OBLIG 2 FOR MA 001

Each student registered for the fall 2000 exam in MA 001 must submit acceptable solutions to two obligatory problem sets. This is the second of those. Each student prepares and hands in an individual solution to his or her "group" teacher, who then corrects and grades the solutions. Those qualifying for a passing grade (4.0 or better) are considered accepted. Remember to write your full name and the number of your MA 001 colloquium group on the solution you hand in.

The absolute deadline for handing in solutions to both obligatory problem sets is Friday November 10th 2000.

Give a mathematical justification for all answers. Give exact answers when possible.

Problem 1

Student A and student B are both going to drink a cup of coffee. The coffee is served black, with volume $V_k = 150$ ml and temperature $T_k = 68°$C. Both students want a dash of milk in the coffee, with volume $V_m = 10$ ml. The milk is stored cold, and has temperature $T_m = 4°$C. Student A pours milk in her coffee, but then the mobile phones of both students ring, and student B does not pour milk in his cup.

(a) When the milk is poured into the coffee, the mixture gets temperature

$$T_b = \frac{V_k T_k + V_m T_m}{V_k + V_m} .$$

Find the mixed temperature in student A's cup of coffee with milk.

(b) As the conversations drag out, the coffee temperature (with or without milk) decreases exponentially towards the room temperature, which is constant at $T_r = 20°$C. After t minutes the temperature is given by the expression

$$T(t) = T_r + (T_0 - T_r)\left(\frac{1}{2}\right)^{t/h} .$$

Give a precise interpretation of the constants T_0 and h.

(c) After 10 minutes ($t = 10$) the temperature in student A's coffee (with milk) has fallen to 42°C. Find the constant h. What is the temperature in student A's coffee after another 10 minutes ($t = 20$) ?

(d) After 20 minutes student B is finished with the phone call. What is then the temperature in student B's coffee (without milk) ?

(e) Student B then pours milk in his coffee. Find the mixed temperature for the coffee with milk added after 20 minutes. Which cup of coffee with milk is warmest after 20 minutes ?

Problem 2

(a) Let a, b, k and ℓ be positive constants. Show that the function

$$F(t) = -(a/k)te^{-kt} - (a/k^2)e^{-kt} - (b/\ell)e^{-\ell t}$$

is an anti-derivative to the function

$$f(t) = ate^{-kt} + be^{-\ell t}.$$

After a rainfall, water runs into a water reservoir. After $t \geq 0$ hours the incoming flow rate is $f(t)$ m^3/h, where $f(t)$ is given as in (a) and $a = 2.0 \cdot 10^5$, $b = 2.0 \cdot 10^4$, $k = 1.0$ and $\ell = 0.1$.

(b) Find $f(0)$ and $\lim_{t\to\infty} f(t)$.

(c) Let $V(t)$ m^3 be the total water volume that has run into the reservoir from time 0 to time t. Then $V'(t) = f(t)$. Find a formula for $V(t)$ for $t \geq 0$.

(d) Find $V(0)$ and $\lim_{t\to\infty} V(t)$.

(e) Water is also let out of the reservoir at a constant outgoing flow rate $c = 2.0 \cdot 10^4$ m^3/h for $t \geq 0$, until the reservoir is empty. Show that the reservoir is emptied within $t = 20$ hours.

Problem 3

A wave in the xy-plane is at time t given as the graph of the function

$$f_t(x) = 4\cos x + 3\cos(x - t).$$

This is the sum of two waves in the xy-plane: one at rest with amplitude 4, and one with amplitude 3 moving to the right as t grows.

(a) Show that $f_t(x) = (4 + 3\cos t)\cos x + (3\sin t)\sin x = C\cos(x - x_0)$ where $C = \sqrt{25 + 24\cos t}$ and $\tan x_0 = 3\sin t/(4 + 3\cos t)$.

(b) Let $g(t) = 3\sin t/(4 + 3\cos t)$ for $0 \leq t \leq \pi$. Find $g'(t)$ and decide where g is increasing and where g is decreasing.

(c) Find the maximum point and the maximal value for $g(t)$. Give exact answers.

(d) At time t the top of the wave (wavecrest) closest to the y-axis has xy-coordinates (x_0, C). Find the xy-coordinates of the wavecrest when $g(t)$ has its maximal value.

Problem 4

(a) The equation $y = cx^r$ is valid for $(x, y) = (4, 5)$ and $(x, y) = (25, 2)$. Find c and r. Is the graph of y as a function of x a straight line in an orthogonal linear coordinate system, in a singly logarithmic coordinate system, or in a doubly logarithmic coordinate system ?

(b) Use L'Hôpital's rule (possibly several times) to find

$$\lim_{h\to 0} \frac{1 - \cos 2h}{h^2}.$$

(c) Find the definite integral

$$\int_{-\pi/2}^{\pi/2} \sin^4 x \cos x \, dx$$

by using the substitution $u = g(x) = \sin x$.

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