Notes for talks on EO_2 resolutions

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Abstract

These are my notes for a series of talks on EO_2 resolutions. The new mathematics presented here is joint with Charles Rezk and has profited from conversations with Paul Goerss and Mike Hopkins

1 Introduction

I would like to discuss an understanding of the Morava stabilizer group at the prime 2 for height 2 formal groups. I will need to introduce this topic by considering some easier examples. Most topologists have some understanding of the image of J and the J spectrum. From Adams and others we know that there is a map $\psi^3 - 1 : KO \to KO$ whose fiber is the J spectrum at 2. This is a spectrum which contains the image of the J homomorphism and some additional classes which are always a part of this homotopy.

We want to understand this case from the formal group point of view. The Morava stabilizer group is the group of automorphisms of a formal group. For the image of J case we want to look at the multiplicative formal group which is a formal group of height 1 and we want to concentrate on the prime 2. Let R be a local ring with a maximal ideal, m, such that $R/m = \mathbb{F}_2$. The Lubin-Tate theory asserts that the lifts of the multiplicative formal group to a formal group over R is co-representable and the co-representing ring is just $\mathbb{Z}_2$. The Morava stabilizer group is just the group of automorphisms of this ring. This group is the multiplicative group of $\mathbb{Z}_2$ which is $\mathbb{Z}/2 \oplus \mathbb{Z}_2$. Algebraic topologists insist on a grading and so the Lubin-Tate ring is $\mathbb{Z}_2[u, u^{-1}]$ where $|u| = 2$. This graded ring is the homotopy of a spectrum, K. The Morava stabilizer group acts on the homotopy. A theorem of Adams gives an action on the spectrum K, itself.

Theorem 1.1 $K^{h\mathbb{Z}/2} = KO$.

There is a spectral sequence converging to the homotopy of $K^{h\mathbb{Z}/2}$ whose E_2 term is $H^*(\mathbb{Z}/2; K_*)$. The non-zero element of $\mathbb{Z}/2$ acts on u by sending it to $-u$. The result is the well known homotopy of KO. It is important to note that the class which corresponds to the generator ξ of $H^*(\mathbb{Z}/2, \mathbb{Z})$ has Adams filtration (2,0) and so is in the minus two stem. The class $u^2\xi = \eta^2$. We should note that in this setting, the connected cover of the spectrum is not so easy to describe.

2 Formal groups of height 2

Now let's look at the case of a formal group of height 2 over $\mathbb{F}_4$. The Lubin-Tate theory is similar. Let R be a local ring so that $R/m = \mathbb{F}_4$. The ring which co-represents the deformation space of lifts is now $\mathbb{W}_{\mathbb{F}_4}[[u_1]][u, u^{-1}]$. Again, this is the graded answer where $|u_1| = 0$ and $|u| = 2$. The Morava stabilizer group is the group of automorphisms of the formal group over $\mathbb{F}_4$. This is a much more complicated non-commutative group that the previous case.

Let's call the group S_2. It is possible to give a description of the group in a more conventional fashion. We begin by considering the integral lattice in $\mathbb{H}$ generated by $\pm 1, \pm i, \pm j, \pm k$. We can extend the lattice by including the 16 elements $(1/2)(\pm 1 \pm i \pm j \pm k)$. In the resulting ring we complete at the ideal generated by 2. The Morava stabilizer group, S_2, is the group of units in this ring. There is a finite group of order 24, G_{24}, the group of units before completing. There is a class S which represents the square root of 2, The torsion free part is a free module over $W_{\mathbb{F}_4}$ on 1 and S.

There is a spectrum, E_2, whose homotopy is the co-representing ring, $\mathbb{W}_{\mathbb{F}_4}[[u_1]][u, u^{-1}]$. The group S_2 acts on this ring. A fundamental result of Hopkins and Miller is the following theorem.

Theorem 2.1 *S_2 acts on the spectrum E_2 as a group of E_∞ operations. $E_2^{hG_{24}} = EO_2$.*

The homotopy of EO_2 is quite interesting. Our story here is less specific. The question which we ask is: Is there a finite resolution of EO_2-modules which calculates $L_{K(2)}(S^0)$. The E_2 term of the Adams-Novikov spectral sequence is $H^*(S_2; E_{2*})$. The steps in doing this would seem to be the following.

Theorem 2.2 *(Ravenel)*

$$\begin{aligned} H^*(S_2; \mathbb{Z}/2[u^{\pm 1}]) &= (K(2)_*[h_{1,0}, h_{1,1}, g]/(h_{1,0}h_{1,1}, v_2 h_{1,0}^3 - h_{1,1}^3) \otimes \Lambda(\beta) \\ &\quad \oplus K(2)_*[\zeta]\langle \xi, \xi^2 \rangle) \otimes \Lambda(\rho) \end{aligned}$$

The filtration of $|g| = (4, 0)$, $|\zeta| = (1, 0)$, $|\xi| = (1, 0)$.

This next theorem gives the connection between G_{24} and this calculation.

Theorem 2.3 $H*(G_{24}; \mathbb{Z}/2[u^{\pm 1}]) = K(2)_*[h_{1,0}, h_{1,1}, g]/(h_{1,0}h_{1,1}v_2 h_{1,0}^3 - h_{1,1}^3)$

This suggests that the cohomology of S_2 is given by $\Lambda(\rho)$ (chain complex of length 4).

$$\Lambda(\rho)(EO_{2*}(V(1)) \to \mathbb{Z}/2[\zeta, v_2^{\pm 1}] \to \xi\mathbb{Z}/2[\zeta, v_2^{\pm 1}] \to \beta EO_{2*}(V(1)))$$

We have two questions about this. First, what spectrum has $\mathbb{Z}/2[\zeta, v_2^{\pm 1}]$ as its homotopy? Second, where do the maps come from?

If we consider the $\psi^3 - 1$ map of the image of J spectrum case as the right source for the maps we could look at the following. Consider the class $i + j + k = \sqrt{-3}$. This conjugates G_{24} to another finite group of order 24, G'_{24}. The intersection of these two groups is a cyclic group of order 6. We can think of this in terms of the following diagram.

$$\begin{tikzcd} E_2^{hG_{24}} \arrow[r, "1"] \arrow[d, "{[x]}"'] & E_2^{hG24} \arrow[d] \\ E_2^{hG'_{24}} \arrow[r] & E_2^{h\mathbb{Z}/6} \end{tikzcd}$$

We would expect that this diagram does not commute and the difference should define a map $1 - [x] : E_2^{hG_{24}} \to E_2^{h\mathbb{Z}/6}$. This is reminiscent to what happened in the K-theory situation. This suggests that $E_2^{h\mathbb{Z}/6}$ should be interesting. We note the following: The $\mathbb{Z}/3$ leaves $u^3 = v_2$ as an invariant. The $\mathbb{Z}/2$ acts in the usual fashion and the result is $H^*(\mathbb{Z}/2; E_{2*}(V(1))) = K(2)_*[\zeta]$ which looks right.

I do not know if the map above $1 - [x]$ is the correct map. Let us go back to the image of J picture again to get some hints.

The image of J spectrum is the fiber of the map $\psi^3 - 1 : KO \to KO$. When restricted to connected covers, this map can be lifted to a map $g : bo \to \Sigma^4 bsp$. Here bo is the connected spectrum constructed from Bott peridocity and bsp is the connected spectrum constructed from Bott peridocity using BSP, the symplectic group. The homotopy groups are:

$$\pi_j(bo) = \begin{cases} \mathbb{Z} & j \equiv 0(4),\ j \geq 0 \\ \mathbb{Z}/2 & j \equiv 1,2(8),\ j > 0 \\ 0 & \text{otherwise} \end{cases} \quad \pi_j(bsp) = \begin{cases} \mathbb{Z} & j \equiv 0(4),\ j \geq 0 \\ \mathbb{Z}/2 & j \equiv 5,6(8),\ j > 0 \\ 0 & \text{otherwise} \end{cases}$$

There is another way to construct a map like g. Consider a bo-resolution.

$$bo \rightrightarrows bo \wedge bo \cdots$$

Theorem 2.4 *$bo \wedge bo = bo \vee \Sigma^4 bsp \vee X$ where X is some other spectrum.*

This allows one to define a map $f : bo \to \Sigma^4 bsp$.

Theorem 2.5 *Any map such as f which induces an isomorphism in cohomology in dimension 4 induces the same map in homotopy as does g above.*

This gives a formally alternative understanding of the J-spectrum and a version of its connective cover.

Corollary 2.6 $J = L_{K(1)} S^0 = v_1^{-1}(fib(bo \to \Sigma^4 bsp))$.

This suggests that we should look at a connected version of EO_2 which we will call eo_2. Hopkins and I have constructed it and Hopkins and others have produced versions with good properties. We need only a minimal version. Here is a table of conparsions between eo_2 and bo.

	bo	eo_2
H^*	$A \otimes_{A(1)} \mathbb{Z}/2$	$A \otimes_{A(2)} \mathbb{Z}/2$
π_*	easy	computed but complicated
periodicity	v_1^4	v_2^{32}
Hurwitz image	$\mathbb{Z}/2 \in \pi_{8k+1,2}$	very large

3 Conjectural part

Now we come to the conjectural part of the talk. I will try to make clear what is a theorem and what is conjectural.

We can form an eo_2-resolution.

$$eo_2 \rightrightarrows eo_2 \wedge eo_2 \cdots$$

We do not have a splitting theorem for $eo_2 \wedge eo_2$. We do have an algebraic version.

Theorem 3.1 *As an A-module, $H^*(eo_2 \wedge eo_2) = \oplus_{i \geq 0} A \otimes_{A(2)} M_i$ where $M_i = H^*(\Sigma^{8i} bo_i)$ and bo_i is the ith bo-Brown-Gitler spectrum. ($H^*(bo_i) = A/A\{Sq^1, Sq^2, \chi Sq^{4j}, j > i\}$)*

We know that $eo_2 \wedge eo_2$ does not split as $\vee \Sigma^{8i} eo_2 \wedge bo_i$. This follows from some particular calculations in $\pi_*(S^0)$ in the 38 and 39 stems.

Conjecture 3.2 *$eo_2 \wedge eo_2 = eo_2 \vee eo_2 \wedge (\Sigma^8 bo_1 \cup \Sigma^{16} bo_2) \vee Y$ where Y is some unspecified spectrum. There is some map connecting the second and third parts.*

There is an interesting map between $\Sigma^{15} bo_2 \to \Sigma^8 bo_1$ which induces the known differential in the Adams spectral sequence. Let D be the cofiber. Then $eo_2 \wedge D$ has an extra homotopy class in dimension 32. If we cone this off with a single copy of eo_2 we get a spectrum C whose Adams spectral sequence can be completly computed. If we invert v_2 in C we get a spectrum whose Adams-Novikov spectral sequence looks very much like a candidate for $E_2^{h\mathbb{Z}/6}$. In particular, they have the same E_1-term.

If we suppose that this all will work, then we need to compute the map between eo_2 and C. Shimomura has calculated something which we can use as describing this map at least if we concentrate on just the Moore space. If we use this picture, we can read off all of the Adams-Novikov differentials from the known eo_2 structrue and the map defining C.

Next we should come back to the short chain complex discussed earlier. Let F be the fiber of the map $v_2^{-1} eo_2 \to v_2^{-1} C$. Let CF be Brown-Commentez dual of the above map. Then $L_{K(2)}(V(0)$ should fit into an exact sequence

$$CF \to L_{K(2)}(V(0) \to F.$$

This is almost right. We have to take into account the two copies of this construction given by $\Lambda(\rho)$.

This understanding gives a conceptual approach to understanding Shimomura's computations and also suggests what the differentials should be in the Adams-Novikov spectral sequence.

The short complex gives a complex like:

$$eo_2 \to C \to C \to eo_2$$

If we invert v_1 we have

$$KO[v_2^4/v_1^{12}] \to (v_2/v_1^3)KO[v_2/v_1^3] \to (v_2/v_1^3)KO[v_2/v_1^3] \to KO[v_2^4/v_1^{12}]$$

It is useful to describe these maps by what happens to various powers of v_2. These results are a consequence of Shimomura's calculations. In each case we assume $n > 1$.

$$\begin{array}{ccccccc}
2^n(2t+1) & \to & 2^{n-1}(4t+1) & & & & \\
& & 2^{n-2}(4t+3) & \to & 2^{n-1}(2t+1) & & \\
& & 2^n(2t+1)+1 & \to & 2^{n-1}(4t+1)+1 & & \\
& & & & 2^{n-1}(4t+3)+1 & \to & 2^n(2t+1)
\end{array}$$

This leaves 4 free copies. The first two represent the image of J and the second two is a kind of dual image of J. This pattern is typical of $n = 2$ phenomena.