UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Examination in:	MA 001 — Matematiske metoder.
Day of examination:	Mandag 11. desember 2000.
Examination hours:	09.00 – 15.00.
This examination set consists of 4 pages.	
Appendices:	Ingen.
Permitted aids:	Alle skriftlige hjelpemidler og lommekalkulator.

Make sure that your copy of the examination set is complete before you start solving the problems.

Problem 1.

a) Find the indefinite integrals

$$\int (3 + 4\sin 2x)\,dx \qquad \text{and} \qquad \int x^2 \cos x\,dx\,.$$

b) Let $D = 1 + i\sqrt{3}$ where $i^2 = -1$. Find the square root $\sqrt{D}$ and give the answer in the form $a + ib$.

Problem 2.

Let a function ℓ be given by the formula $\ell(y) = y + \ln y$.

a) For which real y is $\ell(y)$ defined ? Find $\lim_{y\to 0^+} \ell(y)$ and $\lim_{y\to\infty} \ell(y)$. What is the set of values of ℓ ?

b) Compute $\ell'(y)$. Decide if ℓ is strictly increasing or not. Compute $\ell''(y)$. Decide if the graph of ℓ curves upwards or downwards.

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Let $y = e(x)$ be the inverse function of $x = \ell(y)$, i.e., $y = e(x)$ if and only if $x = \ell(y)$.

c) What is the set where e is defined, and what is the set of values of e ? Find $\lim_{x\to-\infty} e(x)$ and $\lim_{x\to\infty} e(x)$.

The following table gives some values of the function ℓ, correct to two decimal places:

y	0.10	0.40	0.70	1.00	1.30
$\ell(y)$	-2.20	-0.52	0.34	1.00	1.56

d) Let $L(y) = ay + b$ be the linear function determined by $L(0.40) = \ell(0.40) \approx -0.52$ and $L(0.70) = \ell(0.70) \approx 0.34$.
Find y_1 such that $L(y_1) = -0.20$, correct to two decimal places.

e) Is the graph of L lying over or under the graph of ℓ when $0.40 < y < 0.70$? Is the exact solution y_2 to the equation $\ell(y_2) = -0.20$ greater than or less than the solution y_1 to the equation $L(y_1) = -0.20$? Justify both answers.

f) Let K be a positive constant. Show that

$$y + K \ln y = K \cdot \ell(\frac{y}{K}) + K \ln K$$

by calculating $K \cdot \ell(y/K)$.

Problem 3.

In an enzymatic reaction a substance A is converted to a substance B. Due to an inhibitor (poison), that which is expended of substance A is not replaced. Let $y = y(t)$ be the concentration of the substance A at time t. We suppose that $y(t) > 0$ for all t. The reaction rate is assumed to be given by Michaelis and Menten's equation, which leads to the differential equation

$$(*) \qquad \frac{dy}{dt} = -V \cdot \frac{y}{y+K} .$$

Here V and K are positive constants. This is a separable differential equation $dy/dt = f(t)g(y)$ where $f(t) = -V$ and $g(y) = y/(y+K)$.

a) Show that the solutions $y = y(t)$ of the differential equation $(*)$ are precisely the functions that satisfy the equation

$$y + K \ln y = -Vt + C$$

where C is an arbitrary constant.

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b) Use the result in problem 2(f) to show that the solutions to the differential equation (*) can be expressed as

$$y(t) = K \cdot e(-\frac{V}{K}t + D)$$

where e is the function from problem 1 and D is an arbitrary constant.

c) Show that

$$D = \frac{y_0}{K} + \ln\frac{y_0}{K}$$

where $y_0 = y(0)$.

d) Find the limit for the concentration of substance A when $t \to \infty$, i.e., $\lim_{t\to\infty} y(t)$.

e) Suppose that $V = 2.40$, $K = 2.00$ and $y_0 = 2.00$. Assume also that the solution y_1 from problem 2(d) accurately approximates the solution $y_2 = e(-0.20)$ from problem 2(e) to one decimal place. Use this and the result from problem 2(d) to find $y(1.00)$, correct to one decimal place.

Problem 4.

a) Shade the region M in the xy-plane that is given by the inequalities

$$x - 2y \geq 1\,, \qquad x + 3y \geq -4\,, \qquad 2x + y \leq 8\,, \qquad 3x - y \leq 8\,.$$

Use the same unit of length on both axes.

b) Draw, in the same figure, one of the level curves of the function $f(x, y) = x + y$, and find the greatest and the least value of $f(x, y)$ when (x, y) ranges through M.

Problem 5.

a) Find the eigenvalues of the matrix

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$

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b) Find the general solution $(x(t), y(t))$ to the system of differential equations:

$$(**) \qquad \begin{aligned} \frac{dx}{dt} &= x + 2y \\ \frac{dy}{dt} &= x \end{aligned}$$

c) Find the special solution to the system of differential equations $(**)$ that is subject to the initial conditions $x(0) = 1$ and $y(0) = 2$. Also find the other special solution that satisfies $x(0) = 1$ and $\lim_{t\to\infty} y(t) = 0$.

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