Toward a Global Understanding of $\pi_*(S^n)$

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Abstract. This talk will describe recent advances in getting a global picture of the homotopy groups of spheres. These results begin with the work of Adams on the homotopy determined by K-theory. Substantial new information follows from the nilpotence results of Devinatz, Hopkins and Smith.

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1 Introduction

Until about 1960, the primary method used to calculate homotopy groups of spheres was the EHP sequence. This was invented by James at the prime 2 and Toda at odd primes. Early steps in this direction were taken by Freudenthal. Basically, the EHP sequence is a consequence of the result that

$$S^n \to \Omega S^{n+1} \to \Omega S^{2n+1}$$

is a 2 local fibration. At odd primes there is a similar result with some twists. Spectral sequences give a way to organize such calculations. We consider the filtration of $\Omega^{n-1}S^n$ given by

$$S^1 \to \Omega S^2 \to \cdots \to \Omega S^{n-2} S^{n-1} \to \Omega^{n-1} S^n.$$

When we apply homotopy to this filtration we get a spectral sequence in the standard fashion. The E_1 is given by

$$E_1^{s,t} = \pi_{t+1}(\Omega^{s-1} S^{2s-1}) = \pi_{t+s}(S^{2s-1}).$$

The key feature here is that the input to this spectral sequence is the output of an earlier calculation. In particular, once $\pi_1(S^1)$ is determined, no other outside calculation is necessary. This seductive feature attracted a lot of attention early on. This feature caused many to miss some obvious additional structure which is the current focus. If we look only at $E_1^{s,t}$ for $s \leq n$, the spectral sequence converges to $\pi_{t+n}S^n$. If we allow all s, the spectral sequences converges to the stable homotopy groups which we write as $\pi_t(S^0)$. The filtration induced on $\pi_t(S^0)$ refers to the sphere of origin of the class. This means the smallest integer s such that the homotopy class is in the image of the suspension map $\Omega^{s-1}S^s \to \Omega^{\infty-1}S^\infty$. The class in $E_1^{s,t} = \pi_{t+s}(S^{2s-1})$ which projects to a class is called the Hopf invariant of that class. There are a few global results obtained essentially from the EHP sequence approach.

THEOREM 1.1 *(Serre) The groups, $\pi_j S^n$ are finite except if $j = n$ or if $n = 2k$ and $j = 4k - 1$.*

THEOREM 1.2 *(James and Toda) The E_2 term of the EHP spectral sequence is an $\mathbb{F}_p$ vector space.*

James at 2 and Toda at odd primes essentially proved this. This result gives an estimate of the maximum order of the torsion subgroup of $\pi_t S^n$. This result was sharpened to the best possible by the following result.

THEOREM 1.3 *(Cohen, Moore, and Neisendorfer) If $j \neq 2n+1$ then $p^n\pi_j(S^{2n+1}) = 0$ for p an odd prime. There are classes of order p^n.*

At the prime 2 the sharpest estimate is not known. The result of James implies that $2^{2n}\pi_j(S^{2n+1}) = 0$. The maximum known elements would suggest a more complicated formula but approximately $2^{n+1}\pi_j(S^{2n+1}) = 0$. A precise conjecture is made in the next section.

There is another feature of the EHP spectral sequence which should be noted. Since $E_1^{s,t} = \pi_{t+s}(S^{2s-1})$ it is clear that if $t < 3s - 3$ then $E_1^{s,t}$ depends only on the value of $t - s$. In general, $E_r^{s,t} = E_r^{s+2^{r/2+1},t+2^{r/2+1}}$ provided that $2^{r/2+1} + t < 3(s + 2^{r/2+1}) - 3$. This allows one to describe a stable EHP spectral sequence in which $SE_1^{s,t} = \pi_{t-s-1}(S^0)$. This spectral sequence is defined for all $s \in \mathbb{Z}$. It is a consequence of Lin's theorem that this spectral sequence converges to $\pi_t(S^{-1})$. The paper by Mahowald and Ravenel [8] explores the consequences of this observation and gives complete references.

2 v_1 PERIODICITY

Another global result which does not follow from EHP considerations is the following result.

THEOREM 2.1 *(Nishida) Under composition, any element in a positive stem is nilpotent.*

It is this result which leaves one in a quandary as to how to describe an infinite calculation. Adams was the first to notice how to use Bott periodicity to construct infinite families. This is somewhat easier at odd primes but one can accomplish essentially the same thing by considering the finite complex, $Y^6 = \mathbb{C}P^2 \wedge \mathbb{R}P^2$ at $p = 2$ and $Y^k = S^{k-1} \cup_p e^k$ at p odd. We have the following result.

PROPOSITION 2.2 *Let $q = 2(p-1)$. For each prime p and each $k > 6$ there is a map $Y^{k+q} \to Y^k$ such that all composites*

$$Y^{k+qj} \to Y^{k+q(j-1)} \to \cdots \to Y^k$$

are essential for all j and k. We will call this map v for any j and k.

This means that we can consider the homotopy theory, $[Y^*, \_]$ as a module over $\mathbb{Z}[v]$. We will label this homotopy modlule as $\pi_*(\_, Y)$. We can again ask for freeness and exponents with respect to this module. The question about freeness has been completely answered. The exponent question is completely open. A starting point for understanding freeness in this context is the following result. It is possible to compare the fibers of the single suspension map in the EHP sequence. This gives a new sequence of fibrations

$$W(n) \to S^{2n-1} \to \Omega^2 S^{2n+1}$$

In this context, Serre's theorem is equivalent to the assertion that $W(n)$ is rationally acyclic. To get information about $[Y^*, \_]$ for spheres, it is useful to compare $[Y^*, W(n)]$ for various n. The following result allows this.

PROPOSITION 2.3 *There is a a map $W(n) \to \Omega^{2p} W(n+1)$ which induces an isomorphism in $v^{-1}\pi_*(\_, Y)$ homotopy.*

This proposition is key to determining the homotopy which can be detected in some sense by K-theory. In order to state the result we need to recall a small part of the Snaith splitting theorem. We will state the results for the prime 2. Something similar is true for odd primes.

THEOREM 2.4 *(Snaith) There is a map*

$$s_n : \Omega^{2n+1} S^{2n+1} \to \Omega^\infty \Sigma_0^\infty \mathbb{R}P^{2n}$$

which induces a monomorphism in homology.

Using these maps we can prove the following.

THEOREM 2.5 *The Snaith maps, s_n induce isomorphisms in the homotopy theory $v^{-1}\pi_*(\_, Y)$.*

All that remains is to compute this homotopy theory and that is an easy calculation. Thus a summand in $\pi_{k+2n+1}(S^{2n+1})$ for each $k \neq 4, 5 \bmod 8$ is determined. For certain values of n there are non-trivial summands for the other values of k. This aspect of homotopy theory is quite well understood. This material appears in several papers, the last one, [3], contains references to earlier work. The computational aspects is being pursued by Bendersky and Davis.

This discussion works in a very similar fashion at odd primes and the result is much easier to state. Let p be an odd prime and $q = 2p - 2$. Let $\nu(k)$ be the maximum power of p which divides k.

THEOREM 2.6 *([13]) If $j = kq - 1$ or if $j = kq - 2$ then $j > 2n + 1$, then $\pi_{j+2n+1}(S^{2n+1})$ contains a $\mathbb{Z}/p^{\min(n,\nu(k)}$ summand.*

The homotopy detected by K-theory is special at the prime 2. At all odd primes in behaves in a similar fashion with the summand being defined by number theoretic functions as the above Theorem illustrates. In particular, at odd primes the elements of maximal order are found in the homotopy detected by K-theory. Typically, exponent theorems are proved by showing that the loop space power map has a certain order. In particular, we consider $P(r) : \Omega^{2n+1}S^{2n+1} \to \Omega^{2n+1}S^{2n+1}$ given by multiplication by p^r in the loop variable. Theorem 1.3 is proved by showing that $P(n)$ is null if p is odd. A result this simple at 2 is false. The conjectured result is:

CONJECTURE 2.7 *At the prime 2, where $P(n)$ refers to the 2^n power map we expect:*

- *If $n \equiv -1, 0$ mod 4, then $P(n)$ is null.*
- *If $n \equiv 1, 2$ mod 4 and $n > 1$, then $P(n + 1)$ is null.*
- *Among the torsion classes in $\pi_*(S^{2n+1})$, the element of maximal order is detected by K-theory.*

If all parts of this conjecture are correct, the proof would have to be quite different than the proof of Theorem 1.3 since we have the following result.

THEOREM 2.8 *If $n \equiv -1, 0$ mod 4, then there is a homotopy class of order 2^n detected by K-theory. If $n \equiv 1, 2$ mod 4 and $n > 1$, then the maximum order among the classes detected by K-theory is 2^{n-1}.*

A conjecture of this sort was first made by Barratt. This version is due to Barratt and Mahowald.

3 TELESCOPES AND LOCALIZATIONS

In order to understand the next kind of periodicity I want to introduce some additional notation. The first question which needs to be answered is: "For which finite complexes, F, are there maps, $v : \Sigma^k F \to F$, all of whose iterates are essential?" We will find it easier to suppress the suspension variable in this discussion. We are looking for maps like the map described above for Y. Devinatz, Hopkins and Smith [4]answered this question.

THEOREM 3.1 *Let F be a finite complex and $v : \Sigma^k F \to F$. The composite*

$$\Sigma^{k \cdot j} F \to \Sigma^{k(j-1)} F \to \cdots \to F$$

is essential for all j if and only if $MU_*(v) \neq 0$ where MU_* is complex bordism theory.

MU_* splits into a wedge of theories at a fixed prime. These smaller theories are called Brown-Peterson homology theories, BP_*. Their homotopy is given by $\pi_*(BP) = \mathbb{Z}[v_i, i = 1, \cdots]$. The dimension of v_i is $2(p^i - 1)$. In order to understand a particular periodicity family it is useful to localize BP. Consider the theory defined by $v_n^{-1}BP$. It is possible to get a more efficient theory by first killing v_i, for $i > n$ and inverting v_n in this new theory. Call the resulting spectrum, $E(n)$. We have $\pi_*(E(n)) = \mathbb{Z}_{(p)}[v_1, \cdots, v_n, v_n^{-1}]$. Work of Miller, Ravenel and Wilson, [9], show that this spectrum leads to an important localization. Note that $E(1) = K$ at 2. At other primes $E(1)$ is one of the factors into which K splits.

Bousfield, [2] has introduced a notion of localization at a spectrum. An excellent discussion of this is in the paper by Ravenel, [10]. A particularly important family of localizations is that given by localization with respect to $E(n)$. This gives rise to the chromatic tower. Let $L_n(X)$ be the Bousfield localization of X with respect to $E(n)$. Suppose X is a p-complete spectrum. Then there are commutative diagrams

$$\begin{array}{ccc} L_{i+1}(X) & \to & L_i(X) \\ \uparrow & & \uparrow \\ X & \simeq & X \end{array}$$

such that $X \to \text{homlim } L_i(X)$ is a homotopy equivalence.

The computations in the chromatic tower have been done for the stable sphere if $i = 1$ and all primes or $i = 2$ and the prime is larger than 3. For $i = 1$ the results of the previous section describe the answer. For $i = 2$ the result is very complicated and the reader is referred to the paper by Shimomura and Yabe, [12]. It is quite interesting to note that the Shimomura-Yabe result can be stated in terms of number theory functions for all primes $p > 3$. This is analogous to Theorem 2.6. These results suggest that the answer for the infinite prime might be possible. This would give the homotopy information in terms of functions whose argument is the prime and whose value is the order of a summand. In this sense, Theorem 2.6 and the Shimomura-Yabe result [12] are results for the infinite prime. It seems that if $n > p - 1$, then $L_n(S^0)$ should have such a prime independent description.

The situation for unstable spheres and L_2 localizations is still not clear. Bousfield has also defined localizations of spaces with respect to a spectrum. This seems to be a somewhat harder notion than localization of spectra. For S^{2k+1}, Arone and Mahowald, [1], have constructed a tower which reduces to a finite tower for each L_n. Here are some details. Let X be some space (or spectrum) and let F be some functor. Then Goodwillie [5] constructs a tower of functors

$$\begin{array}{ccccccccc} P_1F(X) & \leftarrow & P_2F(X) & \leftarrow & \cdots & \leftarrow & P_nF(X) & \leftarrow & \cdots \\ \downarrow & & \downarrow & & & & \downarrow & & \\ D_2F(X) & & D_3F(X) & & & & D_{n+1}F(X) & & \end{array}$$

and a collection of maps $F(X) \to P_nF(X)$ such that the inverse limit of the tower is equivalent to $F(X)$ and the fibers at each stage, $D_iF(X)$, are infinite loop spaces. For the example of the identity functor and for $X = S^{2k+1}$, this tower is investigated by Arone and Mahowald in [1]. For our purposes, the key result is the following.

THEOREM 3.2 *For each prime and each n, the L_n localization of S^{2k+1} can be represented by a tower of n fibrations. Each of the fibers is an infinite loop space. The fiber at the stage k, D_k, satisfies $L_{k-1}D_k = pt$.*

The key point is the observation that the Goodwillie tower is constant, except when $n = p^k$. In this case the stable spectrum represented by the fiber at the $n = p^k$ stage has acyclic homology with respect to the homology theory $E(k-1)$. This result can be used to compute the homotopy of L_nS^{2n+1} once one knows the stable theory. This has been done for L_1. It represents an interesting problem for L_2 at primes bigger than 3, in view of the Shimomura-Yabe calculations [12].

If $n > 1$, the homotopy theory defined by a finite complex with a self map detected by v_n seems to detect more homotopy than is present in L_nS^{2n+1}. That this should be the same is called the telescope conjecture. Recent work of Ravenel suggest this conjecture is false. Several proofs of the disproof of the telescope conjecture have been circulated but it is not yet clear if the result is proved.

4 Formal groups and homotopy theory

In addition, the connection of MU_* with formal groups has played an important role in understanding higher periodicities. The starting point is the multiplication map, $\mu : \mathbb{C}P\times\mathbb{C}P \to \mathbb{C}P$. Let $\alpha \in MU^2(\mathbb{C}P)$ represent the cohomology class given by $\mathbb{C}P = MU(1) \to \Sigma^2 MU$. Then $MU^*(\mu)(\alpha)$ is a power series in two variables. This power series, F, satisfies the axioms of a one dimensional commutative formal group over the ring $MU^*(pt)$. The key theorem is due to Quillen.

THEOREM 4.1 *(Quillen) The formal group constructed above induces an isomorphism from the Lazard ring to $MU^*(pt)$. All of the constructions in the theory of one dimensional commutative formal groups carry over to this topological setting.*

Hopkins and Miller have discovered a partial converse to this result. Let $\mathcal{FG}$ denote the category having as objects pairs (k, Γ), where k is a perfect field of characteristic p, and Γ is a formal group of height n over k, and with morphisms $\alpha : (k_1, \Gamma_1) \to (k_2, \Gamma_2)$ consisting of a pair (i, f), where i is a map $i : k_1 \to k_2$ of rings and f is an isomorphism $f : \Gamma_1 \to \Gamma_2$ of formal group laws. Then we have:

THEOREM 4.2 *(Hopkins-Miller) There exists a functor $(k, \Gamma) \to E_{k,\Gamma}$ from $\mathcal{FG}^{op}$ to the category of A_∞ ring spectra, such that,*

1. *$E_{k,\Gamma}$ is a commutative ring spectrum;*
2. *there is a unit in $\pi_2 E_{k,\Gamma}$;*
3. *$\pi_{odd}E_{k,\Gamma} = 0$, from which it follows that $E_{k,\Gamma}$ is complex orientable;*
4. *and such that the corresponding formal group law over $\pi_0 E_{k,\Gamma}$ is the the universal deformation of (k, Γ).*

A discussion of this result and related topics of formal groups and universal deformations is in the course notes prepared by Rezk [11].

Hopkins and Miller apply this result to construct higher K-theories, EO_n, at primes p where $(p-1)|n$. At 2 and 3, EO_2 is very interesting. In particular, this spectrum captures the way in which L_2 differs from the calculations of [12]. There is a connected version of EO_2 which is called eo_2. Various constructions of this spectrum yield various properties. In particular, Hopkins and Miller have constructed a version which makes eo_{2*} into an E_∞ ring spectrum. In [7] the homotopy groups eo_{2*} are computed. That paper also discusses the connection that this spectrum has with elliptic curves over $\mathbb{F}_4$ and height 2 elliptic curves. There will be a sequence of papers by Hopkins, Miller and others which expand on this theory. Without writing down specific groups, we observe that using this spectra we can show that a substantial part of the known calculation of the stable stems fit into periodic families. The basic periodicity of eo_2 at 2 is 192 which represents v_2^{32}. At the prime 3, the period is 72. How all of this should work out on unstable spheres is still not clear.

The connection with elliptic curves should be expanded on. Elliptic curves over a ring R can be co-represented by $\mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$. The coefficients, a_i, are the coefficients in the Weierstrass form of the equation for the curve,

$$x^3 + a_2x^2 + a_4x + a_6 = y^2 + a_1xy + a_3y.$$

This equation represents a curve with a single point on the line at infinity. The discriminant, Δ, is a polynomial in the coefficients. If $\Delta \neq 0$, then the curve is non-singular. Coordinate transformations which preserve the curve are

$$\begin{aligned} x &\mapsto x + r \\ y &\mapsto y + sx + t \end{aligned}$$

These substitutions give transformation formulas for the coefficients. We can use these to construct a Hopf algebroid,

$$\mathbb{Z}[a_1, a_2, a_3, a_4, a_6] \rightrightarrows \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, s, r, t]$$

The homology of this Hopf algebroid is the E_2 term of the Adams-Novikov spectral sequence to calculate $\pi_*(eo_2)$. The homology in dimension 0 is isomorphic to the ring of modular forms. There are differentials in the Adams-Novikov spectral sequence. For more details see [7].

The theories, E_{p-1}, are also related to curves. These curves, of genus $\binom{p-1}{2}$, give rise to formal groups in a more complicated fashion. This is discussed in the paper by Gorbounov and Mahowald, [6]. In this case too, the connection is exploited to give an easy calculation of the E_2 term of the Adams-Novikov spectral sequence.

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