Slides for talks on eo_2-resolutions

Mark Mahowald
Mathematics Department
Northwestern University
Evanston IL 60208

December 28, 1998

1 Introduction

My goals in this talk are:

- To discuss the Morava stablizer group, S_2 at the prime 2.
- To understand the calculation of $H^*(S_2)$ with trivial coefficients and for $E_2(V(0))$ as coeficients. This is related to Shimomura's calculations.
- We want to understand both of these items in terms of the Hopkins-Miller spectrum EO_2.
- We begin by discussing the case of S_1 which is quite well understood.

First let's look at S_1 at 2.

The image of J spectrum is given by the fiber of a map

$$\psi^3 - 1 : KO \to KO$$

The formal group approach

- We begin with the multiplicative formal group over $\mathbb{F}_2$ $F(x, y) = x + y + xy$
- The group of automorphisms is $\mathbb{Z}_2^+ = \mathbb{Z}/2 \oplus \mathbb{Z}_2$
- The Lubin-Tate ring is $\mathbb{Z}_2$. As a graded ring it is $\mathbb{Z}_2[u^{\pm 1}]$
- The Lubin-Tate ring is the homotopy of a spectrum K
- S_1 acts on the Lubin-Tate ring **and** on the spectrum K
- $K^{h\mathbb{Z}/2} = KO$; ψ^3 is a topological generator of $\mathbb{Z}_2$
- $J = K^{hS_1} = (K^{h\mathbb{Z}/2})^{h\mathbb{Z}_2}$.

Now let's consider the case $n = 2$

The formal group is not so familiar. Over $\mathbb{F}_4$ one can take the group with 2 series $[2] = x^4$.

Another choice which is interesting is the formal completion of the group structure in the elliptic curve over $\mathbb{F}_4$ defined by $x^3 = y^2 + y$ In this case, the 2 series is $[2] = z^4(\Sigma_{i \geq 0} z^{12(2^i - 1)}$.

The Lubin-Tate ring is $\mathbb{W}_{\mathbb{F}_4}[[u_1]]$

The graded Lubin-Tate ring is the homotopy of a spectrum E_2

S_2 is the automorphism group of the formal group over $\mathbb{F}_4$

S_2 is the group of units in the ring defined by the quaterions. It has a finite subgroup of order 24 given by $\pm 1, \pm i, \pm j, \pm k$ and $(1/2)(\pm 1 \pm i \pm j \pm k)$

Theorem 1.1 (Hopkins and Miller) *S_2 acts on the spectrum E_2 as a group of E_∞ operations. $E_2^{hG_{24}} = EO_2$.*

The homotopy of EO_2 is quite interesting. Our story here is less specific.

The question which we ask is: Is there a finite resolution of EO_2-modules which calculates $L_{K(2)}(S^0)$.

The E_2 term of the Adams-Novikov spectral sequence is $H^*(S_2; E_{2*})$.

The first step in calculating this is the following

Theorem 1.2 *(Ravenel)*

$$H^*(S_2; \mathbb{Z}/2[u^{\pm 1}]) =$$
$$(K(2)_*[h_{1,0}, h_{1,1}, g]/(h_{1,0}h_{1,1}, v_2 h_{1,0}^3 - h_{1,1}^3) \otimes \Lambda(\beta)$$
$$\oplus K(2)_*[\zeta]\langle \xi, \xi^2 \rangle) \otimes \Lambda(\rho)$$

The filtration of $|g| = (4,0)$, $|\zeta| = (1,0)$, $|\xi| = (1,0)$.

This next theorem gives the connection between G_{24} and this calculation.

Theorem 1.3

$$H * (G_{24}; \mathbb{Z}/2[u^{\pm 1}]) =$$
$$K(2)_*[h_{1,0}, h_{1,1}, g]/(h_{1,0}h_{1,1}v_2 h_{1,0}^3 - h_{1,1}^3)$$

This suggests that the cohomology of S_2 is given by $\Lambda(\rho)$(chain complex of length 4).

$$\Lambda(\rho)(EO_{2*}(V(1)) \to \mathbb{Z}/2[\zeta, v_2^{\pm 1}] \to \\ \xi\mathbb{Z}/2[\zeta, v_2^{\pm 1}] \to \beta EO_{2*}(V(1)))$$

We have two questions about this.

- What spectrum has $\mathbb{Z}/2[\zeta, v_2^{\pm 1}]$ as its homotopy?
- Where do the maps come from?

Following the image of J case we consider the diagram where $x = \sqrt{-3}$ and G'_{24} is another finite group obtained by conjugation with x.

$$\begin{array}{ccc} E_2^{hG_{24}} & \xrightarrow{1} & E_2^{hG24} \\ {\scriptstyle [x]}\downarrow & & \downarrow \\ E_2^{hG'_{24}} & \longrightarrow & E_2^{h\mathbb{Z}/6} \end{array}$$

This suggests that $[x] - 1 : E_2^{hG_{24}} \to E_2^{h\mathbb{Z}/6}$ should be an interesting map.

Can we understand the correct map $E_2^{hG_{24}} \to E_2^{h\mathbb{Z}/6}$ which occurs in the resolution from some other point of view?

The map

$$\psi^3 - 1 : KO \to KO$$

when restricted to connected covers can be lifted to a map

$$g : bo \to \Sigma^4 bsp.$$

Consider a bo-resolution.

$$bo \rightrightarrows bo \wedge bo \cdots$$

Theorem 1.4 $bo \wedge bo = bo \vee \Sigma^4 bsp \vee X$ *where* X *is some other spectrum.*

This allows one to define a map $f : bo \to \Sigma^4 bsp$.

Theorem 1.5 *Any map such as* f *which induces an isomorphism in cohomology in dimension 4 induces the same map in homotopy as does* g *above.*

Corollary 1.6 $J = L_{K(1)} S^0 = v_1^{-1}(fib(bo \to \Sigma^4 bsp))$.

This suggests that we should look at a connected version of EO_2 which we will call eo_2.

We can form an eo_2-resolution.

$$eo_2 \rightrightarrows eo_2 \wedge eo_2 \cdots$$

We do not have a splitting theorem for $eo_2 \wedge eo_2$. We do have an algebraic version.

Theorem 1.7 *As an A-module,* $H^*(eo_2 \wedge eo_2) = \oplus_{i\geq 0} A \otimes_{A(2)} M_i$ *where* $M_i = H^*(\Sigma^{8i} bo_i)$ *and* bo_i *is the ith bo-Brown-Gitler spectrum.* ($H^*(bo_i) = A/A\{Sq^1, Sq^2, \chi Sq^{4j}, j > i\}$)

Conjecture 1.8 $eo_2 \wedge eo_2 = eo_2 \vee eo_2 \wedge (\Sigma^8 bo_1 \cup \Sigma^{16} bo_2) \vee Y$ *where* Y *is some unspecified spectrum. There is some map connecting the second and third parts.*

$$D \to \Sigma^{15} bo_2 \to \Sigma^8 bo_1$$

$$\Sigma^{32} eo_2 \to D \to C$$

Conjecture 1.9 $v_2^{-1}C = E_2^{h\mathbb{Z}/6}$ *and the map in the resolution above is obtained from the* eo_2 *resolution.*

- Next we should come back to the short chain complex discussed earlier.
- Let F be the fiber of the map $L_2 eo_2 \to L_2 C$.
- Let CF be Brown-Commentez dual of the above map.
- Then $L_{K(2)}(V(0)$ should fit into an exact sequence

$$CF \to L_{K(2)}(V(0) \to F.$$

- This is not quite the end since there is the exterior generator ρ

Shimomura's calculations of the E_2 term for $L_{K(2)}(V(0))$ fit this resolution. The resolution conjectures all of the differentials in the Adams-Novikov spectral sequence.