \wedge \bar{y})$. Then using

Theorem 2.7 and the discussion following Theorem 3.2, $ep_1^i(\tau) \circ \sigma$ takes z to the point

$$w \rtimes f_{\bar{x}}(\theta_1 + \theta_2) \wedge f_{\bar{y}}(\theta_1 - \theta_2).$$

So suppose we know that some point $z \in \text{Sep}_1^{i+1}(S_2^{n-1})$ has image $w \bowtie t_1 \wedge x \wedge t_2 \wedge y$ under $\text{ep}_1^i(\tau) \circ \sigma$. We would like to determine enough about the image of z under $(SA)_1$ to determine the image of z under $\sigma \circ \tau SA_1$. Assume $x = x_a$ in $\bar{x}$ and $y = y_b$ in $\bar{y}$. Then using the formula for $f_{\bar{z}}$ we get

$$\theta_1 + \theta_2 = t_1 \rho(x_a) + \rho_a(\bar{x})$$

$$\theta_1 - \theta_2 = t_2 \rho(y_b) + \rho_b(\bar{y}).$$

Solving we easily get that $\rho_b(\bar{y}) - \rho_a(\bar{x}) + 2\theta_2$ and hence $\theta_{a,b}$ depend only on t_1, t_2, x and y .

Also note that $2\theta_1 - \rho_a(\bar{x}) - \rho_b(\bar{y})$ also depends only on t_1, t_2, x and y . The proof then follows by the following.

Lemma 3.14 Let $t = \theta_1 - |\theta_2|$, then

$$\rho_{ab}^* \leq t \leq \rho_{ab}^* + \rho^*(w \bowtie \theta_{ab} \wedge x_a \wedge y_b)$$

and

$$t - \rho_{ab}^* = \theta_1 - \frac{\rho_a(\bar{x}) + \rho_b(\bar{y}) + |\rho_b(\bar{y}) - \rho_a(\bar{x}) + 2\theta_2|}{2}$$

The inequalities show (see discussion following 3.2) the map τ at time t is 'using' the $(a,b)^{\text{th}}$ factor of $(SA)_1$ applied to z . The equality shows $t - \rho_{a,b}^*$ depends only on t_1, t_2, x and y . This is enough information to compute the image of z under $\tau(SA)_1$ and so completes the proof of 3.13.

Proof of Lemma 3.14

Fix $z = \theta_1 \wedge w \times \theta_2 \wedge \bar{x} \wedge \bar{y}$. Then $ep^1(\tau) \circ \sigma$ applied to z is

$$w \times f_{\bar{x}}(\theta_1 + \theta_2) \wedge f_{\bar{y}}(\theta_1 - \theta_2).$$

For fixed θ_2 , this defines a loop and we can speak of the length of time (in θ_1) during which $f_{\bar{x}}(\theta_1 + \theta_2)$ is 'using' x_a and $f_{\bar{y}}(\theta_1 - \theta_2)$ is using y_b simultaneously. An easy calculation shows this length of time is $\rho^*(\theta_{ab} \wedge x_a \wedge y_b)$.

Now recall $\rho^*_{a,b} = \sum_{c,d} \rho^*(\theta_{c,d} \wedge x_c \wedge y_d)$ where

$(c,d) < (a,b)$ in the lexicographic order. Since $f_{\bar{x}}(\theta_1 + \theta_2) \wedge f_{\bar{y}}(\theta_1 - \theta_2)$ begins 'using' $x_1 \wedge y_1$ at $\theta_1 = |\theta_2|$, it is easy to see that

$$\rho^*_{a,b} = \max(\rho_a(x), \rho_b(y)) - |\theta_2|$$

or

$$\rho^*_{a,b} = \frac{\rho_a(\bar{x}) + \rho_b(\bar{y}) + |\rho_b(\bar{y}) - \rho_a(\bar{x}) + 2\theta|}{2} - |\theta_2|.$$

This completes the proof of Lemma 3.14 and ends the sequence of Lemmas involved in the proof of Theorem 3.1.

The $\overline{ep}^1$ construction and Toda's Star construction

In the course of Toda's calculation of the homotopy groups of spheres the star construction is defined [25; Lemma 3.5]. This construction is used to settle various group extensions. It is known for instance that $\eta^* = \nu$ and $\nu^* = \sigma$ where η, ν , and σ are the elements of Hopf

invariant one in the 1, 3 and 7 stems respectively. Since $\overline{ep}^1(\eta)$, $\overline{ep}^1(v)$ in the 3 and 7 stems have Hopf invariant one by Theorem 3.1, it is reasonable to conjecture that $\overline{ep}^1$ and Toda's star construction are related. The purpose of this part is to show that these constructions are virtually the same.

Suppose α , β and γ are elements of the homotopy groups of spheres. If $\alpha \circ \beta = 0$ and $\beta \circ \gamma = 0$, the secondary construction $\{\alpha, \beta, \gamma\}$ or Toda bracket is defined. Choose representatives a , b and c of α , β and γ . Suppose γ is defined on S^n and consider S^n included in S^{n+1} as the equator.

Definition 3.15 The Toda bracket $\{\alpha, \beta, \gamma\}$ is a map defined on S^{n+1} with the following properties [25; 1.14].

- a) On the equator S^n , the map is the composition $a \circ b \circ c$.
- b) On the top hemisphere, the map is a composed with a homotopy of $b \circ c$ to zero.
- c) On the lower hemisphere, the map is a homotopy of $a \circ b$ to zero composed with c .

Remark 3.16 The Toda bracket is defined modulo some indeterminacy. When α , β and γ are stable elements, the indeterminacy can be decreased (sometimes) by desuspending the elements to the lowest spheres where the hypotheses $\alpha \circ \beta = 0$ and $\beta \circ \gamma = 0$ hold and then forming the Toda bracket "early". We denote this Toda bracket by $\{\alpha, \beta, \gamma\}_e$.

Toda uses a more exact but messier notation [25; 1.14].

Toda's star construction and $\overline{ep}^1$ are similar in the sense they satisfy similar theorems. The following is a simplified version of [25; Theorem 3.6].

Theorem 3.17 Let $\alpha \in \pi_{n+k} S^n$ be such that $2\alpha = 0$ or k is even. Then there exists an element $\alpha^* \in \pi_{2n+2k+1} S^{2n}$ such that stably $\beta \wedge \alpha^* \in \{\alpha, \beta, \alpha\}$ whenever $\alpha \circ \beta = 0$.

Now let $p: ep^1 S^{n+k} \rightarrow S^{2n+2k+1}$ be the map which pinches the lower cell and let $q: S^{2n} \rightarrow ep^1 S^n$ be the inclusion of the lowest cell.

Theorem 3.18 Let $\alpha \in \pi_{n+k} S^n$, $\beta \in \pi_{m+1} S^m$, and $\alpha \circ \beta = 0$. Then

$$ep^1(\alpha) \wedge \beta \in q \circ \{\alpha, \beta, \alpha\} \circ p.$$

Remark Recall that $\overline{ep}^1(\alpha)$ is $ep^1(\alpha)$ composed with a retraction $r: ep^1 S^n \rightarrow S^{2n}$. An easy corollary is that the above theorem holds when $ep^1(\alpha)$ is replaced by $\overline{ep}^1(\alpha)$ whatever the choice of retraction r . The above theorem holds even when there is no retraction.

Proof of Theorem 3.18

The proof is to give a homotopy of

$$ep^1(\alpha) \wedge \beta: ep^1(S^{n+k}) \wedge S^{m+1} \rightarrow ep^1(S^n) \wedge S^m$$

to a map

$$S^{2n+2k+m+1+1} = ep_1^1(S^{n+k}) \wedge S^{m+1} \rightarrow S^{2n+m}$$

which will be $\{\alpha, \beta, \alpha\}$ by definition 3.15.

Write $ep^1(S^n)$ as $S^1 \rtimes_{\mathbb{Z}_2} S^n \wedge S^n$ and parameterize

S^1 by $\theta \in [0, 2\pi]$ so that the antipodal action is given by $\theta \rightarrow \theta + \pi \pmod{2\pi}$. Let $H[t]: S^{n+k} \wedge S^{m+1} \rightarrow S^n \wedge S^m$ be a homotopy so that $H[0] = *$ and $H[1] = a \wedge b$ where a and b are representatives of α and β . Define a homotopy

$$h_s: (S^1 \rtimes S^{n+k} \wedge S^{n+k}) \wedge S^{m+1} \rightarrow S^1 \rtimes S^n \wedge S^n \wedge S^m$$

where $0 \leq s \leq 1$ by the formula:

$$h_s(\theta \rtimes x \wedge y \wedge z) = \theta \rtimes a(x) \wedge H[(1-s) + \lfloor \frac{2}{\pi} s \theta \rfloor](y \wedge z)$$

$$\text{when } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

This defines h_s where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. To extend to all θ use equivariance. That is

$$h_s(g \cdot (\theta \rtimes x \wedge y \wedge z)) = g \cdot h_s(\theta \rtimes x \wedge y \wedge z)$$

where g generates the Z_2 action.

Now $h_0 = \text{ep}^1(\alpha) \wedge \beta$ and h_1 defines a map on $\text{ep}^1(S^{n+k}) \wedge S^{m+1}$ which is trivial on the bottom cell (i.e. on $(S^0 \rtimes S^{n+k} \wedge S^{n+k}) \wedge S^{m+1}$).

To give a further homotopy so that the image of the map is contained in the bottom cell of $\text{ep}^1(S^n \wedge S^m)$ let for $0 \leq \theta \leq \pi$

$$h_1[r](\theta \rtimes x \wedge y \wedge z) = \theta - r\theta \rtimes \pi_2 h_1(\theta \rtimes x \wedge y \wedge z)$$

where π_2 is the projection on the second factor of the half smash $\rtimes$. Use equivariance to extend $h_1[r]$ to all θ . Notice the formula above is actually equivariant,

because when $\theta = 0$ or $\theta = \pi$, $h_1(\theta \times x \wedge y \wedge z) = *$. Now $h_1[0]$ is plainly equal to h_1 above and $h_1[1]$ has its image in the bottom cell of $ep^1(S^n) \wedge S^m$. Putting h_s and $h_1[r]$ together we get a homotopy of $ep^1(\alpha) \wedge \beta$ to a map from $S^{2n+2k+m+1+1}$ to S^{2n+m} as desired.

It remains to show that $h_1[1]$ is a representative of $\{\alpha, \beta, \alpha\}$. To do this we give a parameterization of the top cell of $ep^1(S^{n+k}) \wedge S^{m+1}$ as follows. Write $S^{2n+2k+m+1+1}$ as $[0, \pi] \times S^{n+k} \wedge S^{n+k} \wedge S^{m+1}$ so that 0 and π are the two poles. Then we can give a map

$$[0, \pi] \times S^{2n+2k+m+1}/\sim \rightarrow ep_1^1(S^{n+k}) \wedge S^{m+1}$$

as follows

$$(\theta, x_1 \wedge x_2 \wedge x_3) \rightarrow \theta \times (x_1 \wedge x_2) \wedge x_3.$$

Plainly, this gives a homotopy equivalence; the map simply collapses a longitude to the basepoint. With this parameterization, the map $h_1[1]$ on the top hemisphere is $a \wedge H$ and on the bottom hemisphere is $H' \circ (1 \wedge a \wedge 1)$ where H is the homotopy of $a \wedge b$ to zero above and H' is a homotopy of $a \wedge 1 \wedge b$ to zero related to H by equivariance. This shows $h_1[1]$ is a representative of $\{\alpha, \beta, \alpha\}$ by definition 3.15 and so completes the proof of Theorem 3.17.

Consider the map $\overline{ep}^1(\alpha)$. This is a map

$$ep^1(S^{n+k}) \xrightarrow{ep'(\alpha)} ep^1 S^n \xrightarrow{r} S^{2n}$$

where r is a retraction. Now by section 2, such a

retraction occurs when n is even. If k is also even, then $ep^1(S^{n+k})$ is the wedge $S^{2(n+k)} \vee S^{2(n+k)+1}$ and $ep^1(\alpha)$ can be regarded as a map of the $2(n+k)+1$ sphere to the $2n$ sphere. In case $n+k$ is odd

$$ep^1(S^{n+k}) = S^{2(n+k)} \cup_2 e^{2(n+k)+1}$$

and the map $\overline{ep}^1(\alpha)$ is an extension over $\mathbb{Z}_2$ of the map α^2 . This extension may have a nontrivial Hopf invariant (Theorem 3.1). In this case, if there is an earlier extension γ of α^2 over $\mathbb{Z}_2$ (i.e. on an earlier sphere than S^{2n}), then the difference of $\overline{ep}^1(\alpha)$ and γ is a map of spheres $S^{2n+2k+1} \rightarrow S^{2n}$ which has the same Hopf invariant as $\overline{ep}^1(\alpha)$ because γ is a suspension.

Proposition 3.19

$$\beta \wedge (ep^1(\alpha) - \gamma) \in \{\alpha, \beta, \alpha\} + \{\beta, \alpha^2, 2\}_e$$

where γ is an early extension of α^2 over $\mathbb{Z}_2$.

Remark Notice that if there were no restriction on the bracket $\{\beta, \alpha^2, 2\}$, the proposition would have no content, because the indeterminacy of the unrestricted bracket includes all compositions with β .

Proof of Proposition 3.19 The proposition follows from Theorem 3.18 and the observation that $\gamma \wedge \beta \in \{\beta, \alpha^2, 2\}_e$ by [25; Prop. 1.9].

SECTION 4. SQUARING OPERATIONS IN EXT AND DIFFERENTIAL FORMULAS

All the results of this section are stable results. Throughout, N will stand for some integer which is 'large enough'; we will neglect to indicate how large N must be for any particular use. Let G denote the group Z_2 and let H_* and H^* denote reduced homology and cohomology with Z_2 coefficients.

There are squaring operations [13], [22], [19]

$$Sq^i: \text{Ext}_{\underline{MA}}^{s,t}(H_*S^k, H_*S^n) \rightarrow \text{Ext}_{\underline{MA}}^{s+i, 2t}(H_*S^{2k}, H_*S^{2n}).$$

The purpose of this section is to extend these operations to operations

$$Sq^i: \text{Ext}^{s,t}(H_*K, H_*S^n) \rightarrow \text{Ext}^{s+i, t+i}(H_*ep_{t-s}^{t-i}K, H_*S^{2n})$$

which can be used to detect the extended powers. Then using the results of section 1, a differential formula (Theorem 4.1) is found in the spectral sequence for the cohomotopy of RP_{k+i+1}^{k+r+i} . This formula is the basis for the calculation of differential formulas in the spectral sequence for the homotopy of spheres (Theorems 4.7, 4.8). At the end of this section some other applications of the extended powers are collected.

If $a \in \text{Ext}^{s,t}$, let $Sq_1(a) = Sq^{s-1}(x)$. Let x_{k+i+j} for $0 \leq j \leq r$ denote the cohomology classes of RP_{k+i}^{k+i+r} . Suppose $a \in E_2^{t-s}(S^N)$ is a permanent cycle and $t-s = k$. Then in the stable spectral sequence for the cohomotopy of RP_{k+i+1}^{k+i+r} the following formula holds using the complex

$V(H^*(RP_{k+i+1}^{k+i+r}), H_*(S^{2N}))$ of section 1 (Theorems 1.2, 1.8).

The generator of $H_{2N}(S^{2N})$ is suppressed.

Theorem 4.1

$$d_{r+1}(Sq_{i+r}(a) \otimes x_{k+i+r}) = \sum_{0 < j \leq r} (i+k, j) Sq_i(a) \lambda_{j-1} \otimes x_{k+i+j}.$$

The proof is postponed.

Suppose there is a retraction $r: ep^1(S^N) \rightarrow S^{2N}$.

Recall [1] the stable Adams spectral sequence $E_r(X)$ is the homotopy spectral sequence of a tower

$$Y_s \rightarrow Y_{s-1} \rightarrow \dots \rightarrow Y_0 = X.$$

The purpose of this first part is to describe how the retraction r behaves with respect to this tower and to describe the relationship between the extended powers ep^1 and the squaring operations in Ext. References for this part are [13], [22], [19].

Now suppose Y_* is a tower for S^N , W_* is a tower for S^{2N} , and r is any retraction $r: ep^1 S^N \rightarrow S^{2N}$.

Proposition 4.2 There is a map

$$\theta: ep^1(Y_*) \rightarrow W_*$$

such that

- (i) $\theta = r$ when restricted to $ep^1(Y_0)$
- (ii) $\theta(ep^j Y_s) \subset W_{2s-j}$
- (iii) The following diagrams commute up to homotopy

$$\begin{array}{ccc}
 \text{ep}^j Y_s & \xrightarrow{\theta} & W_{2s-j} \\
 \downarrow & & \downarrow \\
 \text{ep}^{j+k} Y_s & \xrightarrow{\theta} & W_{2s-j-k}
 \end{array}$$

$$\begin{array}{ccc}
 \text{ep}^j Y_s & \xrightarrow{\theta} & W_{2s-j} \\
 \downarrow & & \downarrow \\
 \text{ep}^j Y_{s-k} & \xrightarrow{\theta} & W_{2s-2k-j}
 \end{array}$$

Proof: To prove the proposition, we need some auxillary spaces. Let

$$Z_s = \bigcup_{i+j=s} Y_i \wedge Y_j$$

Note there is a G action on the spaces Z_s by switching factors. Hence the space

$$S^t \rtimes_G Z_s$$

is defined.

Note that $\text{ep}^1 Y_0 = S^1 \rtimes_G Z_0$. The proposition follows from the following lemma and the fact

$$\text{ep}^i Y_s \subset S^1 \rtimes_G Z_{2s}$$

Lemma 4.3 There is a map $\theta: \text{ep}^i(Z_*) \rightarrow W_*$ such that

- (i) $\theta = r$ when restricted to $S^1 \rtimes_G Z_0$
- (ii) $\theta(S^j \rtimes_G Z_s) \rightarrow W_{s-j}$

(iii) The analogous diagrams (as in 4.2) commute up to homotopy.

Proof: Recall from [1; 3.4] that

$$H_*(Y_s) \rightarrow H_*(Y_{s-1})$$

is the zero map. Also recall $W_s \rightarrow W_{s-1}$ may be regarded as a fibration induced by a map $W_{s-1} \rightarrow K$ where K is a product of Eilenberg-MacLane spaces for Z_2 . Consequently a map $X \rightarrow W_{s-1}$ lifts to a map into W_s precisely when the composition $X \rightarrow W_{s-1} \rightarrow K$ is homotopic to the trivial map. Since K is a generalized Eilenberg-MacLane space, this occurs precisely when the induced map $H_*(X) \rightarrow H_*(K)$ is zero.

For $s = 0$, define θ for all j by condition (i). Inductively, assume the lemma is true for s and all j . Because $(S^0 \rtimes_G Z_s) \rightarrow W_s$ and the map

$$S^0 \rtimes_G Z_{s+1} \longrightarrow S^0 \rtimes_G Z_s$$

is zero in homology, θ can be extended so that

$$(S^0 \rtimes_G Z_{s+1}) \rightarrow W_{s+1}.$$

Now assume θ has been defined for j and $s+1$

$$\theta(S^j \rtimes_G Z_{s+1}) \subset W_{s+1-j}.$$

We wish to show we can extend to $j+1$ and $s+1$. The composition

$$S^{j+1} \rtimes_G Z_{s+1} \rightarrow S^{j+1} \rtimes_G Z_s$$

defines

$$\bar{\theta}: S^{j+1} \rtimes_G Z_{s+1} \rightarrow W_{s-j-1}.$$

The obstruction to lifting this map into W_{s-j} is the map of pairs

$$(S^{j+1} \rtimes_G Z_{s+1}, S^j \rtimes_G Z_{s+1}) \rightarrow (W_{s-j-1}, W_{s-j}).$$

Since $W_{s-j-1} \rightarrow W_{s-j}$ is the fibration induced by a map $W_{s-j} \rightarrow K$ where K is an Eilenberg-MacLane space, this obstruction is equivalent to a map into the pair $(K, *)$.

Hence the obstruction is a collection of elements of

$$H^*(S^{j+1} \wedge Z_{s+1}). \quad (\text{Note } S^{j+1} \rtimes_G Z_{s+1} / S^j \rtimes_G Z_{s+1}$$

$= S^j \wedge Z_{s+1}$). Since $\bar{\theta}$ was defined by a composition, the obstruction factors through the zero map

$$H^*(S^{j+1} \wedge Z_s) \rightarrow H^*(S^{j+1} \wedge Z_{s+1}). \quad \text{This completes the proof.}$$

Proposition 4.4 Suppose $\alpha: K \rightarrow S^N$ has filtration s in the stable spectral sequence $E_r(K, S^N)$. Then

$$\overline{ep}^i(\alpha): ep^i K \rightarrow S^{2N} \quad \text{has filtration } 2s-i.$$

Proof: By hypothesis $\bar{\alpha}: K \rightarrow Y_s$ where the composition $K \rightarrow Y_s \rightarrow S^N$ is α . Hence $ep^i(\bar{\alpha}): ep^i K \rightarrow ep^i Y_s$.

The proposition now follows immediately from Proposition 4.2.

To define the squaring operations in Ext , let $J(S^n)$ be an injective resolution of $H_* S^n$ as in [22]. This means there is a long exact sequence

$$0 \rightarrow H_* S^n \xrightarrow{\epsilon} J^0 \xrightarrow{d} J^1 \xrightarrow{d} J^2 \rightarrow \dots$$

where J^1 is an injective in $\underline{MA}$. The map ε is called the augmentation and i is the homological degree. Each module also has an internal degree and all maps preserve internal degree. All modules in $\underline{MA}$ are connected ($= 0$ in internal degree 0). Similarly let $I(S^{2n})$ be an injective resolution of H_*S^{2n} . Then the following proposition is proved using the fact that I is an acyclic complex and J is injective.

Proposition 4.5 There is a sequence D^i of maps in $\underline{MA}$

$$D^i J \otimes J \rightarrow I$$

so that

(i) D^i has homological degree $-i$. That is,

$$D^i(J_s \otimes J_{s'}) \subset I_{s+s'-i}.$$

(ii) $dD^i + D^i d = D^{i-1} + D^{i-1}T$ where T is the map $J \otimes J \rightarrow J \otimes J$ which switches factors.

(iii) D^0 commutes with augmentations. That is, regarding $H_*S^n \otimes H_*S^n \subset J^0 \otimes J^0$ we have $D^0: H_*S^n \otimes H_*S^n \simeq H_*(S^{2N}) \subset I^0$.

(iv) If D_1 is another sequence of maps satisfying the conditions above, then D_1 and D are equivariantly chain homotopic. That is, there exists a sequence of maps E so that

$$dE^i + E^i d = E^{i-1} + E^{i-1}T + D^i + D_1^i.$$

To define the squaring operations, suppose $a \in \text{Ext}^{s,t}(H_*K, H_*S^n)$. Then a is represented by a map

$$a: H_* S^t K \longrightarrow J^S.$$

Now, since $H_*(S^t K \wedge S^t K) = H_* S^t K \otimes H_* S^t K$ we have

$$a \wedge a: H_*(S^t K \wedge S^t K) \rightarrow J^S \otimes J^S.$$

Definition $Sq^i(a) \in \text{Ext}^{s+i, 2t}(H_* K \wedge K, H_* S^{2n})$ is the element represented by the composition

$D^{s-i}(a \wedge a): H_*(S^t K \wedge S^t K) \rightarrow J^{s+i}$. Using Proposition 4.5, it is easy to check the operations Sq^i are well defined.

Now let $p: \text{ep}^{s-i}(S^{t-s} K) \rightarrow S^{2t-s-i}(K \wedge K)$ be the map which pinches out $\text{ep}^{s-i-1}(S^{t-s} K)$. Recall also the map σ of Theorem 2.7 composing p with the $t-s$ fold iterate of σ gives

$$S^{t-s} \text{ep}_{t-s}^{t-i} K \rightarrow \text{ep}^{s-i} S^{t-s} K \rightarrow S^{2t-s-i}(K \wedge K).$$

Using the naturality of Ext and the induced map from this composition we may regard the squaring operations as

$$Sq^i: \text{Ext}^{s,t}(H_* K, H_* S^n) \rightarrow \text{Ext}^{s+i, t+i}(H_* \text{ep}_{t-s}^{t-i} K, H_* S^{2n}).$$

Now suppose N is chosen so there is a retraction

$r: \text{ep}^{s-i}(S^N) \rightarrow S^{2N}$. If $\alpha: S^{t-s} K \rightarrow S^N$, then as in section 3,

$\overline{\text{ep}}^{s-i}(\alpha)$ is a map $\text{ep}^{s-i}(S^{t-s} K) \rightarrow S^{2N}$.

Theorem 4.6 If $a \in \text{Ext}$ detects α , then stably the map $\overline{\text{ep}}^{s-i}(\alpha)$ is detected by $Sq^i(a)$.

Proof: The main idea of the proof is to use the topological map θ of Proposition 4.2 to produce the algebraic maps D^i of Proposition 4.5.

Recall [1] that if $J(S^N)$ is an injective resolution of $H_*(S^N)$, then an Adams tower Y_* so that $H_*(Y_s/Y_{s+1}) \simeq J^s$ is called a realization of the resolution J^* . It is a theorem [1] that any resolution can be realized through a stable range. Now suppose Adams towers Y_* and W_* have been chosen as realizations of injective resolutions JS^N and IS^{2N} . Note in the stable range

$$H_*((Y_s \wedge Y_s)/(Y_{s+1} \wedge Y_{s+1})) \simeq J^s \otimes J^s.$$

Consider the composite

$$S^i/S^{i-1} \wedge (Y_s \wedge Y_s/Y_{s+1} \wedge Y_{s+1}) \rightarrow \text{ep}_{i-1}^i(Y_s/Y_{s+1}) \rightarrow W_{s-i}/W_{s-i-1}$$

where the first map comes from the representation of $\text{ep}^i X$ as $S^i \rtimes_G X \wedge X$ and the second map is induced by θ as in Proposition 4.2. The homology of the left hand space is $H_*(S^i/S^{i-1}) \otimes J^s \otimes J^s$ and the homology of the right hand space is I^{2s-i} . Then letting $g \in H_*(S^i/S^{i-1})$ be the class carried by the top hemisphere of S^i , let D^i be the composite

$$J^s \otimes J^s \rightarrow \{g\} \otimes J^s \otimes J^s \rightarrow I^{2s-i}.$$

Then it is easy to show the maps D^i satisfy the properties in Proposition 4.5 by using the decomposition of S^j with two cells in each dimension. The Theorem then follows from the definitions by factoring the map $\overline{\text{ep}}^{s-i}(\alpha)$ as

$$\text{ep}^{s-i}(S^{t-s}K) \rightarrow \text{ep}^{s-i}(Y_*) \rightarrow \text{ep}^{s-i}(W_*) \rightarrow S^{2N}.$$

Proof of Theorem 4.1

Let X'_* and X_* be Adams towers for S^N and S^{2N} respectively. Since a is a permanent cycle, there is a map $\alpha: S^{N+k} \rightarrow X'_S$ realizing a . The

$$\overline{ep}^{r+i}(\alpha): S^{N+k} \text{RP}_{N+k}^{N+k+i+r} \rightarrow X_{2s-i-r} \text{ by Proposition 4.4.}$$

Assume N is divisible by a large enough power of 2 so there is a homotopy equivalence $S^{N+k} \text{RP}_{N+k}^{N+k+i+r} \rightarrow S^{k+2N} \text{RP}_{k+i+1}^{k+i+r}$. In what follows, we consider the

construction with respect to the spectral sequence associated with the truncated tower

$$X_{2s} \rightarrow X_{2s-1} \rightarrow \cdots \rightarrow X_{2s-r}$$

in order to reduce indeterminacy.

Let $Y \xrightarrow{f} W \xrightarrow{g} Z$ denote the cofibration given by

$$S^M \text{RP}_k^{k+i} \rightarrow S^M \text{RP}_k^{k+i+r} \rightarrow S^M \text{RP}_{k+i+1}^{k+i+r}.$$

Also let y , w , and z denote the top cohomology classes of these spaces. Then by 4.6, $\overline{ep}^{i+r}(\alpha)$ is detected by $\text{Sq}_{i+r}(a) \otimes w$.

Now the construction $f^{*,*}: E_r(W, S^{2N}) \rightarrow E_*(Y, S^{2N})$ (see 1.22) gives by 4.6

$$\text{Sq}_{i+r}(a) \otimes w \rightarrow \text{Sq}_i(a) \otimes y.$$

The coboundary $\delta: E(Y, S^{2N}) \rightarrow E(Z, S^{2N})$ gives

$\text{Sq}_i(a) \otimes y \rightarrow \Sigma$ where Σ is the sum in the statement of

the Theorem, using the E_1 term of section 1. Applying Theorem 1.24 in the modified-truncated tower spectral sequence, letting $\Sigma = \gamma$ in the statement of 1.24, we see that the only possibility for β is $Sq_{i+r}(a) \otimes x_{k+i+r}$.

This gives the differential in the spectral sequence for the cohomotopy of Z as desired.

Notation: Elements in $E_2(S^n)$ will be denoted by the leading terms of their representatives in the E_1 term of section 1. For instance $x_n \otimes \lambda_6 \lambda_1 + x_n \otimes \lambda_4 \lambda_3$ will be denoted by λ_{61} or simply 61. Similarly $\lambda_{15} \lambda_1$ may be denoted by 15,1. See [10] and also [27] for a partial calculation of $E_2(S^n)$ using the Λ algebra.

Suppose $a \in E_2^{s,t}$ is a permanent cycle in the Adams spectral sequence for the homotopy of a sphere.

Theorem 4.7 (Kahn, Milgram) There are differentials in the stable Adams spectral sequence for the homotopy of spheres.

$$d_2 Sq_{i+1}(a) = Sq_i(a) \lambda_0 \quad \text{whenever } t-s+i = -1 \bmod 2$$

$$d_3 Sq_{i+2}(a) = Sq_i(a) \lambda_1 \quad t-s+i = -1 \bmod 4$$

$$d_5 Sq_{i+4}(a) = Sq_i(a) \lambda_3 \quad t-s+i = -1 \bmod 8$$

$$d_9 Sq_{i+8}(a) = Sq_i(a) \lambda_7 \quad t-s+i = -1 \bmod 16$$

Proof: By Theorem 4.1

$$d_{r+1} Sq_{i+r}(a) \otimes x_{t-s+i+r} = Sq_i(a) \lambda_{r-1} \otimes x_{t-s+i+r} + \Sigma$$

where Σ is a sum of terms in lower cells x . By 2.21, the top cell of $RP_{t-s+i+1}^{t-s+i+r}$ splits off, so this differential

actually occurs in the spectral sequence for a sphere. Notice Theorem 4.1 also shows Σ is in the image of an earlier differential. This completes the proof.

Theorem 4.1 can be used to generalize Theorem 4.7 to find differential formulas involving elements which are related by their names in Λ .

Theorem 4.8 The following differential formulas hold in the stable Adams spectral sequence. Notation $Sq_i = Sq_i(a)$ where $E_2^{s,t}$ is a permanent cycle as above.

$$\begin{aligned}
 d_3(Sq_{i+2}\lambda_1) &= Sq_i\lambda_1\lambda_1 \text{ whenever } t-s+i = 2 \text{ or } 3 \bmod 4 \\
 d_3 Sq_{i+2}\lambda_{233} &= Sq_i\lambda_{1233} && " \\
 d_3 Sq_{i+2}\lambda_{15} &= Sq_i\lambda_{1,15} = Sq_i\lambda_{13,3} && " \\
 d_3 Sq_{i+2}\lambda_{24111} &= Sq_i\lambda_{124111} && " \\
 d_3 Sq_{i+2}\lambda_{53} &= Sq_i\lambda_{153} = Sq_i\lambda_{333} && " \\
 d_4 Sq_{i+3}\lambda_{33} &= Sq_i\lambda_{233} && t-s+i = 3 \bmod 4 \\
 d_4 Sq_{i+3}\lambda_{4111} &= Sq_i\lambda_{24111} && " \\
 d_5 Sq_{i+4}\lambda_3 &= Sq_i\lambda_{33} && t-s+i = 5 \text{ or } 7 \bmod 8 \\
 d_5 Sq_{i+4}\lambda_{33} &= Sq_i\lambda_{333} && t-s+i = 4, 5, 6 \text{ or } 7 \bmod 8 \\
 d_6 Sq_{i+5}\lambda_{4111} &= Sq_i\lambda_{44111} && t-s+i = 7 \bmod 8 \\
 d_8 Sq_{i+7}\lambda_{233} &= Sq_i\lambda_{6233} && t-s+i = 7 \bmod 8 \\
 d_9 Sq_{i+8}\lambda_7 &= Sq_i\lambda_{77} && t-s+i = 11 \bmod 16 \\
 d_{10} Sq_{i+9}\lambda_{333} &= Sq_i\lambda_{8333} && t-s+i = -1 \bmod 8
 \end{aligned}$$

Remark The d_5 whose image is $Sq_1 \lambda_{333}$ is of special interest because its image is the same as the image of a d_{13} given by Milgram [19, p. 172]. The d_{13} occurs only when $t-s+i = 21 \bmod 32$ and so the d_5 has already killed its image. The differential involving λ_{15} and the last differential involving λ_{8333} are interesting because λ_{15} and λ_{8333} are not d_2 cycles.

Proof: We give the proof for the d_3 involving 24111 , the others are similar. Using the notation of Theorem 4.12, we get

$$d_3(Sq_{i+2} \otimes x_{k+i+2}) = Sq_1 \lambda_1 \otimes x_{k+i+2} + \text{terms in lower cells } x.$$

Let x_{k+i+2}^* be the homology class dual to x_{k+i+2} . Then we have consulting Mahowald's tables [15, p. 52] that $x_{k+i+2}^* \otimes 24111$ is the leading term of a permanent cycle in the spectral sequence $E(S, RP_{k+i+1})$. Using the composition $E(RP, S) \otimes E(S, RP) \rightarrow E(S, S)$ we get

$$d_3(Sq_{k+i+2} \lambda_{24111}) = Sq_1 \lambda_{124111} + \text{other terms}.$$

Because the lower terms of $x_{k+i+2}^* \otimes \lambda_{24111}$ might contain terms $x_{k+i+1}^* \otimes \lambda_I$ where $\lambda_0 \lambda_I$ is not admissible, we do not know yet that $Sq_1 \lambda_1 \lambda_{24111}$ is the leading term of the image of the differential. This could be settled by an extended Λ calculation. However, in many cases there is an easier method. Let Σ denote the image of the d_3 . Then it is not hard to see that in E_1 of section 1

$$d_1(x_{k+i+2}^* \otimes \lambda_{24111} + \text{lower terms}) = x_{k+i}^* \otimes \Sigma.$$

Consulting Mahowald's tables we see the image of this d_1 must be nonzero in E_2 , because $x_{k+i+2}^* \lambda_{24111}$ is not a class in $E_2(\mathbb{RP}_{k+i})$. Thus Σ is not zero and so must represent 124111 , since this is the only element in the 'box'. This completes the proof.

The rest of this section collects some examples of applications of the extended powers to the homotopy of spheres.

Theorem 4.9 In the stable 2-component of the homotopy of spheres the following relations hold.

$$\begin{aligned} 2\alpha^2 &= 0 && \text{if stem } \alpha = -1 \bmod 2 \\ \eta\alpha^2 &= 0 && \text{if stem } \alpha = -1 \bmod 4 \\ v\alpha^2 &= 0 && \text{if stem } \alpha = -1 \bmod 8 \\ \sigma\alpha^2 &= 0 && \text{if stem } \alpha = -1 \bmod 16 \\ \beta_j\alpha^2 &= 0 && \text{if stem } \alpha = 2^{j+2} - \text{stem } \beta_j - 2 \bmod 2^{j+3} \end{aligned}$$

where β_j is the j^{th} generator of the image of J ($\beta_1 = v$ in the 3 stem).

Proof: Consider $ep(\alpha): ep(S^{k+N}) \rightarrow S^{2N}$ where k is the stem of α . Here N is assumed to be divisible by a sufficiently large power of 2 so that there is a retraction $ep(S^N) \rightarrow S^{2n}$ in the range of interest.

(Proposition 2.17). Now by corollary 2.8,

$ep(S^{k+N}) = S^{k+N}_{\mathbb{RP}_{k+n}}$. By Theorem 2.2, (and the fact N is

divisible by a large power of 2) there is a map which is an inclusion on $S^{2(k+N)}$

$$S^{2(k+N)} \bigcup_{\beta} e^{2k+2N+\text{stem } \beta+1} \rightarrow S^{k+N} \text{RP}_{k+N}$$

provided $\text{stem } \beta+1 = (k + \text{stem } \beta+1)$ where β is an odd multiple of the generator of the image of J . Equivalently $\text{stem } \beta + 1 = 8c + 2^d$ and $k + 8c + 2^d + 1 = (\text{odd}) 2^{4d+d}$. Now if $\text{stem } \beta + 1 = 8c + 2^d$, then $4c + d = j + 2$. Finally we get $k + \text{stem } \beta_j + 2 = (\text{odd}) 2^{j+2}$ and so $\text{stem } \alpha = k = 2^{j+2} - \text{stem } \beta_j - 2 \bmod 2^{j+3}$. Now on S^{2k+2N} , the map $ep(\alpha)$ is equal to α^2 . Since β is evidently zero into the bottom cell of the two cell complex, we have $\beta\alpha^2 = 0$ as desired.

The theorem can be extended to elements not in the image of J as follows. Suppose $\gamma \in \pi_*(S^{k+1})$ and γ goes to zero under the map

$$S^{N+k+1} \rightarrow S^{N+1} \text{RP}_k,$$

then $\gamma\alpha^2 = 0$ when α is in stem k . For example let $\epsilon \in \pi_{N+8}(S^N)$ be the element which is detected by λ_{233} in $E_2(S^N)$.

Proposition 4.10 $\epsilon\alpha^2 = 0$ when $\text{stem } \alpha = 2, 3$ or $7 \bmod 8$.

Proof: Consult Mahowalds tables [15, p. 52] to see that ϵ goes to zero under the map $S^{k+1} \rightarrow S^1 \text{RP}_k$ when k is as specified. When $k = 2 \bmod 8$, there is nothing in the 'box'. When $k = -1 \bmod 4$, the only possibility is

that the element has jumped to filtration 4. Since there is no homotopy element in filtration 4, stem 8, for the lowest cell, we are done.

Remark: Mahowalds tables list elements according to the skeletal filtration. Thus any element in the image $\pi^S S^k \rightarrow \pi_*^S RP_k$ must be detected on the lowest cell.

Example 4.11 If $k = \text{stem } \alpha = 1 \bmod 4$ then

$$2x \in \{\alpha^2, 2, \eta\} \text{ for some } x.$$

Proof: Stably the following composition is in the Toda bracket

$$S^{2k+2N+2} \xrightarrow{\tilde{\eta}} S^{k+N}_{RP_{k+N}} \xrightarrow{ep(\alpha)} S^{2N}$$

where $\tilde{\eta}$ is a coextension of η over the attaching map of the second cell of RP_{k+N} . Consulting M.T. again gives the result.

Proposition 4.12 Suppose $k = \text{stem} = 2 \bmod 4$. Then

$$\eta \alpha^2 = 2 \overline{ep}^1(\alpha).$$

Proof: The first three cells of RP_k are

$$(S^k \vee S^{k+1}) \cup_{\eta+2} e^{k+2}$$

where the attaching map of the top cell is the sum of η into the $k = \text{sphere}$ and 2 into the $k+1$ sphere.

Proposition 4.13 Suppose $2\alpha = 0$ and $k = \text{stem} = 2 \bmod 4$.

Then

$$0 \in \{\alpha, 2, \alpha\}.$$

Proof: By Theorem 3.13, $2ep^1(\alpha) \in \{\alpha, 2, \alpha\}$. Thus by Proposition 4.12, $\eta\alpha^2 \in \{\alpha, 2, \alpha\}$. Since $\eta\alpha^2$ is in the indeterminacy of the bracket, $0 \in \{\alpha, 2, \alpha\}$ as desired.

Remark Proposition 4.13 shows $0 \in \{\sigma^2, 2, \sigma^2\}$. Theorem 4.9 shows $\eta\sigma^2 = 0$. Since $ep^1(2) = \eta$, Theorem 3.13 shows $0 \in \{2, \sigma^2, 2\}$. These facts are enough to construct the bracket $\{2, \sigma^2, 2, \sigma^2\}$. One can show if $x \in \{2, \sigma^2, 2, \sigma^2\}$ then x is detected by h_4^2 in E_2 in the 30 stem. Similarly using $2\theta_4 = 0$, we get $0 \in \{\theta_4, 2, \theta_4\}$ where $\theta_4 = x$.

Proposition 4.14 Suppose $k = \text{stem } \alpha = 2 \bmod 4$ and $2\alpha = 0$. Then $\eta\alpha^2 = 0$.

Proof: $2\alpha = 0$ implies $\eta\alpha \in \{2, \alpha, 2\}$ because $ep^1(2) = \eta$. So $\eta\alpha^2 \in \{2, \alpha, 2\} = 2\{\alpha, 2, \alpha\}$. By 4.13, $\{\alpha, 2, \alpha\}$ consists of its indeterminacy, that is compositions with α . So, since $2\alpha = 0$, we are done.

Finally the following Proposition is an interesting example.

Proposition 4.15 Given that $h_i h_1$ detects an element in the homotopy of spheres of Hopf invariant v , then $h_i h_i h_2 + h_{i+1} h_1 h_1$ is a permanent cycle of Hopf invariant v^2 .

Proof: Using the known results [13] it is easy to see $Sq^1(h_i h_1) = h_i h_i h_2 + h_{i+1} h_1 h_1$. Now $\overline{ep}^1(h_i h_1)$ is a map from a wedge of spheres. Hence the element detects a map from a sphere to a sphere as asserted. The statement about

the Hopf invariant follows from Theorem 3.1. It is known [16] that there is no unstable element of Hopf invariant v^2 in this stem.

Corollary 4.16 Since $h_i h_1$ detects an element in the homotopy of spheres for any i according to Mahowald, so does $h_i h_1 h_2$ (when not zero in E_2).

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