MAT1300
MANDATORY ASSIGNMENT I
DUE BY MARCH 13TH 2009

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The 15 parts I(a-f), II(a-e) and III(a-d) of this assignment all carry equal weight. To pass, at least 40% of the answers need to be correct. In each case you may use the conclusions of previous questions, even if you have not answered them. Parts I(d) and III(b) may be a little harder than the other questions.

Most parts ask for a proof that a certain claim is correct. The emphasis is on providing clear and valid arguments for these claims. No credit is given for only stating the given hypothesis and then asserting the desired conclusion, without indication of proof. In problems III(c-d), references may be given to the textbook, if appropriate.

Students may work together to solve these problems, but your written answers should reflect your own understanding. If it is unclear whether you understand the answers you hand in, you may be asked to give an oral presentation of your work.

Problem I

It is well known that $x+y=y+x$ and $xy=yx$. In this problem we investigate when it is the case that $x^y=y^x$. This happens when $x=y$, but also, for instance, when $x=2$ and $y=4$.

Let $f\colon (0,\infty)\to\mathbb{R}$ be the function

$$f(x)=\frac{\ln(x)}{x}\,,$$

where ln denotes the natural (base e) logarithm.

(a) For positive real numbers x and y, show that

(1) $x^y<y^x$ if and only if $f(x)<f(y)$,
(2) $x^y=y^x$ if and only if $f(x)=f(y)$, and
(3) $x^y>y^x$ if and only if $f(x)>f(y)$.

[Hint: Use that $\ln(x^y)=y\ln(x)$, and that ln is a strictly increasing function.]

(b) Compute $f'(x)$, show that $f'(x)>0$ for $x\in(0,e)$, $f'(e)=0$, and $f'(x)<0$ for $x\in(e,\infty)$. State the mean value theorem (= middelverdisetningen). Carefully explain how to deduce that f is strictly increasing on the subset $(0,e]\subset(0,\infty)$, and strictly decreasing on the subset $[e,\infty)\subset(0,\infty)$.

(c) We know that $e<\pi<\sqrt{10}$. Decide which of the numbers $\pi^{\sqrt{10}}$ and $(\sqrt{10})^\pi$ is the greater, without actually computing the numbers.

(d) Compute $f(e)$ and $\lim_{x\to\infty} f(x)$. State the intermediate value theorem (= skjæringssetningen). Carefully explain how to deduce that f maps (e,∞) onto $(0,1/e)$, so that $f((e,\infty)) = (0,1/e)$.

(e) Compute $f(1)$, and show that for $x \in (0,1]$ the equation $x^y = y^x$ (in the unknown $y \in (0,\infty)$) only has the solution $y = x$. Also show that when $x = e$, the equation only has the solution $y = e$.

(f) Show that for $x \in (1,e)$ the equation $x^y = y^x$ (in the unknown $y \in (0,\infty)$) has exactly two solutions: one being $y = x$ and the other with $y \in (e,\infty)$.

Remark. This shows that there is a well-defined function $g\colon (1,e) \to \mathbb{R}$ that takes each $x \in (1,e)$ to the unique $y = g(x)$ in (e,∞) with $x^y = y^x$. By symmetry, the function extends to a function $g\colon (1,\infty) \to \mathbb{R}$ such that for each $x \in (1,\infty)$ we have $x^y = y^x$ for $y = g(x)$, and $y \neq x$ except for $x = e$. You are not asked to prove the claims in this remark.

Problem II

In this problem we study a method for computing square roots. For simplicity, we specialize to the case of computing $\sqrt{10}$.

Define a sequence $(x_n)_{n=1}^\infty$ of positive real numbers by setting $x_1 = 3$ and letting

$$x_{n+1} = \frac{1}{2}\big(x_n + \frac{10}{x_n}\big)$$

for all natural numbers n.

(a) Express x_2 and x_3 as rational numbers (integer fractions), and then compute their decimal expansions to six decimal places.

(b) Let $y_n = x_n - \sqrt{10}$, so that $x_n = \sqrt{10} + y_n$ for all $n \geq 1$. Show that

$$x_{n+1} = \sqrt{10} + \frac{1}{2} \cdot \frac{y_n^2}{\sqrt{10} + y_n},$$

so that

$$y_{n+1} = \frac{1}{2} \cdot \frac{y_n^2}{\sqrt{10} + y_n}$$

for all $n \geq 1$. [Hint: You may use that $10 - y_n^2 = (\sqrt{10} + y_n)(\sqrt{10} - y_n)$.] Explain why it follows that $y_n \geq 0$ for all $n \geq 2$, so that $x_n \geq \sqrt{10}$ for all $n \geq 2$.

(c) Show that $10/x_n \leq x_n$ for all $n \geq 2$, and deduce that $(x_n)_n$ is a decreasing sequence for $n \geq 2$. State the fundamental axiom of analysis (= analysens fundamentalaksiom). Carefully explain why $(x_n)_{n=1}^\infty$ is a convergent sequence.

(d) Let $r = \lim_{n\to\infty} x_n$ be the limit of this sequence. Explain why

$$r = \frac{1}{2}(r + \frac{10}{r})$$

and deduce that $r = \sqrt{10}$. You may use that the formula $h(x) = \frac{1}{2}(x + (10/x))$ defines a continuous function $h\colon (0,\infty) \to \mathbb{R}$, but you should explain how this is used.

(e) For a fixed n, suppose that $0 \leq y_n < 10^{-d}$ for some natural number d, so that x_n approximates $\sqrt{10}$ to d decimal places. Show that $0 \leq y_{n+1} < 10^{-2d}$, so that x_{n+1} approximates $\sqrt{10}$ to $2d$ decimal places.

Remark. This shows that the precision of the estimate $x_n \approx \sqrt{10}$ doubles after each iteration, when measured by the number of correct decimal places. After ten iterations the number of correct decimal places is multiplied by over one thousand. You are not asked to prove the claims in this remark.

Problem III

Let $(\mathbf{z}_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^m$, converging to a limit $\mathbf{z}$ as $k \to \infty$, and let

$$E = \{\mathbf{z}_k \mid k \geq 1\} \cup \{\mathbf{z}\}$$

be a subset of $\mathbb{R}^m$.

(a) Show that E is a bounded subset of $\mathbb{R}^m$. [Hint: See the proof of Theorem 4.68.]

(b) Show that E is a closed subset of $\mathbb{R}^m$. [Hint: First consider sequences $(\mathbf{x}_n)_n$ in E that take the same value $\mathbf{y} \in E$ infinitely often. Then consider other sequences.]

(c) Let $\mathbf{f}\colon E \to \mathbb{R}^p$ be a continuous function. Is it true that $\mathbf{f}(E)$ is closed and bounded in $\mathbb{R}^p$? If yes, give a proof or a suitable reference. If no, give a counterexample.

(d) Let $\mathbf{f}\colon E \to \mathbb{R}^p$ be a continuous function. Is it true that $\mathbf{f}$ is a uniformly continuous function? If yes, give a proof or a suitable reference. If no, give a counterexample.