

Welcome to the course MAT3500/MAT4500 TOPOLOGY.

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TOPOS τόπος means place/location. LOGOS λόγος means study/reason. Topology provides a language for proving results about shapes.

Early examples in the work of Euler:

Bridges of Königsberg (1736). Seek continuous path traversing each bridge exactly once. Distances immaterial. Reduces to a counting problem in graph theory, which is easier to settle.

Euler characteristic (1758). The surface of a polyhedron consists of V vertices, E edges and F faces. The alternating sum $\chi = V - E + F$ is invariant under subdivision of the surface, and is in fact a topological invariant. Compare cube, cube with a truncated vertex, octahedron, and a toric solid. Suggests spheres/surfaces of convex bodies and tori are topologically distinct.

Topology emphasizes continuity, which ensures preservation of limits. Recall continuity in the context of metric spaces, before generalizing to topological spaces.

Metric spaces $(X, d) \longrightarrow$ Topological spaces $(X, \mathcal{T})$.

Definition: A *metric* d on a set X is a (real-valued) function

$$\begin{aligned} d: X \times X &\longrightarrow \mathbb{R} \\ (x, y) &\longrightarrow d(x, y) \end{aligned}$$

satisfying

(a) $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.

[We could have written $d: X \times X \rightarrow [0, \infty) = \mathbb{R}_+$.]

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$ [symmetry].

(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ [triangle inequality].

Draw a triangle $\triangle xyz$. We call the pair (X, d) a *metric space*. Elements of X are called *points*.

Example: $X = \mathbb{R}^n = \{\vec{x} = (x_1, \dots, x_n) \mid x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n\}$ with the *Euclidean distance*

$$d(\vec{x}, \vec{y}) = \sqrt{(y_1 - x_1)^2 + \dots + (y_n - x_n)^2} = \left(\sum_{i=1}^n (y_i - x_i)^2 \right)^{1/2}.$$

Example: For $a < b$ and $p \geq 1$ real numbers, $X = C([a, b]) = \{f: [a, b] \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$ with

$$d_p(f, g) = \left(\int_a^b |g(t) - f(t)|^p dt \right)^{1/p}$$

the L^p -distance, equal to

$$= \|g - f\|_p$$

given by the L^p -norm. The metric space (X, d_p) comes from the *normed vector space* $(X, \|\cdot\|_p)$. In the case $p = 2$, the L^2 -norm

$$\|h\|_2 = \langle h, h \rangle^{1/2}$$

comes from the inner product

$$\langle f, g \rangle = \int_a^b f(t)g(t) dt$$

making $(X, \langle -, - \rangle)$ an *inner product space*.

Inner product spaces $\longrightarrow$ normed vector spaces $\longrightarrow$ metric spaces.

Consider two metric spaces (X, d_X) and (Y, d_Y) . A function $f: X \rightarrow Y$ is an *isometry* if

$$d_X(x, y) = d_Y(f(x), f(y))$$

for all $x, y \in X$.

Example: The isometries $f: \mathbb{R} \rightarrow \mathbb{R}$ are the translations

$$f(x) = x + b$$

and the reflections

$$f(x) = -x + b.$$

These are *isometric isomorphisms*, meaning that $f: X \rightarrow Y$ is bijective, hence admits an inverse function $g: Y \rightarrow X$, such that $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$. We write $g \circ f = 1_X$ and $f \circ g = 1_Y$. Note that g is automatically an isometry.

If there exists an isometric isomorphism $f: (X, d_X) \rightarrow (Y, d_Y)$ then (X, d_X) and (Y, d_Y) are metrically equivalent. Any property about the points of X and their distances can be translated to a logically equivalent statement about the points of Y and their distances, by sending each $x \in X$ to its image $f(x) \in Y$.

The study of metric spaces up to metric equivalence may (somewhat superfluously) be called *metric geometry*. Trigonometry is a simple example. Euler's studies do not depend on the full metric structure.

GEOMETRY means earth measurement.

Topology focuses on a less rigid class of functions, namely the continuous functions, briefly called *maps*.

Definition: Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $f: X \rightarrow Y$ be a function. Fix a point $x \in X$. We say that f is *continuous at x* if for any positive real number ϵ there is a positive real number δ such that for each $y \in X$ with $d_X(x, y) < \delta$ it is true that $d_Y(f(x), f(y)) < \epsilon$. "In order to surely be mapped within ϵ of $f(x)$ in Y , it is enough to start within δ of x in X ."

If $f: X \rightarrow Y$ is continuous at x , for each point $x \in X$, we say that f is *continuous*, or that f is a *map*.

Here are some sample results about continuous functions $f: [a, b] \rightarrow \mathbb{R}$.

Theorem: f achieves its minimum c and its maximum d [extreme value theorem].

This ensures that optimization problems have a solution.

Theorem: f achieves all values between c and d [intermediate value theorem].

Corollary: If $f: [0, 1] \rightarrow [0, 1]$ is continuous, then there exists a point $x \in [0, 1]$ with $f(x) = x$, i.e., a *fixed point*.

Proof: Consider $g: [0, 1] \rightarrow \mathbb{R}$ given by $g(x) = f(x) - x$. Then $g(0) \geq 0$ and $g(1) \leq 0$, so $g(x) = 0$ for some $x \in [0, 1]$. QED

We will extend these kinds of results to all metric spaces (X, d) , and beyond, to topological spaces $(X, \mathcal{T})$.

The ϵ - δ definition of continuity can be reformulated in terms of the notion of *open subsets*. We first review this for metric spaces, and then turn this into a definition of topological spaces.

Definition: Let (X, d) be a metric space, $x \in X$ a point and $\epsilon > 0$ a positive real number. Let

$$B_d(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

be the (open) ϵ -ball around x in X . We simply write $B(x, \epsilon)$ if d is implicit.

A subset U of X is *open* if for each point $x \in U$ there is an $\epsilon > 0$ such that $B(x, \epsilon) \subset U$.

Example: An open interval (a, b) in $\mathbb{R}$ is open, but a half-open interval $(a, b]$ or $[a, b)$, or a closed interval $[a, b]$, is not. Note: To be ‘not open’ is not the same as being ‘closed’.

Lemma: An ϵ -ball $U = B(x, \epsilon)$ is open.

Proof: For each $y \in B(x, \epsilon)$ we must find a $\delta > 0$ such that $B(y, \delta) \subset B(x, \epsilon)$. Let $\delta = \epsilon - d(x, y)$. Then for each $z \in B(y, \delta)$ we have $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \delta = d(x, y) + \epsilon - d(x, y) = \epsilon$. QED

Proposition: Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f: X \rightarrow Y$ be a function. Then f is continuous if and only if for each open subset $V \subset Y$ the preimage

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$$

is open in X .

Moral: Continuity of $f: (X, d_X) \rightarrow (Y, d_Y)$ only depends on which subsets of X and Y are open, not on the precise metrics d_X and d_Y .

Basic properties of open subsets in (X, d) :

Proposition: Let (X, d) be a metric space.

(a) The empty and full subsets $U = \emptyset$ and $U = X$ are open.

(b) If $U_1, \dots, U_n$ are open subsets in X , then the finite intersection

$$U_1 \cap \dots \cap U_n = \bigcap_{i=1}^n U_i$$

is also open in X .

(c) If $U_\alpha, U_\beta, \dots$ is any collection of open subsets in X , where $\alpha, \beta, \dots \in J$ lie in any indexing set, then the union

$$U_\alpha \cup U_\beta \cup \dots = \bigcup_{\alpha \in J} U_\alpha$$

is also open in X .

Proof: ((ETC))

§12 Topological Spaces: In *point set topology* we interpret shape to mean a set X , whose elements we call the points of X , together with enough information to determine which functions mapping to or from X are continuous. This means that we must know exactly which subsets $U \subset X$ are open. We do this by *axiomatically declaring* some or all of the subsets to be open. The remaining subsets are then not open. To obtain a reasonable theory, we insist that the specification of which subsets of X are open must satisfy the three conditions of the proposition above.

Definition: A *topological space* is a set X together with a specification of which subsets $U \subset X$ are said to be open. We assume that

(a) The subsets $\emptyset$ and X are open.

(b) The intersection

$$U_1 \cap \dots \cap U_n$$

of any finite collection $\{U_1, \dots, U_n\}$ of open subsets of X is open.

(c) The union

$$\bigcup_{\alpha \in J} U_\alpha$$

of any (finite or infinite) collection $\{U_\alpha\}_{\alpha \in J}$ of open subsets of X is open.

Remark: The ordering of (b) and (c) is often switched. We think finite intersections are easier than arbitrary unions. By induction on n it suffices to check (b) for $n = 2$. The cases $n = 1$ and $J = \{\alpha\}$ (a single element) are tautological. The case $J = \emptyset$ of (c) amounts to the case $U = \emptyset$ of (a), while the case $n = 0$ of (b) can be interpreted as the case $U = X$ of (a).]

Example: A metric space (X, d) determines a topological space, with the same set X of points, where we declare a subset $U \subset X$ to be open in the topological sense if and only if it is open in the metric sense. We call this the topological space *underlying* the given metric space.

Consider a topological space X , defined as above. We need notation for the specification of which subsets U of X are open, and which are not. We do this by letting

$$\mathcal{T} = \{U \mid U \text{ is an open subset of } X\}$$

denote the collection of all open subsets. This is a set whose elements are the sets U that are open subsets of X , i.e., $\mathcal{T}$ is a set of sets. To avoid confusion, we will refer to a set of sets as a collection of sets, so that a collection is a set whose elements are themselves sets. Knowledge of the collection $\mathcal{T}$ then determines which subsets of X are open, since $U \subset X$ is open if and only if $U \in \mathcal{T}$. We reformulate the definition as follows:

Definition: A *topological space* $(X, \mathcal{T})$ is a set X and a collection $\mathcal{T}$ of subsets of X , such that

- (a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
- (b) If $U_1, \dots, U_n \in \mathcal{T}$ then $U_1 \cap \dots \cap U_n \in \mathcal{T}$.
- (c) If $U_\alpha \in \mathcal{T}$ for each $\alpha \in J$, then $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}$.

We call the elements $x \in X$ the *points* of X , and we call the elements $U \in \mathcal{T}$ the *open subsets* of X . We call the collection $\mathcal{T}$ the *topology* on X .

Example: Let X be any set. In the *trivial topology* on X only $\emptyset$ and X are open subsets, so that

$$\mathcal{T}_{triv} = \{\emptyset, X\}.$$

Example: Let X be any set. In the *discrete topology* on X every subset $U \subset X$ is open, so that

$$\mathcal{T}_{disc} = \{U \mid U \subset X\} = \mathcal{P}(X)$$

equals the *power set* on X .

Example: The *empty space* $X = \emptyset$ admits only one topology, which is both trivial and discrete:

$$\mathcal{T}_{triv} = \mathcal{T}_{disc} = \{\emptyset\}.$$

Example: Each *one point space* $X = \{a\}$ also admits only one topology, which is both trivial and discrete:

$$\mathcal{T}_{triv} = \mathcal{T}_{disc} = \{\emptyset, X\}.$$

Example: A *two point space* $X = \{a, b\}$ admits four different topologies.

- (1) The trivial topology $\mathcal{T}_{triv} = \{\emptyset, X\}$.
- (2) A Sierpinski topology $\mathcal{T}_a = \{\emptyset, \{a\}, X\}$ where $\{a\}$ is open but $\{b\}$ is not.

(3) Another Sierpinski topology $\mathcal{T}_b = \{\emptyset, \{b\}, X\}$ where $\{b\}$ is open but $\{a\}$ is not.

(4) The discrete topology $\mathcal{T}_{disc} = \{\emptyset, \{a\}, \{b\}, X\}$.

Example: A *three point space* $X = \{a, b, c\}$ admits 29 different topologies. We omit to list them.

Definition: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f: X \rightarrow Y$ be a function. Then f is *continuous* if for each open subset V in Y the preimage

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$$

is open in X .

Equivalently, f is *continuous* if for each $V \in \mathcal{T}_Y$ the preimage $f^{-1}(V) \in \mathcal{T}_X$.

[In terms of power sets, continuity asks that the function $V \mapsto f^{-1}(V)$ takes $\mathcal{T}_Y \subset \mathcal{P}(Y)$ to $\mathcal{T}_X \subset \mathcal{P}(X)$.]

If (X, d_X) and (Y, d_Y) are metric spaces, with underlying topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, then a function $f: X \rightarrow Y$ is continuous in the metric sense if and only if it is continuous in the topological sense. Hence continuity of $f: X \rightarrow Y$ only depends on the topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, not on the specific metrics d_X and d_Y .

For metric spaces, we can talk about isometries (for metric geometry) or continuous maps (for topology), while for topological spaces we can (only) talk about continuous maps. However, for these, the topological spaces are more closely adapted to the essence of that notion.

Different metric spaces can have the same underlying topological space. Since the point sets are the same, this means that two metric spaces (X, d_1) and (X, d_2) , with different metrics d_1 and d_2 , can have the same notion of open subset, and hence determine the same topology $\mathcal{T}$ and underlying topological space $(X, \mathcal{T})$.

Consider the *unit circle*

$$X = S^1 = \{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$$

in the Euclidean plane. We can give S^1 the *subspace metric* d_1 , where

$$d_1(\vec{x}, \vec{y}) = \|\vec{y} - \vec{x}\|_2$$

for $\vec{x}, \vec{y} \in S^1$ equals the distance between these points in $\mathbb{R}^2$, i.e., the length of the chord connecting them.

We can also give S^1 the *Riemannian metric* d_2 , where

$$d_2(\vec{x}, \vec{y}) = \inf \|\gamma\|$$

is the length of the shortest path γ in S^1 connecting $\vec{x}$ to $\vec{y}$, i.e., the length of the arc connecting the two points. If $\gamma: [a, b] \rightarrow S^1 \subset \mathbb{R}^2$ is a continuously differentiable path, its length is defined as

$$\|\gamma\| = \int_a^b \|\gamma'(t)\|_2 dt,$$

where $\gamma'(t) \in \mathbb{R}^2$ gives the velocity and $\|\gamma'(t)\|_2$ gives the speed of the curve at time t .

These metrics differ, but give rise to the same open subsets. This follows from

$$\frac{d_1(\vec{x}, \vec{y})}{2} = \sin \frac{d_2(\vec{x}, \vec{y})}{2}.$$

Fortunately, even if we people may disagree on what is the correct measure of distance between two points on S^1 , we will agree on what it means for functions $f: S^1 \rightarrow Y$ or $g: W \rightarrow S^1$ to be continuous, for either one of these two metrics.

In the analogous situation for the unit sphere $S^2 \subset \mathbb{R}^3$, modeling the Earth's surface, moving one Earth radius from the North pole with respect to the subspace metric from $\mathbb{R}^3$ brings you to 30 degrees north of the equator, near Cairo, while moving one Earth radius with respect to the Riemannian metric only brings you to 32.7 degrees north, near Nazareth.

The rule (Metric spaces $\longrightarrow$ Topological spaces) is thus not one-to-one (= injective), and loses some information, but this information is irrelevant for the purpose of understanding continuity.

The passage to topological spaces also permits us to generalize the theory and consider new examples, because not every topological space comes from a metric space.

Example: In any metric space (X, d) , distinct points $x \neq y$ can be separated by disjoint open subsets $U = B(x, \epsilon)$ and $V = B(y, \epsilon)$, where we set $\epsilon = d(x, y)/2$. This is called the *Hausdorff property*. However, not every topological space has the Hausdorff property. For example, this happens for the trivial topology and the two Sierpinski topologies on $X = \{a, b\}$. Hence these topological spaces cannot come from a metric space, and are not *metrizable*.

A less obscure topology that does not come from a metric is the *topology $\mathcal{T}_p$ of pointwise convergence* on $X = C([a, b])$. Each topology determines a notion of convergence, and for $\mathcal{T}_p$ a sequence of functions

$$f_1, f_2, \dots, f_n, \dots$$

converges in $(X, \mathcal{T}_p)$ to a function g if and only if the sequence of real numbers

$$f_1(t), f_2(t), \dots, f_n(t), \dots$$

converges in $\mathbb{R}$ to $g(t)$, for each $t \in [a, b]$. This is a sensible topology on X , and it has the Hausdorff property, but it does nonetheless not come from a metric d on $X = C([a, b])$.

The rule (Metric spaces $\longrightarrow$ Topological spaces) is thus also not onto (= surjective). To view pointwise convergence as an instance of a general notion of convergence, we must go beyond metric spaces.

We will study the Urysohn/Tychonoff *metrization theorem*, which gives sufficient (but not necessary) conditions for a topological space to come from a metric space.

We shall define what it means for a topological space $(X, \mathcal{T}_X)$ to be *compact*. A subset $A \subset \mathbb{R}$, with the *subspace topology* is compact if and only if it is closed and bounded, in the metric sense. In particular, a closed interval $[a, b]$ is compact. Moreover, the continuous image of a compact space is compact. Hence, if $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then its image $A = f([a, b])$ is a closed and bounded subset of $\mathbb{R}$. In particular, $c = \inf A$ and $d = \sup A$ are both elements of A , which implies that f achieves its minimum and its maximum. The general notion of compactness, and the theorem that the continuous image of a compact space is compact, vastly generalize the extreme value theorem.

We shall also define what it means for a topological space $(X, \mathcal{T}_X)$ to be *connected*. A subset $A \subset \mathbb{R}$ is connected if and only if it is an interval. In particular, a closed interval $[a, b]$ is connected. Moreover, the continuous image of a connected space is connected. Hence, if $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then its image $A = f([a, b]) \subset [c, d]$ is an interval in $\mathbb{R}$, hence must be equal to $[c, d]$. In particular, each intermediate value between c and d is also achieved by f . The general notion of connectedness, and the theorem that the continuous image of a connected space is connected, vastly generalize the intermediate value theorem.

The union $[0, 1] \cup [2, 3]$ is compact but not connected. The interval $[1, \infty)$ is connected but not compact.

Theorem: If $f: [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1]$ is any continuous map, then there exists a fixed point $\vec{x} = (x_1, x_2) \in [0, 1] \times [0, 1]$ with $f(\vec{x}) = \vec{x}$.

We prove this in Section 55, using the *fundamental group*. More generally, Brouwer proved:

Theorem: If $f: [0, 1]^n \rightarrow [0, 1]^n$ is any continuous map, then there exists a fixed point $\vec{x} \in [0, 1]^n$ with $f(\vec{x}) = \vec{x}$.

The proof for $n \geq 3$ uses homology, discussed in MAT4530 – Algebraic Topology I.

Two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are *topologically equivalent*, if there exists mutually inverse continuous functions $f: X \rightarrow Y$ and $g: Y \rightarrow X$. We then call f (or g) a *homeomorphism*, and say that X and Y are *homeomorphic*. We often write $X \cong Y$ or

$$f: X \xrightarrow{\cong} Y$$

to indicate that f is a homeomorphism.

HOMEOMEO means similar.

Example: If $X = (0, 1]$ and $Y = [1, \infty)$, the functions $f: X \rightarrow Y$ with $f(x) = 1/x$ and $g: Y \rightarrow X$ with $g(y) = 1/y$ are mutually inverse and both continuous. Hence $f: (0, 1] \rightarrow [1, \infty)$ is a homeomorphism.

In general, a homeomorphism f and its inverse g specify a one-to-one correspondence (= bijection) between the points of X and the points of Y , sending $x \in X$ to $f(x) \in Y$. Under this correspondence, a subset $U \subset X$ is open if and only if

$$f(U) = \{f(x) \in Y \mid x \in U\}$$

is open in Y . Note that this is equal to the set

$$g^{-1}(U) = \{y \in Y \mid g(y) \in U\}.$$

Hence any statement about $(X, \mathcal{T}_X)$ that only refers to the points and open subsets of X is logically equivalent to the corresponding statement about $(Y, \mathcal{T}_Y)$, under the substitution $x \mapsto f(x)$, $U \mapsto f(U) = g^{-1}(U)$. Properties given by such statements are said to be *topological properties*.

For example, for a space to be discrete, Hausdorff, metrizable, compact or connected, are all examples of topological properties.

The surface of a doughnut, the surface of a mug with one handle, the space of possible placements for the hour hand and the minute hand of an analog clock, the space of configurations for a suitable linkage, the *product space* $S^1 \times S^1$, a quotient space of the square $I \times I$, a quotient space of a hexagon, and the complex points on an elliptic curve, are all examples of topologically equivalent spaces. These are all Hausdorff, metrizable, compact and connected, but are not discrete.

Metric properties. Completeness and completion. Metrics on the rational numbers. The reals and the p -adic numbers. Product relation. The functional equation for the Riemann ζ -function (in Landau ξ -form).

Sequential spaces, compactly generated (weak) Hausdorff spaces.

Grothendieck topoi, quasi-topological spaces.

Local vs. global. Manifold geometry. Topological, PL, differentiable, (complex) analytic, algebraic categories. Singularities.

Projective spaces. Zariski topology.

§13 Bases for Topologies: Explicitly specifying a topology $\mathcal{T}$ can be cumbersome.

Example: For $X = \mathbb{R}$ with the metric topology, the open subsets $U \subset X$ are (finite or) countable unions of disjoint open intervals:

$$U = \coprod_i (a_i, b_i).$$

For $X = \mathbb{R}^2$ with the metric topology, an explicit enumeration of the open subsets (without repetitions) is difficult to spell out.

In the case of metric spaces (X, d) , the ϵ -balls $B(x, \epsilon)$ help us present open sets.

Lemma: Each open $U \subset X$ is a union of ϵ -balls (for varying $x \in X$ and $\epsilon > 0$), and each such union is open.

Proof: Let U be (metrically) open. For each $x \in U$ choose $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset U$. Then

$$U = \bigcup_{x \in U} B(x, \epsilon_x).$$

QED

Hence the metrically open subsets of X are precisely the unions of (arbitrary collections of) ϵ -balls. The key feature of ϵ -balls that ensures that the metrically open sets define a topology is the following.

Lemma: Let $B_1 = B(x_1, \epsilon_1)$ and $B_2 = B(x_2, \epsilon_2)$ be any two ϵ -balls. For each $x \in B_1 \cap B_2$ there is an $\epsilon_x > 0$ such that

$$B(x, \epsilon_x) \subset B_1 \cap B_2.$$

Equivalently, the intersection $B_1 \cap B_2$ is a union of ϵ -balls.

Proof: Let

$$\epsilon_x = \min\{\epsilon_1 - d(x_1, x), \epsilon_2 - d(x_2, x)\}.$$

Then $y \in B(x, \epsilon_x)$ satisfies $d(x_1, y) \leq d(x_1, x) + d(x, y) < d(x_1, x) + \epsilon_1 - d(x_1, x) = \epsilon_1$, so $y \in B_1$. Likewise $y \in B_2$. Hence

$$B_1 \cap B_2 = \bigcup_{x \in B_1 \cap B_2} B(x, \epsilon_x).$$

QED

We carry these ideas over to general topological spaces. Let X be any set.

Definition: A collection $\mathcal{B}$ of subsets of X is called a *basis* if

- (a) for each $x \in X$ there is a *basis element* $B \in \mathcal{B}$ with $x \in B$, and
- (b) for any two basis elements $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \cap B_2$ there is a $B_x \in \mathcal{B}$ with

$$x \in B_x \subset B_1 \cap B_2.$$

Lemma: Condition (b) is equivalent to asking that the intersection of any two basis elements is a union

$$B_1 \cap B_2 = \bigcup_{\alpha \in J} B_\alpha$$

of basis elements.

Proof: For each $x \in B_1 \cap B_2$ choose B_x as in (b). Then

$$B_1 \cap B_2 = \bigcup_{x \in B_1 \cap B_2} B_x.$$

QED

Definition: A basis $\mathcal{B}$ *generates* a topology $\mathcal{T}$, with open subsets given by arbitrary unions

$$U = \bigcup_{\alpha \in J} B_\alpha$$

of basis elements $B_\alpha \in \mathcal{B}$.

Lemma 13.1': A subset $U \subset X$ is open if and only if for each $x \in U$ there exists a basis element $B \in \mathcal{B}$ with

$$x \in B \subset U.$$

Proof: If

$$U = \bigcup_{\alpha \in J} B_\alpha$$

is open, and $x \in U$, then there exists an α with

$$x \in B_\alpha \subset U$$

and $B_\alpha \in \mathcal{B}$. Conversely, if we for each $x \in U$ can choose a $B_x \in \mathcal{B}$ with $x \in B_x \subset U$ then

$$U = \bigcup_{x \in U} B_x$$

is a union of basis elements, hence is open. QED

We must justify the term “topology”.

Lemma: The collection $\mathcal{T}$ is a topology on X .

Proof: (a) The empty subset $U = \emptyset$ is the union over $J = \emptyset$. The full subset $U = X$ is the union over $J = X$, where $B_x \in \mathcal{B}$ is any basis element containing x .

(b) If $U = \bigcup_{\alpha \in J} B_\alpha$ and $V = \bigcup_{\beta \in K} B_\beta$ are open, with each $B_\alpha, B_\beta \in \mathcal{B}$, then

$$U \cap V = \bigcup_{\alpha \in J, \beta \in K} B_\alpha \cap B_\beta$$

is a union of intersections $B_\alpha \cap B_\beta$, each of which is a union of basis elements. Hence $U \cap V$ is a union of basis elements, and is open.

(c) Any union of unions of basis elements is a union of basis elements, hence is open. QED

In this situation, we say that $\mathcal{B}$ is a *basis for the topology* $\mathcal{T}$. This is similar, but not identical, to the notion of a basis for a vector space. In a vector space, each vector can be uniquely written as a finite linear combination of basis vectors. In a topological space, each open subset can be written as a union of basis elements. However, the presentation is not assumed to be unique, and the union may well be over an infinite indexing set. In both situations, the choice of a basis is hardly ever unique, so different bases can generate the same topology.

Example: In a metric space (X, d) , the ϵ -balls

$$\mathcal{B} = \{B(x, \epsilon) \mid x \in X, \epsilon > 0\}$$

form a basis for the metric topology $\mathcal{T}$.

Example: The open intervals

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

form a basis for the metric topology on $\mathbb{R}$.

Example: Circular disc $\mathcal{B}_{circ}$ and rectangular $\mathcal{B}_{rect}$ bases for topologies on $\mathbb{R}^2$.

$$(a, b) \times (c, d) \cap (a', b') \times (c', d') = (a', b) \times (c', d)$$

if $a \leq a' < b \leq b', c \leq c' < d \leq d'$.

Example: The single set X forms a basis for the trivial topology on X .

Example: The singleton sets $\{x\}$ for $x \in X$ form a basis for the discrete topology on X .

How can we recognize that a collection $\mathcal{C}$ is a basis for a given topology $\mathcal{T}$?

Lemma 13.2: Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{C}$ be a collection of open subsets in X . Suppose that for each open $U \subset X$ and each point $x \in U$ there is a $C \in \mathcal{C}$ with $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology $\mathcal{T}$.

Proof: (1) Check that $\mathcal{C}$ is a basis. (2) Check that the topology $\mathcal{T}'$ generated by $\mathcal{C}$ is equal to $\mathcal{T}$. ((ETC))

Coarser and finer topologies:

When X has two or more elements, there exist several different topologies on X . Some of these are *comparable* in the following sense. If $\mathcal{T}'$ and $\mathcal{T}$ are topologies on the same set X , we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$ if $\mathcal{T}' \supset \mathcal{T}$, i.e., if each open subset in $(X, \mathcal{T})$ is also open in $(X, \mathcal{T}')$. We then also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$.

Example: The discrete topology is finer than any other topology. The trivial topology is coarser than any other topology. The Sierpinski topology $\mathcal{T}_a$ is not comparable with the Sierpinski topology $\mathcal{T}_b$.

Clearly $\mathcal{B} \subset \mathcal{T}$.

Lemma: The topology $\mathcal{T}$ generated by a basis $\mathcal{B}$ is the coarsest topology that contains $\mathcal{B}$.

Lemma 13.3: Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively, on X . The following are equivalent:

- (1) The topology $\mathcal{T}'$ is finer than the topology $\mathcal{T}$.
- (2) For each point $x \in X$ and basis element $B \in \mathcal{B}$ with $x \in B$ there is a basis element $B' \in \mathcal{B}'$ with

$$x \in B' \subset B.$$

Proof: ((ETC))

Example: The circular disc basis $\mathcal{B} = \mathcal{B}_{circ}$ and the rectangular basis $\mathcal{B}' = \mathcal{B}_{rect}$ define the same topology, $\mathcal{T} = \mathcal{T}'$, on $\mathbb{R}^2$.

Definition: For any set X , the identity function $id = id_X: X \rightarrow X$ is given by $id(x) = x$ for each $x \in X$.

Lemma: Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X . The identity function $id_X: (X, \mathcal{T}_1) \rightarrow (X, \mathcal{T}_2)$ is continuous if and only if $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

Proof: Continuity asks that for each $V \in \mathcal{T}_2$ the preimage

$$id_X^{-1}(V) = V$$

lies in $\mathcal{T}_1$, i.e., that $\mathcal{T}_2 \subset \mathcal{T}_1$.

Roughly speaking, continuous functions map from finer to coarser topologies. Functions going the other way, from coarser to finer topologies, are usually discontinuous.

Subbases:

We obtain a topology from a basis by imposing closure under arbitrary unions. We can obtain a basis from a smaller collection, called a subbasis, by imposing closure under finite intersections.

Definition: A *subbasis* $\mathcal{S}$ is any collection of subsets of X . The basis $\mathcal{B}$ generated by $\mathcal{S}$ is the collection of finite intersections

$$B = S_1 \cap \cdots \cap S_n$$

of subbasis elements $S_1, \dots, S_n \in \mathcal{S}$. The topology $\mathcal{T}$ generated by the subbasis $\mathcal{S}$ is the topology generated by this basis. It consists of arbitrary unions of finite intersections of subbasis elements.

Lemma: The collection $\mathcal{B}$ is a basis.

Proof: (a) The empty intersection of subset of X (with $n = 0$) is X , so $X \in \mathcal{B}$.

(b) The intersection of two basis elements $B_1 = S_1 \cap \cdots \cap S_n$ and $B_2 = S_{n+1} \cap \cdots \cap S_{n+m}$ is the basis element

$$B_1 \cap B_2 = S_1 \cap \cdots \cap S_{n+m}.$$

QED

Munkres assumes that for each $x \in X$ there is some $S \in \mathcal{S}$ with $x \in S$. This appears to be superfluous. Clearly $\mathcal{S} \subset \mathcal{B} \subset \mathcal{T}$.

Lemma: The topology $\mathcal{T}$ generated by $\mathcal{S}$ is the coarsest topology that contains $\mathcal{S}$.

We will see examples of subbases when we discuss product topologies in Section 15. The following example appears in Section 46.

Example: A subbasis for the topology of pointwise convergence $\mathcal{T}_p$ on $X = C([a, b])$ is the collection of subsets

$$S_{t,(c,d)} = \{f: [a, b] \rightarrow \mathbb{R} \mid f(t) \in (c, d)\},$$

where $t \in [a, b]$ and $(c, d) \subset \mathbb{R}$.

Continuity in terms of bases and subbases in the target:

Lemma: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $\mathcal{B}$ be a basis for the topology on Y . A function $f: X \rightarrow Y$ is continuous if and only if for each basis element $B \in \mathcal{B}$ the preimage $f^{-1}(B)$ is open in X .

Proof: Each open subset in Y is a union

$$V = \bigcup_{\alpha \in J} B_\alpha$$

of basis elements $B_\alpha \in \mathcal{B}$, and

$$f^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha),$$

so $f^{-1}(V)$ is open if each $f^{-1}(B_\alpha)$ is open in X .

Example: If (Y, d) is a metric space, a function $f: X \rightarrow Y$ is continuous if and only if the preimage

$$f^{-1}(B_d(y, \epsilon))$$

of each ϵ -ball in (Y, d) is open in X .

Lemma: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $\mathcal{S}$ be a subbasis for the topology on Y . A function $f: X \rightarrow Y$ is continuous if and only if for each subbasis element $S \in \mathcal{S}$ the preimage $f^{-1}(S)$ is open in X .

Proof: Let $\mathcal{B}$ be the basis generated by $\mathcal{S}$. Each basis element is a finite intersection

$$B = S_1 \cap \cdots \cap S_n$$

of subbasis elements $S_i \in \mathcal{S}$, and

$$f^{-1}(B) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n),$$

so $f^{-1}(B)$ is open if each $f^{-1}(S_i)$ is open in X . Now use the previous lemma.

§15 The Product Topology on $X \times Y$: If X and Y are sets, the Cartesian product

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

is the set of ordered pairs (x, y) , with x in X and y in Y .

Example: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the Cartesian plane.

Let Z be any set. To give a function

$$f: Z \longrightarrow X \times Y$$

is equivalent to give two functions $g: Z \rightarrow X$ and $h: Z \rightarrow Y$, related by

$$f(z) = (g(z), h(z))$$

for each $z \in Z$. If X , Y and Z are topological spaces, we will now give the set $X \times Y$ a topology, so that the function f is continuous if and only if the functions g and h are continuous. To give a map

$$f: Z \longrightarrow X \times Y$$

will thus be equivalent to give two maps $g: Z \rightarrow X$ and $h: Z \rightarrow Y$, related by

$$f(z) = (g(z), h(z))$$

for each $z \in Z$. This so-called *universal property* characterizes the product topology on $X \times Y$, and could be taken as a definition, once one knows that such a topology exists.

Definition: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. The *product basis* on $X \times Y$ is the collection

$$\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$$

of subsets of $X \times Y$. The *product topology* $\mathcal{T}$ on $X \times Y$ is the topology generated by $\mathcal{B}$.

((Draw picture of $U \times V \subset X \times Y$.)

We check that $\mathcal{B}$ is a basis (for a topology) on $X \times Y$. (a) Since $X \in \mathcal{T}_X$ and $Y \in \mathcal{T}_Y$ we have $X \times Y \in \mathcal{B}$, so each $(x, y) \in X \times Y$ lies in a basis element. (b) Let $B_1 = U_1 \times V_1$ and $B_2 = U_2 \times V_2$ be basis elements. Then

$$B_1 \cap B_2 = (U_1 \cap U_2) \times (V_1 \cap V_2)$$

is also a basis element, since $U_1 \cap U_2 \in \mathcal{T}_X$ and $V_1 \cap V_2 \in \mathcal{T}_Y$.

The union of two product basis elements

$$(U_1 \times V_1) \cup (U_2 \times V_2)$$

is usually not a basis element. The product basis $\mathcal{B}$ is therefore usually not a topology. The topology it generates instead consists of arbitrary unions

$$W = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$$

of basis elements.

Proposition (= Theorem 15.1): Let $\mathcal{B}$ be a basis for a topology on X , and let $\mathcal{C}$ be a basis for a topology on Y . Then the collection

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for the product topology on $X \times Y$.

Proof: Apply the recognition Lemma 13.2. For each open $W \subset X \times Y$ and point $(x, y) \in W$ we need to find a $D \in \mathcal{D}$ with $(x, y) \in D \subset W$. Since W is open, there is a product basis element $U \times V$ with $(x, y) \in U \times V \subset W$, where U is open in X and V is open in Y . Then $x \in U$ and $y \in V$. Since U is open, there is a basis element $B \in \mathcal{B}$ with $x \in B \subset U$. Likewise there is a basis element $C \in \mathcal{C}$ with $y \in C \subset V$. Then $D = B \times C$ is as required. QED

Example: The open rectangles

$$(a, b) \times (c, d)$$

form a basis for the product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. This equals the “rectangular” basis. We have seen that this defines the same topology as the “circular” or “metric” basis with elements

$$B((x, y), \epsilon)$$

where $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$. Hence the product topology and the metric topology on $\mathbb{R}^2$ are the same.

Example: $S^1 \times S^1$ with the product topology is the *torus* surface. (This is not homeomorphic to the unit sphere S^2 .)

The product basis comes from a smaller subbasis. This point of view will be useful when we consider products of an arbitrary collection of topological spaces.

Definition: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. The *product subbasis* on $X \times Y$ is the collection

$$\mathcal{S} = \{U \times Y \mid U \in \mathcal{T}_X\} \cup \{X \times V \mid V \in \mathcal{T}_Y\}$$

of subsets of $X \times Y$.

Lemma: The product subbasis $\mathcal{S}$ generates the product basis $\mathcal{B}$, hence also the product topology $\mathcal{T}$, on $X \times Y$.

Proof: Finite intersections of subbasis elements are all of the form

$$(U \times Y) \cap (X \times V) = U \cap V$$

with $U \subset X$ and $V \subset Y$ open. Hence $\mathcal{S}$ generates the product basis $\mathcal{B}$. QED

Let

$$\pi_1: X \times Y \longrightarrow X$$

$$(x, y) \longmapsto x$$

be the *projection to the first coordinate*, and let

$$\pi_2: X \times Y \longrightarrow Y$$

$$(x, y) \longmapsto y$$

be the *projection to the second coordinate*.

Lemma: Give $X \times Y$ the product topology. Then the projections

$$\pi_1: X \times Y \longrightarrow X$$

and

$$\pi_2: X \times Y \longrightarrow Y$$

are continuous.

Proof: For each open $U \subset X$ the preimage

$$\pi_1^{-1}(U) = U \times Y$$

lies in the product (sub-)basis, hence is open in the product topology. Likewise, for each open $V \subset Y$ the preimage

$$\pi_2^{-1}(V) = X \times V$$

lies in the product (sub-)basis, hence is open in the product topology. QED

We can now verify the universal property of the product topology.

Proposition: Let Z , X and Y be topological spaces, and give $X \times Y$ the product topology. A function

$$f: Z \longrightarrow X \times Y$$

is continuous if and only if its component functions

$$g = \pi_1 f: Z \longrightarrow X$$

and

$$h = \pi_2 f: Z \longrightarrow Y$$

are both continuous.

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow \pi_1 f & \downarrow f & \searrow \pi_2 f & \\ X & \xleftarrow{\pi_1} & X \times Y & \xrightarrow{\pi_2} & Y \end{array}$$

Proof: We can check continuity of f using the product subbasis

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \in \mathcal{T}_X\} \cup \{\pi_2^{-1}(V) \mid V \in \mathcal{T}_Y\}.$$

Hence f is a map if and only if

$$f^{-1}(\pi_1^{-1}(U)) = g^{-1}(U)$$

is open in Z for each open $U \subset X$, and

$$f^{-1}(\pi_2^{-1}(V)) = h^{-1}(V)$$

is open in Z for each open $V \subset Y$. This is equivalent to g and h being continuous. QED

§16 The Subspace Topology on $Y \subset X$: Let Y be a subset of X . Let Z be any set. To give a function

$$f: Z \longrightarrow Y$$

is equivalent to giving a function $g: Z \rightarrow X$ with $g(z) \in Y$ for each $z \in Z$. If X and Z are topological spaces, we will now give the subset Y a topology such that the function f is continuous if and only if g is continuous.

Definition: Let $(X, \mathcal{T}_X)$ be a topological space, and let $Y \subset X$ be a subset. The *subspace topology* on Y is the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}_X\}.$$

With this topology, we say that Y is a *subspace* of X .

Lemma: The subspace topology is a topology.

Proof: $(Y \cap U_1) \cap \cdots \cap (Y \cap U_n) = Y \cap (U_1 \cap \cdots \cap U_n)$ and

$$\bigcup_{\alpha \in J} (Y \cap U_\alpha) = Y \cap \bigcup_{\alpha \in J} U_\alpha.$$

QED

Definition: Let $i: Y \rightarrow X$ denote the *inclusion map*, with $i(y) = y$ for all $y \in Y$.

Lemma: The inclusion map is a map.

Proof: $i^{-1}(U) = Y \cap U$ for each $U \subset X$. QED

Lemma: $f: Z \rightarrow Y$ is continuous if and only if $g = if: Z \rightarrow X$ is continuous.

Proof: $f^{-1}(Y \cap U) = (if)^{-1}(U)$ for all $U \subset X$. QED

Example: Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, with the subspace topology. The product space

$$S^2 \times \mathbb{R}^3$$

consists of pairs (p, v) , with $p \in S^2$ and $v \in \mathbb{R}^3$. The tangent plane $T_p S^2$ to S^2 at p consists of the vectors orthogonal to p , i.e., the vectors v satisfying $p \cdot v = 0$. Let the *tangent bundle*

$$TS^2 = \{(p, v) \in S^2 \times \mathbb{R}^3 \mid p \cdot v = 0\} = \coprod_{p \in S^2} T_p S^2$$

be the disjoint union of all of the tangent planes, topologized as a subspace of $S^2 \times \mathbb{R}^3$. The *bundle projection*

$$\pi: TS^2 \longrightarrow S^2$$

is the map given by $\pi(p, v) = p$. The point preimages

$$\pi^{-1}(p) = T_p S^2$$

are the tangent planes. A *continuous vector field* on S^2 is a map $f: S^2 \rightarrow TS^2$ such that $\pi f = id_{S^2}$.

$$\begin{array}{ccc} & & TS^2 \\ & \nearrow f & \downarrow \pi \\ S^2 & \xrightarrow{id} & S^2 \end{array}$$

To each point $p \in S^2$ the vector field associates a tangent vector $v \in T_p S^2$, with $f(p) = (p, v)$. The subspace

$$S^2 \times S^2 \subset S^2 \times \mathbb{R}^3$$

consists of pairs (p, u) with $p \in S^2$ a point and $u \in \mathbb{R}^3$ a unit vector. Let the *unit tangent bundle* be

$$US^2 = \{(p, u) \in S^2 \times S^2 \mid p \cdot u = 0\},$$

topologized as a subspace of $S^2 \times S^2$. The bundle projection

$$\pi: US^2 \longrightarrow S^2$$

is the map given by $\pi(p, u) = p$. The point preimages are the unit circles in the tangent planes. A *continuous unit vector field* on S^2 is a map $f: S^2 \rightarrow US^2$ such that $\pi f = id_{S^2}$.

$$\begin{array}{ccc} & & US^2 \\ & \nearrow f & \downarrow \pi \\ S^2 & \xrightarrow{id} & S^2 \end{array}$$

In fact, no such unit vector field exists: each continuous vector field f on the sphere has at least one zero, i.e., at least one point $p \in S^2$ with $f(p) = (p, 0)$.

Example: Recall that $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ denotes Euclidean n -space. Its metric and product topologies agree. Let the space of $n \times n$ real matrices $M_n(\mathbb{R}) = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ be the product space given by ordered n -tuples of (column) vectors in $\mathbb{R}^n$:

$$A = (\vec{v}_1, \dots, \vec{v}_n) = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}$$

A matrix is *orthogonal* if $A^t A = I_n$, so that

$$\vec{v}_i \cdot \vec{v}_j = \delta_{i,j} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Equivalently, A is invertible with $A^{-1} = A^t$. Let

$$O(n) = \{A \in M_n(\mathbb{R}) \mid A \text{ is orthogonal}\}$$

be the subset of orthogonal matrices. With the subspace topology, this is a subspace of $M_n(\mathbb{R})$. For example,

$$O(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mid 0 \leq \theta < 2\pi \right\} \amalg \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \mid 0 \leq \theta < 2\pi \right\}$$

is the disjoint union of two circles inside $M_2(\mathbb{R}) = \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{R}^4$, one consisting of rotations and one of reflections.

The matrix product AB of two orthogonal matrices A and B is again orthogonal, as is the inverse A^{-1} , so the matrix product makes $O(n)$ a group, called the *orthogonal group*. The product operation

$$\begin{aligned} \mu: O(n) \times O(n) &\longrightarrow O(n) \\ (A, B) &\longmapsto AB \end{aligned}$$

and the inverse operation

$$\begin{aligned} \iota: O(n) &\longrightarrow O(n) \\ A &\longmapsto A^{-1} = A^t \end{aligned}$$

are both continuous, so $O(n)$ is a *topological group*. (When we discuss manifolds, we will say more precisely that $O(n)$ is a *Lie group*.) To verify the continuity of these operations, it suffices to check the continuity of the composites

$$\begin{aligned} O(n) \times O(n) &\xrightarrow{\mu} O(n) \subset M_n(\mathbb{R}) \\ O(n) &\xrightarrow{\iota} O(n) \subset M_n(\mathbb{R}). \end{aligned}$$

This, in turn, follows from the continuity of each of the n^2 components of $M_n(\mathbb{R})$:

$$(A, B) \longmapsto (AB)_{i,j} = \sum_{k=1}^n a_{i,k} b_{k,j}$$

and

$$A \longmapsto (A^{-1})_{i,j} = A_{i,j}^t = a_{j,i}.$$

Finally, this follows from the fact that polynomials define continuous functions.

Open in a subspace:

Let Y be a subspace of X . A subset $V \subset Y$ is also a subset of X . If $\mathcal{T}_X$ and $\mathcal{T}_Y$ are the topologies on X and Y , then “ V is open in X ” means $V \in \mathcal{T}_X$, while “ V is open in Y ” means $V \in \mathcal{T}_Y$. If V is open in X , then $V = Y \cap V$ is also open in Y , but the converse needs not hold. The converse does hold if Y is open in X :

Lemma: Let Y be a subspace of X . If V is open in Y , and Y is open in X , then V is open in X .

Proof: First, $V = Y \cap U$ for some open U in X . Second, Y is open in X . Hence the intersection $Y \cap U = V$ is open in X . QED

Subspaces and bases:

Lemma: Let $Y \subset X$. If $\mathcal{B}$ is a basis for the topology on X , then

$$\mathcal{B}_Y = \{Y \cap B \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y .

Proof: We use the recognition lemma. Each $Y \cap B$ for $B \in \mathcal{B}$ is open in Y . For each $y \in Y \cap U$ with U open in X there is a $B \in \mathcal{B}$ with $y \in B \subset U$, and then $y \in Y \cap B \subset Y \cap U$. QED

Subspaces and products:

Theorem: Let A be a subspace of X and B a subspace of Y . Then the product topology on $A \times B$ is equal to the subspace topology from $X \times Y$.

Proof: Let U range over the open subsets of X and let V range over the open subsets of Y , so that $U \times V$ ranges over the product basis for $X \times Y$. Then $A \cap U$ ranges over the open subsets of A and $B \cap V$ ranges over the open subsets of B . Hence

$$(A \cap U) \times (B \cap V) = (A \times B) \cap (U \times V)$$

ranges over the product basis for $A \times B$, as well as over a basis for the subspace topology on $A \times B$. Hence these bases are equal, and generate the same topology. QED

§17 Closed Sets: Definition: A subset A in a topological space X is said to be *closed* precisely when $X - A$ is open.

Lemma: The closed subsets of X are the complements $A = X - U$ of the open subsets U in X .

Proof: $X - A = U$ if and only if $A = X - U$. QED

Example: The intervals $[a, b]$ in $\mathbb{R}$ are closed, because $(-\infty, a) \cup (b, \infty)$ is open.

Example: Let $a < b$. As subsets of $\mathbb{R}$, $\emptyset$ is open and closed, (a, b) is open but not closed, $[a, b]$ is not open but closed, and $[a, b)$ is neither open nor closed. Hence there is no logical implication between the statements “ A is open”, “ A is not open”, “ A is closed” and “ A is not closed”.

There is also, in general, no relation between the terms “a subset of a topological space is closed” and “a collection is closed under certain operations”.

Example: The first quadrant $[0, \infty) \times [0, \infty)$ in $\mathbb{R}^2$ because its complement

$$(-\infty, 0) \times \mathbb{R} \cup \mathbb{R} \times (-\infty, 0)$$

is open in $\mathbb{R}^2$.

Example: In the trivial topology on X , the closed subsets are $\emptyset$ and X .

Example: In the discrete topology on X , each subset is closed.

Example: In the subspace $Y = [0, 1] \cup (2, 3)$ of $\mathbb{R}$ the subsets $A = [0, 1] = Y \cap (-1/2, 3/2)$ and $B = (2, 3) = Y \cap (3/2, 7/2)$ are both open. Since $A = Y - B$ and $B = Y - A$, both

subsets are also closed. (In Section 23 we will say that A and B form a *separation* of Y , exhibiting Y as a *disconnected*, i.e., not *connected*, space.)

Theorem: Let X be a topological space. The collection $\mathcal{C}$ of closed subsets satisfies

(a) The subsets $\emptyset$ and X are closed.

(b) The union

$$C_1 \cup \cdots \cup C_n$$

of any finite collection of closed subsets is closed.

(c) The intersection

$$\bigcap_{\alpha \in J} C_\alpha$$

of any (finite or infinite) collection of closed subsets is closed.

Proof:

$$X - C_1 \cup \cdots \cup C_n = (X - C_1) \cap \cdots \cap (X - C_n)$$

is a finite intersection of open sets, hence is open.

$$X - \bigcap_{\alpha \in J} C_\alpha = \bigcup_{\alpha \in J} (X - C_\alpha)$$

is a union of open sets, hence is open. QED

Let X be any set. Given a collection $\mathcal{C}$ of subsets C of X satisfying (a), (b) and (c) above, the collection $\mathcal{T}$ of complements $U = X - C$ satisfies the conditions to define a topology on X , so that $\mathcal{C}$ is the collection of closed subsets. Hence we could equally well define a topological space in terms of the set X and the collection $\mathcal{C}$ of closed subsets.

Zariski topology: Consider the complex polynomials

$$p(x) = \sum_{i=0}^d a_i x^i$$

in one variable, where $d \geq 0$ varies freely. These are the elements $p(x) \in \mathbb{C}[x]$. The zero set

$$Z(p) = \{z \in \mathbb{C} \mid p(z) = 0\}$$

is a finite subset of $\mathbb{C}$, unless $p = 0$, in which case $Z(0) = \mathbb{C}$. Conversely, each finite subset

$$\{z_1, \dots, z_d\}$$

of $\mathbb{C}$ arises as the zero set of a polynomial

$$\begin{aligned} p(x) &= (x - z_1) \cdots (x - z_d) \\ &= x^d - (z_1 + \cdots + z_d)x^{d-1} + \cdots + (-1)^d z_1 \cdots z_d \end{aligned}$$

François Viète/Vieta (1540–1603). This collection $\mathcal{C}$ of subsets of $\mathbb{C}$, i.e., the finite subsets and the set $\mathcal{C}$ itself, satisfy (a), (b) and (c) of the theorem. Hence their complements

$$U(p) = \mathbb{C} - Z(p) = \{z \in \mathbb{C} \mid p(z) \neq 0\}$$

and $\emptyset = \mathbb{C} - \mathbb{C}$ form the open sets in a topology $\mathcal{T}_{Zar}$ on $\mathbb{C}$, called the *Zariski topology*. Oscar Zariski (1899–1986). This topology is coarser than the metric topology on $\mathbb{C} \cong \mathbb{R}^2$, but is finer than the discrete topology.

Cofinite topology: A similar construction works for any set X . The collection

$$\mathcal{C}_{cof} = \{F \subset X \mid F \text{ is finite}\} \cup \{X\}$$

of finite subset $F \subset X$, together with X itself, is closed under finite unions and arbitrary intersections. Hence the collection

$$\mathcal{T}_{cof} = \{X - F \mid F \text{ is finite}\} \cup \{\emptyset\}$$

of complements of finite sets, together with $\emptyset$, define a topology on X . We call this the *cofinite* or *finite complement* topology on X .

If the set X is finite then each subset is finite, so $\mathcal{C} = \mathcal{T}_{cof} = \mathcal{P}(X)$ and the cofinite topology equals the discrete topology. If X is infinite then each singleton set $\{x\}$ is closed, but not open, in X , so in this case the cofinite topology is strictly coarser than the discrete topology.

Zariski topology on $\mathbb{C}^n$:

Consider the complex polynomials

$$p(x_1, \dots, x_n) = \sum_{i_1, \dots, i_n \geq 0} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n}$$

in n variables, where only finitely many coefficients $a_{i_1, \dots, i_n}$ are nonzero. These are the elements

$$p(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n].$$

The collection $\mathcal{C}$ of zero sets

$$Z(p) = \{(z_1, \dots, z_n) \mid p(z_1, \dots, z_n) = 0\}$$

are closed under finite unions, since $Z(p) \cup Z(q) = Z(pq)$ where pq is the product of p and q . However, for $n \geq 2$ it is not generally closed under intersections. Hence the collection $\mathcal{B}$ of complements

$$\mathbb{C}^n - Z(p) = \{(z_1, \dots, z_n) \mid p(z_1, \dots, z_n) \neq 0\}$$

is not a topology. However, it is closed under finite intersections, and defines a *basis* for a Zariski topology, with closed subsets the intersections

$$\begin{aligned} \bigcap_{\alpha \in J} Z(p_\alpha) &= \{(z_1, \dots, z_n) \mid p_\alpha(z_1, \dots, z_n) = 0 \text{ for all } \alpha \in J\} \\ &= \{(z_1, \dots, z_n) \mid p(z_1, \dots, z_n) = 0 \text{ for all } p \in I\} \end{aligned}$$

where $I = (p_\alpha \mid \alpha \in J) \subset \mathbb{C}[x_1, \dots, x_n]$ is the *ideal* generated by the p_α . The Zariski open sets are the unions

$$\bigcup_{\alpha \in J} (\mathbb{C}^n - Z(p_\alpha)) = \mathbb{C}^n - \bigcap_{\alpha \in J} Z(p_\alpha).$$

In fact (David Hilbert, Emmy Noether), each Zariski open subset arises are a finite such union, since each ideal I in $\mathbb{C}[x_1, \dots, x_n]$ is finitely generated.

Closed sets in subspaces:

Lemma: Let $Y \subset X$ be a subspace. A subset A of Y is closed in the subspace topology if and only if $A = Y \cap C$ for some closed $C \subset X$.

Proof: The open subsets of Y are those of the form $Y \cap U$ for U open in X . Hence the closed subsets of Y are those of the form

$$Y - Y \cap U = Y \cap (X - U)$$

for U open in X , which are those of the form $Y \cap C$ for C closed in X . QED

Lemma: Let $Y \subset X$ be a subspace and let $A \subset Y$. If A is closed in X then A is closed in Y . The converse holds if Y is closed in X .

Proof: If A is closed in X then $A = Y \cap A$ is closed in Y . If $A = Y \cap C$ with Y and C closed in X then A is closed in X . QED

Interior, closure and boundary:

Definition: Let X be a space and $A \subset X$ any subset. The union

$$\text{Int } A = \bigcup \{U \mid U \subset A, U \text{ open in } X\}$$

of the subsets of A that are open in X is an open subset of A called the *interior* of A in X .

Lemma: $\text{Int } A$ is the largest open subset of A , in the sense that it is open in X and contained in A , and is maximal with this property, so that for each open U in X with $U \subset A$ we have $U \subset \text{Int } A$.

Note that $\text{Int } A = A$ is and only if A is open in X .

Example: The interior of $[a, b]$ in $\mathbb{R}$ is (a, b) , while the interior of $[a, b] \times \{0\}$ in $\mathbb{R}^2$ is $\emptyset$. Hence the interior depends not only on A , but also on the containing space X .

If $A \subset C$ we say that A is a subset of C , or that C is a *superset* of A .

Definition: Let X be a space and $A \subset X$ any subset. The intersection

$$\text{Cl } A = \bigcap \{C \mid A \subset C, C \text{ closed in } X\}$$

of the supersets of A that are closed in X is a closed superset of A called the *closure* of A in X .

Lemma: $\text{Cl } A$ is the smallest closed superset of A , in the sense that it is closed in X and contains A , and is minimal with this property, so that for each closed C in X with $A \subset C$ we have $\text{Cl } A \subset C$.

Note that $A = \text{Cl } A$ is and only if A is closed in X .

Lemma: $X - \text{Cl } A = \text{Int}(X - A)$ and $X - \text{Int } A = \text{Cl}(X - A)$.

Proof: $U \subset X - A$ is open if and only if $X - U \supset A$ is closed, and $U \subset A$ is open if and only if $X - U \supset X - A$ is closed. QED

Definition: For $A \subset X$, with X a topological space, the (*topological*) *boundary* of A in X is

$$\text{Bd } A = \text{Cl } A - \text{Int } A.$$

This is a closed subset of X .

Common alternative notations are $\overset{\circ}{A} = \text{Int } A$, $\bar{A} = \text{Cl } A$ and $\partial A = \text{Bd } A$. We might write $\text{Cl}_X A$ for the closure on A in X , to emphasize X .

Closure in subspaces:

The dependence of the closure of A on the containing space is clarified by the following result.

Theorem: Let $Y \subset X$ be a subspace, and $A \subset Y$ a subset. Let $\bar{A}$ denote the closure of A in X . Then the closure of A in Y equals $Y \cap \bar{A}$:

$$\text{Cl}_Y A = Y \cap \text{Cl}_X A.$$

Proof: Letting $\{C_\alpha\}_{\alpha \in J}$ range over the closed sets in X that contain A , the intersections $Y \cap C_\alpha$ ranges over the closed sets in Y that contain A . Taking intersections we find

$$\text{Cl}_Y A = \bigcap_{\alpha \in J} (Y \cap C_\alpha) = Y \cap \bigcap_{\alpha \in J} C_\alpha = Y \cap \text{Cl}_X A.$$

QED

This suggests reserving the notation $\bar{A}$ for the closure in X , since $Y \cap \bar{A}$ is then available as notation for the closure in Y .

Neighborhoods:

Open sets express nearness (or proximity) conditions, in the sense that for $x \in U \subset X$ with U open in the topological space X , the condition “for all $y \in U$ ” is one way of saying “for all $y \in X$ that are sufficiently close to x ”. For example, in a metric space (X, d) the condition “for all $y \in B(x, \epsilon)$ ” is equivalent to saying “for all $y \in X$ with $d(x, y) < \epsilon$ ”.

Definition: If $x \in U \subset X$ with U open in X , then we say that U is a *neighborhood* of x in X .

Replacing U by a smaller open subset V containing x , with $x \in V \subset U$, then expresses a more restrictive closeness condition, and “for y sufficiently close to x ” usually quantifies a property that holds for each (arbitrarily small) neighborhood of x in X .

Some authors refer to our neighborhoods as “open neighborhood”, and call any $N \subset X$ containing an open neighborhood a “neighborhood”, without requiring N itself to be open:

$$x \in U \subset N \subset X.$$

If $A \cap U$ is nonempty, so that $A \cap U \neq \emptyset$, we say that A *intersects* U , or that A *meets* U . The following theorem expresses the closure of A in terms of A meeting neighborhoods.

Theorem 17.5(a): Let A be a subset of a space X . A point $x \in X$ lies in $\bar{A} = \text{Cl } A$ if and only if for each neighborhood U of x in X the intersection $A \cap U$ is nonempty.

In other words, $x \in \bar{A}$ if and only if each neighborhood of x intersects A .

Proof: Let $K = X - A$ be the complement of A . A point $x \in X$ lies in $\text{Int } K$ if and only if there is an open $U \subset X$ with $x \in U \subset K$. Equivalently, $x \in \text{Int } K$ if and only if for some neighborhood U of x in X we have $A \cap U = \emptyset$. Then the negated statements must also be equivalent, so that $x \in X - \text{Int } K$ if and only if there exists a neighborhood U of x in X with $A \cap U \neq \emptyset$. Here $X - \text{Int } K = \text{Cl}(X - K) = \text{Cl } A$.

Theorem 17.5(b): Let $\mathcal{B}$ be a basis for the topology on a space X , and let $A \subset X$ be a subset. A point $x \in X$ lies in $\bar{A} = \text{Cl } A$ if and only if for each basis element B containing x the intersection $A \cap B$ is nonempty.

Proof: There is an open $U \subset X$ with $x \in U \subset K$ if and only if there is a basis element $B \in \mathcal{B}$ with $x \in B \subset K$. QED

Example: The closure of $A = (a, b)$ in $\mathbb{R}$ is $\bar{A} = [a, b]$.

Example: The closure of $A = \{1/n \mid n \in \mathbb{N}\}$ in $\mathbb{R}$ is $\bar{A} = \{0\} \cup A$.

Example: The closure of the set $A = \mathbb{Q}$ of rational numbers in $\mathbb{R}$ is $\bar{A} = \mathbb{R}$. (This is not the same as the *algebraic closure*, which consists of the algebraic numbers, omitting the transcendental numbers.)

Proof: For each $x \in \mathbb{R}$ and each $\epsilon > 0$ the neighborhood $B(x, \epsilon) = (x - \epsilon, x + \epsilon)$ contains (many) rational numbers.

Example: The closure of $A = (-\infty, 1) \cup (1, 2) \cup (\mathbb{Q} \cap (2, 4)) \cup \{5\}$ is $\bar{A} = (-\infty, 4] \cup \{5\}$.

Example: If A is closed and U is open, then $A - U = A \cap (X - U)$ is closed and $U - A = U \cap (X - A)$ is open.

Dense subsets:

Definition: A subset A of a space X is *dense* in X if $\bar{A} = X$.

Lemma: A is dense in X if and only if each nonempty open $\emptyset \neq U \subset X$ intersects A .

Proof: Each $x \in X$ lies in $\bar{A}$ if and only if for each $x \in X$ and each open U containing x the intersection $A \cap U$ is nonempty. This is equivalent to asking that for each nonempty open U the intersection $A \cap U$ is nonempty.

Example: $\mathbb{Q}$ is dense in $\mathbb{R}$.

(We omit the discussion of “limit points”, less confusingly known as “cluster points”.)

Convergence:

Definition: Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of (positive) natural numbers. (Munkres writes $\mathbb{Z}_+$ for this set.) A *sequence* in a set X is a function

$$\begin{aligned}\mathbb{N} &\longrightarrow X \\ n &\longmapsto x_n.\end{aligned}$$

We often describe the sequence by an infinite (ordered) list

$$x_1, x_2, x_3, \dots$$

of elements in X , or write $(x_n)_n = (x_n)_{n=1}^\infty$ for this sequence. It is also common to write $\{x_n\}_n = \{x_n\}_{n=1}^\infty$, even if this suggests an unordered list, or just the set of values of the function $n \mapsto x_n$.

Definition: Let $(X, \mathcal{T})$ be a topological space. A sequence

$$x_1, x_2, x_3, \dots$$

converges to a point $x \in X$ if for each neighborhood U of x there is an integer N such that $x_n \in U$ for all $n \geq N$.

Lemma: Let $\mathcal{B}$ be a basis for the topology on X . A sequence $(x_n)_n$ converges to x if and only if for each basis neighborhood $x \in B \in \mathcal{B}$ there is an N such that $x_n \in B$ for all $n \geq N$.

If $(x_n)_n$ converges to x , then we say that x is a *limit* of the sequence.

Example: Let (X, d) be a metric space. Then $(x_n)_n$ converges to x if and only if for each $\epsilon > 0$ there is an N such that $d(x_n, x) < \epsilon$ for each $n \geq N$. Many sequences do not converge, but a sequence in a metric space can converge to at most one point.

Example: In $X = \{a, b\}$ with the Sierpinski topology $\mathcal{T}_a = \{\emptyset, \{a\}, X\}$, the constant sequence $x_n = a$ for all $n \geq 1$ converges to a . It also converges to b , because the only neighborhood of b is $U = X$, which contains x_n for all n . A sequence in a general topological space can thus converge to multiple points.

Hausdorff spaces:

To ensure the uniqueness of limits, when they exist, Felix Hausdorff (1868–1942) identified the following property.

Definition: A topological space X is *Hausdorff* if for each pair $x \neq y$ of distinct points in X there are neighborhoods U and V of x and y , respectively, with $U \cap V = \emptyset$.

Lemma: Any metric space (X, d) is Hausdorff.

Proof: Given $x \neq y$ in X let $\epsilon = d(x, y)/2$. Then $U = B(x, \epsilon)$ and $V = B(y, \epsilon)$ are neighborhoods of x and y , respectively, with $U \cap V = \emptyset$ (by the triangle inequality). QED

Example: The Sierpinski space $(X, \mathcal{T}_a)$ is not Hausdorff, since the only neighborhood of b is $V = X$, which contains a , so no neighborhood U of a is disjoint from V .

Theorem 17.10: If X is a Hausdorff space, then a sequence $(x_n)_n$ in X can converge to at most one point of X .

Proof: Suppose that $(x_n)_n$ converges to x and to y , with $x \neq y$. Choose disjoint neighborhoods U and V of x and y . Then there are integers N and M so that $x_n \in U$ for all

$n \geq N$ and $x_n \in V$ for all $n \geq M$. For $n = \max\{M, N\}$ this asserts that $x_n \in U \cap V$, which is impossible, since $U \cap V = \emptyset$. QED

Hence, in Hausdorff spaces we can speak of *the limit* of a sequence, granting that one such exists.

Separation axioms:

Theorem 17.8: Let X be a Hausdorff space. Then each finite subset $F \subset X$ is closed.

Proof: It suffices to prove that “points are closed” in Hausdorff spaces, i.e., that each singleton set $\{y\} \subset X$ is closed. For each point $x \neq y$ there exist disjoint separating neighborhoods U_x and V_x of x and y , respectively. Then

$$X - \{y\} = \bigcup_{x \neq y} U_x$$

is the union of the open sets U_x , hence is open. Thus $\{y\}$ is closed. QED

Example: The opposite implication does not hold, since any infinite set X with the finite complement topology is not Hausdorff. If U and V are neighborhoods of x and y , then $X - U$ and $X - V$ are finite, so $(X - U) \cup (X - V) = X - (U \cap V)$ is a union of two finite sets, hence is finite. Since X is infinite, we cannot have $U \cap V = \emptyset$.

The Hausdorff property is one of a list of additional hypotheses one might make about a topological space, called *separation axioms*, or *Trennungsaxiome*. The condition (T1) that points are closed is weaker than the Hausdorff property (T2), and stronger conditions, called (T3), (T4), regularity and normality, appear in Section 31.

§18 Continuous Functions: Recall:

Definition: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f: X \rightarrow Y$ is *continuous* if and only if for each open V in Y the preimage $f^{-1}(V)$ is open in X .

Note that continuity depends on the chosen topologies on X and Y , and that it suffices to check that $f^{-1}(V)$ is open in X for V ranging through a basis, or through a subbasis, for the topology on Y .

We can reexpress continuity in terms of the closure operator $A \mapsto \bar{A}$, or in terms of preimages of closed sets.

Theorem 18.1: Let X and Y be topological spaces, and $f: X \rightarrow Y$ a function. The following are equivalent.

- (1) The function f is continuous.
- (2) For each $A \subset X$ one has $f(\bar{A}) \subset \overline{f(A)}$.
- (3) For each closed $B \subset Y$ the preimage $f^{-1}(B)$ is closed in X .

Proof: If f is continuous and $x \in \bar{A}$ we show that $f(x) \in \overline{f(A)}$. Let V be a neighborhood of $f(x)$ in Y . Then $f^{-1}(V)$ is a neighborhood of x in X , so $A \cap f^{-1}(V) \neq \emptyset$. Hence $f(A) \cap V \neq \emptyset$, so $f(x) \in \overline{f(A)}$.

If $f(\bar{A}) \subset \overline{f(A)}$ for each A , and B is closed in Y , let $A = f^{-1}(B)$. Then $f(A) \subset B$, so $\overline{f(A)} \subset B$, since B is closed. The hypothesis then implies $f(\bar{A}) \subset B$ and $\bar{A} \subset f^{-1}(B) = A$. Hence $A = \bar{A}$ is closed.

If $f^{-1}(B)$ is closed for each closed B , then for each open V let $B = Y - V$, so that $V = Y - B$ and

$$f^{-1}(V) = f^{-1}(Y - B) = X - f^{-1}(B)$$

is the complement of a closed in X , hence is open in X . QED

Definition: Let $x \in X$. We say that $f: X \rightarrow Y$ is *continuous at x* if for each neighborhood V of $f(x)$ there is a neighborhood U of x with $f(U) \subset V$.

Theorem 18.1 (cont.): Let X and Y be topological spaces, and $f: X \rightarrow Y$ a function. The following are equivalent.

(1) The function f is continuous.

(4) For each $x \in X$ the function f is continuous at x .

Proof: If f is continuous, $x \in X$ and $f(x) \in V$ is open in Y , then $x \in f^{-1}(V)$ is open in X , so $U = f^{-1}(V)$ is a neighborhood of x with $f(U) \subset V$.

Conversely, if f is continuous at each $x \in X$, and $V \subset Y$ is open, then for each $x \in f^{-1}(V)$ we can choose a neighborhood U_x of x with $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$, and

$$U = \bigcup_{x \in f^{-1}(V)} U_x$$

is a union of open subsets, hence is open. Moreover $f^{-1}(V) \subset U \subset f^{-1}(V)$, so $f^{-1}(V) = U$ is open. QED

The category of topological spaces:

Theorem 18.2: Let X, Y, Z and W be topological spaces, and let $f: X \rightarrow Y, g: Y \rightarrow Z$ and $h: Z \rightarrow W$ be continuous. Then the composite function

$$g \circ f: X \longrightarrow Z$$

is continuous. Moreover, each identity function

$$id_X: X \longrightarrow X$$

is continuous. These satisfy

$$\begin{aligned} (h \circ g) \circ f &= h \circ (g \circ f): X \longrightarrow W \\ f \circ id_X &= f: X \longrightarrow Y \\ id_Y \circ f &= f: X \longrightarrow Y. \end{aligned}$$

Proof: For each open $V \subset Z$ the preimage $g^{-1}(V)$ is open in Y , so the preimage

$$\begin{aligned} (g \circ f)^{-1}(V) &= \{x \in X \mid g(f(x)) \in V\} \\ &= \{x \in X \mid f(x) \in g^{-1}(V)\} \\ &= f^{-1}(g^{-1}(V)) \end{aligned}$$

is open in X . For each open $V \subset X$, the preimage $(id_X)^{-1}(V) = V$ is of course open in X . QED

Following Samuel Eilenberg (1913–1998) and Saunders MacLane (1909–2005), we say that the collection of topological spaces and the sets of continuous functions between them, with the composition law and identity functions given above, form a *category*.

Theorem 18.2 (cont.): The unique function $\eta: \emptyset \rightarrow Y$ is continuous. For each singleton space $\{y\}$ the unique function $\epsilon: X \rightarrow \{y\}$ is continuous.

Proof: The requisite preimages are empty, or all of X , hence are open. QED

We say that $\emptyset$ is the *initial* topological space, and each $\{y\}$ is a *terminal* topological space. More generally, each function $f: X \rightarrow Y$ from a space $(X, \mathcal{T}_{disc})$ with the discrete topology, or to a space $(Y, \mathcal{T}_{triv})$ to a space with the trivial topology, will be continuous.

Theorem 18.2 (cont.): For each subspace $A \subset X$, the inclusion $i: A \rightarrow X$ is continuous. Hence the *restriction*

$$f|A = f \circ i: A \longrightarrow Y$$

of a continuous function $f: X \rightarrow Y$ is continuous.

Proof: For each open $V \subset Y$ the preimage $i^{-1}(V) = A \cap V$ is open in the subspace topology on A . QED

The converse does not hold: If a restriction $f|A$ is continuous for some subspace A of X , it does not generally hold that f is continuous on all of X . For the dual setting of a “corestriction”, there is a logical equivalence.

Theorem 18.2 (cont.): For each subspace $B \subset Y$, with inclusion $j: B \rightarrow Y$, a function $f: X \rightarrow B$ is continuous if and only if

$$g = j \circ f: X \longrightarrow Y$$

is continuous. We call f the *corestriction* of g , but this terminology is not fully standard.

Proof:

$$f^{-1}(B \cap V) = g^{-1}(V)$$

for all (open) $V \subset Y$. QED

Example: A constant function $g: X \rightarrow Y$, with $g(X) = \{y\}$ a singleton set, is continuous. This is the case $B = \{y\}$ and $f = \epsilon: X \rightarrow \{y\}$ of the above.

It is more plausible that if $X = A \cup B$ and the restrictions $f|A: A \rightarrow Y$ and $f|B: B \rightarrow Y$ of a function $f: X \rightarrow Y$ are continuous, then f might be continuous. Still, some hypotheses are needed.

Theorem 18.2 (cont.): Let $X = \bigcup_{\alpha \in J} U_\alpha$, where each U_α is open in X . Then $f: X \rightarrow Y$ is continuous if and only if $f|U_\alpha: U_\alpha \rightarrow Y$ is continuous for each $\alpha \in J$.

Proof: Let $V \subset Y$ be open.

$$f^{-1}(V) = \bigcup_{\alpha \in J} U_\alpha \cap f^{-1}(V) = \bigcup_{\alpha \in J} (f|U_\alpha)^{-1}(V).$$

If $f|U_\alpha$ is continuous, then $(f|U_\alpha)^{-1}(V)$ is open in U_α , hence also in X since U_α is open. If this holds for each $\alpha \in J$, then the union above, which equals $f^{-1}(V)$, is also open. QED

A similar result holds if X is viewed as a finite union of closed subspaces. The case of two subspaces implies the general finite case, by induction.

Theorem 18.3 (The pasting lemma): Let $X = A \cup B$, where A and B are closed in X . Then $f: X \rightarrow Y$ is continuous if and only if $f|A: A \rightarrow Y$ and $f|B: B \rightarrow Y$ are continuous.

Proof: Let $C \subset Y$ be closed. Then

$$f^{-1}(C) = (f|A)^{-1}(C) \cup (f|B)^{-1}(C).$$

Here $(f|A)^{-1}(C)$ is closed in the subspace A , since $f|A$ is continuous, and is therefore also closed in X , since A is closed. Likewise $(f|B)^{-1}(C)$ is closed in the subspace B , since $f|B$ is continuous, and is therefore also closed in X , since B is closed. Hence $f^{-1}(C)$ is a finite union of closed subsets of X , and is therefore closed. QED

Example: If $X = [0, 1]$ is written as the infinite union of closed subsets

$$X = \{0\} \cup \bigcup_{n \geq 1} \left[\frac{1}{n+1}, \frac{1}{n} \right],$$

a function $f: [0, 1] \rightarrow \mathbb{R}$ can be continuous on $\{0\}$ (automatically) and on each $[\frac{1}{n+1}, \frac{1}{n}]$ without being continuous at $x = 0$ in $[0, 1]$. For example, $f(0) = 1$ and $f(x) = 0$ for $x \in (0, 1]$ is such a function.

Sums of spaces: In the special case when $X = A \sqcup B$ with $A \cap B = \emptyset$ is a disjoint union of two closed subspaces, or, equivalently, of two open subspaces, we say that X is the *sum* of the spaces A and B , and often write $X = A \sqcup B$. Let $\iota_1: A \rightarrow A \sqcup B$ and $\iota_2: B \rightarrow A \sqcup B$ be the two inclusion maps. Then $f: A \sqcup B \rightarrow Y$ is continuous if and only if $f|_A: A \rightarrow Y$ and $f|_B: B \rightarrow Y$ are continuous.

$$\begin{array}{ccccc} A & \xrightarrow{\iota_1} & A \sqcup B & \xleftarrow{\iota_2} & B \\ & \searrow f\iota_1 & \downarrow f & \swarrow f\iota_2 & \\ & & Y & & \end{array}$$

This universal property characterizes the sum $A \sqcup B$ of the topological spaces A and B in manner that is dual to the characterization of the product space $A \times B$ from Section 15.

Theorem 18.4: A function $f: X \rightarrow A \times B$ is continuous if and only if $\pi_1 f: X \rightarrow A$ and $\pi_2 f: X \rightarrow B$ are both continuous.

$$\begin{array}{ccccc} & & X & & \\ \pi_1 f \swarrow & & \downarrow f & & \searrow \pi_2 f \\ A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \end{array}$$

In Section 22 we will discuss the quotient topology on Y induced from a surjection $p: X \rightarrow Y$ from a space X , with properties dual to the subspace topology on Y induced from an inclusion $i: Y \rightarrow X$.

In the category of sets and functions, there is a natural bijection

$$\{\text{functions } f: X \times Y \rightarrow Z\} \xrightarrow{\cong} \{\text{functions } g: X \rightarrow Z^Y\}$$

where

$$Z^Y = \{\text{functions } h: Y \rightarrow Z\}$$

is the set of functions from Y to Z . The bijection takes the function $f: X \times Y \rightarrow Z$ of two variables to the function $g: X \rightarrow Z^Y$ of its first variable, given by mapping $x \in X$ to the function $g(x): Y \rightarrow Z$ in Z^Y given by $g(x)(y) = f(x, y) \in Z$ for $y \in Y$. If X , Y and Z are topological spaces, and $f: X \times Y \rightarrow Z$ is continuous, then the “adjoint” function $X \rightarrow Z^Y$ takes values in the subset

$$C(Y, Z) = \{\text{continuous functions } h: Y \rightarrow Z\} \subset Z^Y.$$

In Section 46 we consider the “compact-open” topology on $C(Y, Z)$, making this subset a topological space, and prove that $f: X \times Y \rightarrow Z$ is continuous if and only if $g: X \rightarrow C(Y, Z)$ is continuous, but only under a significant hypothesis on Y (that it is locally compact and Hausdorff).

Homeomorphism:

A map (= continuous function) $f: X \rightarrow Y$ is a *homeomorphism* if it has a continuous inverse $g: Y \rightarrow X$, i.e., a map with $gf = id_X$ and $fg = id_Y$.

We say that X and Y are *homeomorphic* if there exists a homeomorphism $f: X \rightarrow Y$. This defines an equivalence relation: Clearly $id_X: X \rightarrow X$ is a homeomorphism, with inverse

id_X . If $f: X \rightarrow Y$ is a homeomorphism, with inverse $g = f^{-1}$, then $g: Y \rightarrow X$ is a homeomorphism with inverse f . If $f: X \rightarrow Y$ and $f': Y \rightarrow Z$ are homeomorphisms, with inverses $g: Y \rightarrow X$ and $g': Z \rightarrow Y$, then $f'f: X \rightarrow Z$ is a homeomorphism with inverse gg' . This is the notion of *isomorphism* in the category of topological spaces and maps.

If $f: (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ is a homeomorphism, with inverse g , then the rule

$$\begin{aligned}\mathcal{T}_X &\xleftarrow{\cong} \mathcal{T}_Y \\ U &\mapsto f(U) = g^{-1}(U)\end{aligned}$$

defines a bijection between the two topologies, with inverse $V \mapsto g(V) = f^{-1}(V)$. Any *topological property* of $(X, \mathcal{T})$, i.e., one expressed in terms of the points and open sets of X is then logically equivalent to the corresponding property of $(Y, \mathcal{T}_Y)$, with x replaced by $f(x)$ and U replaced by $f(U) = g^{-1}(U)$.

Example: The function

$$\begin{aligned}f: B(0, \pi/2) &\longrightarrow \mathbb{R}^2 \\ r \cdot (\cos \theta, \sin \theta) &\longmapsto \tan r \cdot (\cos \theta, \sin \theta)\end{aligned}$$

is a homeomorphism from the open disc of radius $\pi/2$ to the full plane.

Non-example: The continuous bijection

$$\begin{aligned}f: [0, 2\pi) &\longrightarrow S^1 \\ x &\longmapsto (\cos x, \sin x)\end{aligned}$$

is not a homeomorphism, because the inverse function

$$g: S^1 \longrightarrow [0, 2\pi)$$

is discontinuous at $(1, 0)$. The preimage $f^{-1}(V)$ of the neighborhood $V = [0, \pi)$ of $g(1, 0) = 0$ is a half-open semicircle, from $(1, 0)$ through $(0, 1)$ to $(-1, 0)$, including the endpoint $(1, 0)$. This is not open in the subspace topology on S^1 , since any neighborhood of $(1, 0)$ will contain points (x, y) with $y < 0$.

Topological embeddings (= imbeddings): Given a map $e: X \rightarrow Y$ we can give the image subset $e(X)$ the subspace topology from Y . There is then a surjective corestricted map $f: X \rightarrow e(X)$. If e is injective, then f is a continuous bijection. If, moreover, f is a homeomorphism, then we say that $e: X \rightarrow Y$ is an *embedding*.

Example: An embedding $k: S^1 \rightarrow \mathbb{R}^3$ is called a *knot*.

Example: View S^1 as the unit circle in $\mathbb{R}^2 \cong \mathbb{C}$. The map

$$\begin{aligned}e: S^1 &\longrightarrow S^1 \times S^1 \\ e(z) &= (z^2, z^3)\end{aligned}$$

defines an embedding with image a *trefoil knot* in the torus surface.

Manifolds: Geometric topology often means the study of manifolds.

Definition: Let $n \geq 0$. A *topological n -manifold* is a locally Euclidean Hausdorff space, i.e., a Hausdorff space M such that each point $p \in M$ has a neighborhood U that is homeomorphic to (an open subset of) $\mathbb{R}^n$.

If $h: U \cong h(U)$ with $h(U) \subset \mathbb{R}^n$ open, then $V = B(h(p), \epsilon) \subset h(U)$ for some $\epsilon > 0$, and $U' = h^{-1}(V)$ is a neighborhood of p that is homeomorphic to $B(h(p), \epsilon)$. Since $B(h(p), \epsilon) \cong \mathbb{R}^n$, we might replace U with U' and assume that it is homeomorphic to (all of) $\mathbb{R}^n$.

In Section 36 we add the hypothesis that a manifold admits a countable basis $\mathcal{B} = \{B_1, B_2, B_3, \dots\}$ for its topology, and prove that each compact n -manifold can be embedded in a Euclidean space $\mathbb{R}^N$ for some finite N .

Manifolds often arise with more structure than a topology. In MAT4520 we study *differentiable manifolds*, where we also specify what it means for a function $f: M \rightarrow \mathbb{R}$ to be differentiable. This means that each differentiable curve $\gamma: [a, b] \rightarrow M$ has a *tangent vector* $\gamma'(t)$ at each point $\gamma(t)$, and the tangent vectors at a point $p \in M$ form a *tangent space* $T_p M$, which is an n -dimensional vector space. This lets us use linear algebra and multivariable calculus on M , where we in undergraduate courses usually had to restrict to working over an open subset of $\mathbb{R}^n$. By equipping each tangent space with an inner product, we obtain a *Riemannian manifold*, where the length of a curve γ can be measured by integrating the length of its tangent vectors.

$$\|\gamma\| = \int_a^b |\gamma'(t)| dt$$

A 1-manifold is a *curve* and a 2-manifold is a *surface*. Each topological manifold of dimension $n \leq 3$ admits an essentially unique differentiable structure, so in this range of dimensions the study of differentiable manifolds is the same as the study of topological manifolds. Each compact connected curve is homeomorphic to S^1 , and each compact connected surface can be obtained from S^2 by forming a “connected sum” with finitely many tori T^2 , or with finitely many projective planes $\mathbb{R}P^2$. These are classified by their Euler characteristic and their (non-)orientability. William Thurston (1946–2012) formulated and partially proved the *geometrization conjecture*, saying that each compact connected 3-manifold is a connected sum of “geometric pieces”, i.e., prime 3-manifolds that admit one of eight explicit geometries, including the spherical, Euclidean and hyperbolic geometries. This conjecture was proved by Grigori Perelman (1966–) in 2003, using the *Ricci flow* from Riemannian geometry. This includes a proof of the very famous *Poincaré conjecture*, formulated correctly in 1904 by Henri Poincaré (1854–1912), that each compact simply-connected 3-manifold is homeomorphic to S^3 .

In dimension 4, there is a vast difference between topological manifolds and differentiable manifolds. Not every topological 4-manifold admits a differentiable structure, and other manifolds, including $\mathbb{R}^4$, admit an uncountable number of essentially distinct (non-diffeomorphic), differentiable structures. This follows from the work of Simon Donaldson (1957–) on “instanton gauge theory” (ask Kim Frøyshov), and Michael Freedman (1951–) on “topological surgery”. Freedman proved the 4-dimensional generalized Poincaré conjecture in 1981 (and an book aiming to make the long proof accessible to current readers was recently published).

In dimensions $n \geq 5$ the difference is more moderate, in that each compact connected topological n -manifold admits a finite number of distinct differentiable structure. The first examples were found in 1956 by John Milnor (1931–), who showed that S^7 admits 14 other “exotic” differentiable structures than the usual one, for a total of 15, or of 28 if orientations are taken into account. The “surgeries” used by Milnor are far easier to implement in the differentiable category than in the topological category, and led, in the 1960s and 1970s, to a quite good classification strategy for manifolds of dimension $n \geq 5$, called “surgery theory”, mainly due to William Browder (1934–), Sergei Novikov (1938–), Dennis Sullivan (1941–) and C.T.C. Wall (1936–). A key first step was the proof in 1961 by Stephen Smale (1930–) of

the generalized Poincaré conjecture, in all dimensions $n \geq 5$. The broader theory transforms the geometric problem into algebraic problems about quadratic forms over integral group rings of the form $\mathbb{Z}[\pi]$, where π can be the fundamental group of a manifold.

The classification of differentiable structures on spheres was reduced by Milnor and Michel Kervaire (1927–2007) to questions in homotopy theory within algebraic topology (MAT4530, MAT4540, MAT4580/9580), in areas where progress is still being made (Hill–Hopkins–Ravenel, 2016). The parametrized refinement of the surgery classification of manifolds from being “up to h -cobordism” to being “up to diffeomorphism” was expressed by Friedhelm Waldhausen (with Bjørn Jahren and JR) in terms of the algebraic K -theory of so-called “brave new rings”, being homotopy theoretic analogues of group rings.

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Complex manifolds.

Algebraic varieties.

Singularity theory.
—

§19 The Product Topology: Finite products of topological spaces are defined inductively, so that

$$X_1 \times \cdots \times X_n = (X_1 \times \cdots \times X_{n-1}) \times X_n.$$

Infinite products will be defined directly.

Let J and X be sets. A function

$$\begin{aligned} x: J &\longrightarrow X \\ \alpha &\longmapsto x(\alpha) \end{aligned}$$

is an *ordered J -tuple* of elements in X . We call $x_\alpha = x(\alpha)$ the α -th *coordinate* of x . Let X^J be the set of such functions/ordered J -tuples, $x = (x_\alpha)_{\alpha \in J}$.

Definition: Let J be any set, and for each $\alpha \in J$ let X_α be a set. Let $X = \bigcup_{\alpha \in J} X_\alpha$. The *Cartesian product*

$$\prod_{\alpha \in J} X_\alpha$$

is the subset of X^J consisting of $(x_\alpha)_{\alpha \in J}$ with $x_\alpha \in X_\alpha$ for each $\alpha \in J$.

In the special case when the X_α are all equal (to X),

$$\prod_{\alpha \in J} X = X^J.$$

For each $\beta \in J$, let

$$\begin{aligned} \pi_\beta: \prod_{\alpha \in J} X_\alpha &\longrightarrow X_\beta \\ (x_\alpha)_{\alpha \in J} &\longmapsto x_\beta \end{aligned}$$

be the *projection* to the β -th coordinate.

Definition: Let X_α be a topological space, for each $\alpha \in J$. Let

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\}$$

be the collection of subsets of the form

$$\pi_\beta^{-1}(U_\beta) = \prod_{\alpha \in J} U_\alpha,$$

where

$$U_\alpha = \begin{cases} U_\beta & \text{for } \alpha = \beta, \\ X_\alpha & \text{otherwise.} \end{cases}$$

Let

$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$$

be the union of these collections. This collection $\mathcal{S}$ is a subbasis, generating a basis $\mathcal{B}$ and a topology $\mathcal{T}$ on $\prod_{\alpha \in J} X_\alpha$. This is the *product topology* on this set.

Theorem 19.6: A function $f: Y \rightarrow \prod_{\alpha \in J} X_\alpha$ is continuous if and only if the component $\pi_\beta f: Y \rightarrow X_\beta$ is continuous for each $\beta \in J$.

Proof: We can check continuity of f using the subbasis $\mathcal{S}$, so f is continuous if and only if $f^{-1}(S)$ is open in Y for each $S \in \bigcup_{\beta \in J} \mathcal{S}_\beta$, which happens if and only if

$$f^{-1}(\pi_\beta^{-1}(U)) = (\pi_\beta f)^{-1}(U)$$

is open in Y for each $\beta \in J$ and each open $U \subset X_\beta$, which holds if and only if each $\pi_\beta f$ is continuous. QED

The basis elements in $\mathcal{B}$ are the finite intersections of subbasis elements. These are of the form

$$B = \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})$$

for $n \geq 0$ finite, $\beta_1, \dots, \beta_n \in J$, and U_{β_i} open in X_{β_i} for $1 \leq i \leq n$. Equivalently,

$$B = \prod_{\alpha \in J} U_\alpha$$

where $U_\alpha = U_{\beta_i}$ is open in X_α for the finitely many $\alpha \in \{\beta_1, \dots, \beta_n\}$, and $U_\alpha = X_\alpha$ for all remaining $\alpha \in J$. The open sets in $\prod_{\alpha \in J} X_\alpha$ are arbitrary unions of such basis elements B .

There is a finer topology on $\prod_{\alpha \in J} X_\alpha$, called the *box topology*, with basis given by arbitrary products

$$B' = \prod_{\alpha \in J} U_\alpha$$

where U_α is open in X_α for all $\alpha \in J$. This is simpler to describe, but is less important than the product topology.

Example: Let

$$\mathbb{R}^\omega = \mathbb{R}^\mathbb{N} = \prod_{n=1}^{\infty} \mathbb{R}$$

be the set of all real sequences $(x_n)_{n=1}^\infty$. The *diagonal* function

$$\Delta: \mathbb{R} \longrightarrow \mathbb{R}^\omega$$

given by $\Delta(t)_n = t$ for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$ is continuous when $\mathbb{R}^\omega$ is given the product topology, but not for the box topology, since the preimage of the box product basis element

$$B' = \prod_{n=1}^{\infty} (-1/n, 1/n)$$

is $\bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}$, which is not open in $\mathbb{R}$.

Theorem 19.2: Let $\mathcal{B}_\alpha$ be a basis for the topology on X_α , for each $\alpha \in J$. Then the collection of products

$$B = \prod_{\alpha \in J} U_\alpha$$

where $U_\alpha \in \mathcal{B}_\alpha$ for finitely many $\alpha \in J$, and $U_\alpha = X_\alpha$ for all of the remaining α , is a basis for the product topology on $\prod_{\alpha \in J} X_\alpha$.

Theorem 19.4: If X_α is Hausdorff for each $\alpha \in J$, then so is $\prod_{\alpha \in J} X_\alpha$.

Theorem 19.3: Let A_α be a subspace of X_α for each $\alpha \in J$. Then the product topology on $\prod_{\alpha \in J} A_\alpha$ is equal to the subspace topology inherited from $\prod_{\alpha \in J} X_\alpha$.

Theorem 19.5: Let A_α be a subspace of X_α for each $\alpha \in J$. Then

$$\prod_{\alpha \in J} \bar{A}_\alpha = \overline{\prod_{\alpha \in J} A_\alpha}.$$

Proof: ($\subset$) If $x = (x_\alpha)_{\alpha \in J}$ lies in $\prod_{\alpha \in J} \bar{A}_\alpha$, then $x_\alpha \in \bar{A}_\alpha$ for each $\alpha \in J$. Any neighborhood U of x in $\prod_{\alpha \in J} X_\alpha$ contains a basis neighborhood $B = \prod_{\alpha \in J} U_\alpha$ with each U_α open in X_α (and only finitely many $U_\alpha \neq X_\alpha$). Then U_α is a neighborhood of $x_\alpha \in \bar{A}_\alpha$, for each $\alpha \in J$, so $A_\alpha \cap U_\alpha \neq \emptyset$. Choose $y_\alpha \in A_\alpha \cap U_\alpha$. Then $y = (y_\alpha)_{\alpha \in J}$ lies in $\prod_{\alpha \in J} A_\alpha \cap \prod_{\alpha \in J} U_\alpha$, so B meets $\prod_{\alpha \in J} A_\alpha$. Hence x lies in the closure of $\prod_{\alpha \in J} A_\alpha$.

($\supset$) Since $\bar{A}_\alpha \supset A_\alpha$ we have

$$\prod_{\alpha \in J} \bar{A}_\alpha \supset \prod_{\alpha \in J} A_\alpha.$$

We claim that the left hand product is closed, which implies that it contains the closure of the right hand product. The claim follows from

$$\prod_{\alpha \in J} \bar{A}_\alpha = \bigcap_{\alpha \in J} \pi_\alpha^{-1}(\bar{A}_\alpha),$$

since $\bar{A}_\alpha$ is closed in X_α for each $\alpha \in J$, so that $\pi_\alpha^{-1}(\bar{A}_\alpha)$ is closed in $\prod_{\alpha \in J} X_\alpha$ by continuity of π_α , and therefore the displayed intersection of closed sets is itself closed. QED

Pointwise convergence:

Let J be a set and X a topological space. We can view the product topology on

$$X^J = \{(x_\alpha)_{\alpha \in J}\}$$

as giving a topology on the set of functions $f: J \rightarrow X$. In this guise, the product topology is also known as the *pointwise convergence topology*, for the following reason.

Definition: Let J be a set and X a topological space. A sequence $(f_n)_n$ of functions $f_n: J \rightarrow X$ converges pointwise to a function $f: J \rightarrow X$ if for each $\alpha \in J$ the sequence $(f_n(\alpha))_n$ in X converges to $f(\alpha)$.

Proposition: Let J be a set and X a topological space. A sequence $(f_n)_n$ of functions $f_n: J \rightarrow X$ converges to a function $f: J \rightarrow X$, with respect to the product topology on $X^J = \prod_J X$, if and only if the sequence (f_n) converges pointwise to f .

(Compare Exercise 19.6 in Munkres' book, or Proposition 2.7.5 in my course notes.)

§20 The Metric Topology: Recall the definition of a metric d on a set X . The ϵ -balls $B_d(x, \epsilon)$ form a basis for the metric topology $\mathcal{T}_d$ on X . Such topologies are said to be *metrizable*.

Example: The *discrete metric*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y, \end{cases}$$

induces the discrete topology, since each singleton set $\{x\} = B_d(x, 1)$ is open.

Consider $X = \mathbb{R}^n$, with the *Euclidean metric*

$$d(x, y) = d_2(x, y) = \sqrt{\sum_{i=1}^n (y_i - x_i)^2}$$

and the *square metric*

$$\rho(x, y) = d_\infty(x, y) = \max_{1 \leq i \leq n} |y_i - x_i|.$$

We have

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n} \rho(x, y)$$

so

$$B_\rho(x, \epsilon/\sqrt{n}) \subset B_d(x, \epsilon) \subset B_\rho(x, \epsilon)$$

for all $\epsilon > 0$.

Theorem 20.3: The Euclidean topology $\mathcal{T}_d$, the square topology $\mathcal{T}_\rho$ and the product topology $\mathcal{T}_{prod}$ on $\mathbb{R}^n$ are all equal.

Proof: Each Euclidean basis neighborhood $B_d(x, \epsilon)$ of x contains a square basis neighborhood $B_\rho(x, \epsilon/\sqrt{n})$ of x , so $\mathcal{T}_d \subset \mathcal{T}_\rho$.

Each square basis neighborhood

$$B_\rho(x, \epsilon) = \prod_{i=1}^n (x_i - \epsilon, x_i + \epsilon)$$

is a product basis neighborhood, so $\mathcal{T}_\rho \subset \mathcal{T}_{prod}$.

The open intervals (a, b) form a basis for the metric topology on $\mathbb{R}$, so the

$$B = \prod_{i=1}^n (a_i, b_i)$$

form a basis for the product topology. If $x \in B$ then $B_d(x, \epsilon) \subset B$ with

$$\epsilon = \min_{1 \leq i \leq n} \{x_i - a_i, b_i - x_i\}.$$

Hence $\mathcal{T}_{prod} \subset \mathcal{T}_d$. QED

More generally, any two norms on a finite-dimensional real vector space define the same topology. Similar proofs show that any finite product of metric spaces is metrizable.

Consider $X = \mathbb{R}^\omega$. The expression

$$d(x, y) = \sqrt{\sum_{i=1}^{\infty} (y_i - x_i)^2}$$

does not always converge, hence does not define a metric on $\mathbb{R}^\omega$. It does, however, define a metric on the subset

$$\ell^2 = \{x \in \mathbb{R}^\omega \mid \sum_{i=1}^{\infty} x_i^2 < \infty\}$$

of *square summable sequences*. In fact, this is a vector subspace, and the metric comes from an inner product (and a norm) on that vector space.

Likewise, the expression

$$\rho(x, y) = \sup_{i \in \mathbb{N}} |y_i - x_i|$$

is not always finite, hence does not define a metric on $\mathbb{R}^\omega$. It does, however, define a metric on the subset

$$\ell^\infty = \{x \in \mathbb{R}^\omega \mid \sup_{i \in \mathbb{N}} |x_i| < \infty\}$$

of *bounded sequences*. In fact, this is a vector subspace, and the metric comes from a norm (but not an inner product) on that vector space.

Bounded metric spaces:

Definition: A metric space (X, d) is *bounded* if there is an $M \in \mathbb{R}$ such that $d(x, y) \leq M$ for all $x, y \in X$.

Being bounded is a property of the metric, not of the topology.

Theorem 20.1: Given a metric space (X, d) , let

$$\bar{d}(x, y) = \min\{d(x, y), 1\}.$$

Then $\bar{d}$ defines a bounded metric on X , giving the same topology as d .

Proof: We check the triangle inequality for $\bar{d}$. Consider $x, y, z \in X$. If $d(x, y) \geq 1$ or $d(y, z) \geq 1$ then

$$\bar{d}(x, z) \leq 1 \leq \bar{d}(x, y) + \bar{d}(y, z).$$

Otherwise

$$\bar{d}(x, z) \leq d(x, z) \leq d(x, y) + d(y, z) \leq \bar{d}(x, y) + \bar{d}(y, z).$$

The distances in the metric space $(X, \bar{d})$ are bounded by $M = 1$. For $x \in X$ and $0 < \epsilon \leq 1$ we have

$$B_d(x, \epsilon) = B_{\bar{d}}(x, \epsilon),$$

and these form bases for the topologies $\mathcal{T}_d$ and $\mathcal{T}_{\bar{d}}$, which must therefore be equal. QED

The supremum $\rho(x, y) = \sup_{i \in \mathbb{N}} |y_i - x_i|$ would always be defined if we replaced the distance $d(a, b) = |b - a|$ in $\mathbb{R}$ with its bounded variant $\bar{d}$. This works for arbitrary metric spaces and index sets.

Uniform metric: Let (X, d) be any metric space, and let J be any set. For $x, y \in X^J$ let

$$\bar{\rho}(x, y) = \sup_{\alpha \in J} \bar{d}(x_\alpha, y_\alpha),$$

where $\bar{d}(x_\alpha, y_\alpha) = \min\{d(x_\alpha, y_\alpha), 1\}$.

Lemma: $\bar{\rho}$ defines a metric on X^J , and the associated topology $\mathcal{T}_{\bar{\rho}}$ is called the *uniform topology*.

Proof: The triangle inequality follows from

$$\sup_{\alpha \in J} (\bar{d}(x_\alpha, y_\alpha) + \bar{d}(y_\alpha, z_\alpha)) \leq \sup_{\alpha \in J} \bar{d}(x_\alpha, y_\alpha) + \sup_{\alpha \in J} \bar{d}(y_\alpha, z_\alpha).$$

QED

Theorem 20.4: The uniform topology on X^J is finer than the product topology.

Proof: The product topology on X^J is the coarsest topology for which all of the projections

$$\pi_\beta: X^J \longrightarrow X$$

are continuous. Hence it suffices to show that π_β is continuous when X^J is given the uniform topology, for each $\beta \in J$. Fix a point $x = (x_\alpha)_{\alpha \in J} \in X^J$ and a $\beta \in J$, so that $\pi_\beta(x) = x_\beta$. The metric basis neighborhoods of x_β in $(X, \bar{d})$ are of the form $B_{\bar{d}}(x_\beta, \epsilon)$ for $\epsilon > 0$. For any $y \in X^J$ we have

$$\bar{d}(x_\beta, y_\beta) \leq \bar{\rho}(x, y),$$

so

$$B_{\bar{\rho}}(x, \epsilon) \subset \pi_\beta^{-1}(B_{\bar{d}}(x_\beta, \epsilon)),$$

and π_β is continuous. QED

Addendum: When J is infinite, the uniform topology is strictly finer than the product topology.

The (proof of the) next theorem shows that a countable product of metric spaces, such as $\mathbb{R}$, is also metrizable. An example in the next section shows that for uncountable J , the product space $\mathbb{R}^J$ will not be metrizable.

Theorem 20.5: Let $\bar{d}(a, b) = \min\{|b - a|, 1\}$ be the standard bounded metric on $\mathbb{R}$. For $x, y \in \mathbb{R}^\omega$ let

$$D(x, y) = \sup_{i \in \mathbb{N}} \frac{\bar{d}(x_i, y_i)}{i}.$$

Then D is a metric that induces the product topology on $\mathbb{R}^\omega$.

Proof: The triangle inequality for D follows from that for $\bar{d}$ and

$$\sup_i \frac{\bar{d}(x_i, y_i) + \bar{d}(y_i, z_i)}{i} \leq \sup_i \frac{\bar{d}(x_i, y_i)}{i} + \sup_i \frac{\bar{d}(y_i, z_i)}{i}.$$

To show that the product topology is finer than the metric topology, let U be a metric neighborhood of x . Then there exists $\epsilon > 0$ with $x \in B_D(x, \epsilon) \subset U$. We will find a product neighborhood V of x with $V \subset B_D(x, \epsilon)$. To do so, choose an $N \in \mathbb{N}$ with $1/N < \epsilon$. Let

$$V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots$$

be a product basis neighborhood of $x = (x_i)_i$. For any $y \in \mathbb{R}^\omega$ we have

$$D(x, y) = \sup_i \frac{\bar{d}(x_i, y_i)}{i} \leq \max\left\{\frac{\bar{d}(x_1, y_1)}{1}, \dots, \frac{\bar{d}(x_N, y_N)}{N}, \frac{1}{N}\right\}$$

since

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{i} \leq \frac{1}{N}$$

for all $i \geq N$. If $y \in V$ then $\bar{d}(x_i, y_i)/i < \epsilon/i \leq \epsilon$ for $1 \leq i \leq N$, and $1/N < \epsilon$ by our choice of N , so $D(x, y) < \epsilon$. Hence $V \subset B_D(x, \epsilon)$.

Conversely, to show that the metric topology is finer than the product topology, consider a product basis neighborhood

$$W = \prod_{i \in \mathbb{N}} U_i$$

of x , with U_i open in $\mathbb{R}$ for $i \in \{\alpha_1, \dots, \alpha_n\}$ and $U_i = \mathbb{R}$ for the remaining $i \in \mathbb{N}$. Then $x_i \in (x_i - \epsilon_i, x_i + \epsilon_i) \subset U_i$ for some $0 < \epsilon_i \leq 1$, for each $i \in \{\alpha_1, \dots, \alpha_n\}$. We will find an $\epsilon > 0$ with $B_D(x, \epsilon) \subset W$. To do so, choose

$$\epsilon = \min\left\{\frac{\epsilon_i}{i} \mid i = \alpha_1, \dots, \alpha_n\right\}.$$

If $y \in B_D(x, \epsilon)$ then

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(x, y) < \epsilon \leq \frac{\epsilon_i}{i}$$

for each $i \in \{\alpha_1, \dots, \alpha_n\}$. Hence $\bar{d}(x_i, y_i) < \epsilon_i \leq 1$ and $|y_i - x_i| < \epsilon_i$ for each of these i . This ensures that $y \in W$, so $B_D(x, \epsilon) \subset W$. QED

§21 The Metric Topology (continued): We suppose known:

Theorem 21.1: If (X, d_X) and (Y, d_Y) are metric spaces, then a function $f: X \rightarrow Y$ is continuous with respect to the associated topologies if and only if for each $x \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$.

Lemma 21.4: The functions

$$\begin{aligned} \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto x + y \\ (x, y) &\longmapsto x - y \\ (x, y) &\longmapsto x \cdot y \end{aligned}$$

and

$$\begin{aligned} \mathbb{R} \times (\mathbb{R} - \{0\}) &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto x/y \end{aligned}$$

are continuous.

The proof can be carried out with the ϵ - δ characterization of continuity. This implies:

Theorem 21.5: If X is a topological space and $f, g: X \rightarrow \mathbb{R}$ are continuous functions, then so are $f + g$, $f - g$ and $f \cdot g$. If $g(x) \neq 0$ for all $x \in X$, then f/g is continuous.

Proof: We can view g as a map to $\mathbb{R} - \{0\}$ and write f/g as the composition

$$X \xrightarrow{(f, g)} \mathbb{R} \times (\mathbb{R} - \{0\}) \xrightarrow{/} \mathbb{R},$$

where the function (f, g) to the product is continuous since each component function has this property. The other cases are similar. QED

Subspace metric:

If (X, d) is a metric space and $A \subset X$ any subset, then the restriction

$$d_A = d|_{A \times A}: A \times A \longrightarrow \mathbb{R}$$

defines a metric on A called the *subspace metric*. The metric topology on A associated to d_A is equal to the subspace topology.

The sequence lemma:

In a metric space (X, d) , the (countable) collection

$$\{B_d(x, 1/n) \mid n \in \mathbb{N}\}$$

forms a *neighborhood basis at x* , in the sense that each open $U \subset X$ with $x \in U$ contains (at least) one of these neighborhoods $B_d(x, 1/n)$. After all, each such open U contains $B_d(x, \epsilon)$ for some $\epsilon > 0$, and then it also contains $B_d(x, 1/n)$ for (all) n with $1/n < \epsilon$.

This lets us characterize the points in the closure $\bar{A}$ of a subset $A \subset X$ as the limits (in X) of sequences in A . (Do not confuse these limits with the limit points = cluster points of Section 17.)

Lemma 21.2 (The sequence lemma): Let X be a topological space, and let $A \subset X$.

(a) If $(a_n)_n$ is a sequence of points in A , and $a_n \rightarrow x$ in X as $n \rightarrow \infty$, then $x \in \bar{A}$.

(b) If X is metrizable, then the converse holds: For each $x \in \bar{A}$ there is a sequence $(a_n)_n$ in A with $a_n \rightarrow x$ in X as $n \rightarrow \infty$.

Proof: (a) Convergence of $(a_n)_n$ to x means that for each neighborhood U of x in X there is an N such that $a_n \in U$ for all $n \geq N$. Hence $a_N \in A \cap U$, so U meets A . This implies that $x \in \bar{A}$.

(b) Suppose $x \in \bar{A}$. Then each neighborhood $B_d(x, 1/n)$ meets A , so we can choose $a_n \in A \cap B_d(x, 1/n)$. We claim that $(a_n)_n$ converges to x in X . To check this, consider any neighborhood U of x . It contains $B_d(x, \epsilon)$ for some $\epsilon > 0$, hence also contains $B_d(x, 1/n)$ for each $1/n < \epsilon$. Thus $a_n \in B_d(x, 1/n) \subset U$ for all $n \geq N$, where N is chosen so that $1/N < \epsilon$. This implies that $a_n \rightarrow x$ as $n \rightarrow \infty$. QED

Example: If J is uncountable, then $\mathbb{R}^J = \prod_J \mathbb{R}$ is not metrizable. See Munkres (page 133) for a proof that $\mathbb{R}^J$ does not satisfy part (b) of the sequence lemma.

Sequential characterization of continuity:

For metric spaces, we can characterize continuity in terms of preservation of limits of sequences.

Theorem 21.3: Let $f: X \rightarrow Y$ be a function.

(a) If f is continuous, and $x_n \rightarrow x$ in X as $n \rightarrow \infty$, then $f(x_n) \rightarrow f(x)$ in Y as $n \rightarrow \infty$.

(b) If X is metrizable, then the converse holds: Suppose that whenever $(x_n)_n$ is a sequence with $x_n \rightarrow x$ in X as $n \rightarrow \infty$, then $f(x_n) \rightarrow f(x)$ in Y as $n \rightarrow \infty$. Then f is continuous.

Proof: (a) Let V be any neighborhood of $f(x)$. Then $f^{-1}(V)$ is a neighborhood of x , so there is an N with $x_n \in f^{-1}(V)$ for all $n \geq N$. Then $f(x_n) \in V$ for all $n \geq N$. Hence $f(x_n) \rightarrow f(x)$ in Y as $n \rightarrow \infty$.

(b) Let $A \subset X$ be arbitrary. We prove that $f(\bar{A}) \subset \overline{f(A)}$ and appeal to Theorem 18.1. Let $x \in \bar{A}$. Since X is metrizable, there exists a sequence $(a_n)_n$ in A with $a_n \rightarrow x$ in X as $n \rightarrow \infty$. By assumption, $(f(a_n))_n$ then converges to $f(x)$ in Y as $n \rightarrow \infty$. Since this is a sequence in $f(A)$, it follows that $f(x) \in \overline{f(A)}$. Hence $f(\bar{A}) \subset \overline{f(A)}$, and f must be continuous. QED

Note that only the existence of the countable neighborhood basis around each point $x \in X$ was needed for the sequence lemma and the convergent sequence characterization of continuity, not the full existence of a metric d . We return to this in Section 30.

Uniform convergence:

For a set J and metric space (X, d) we can think of the product topology and the uniform topology on

$$\prod_J X = \{(x_\alpha)_{\alpha \in J}\}$$

as giving topologies on the function set

$$X^J = \{f: J \longrightarrow X\}.$$

This leads to two notions of convergence on the set X^J . As noted in Section 19, the first of these is that of pointwise convergence, which makes sense for arbitrary topological spaces X . The second notion is known as *uniform convergence*, and only makes sense for metric spaces (X, d) .

Definition: Let J be a set and (X, d) a metric space. A sequence $(f_n)_n$ of functions $f_n: J \rightarrow X$ *converges uniformly* to a function $f: J \rightarrow X$ if for each $\epsilon > 0$ there is an integer N such that $d(f_n(\alpha), f(\alpha)) < \epsilon$ for all $n \geq N$ and $\alpha \in J$.

Proposition: Let J be a set and (X, d) a metric space. Let

$$\bar{\rho}(f, g) = \sup_{\alpha \in J} \bar{d}(f(\alpha), g(\alpha))$$

be the associated uniform metric on X^J . A sequence $(f_n)_n$ of functions $f_n: J \rightarrow X$ converges to a function $f: J \rightarrow X$, with respect to the uniform topology on X^J , if and only if the sequence (f_n) converges uniformly to f .

(See Section 21, Exercise 7 in Munkres' book.)

Uniform limit theorem:

In the case where J is also a topological space, it makes sense to restrict attention to the functions $f: J \rightarrow X$ that are continuous. Let us switch from J and X to X and Y , so that we are considering the subset $C(X, Y)$ of Y^X .

Theorem 21.6 (Uniform limit theorem): Let X be a topological space and (Y, d) a metric space. Let $f_n: X \rightarrow Y$ for $n \geq 1$ be a sequence of continuous functions. If $(f_n)_n$ converges uniformly to a function $f: X \rightarrow Y$, then f is continuous.

Proof: We show that f is continuous at each point $x_0 \in X$. Let V be a neighborhood of $f(x_0)$ in Y . Choose $\epsilon > 0$ so that $B(f(x_0), \epsilon) \subset V$. By uniform convergence $f_n \rightarrow f$ as $n \rightarrow \infty$ we can choose $N \in \mathbb{N}$ so that for all $x \in X$ we have

$$d(f_N(x), f(x)) < \epsilon/3.$$

By continuity of $f_N: X \rightarrow Y$ at x_0 we can choose a neighborhood U of x_0 in X with

$$d(f_N(x_0), f_N(x)) < \epsilon/3$$

for all $x \in U$. Then

$$\begin{aligned} d(f(x_0), f(x)) &\leq d(f(x_0), f_N(x_0)) + d(f_N(x_0), f_N(x)) + d(f_N(x), f(x)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

for all $x \in U$. QED

§22 The Quotient Topology: We turn to how to paste topological spaces together. If $X = A \sqcup B$ is a union of two subsets, and A and B have assigned topologies, we can form their disjoint union $A \sqcup B$ with the sum topology, and there is a surjective function

$$p: A \sqcup B \longrightarrow X.$$

A subset $U \subset A \sqcup B$ is open in the sum topology if and only if $A \cap U$ is open in A and $B \cap U$ is open in B . We will now see how to use the surjection p to specify a topology on X , called the quotient topology from $A \sqcup B$.

First review the dual situation of subspaces: Let X be a topological space. If $Y \subset X$ is a subset, we can give Y the subspace topology with U open in X if and only if $Y \cap U$ is open in Y . Letting $i: Y \rightarrow X$ denote the inclusion map, we have $Y \cap U = i^{-1}(U)$, so that U is open in X if and only if $i^{-1}(U)$ is open in Y . Let Z be any topological space. A function $g: Z \rightarrow Y$ is continuous if and only if $ig: Z \rightarrow X$ is continuous. More generally, for any topological space X and injective function $i: Y \rightarrow X$ we can give Y a topology such that U is open in X if and only if $i^{-1}(U)$ is open in Y , and then a function $g: Z \rightarrow Y$ is continuous if and only if $ig: Z \rightarrow X$ is continuous. The injective function i is then an embedding of Y in X , so that $Y \cong i(Y) \subset X$.

$$\begin{array}{ccc} Z & & \\ g \downarrow & \searrow ig & \\ Y & \xrightarrow{i} & X \end{array}$$

We return to the quotient space situation.

Definition: Let X be a topological space, and let $p: X \rightarrow Y$ be a surjective function. A subset $V \subset Y$ is open in the *quotient topology* on Y if and only if the preimage $p^{-1}(V)$ is open in X . We call Y with the quotient topology a *quotient space* of X . The surjective function $p: X \rightarrow Y$ is called a *quotient map*.

Lemma: The quotient topology is a topology on Y .

Proof: $p^{-1}(V_1 \cap \cdots \cap V_n) = p^{-1}(V_1) \cap \cdots \cap p^{-1}(V_n)$ and $p^{-1}(\bigcup_{\alpha \in J} V_\alpha) = \bigcup_{\alpha \in J} p^{-1}(V_\alpha)$. QED

Lemma: The quotient topology on Y is the finest topology for which the projection $p: X \rightarrow Y$ is continuous.

Proof: Continuity of p is equivalent to asking that $p^{-1}(V)$ is open in X for each open V in Y , which holds if and only if each open in Y is open in the quotient topology. QED

Lemma: Let Y be a quotient space of X , and let Z be any topological space. A function $g: Y \rightarrow Z$ is continuous if and only if $gp: X \rightarrow Z$ is continuous.

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ & \searrow gp & \downarrow g \\ & & Z \end{array}$$

Proof: For each W open in Z , the preimage $g^{-1}(W)$ is open in Y if and only if $p^{-1}g^{-1}(W) = (gp)^{-1}(W)$ is open in X . QED

Lemma: Let $p: X \rightarrow Y$ be a quotient map. A subset $L \subset Y$ is closed if and only if $p^{-1}(L)$ is closed in X .

Proof: $X - p^{-1}(L) = p^{-1}(Y - L)$. QED

Example: Let $X = \mathbb{R}$ and $Y = \{N, Z, P\}$. Define $p: X \rightarrow Y$ by

$$p(x) = \begin{cases} N & \text{if } x < 0, \\ Z & \text{if } x = 0, \\ P & \text{if } x > 0. \end{cases}$$

The open subsets in the quotient topology on Y are $\emptyset$, $\{N\}$, $\{P\}$, $\{N, P\}$ and Y .

Equivalence relations:

One source of surjective functions $p: X \rightarrow Y$ is given by the equivalence relations on X .

Definition: A *relation* on X is a subset $R \subset X \times X$. We write xRy if $(x, y) \in R$. We say that R is an *equivalence relation* if

- (1) xRx for all $x \in X$;
- (2) If xRy then yRx ;
- (3) If xRy and yRz then xRz .

We then write $[x] = \{y \in X \mid xRy\}$ for the *equivalence class* of x with respect to R . This is a subset of X . Note that $[x] = [y]$ if and only if xRy . Let

$$X/R = \{[x] \mid x \in X\}$$

be the collection of equivalence classes with respect to R . The *canonical projection*

$$\begin{aligned} q: X &\longrightarrow X/R \\ x &\longmapsto [x] \end{aligned}$$

is surjective.

If X is a topological space and R an equivalence relation on X , then we can give X/R the quotient topology from the canonical projection $q: X \rightarrow X/R$, making X/R a quotient space of X and making q a quotient map. Here $V \subset X/R$ is open if and only if

$$q^{-1}(V) = \{x \in X \mid [x] \in V\}$$

is open in X .

Given any surjection $p: X \rightarrow Y$, we can define an equivalence relation R by setting xRy for $(x, y) \in X \times X$ if and only if $p(x) = p(y)$ in Y . Then p factors as the canonical projection $q: X \rightarrow X/R$ followed by the bijection $X/R \cong Y$ sending $[x]$ to $p(x)$. If $p: X \rightarrow Y$ is a quotient map, and we give X/R the quotient topology, then $X/R \cong Y$ is a homeomorphism. (This is dual to the situation for embeddings.)

Example: Let $X = S^2$ be the unit sphere in $\mathbb{R}^3$, and define an equivalence relation R by xRy if $y = \pm x$. The equivalence classes $[x] = \{x, -x\}$ are the (unordered) pairs of antipodal points. The set $\mathbb{R}P^2 = X/R$ equals the collection of such pairs, and is called the (*real*) *projective plane*. The canonical projection

$$q: S^2 \longrightarrow \mathbb{R}P^2$$

maps $x \in S^2$ to $[x] = \{x, -x\}$. The quotient topology on $\mathbb{R}P^2$ has as open subsets the $V \subset \mathbb{R}P^2$ for which $q^{-1}(V)$ is open in S^2 .

Can also view $\mathbb{R}P^2$ as the space of lines $L \subset \mathbb{R}^3$ through the origin. For each fixed plane $\alpha \subset \mathbb{R}^3$ through the origin, the set of $L \subset \alpha$ define a subset of $\mathbb{R}P^2$ called a *projective line*. The projective points and lines satisfy similar axioms to those of the Euclidean plane, but there are no parallel lines. Any two projective points lie on precisely one projective line, and any two projective lines meet in precisely one projective point. (Compare MAT4510 – Geometric Structures.)

(We could also construct $\mathbb{R}P^2$ as a quotient space of $\mathbb{R}^3 - \{0\}$, with $p: \mathbb{R}^3 - \{0\} \rightarrow \mathbb{R}P^2$ identifying nonzero $x, y \in \mathbb{R}P^3$ if $\mathbb{R}x = \mathbb{R}y$, i.e., if there is a nonzero $\lambda \in \mathbb{R}$ with $y = \lambda x$. In this case, the quotient topologies from S^2 and from $\mathbb{R}P^3 - \{0\}$ agree, as we may check later.)

Open maps and closed maps:

Definition: A continuous function $f: X \rightarrow Y$ is an *open map* if for each open $U \subset X$ the image $f(U)$ is open in Y . It is a *closed map* if for each closed $K \subset X$ the image $f(K)$ is closed in Y .

Lemma: Let $p: X \rightarrow Y$ be continuous and surjective. If p is an open map, or a closed map, then p is a quotient map.

Proof: We must show that if $V \subset Y$ and $p^{-1}(V)$ is open in X , then V is open. By surjectivity, $p(p^{-1}(V)) = V$, so if $p^{-1}(V)$ is open in X and p is an open map, then V is open in Y .

In the closed map case, we show that if $L \subset Y$ and $p^{-1}(L)$ is closed in X , then L is closed in Y . The converse holds by the closed set criterion for the continuity of p , so this will characterize the topology on Y as being the quotient topology induced from X along $p: X \rightarrow Y$. By surjectivity, $p(p^{-1}(L)) = L$, so if $p^{-1}(L)$ is closed in X and p is a closed map, then L is closed in Y . QED

Some maps are both open and closed. Some quotient maps are neither open nor closed.

Example: Let $\pi_1: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be the first projection. This is continuous and surjective. Moreover, it is an open map, since $\pi_1(U \times V) = U$ is open in $\mathbb{R}$ for each element $U \times V$ in the product basis for $\mathbb{R} \times \mathbb{R}$. Hence it is also a quotient map. It is not a closed map, because

$$C = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid xy = 1\}$$

is closed in $\mathbb{R} \times \mathbb{R}$, while the image $\pi_1(C) = \mathbb{R} - \{0\}$ is not closed in $\mathbb{R}$.

See Munkres' Section 22 for more on the (lack of) compatibility between quotient topologies and other constructions, such as products and subspaces.

§23 Connected Spaces: The Intermediate Value Theorem asserts that for a continuous function $f: [a, b] \rightarrow \mathbb{R}$ each r between $f(a)$ and $f(b)$ arises as a value $r = f(c)$ for some $c \in [a, b]$. This is a consequence of the fact that $[a, b]$ is connected, which is a topological property that we now discuss for general topological spaces.

We will say that a topological space X is disconnected if there is a continuous surjection (= surjective map) $f: X \rightarrow \{a, b\}$, where $\{a, b\}$ has the discrete topology. Letting $U = f^{-1}(a)$ and $V = f^{-1}(b)$, this is equivalent to asking that $U \cup V = X$, $U \cap V = \emptyset$, U and V are open in X , and U and V are nonempty.

Definition: A *separation* of a topological space X is a pair of nonempty open subsets $U, V \subset X$ with $U \cup V = X$ and $U \cap V = \emptyset$. If a separation of X exists, then X is *disconnected*. If no separation of X exists, then X is *connected*.

By this definition, the empty space $\emptyset$ is connected. Some authors require connected spaces to be nonempty, so that there is a three-case distinction: empty, connected or disconnected. This is similar to the classification of natural numbers as the unit, the primes and the composite numbers.

Lemma: A (nonempty) space X is connected if and only if the only open and closed subsets $U \subset X$ are $\emptyset$ and X .

Proof: If U and V are a separation of X , then U is open and closed, and $U \neq \emptyset$, $U \neq X$. The converse also holds, with $V = X - U$. QED

Example: $X = \{a, b\}$ with the trivial topology is connected, since the only open and closed subsets are $\emptyset$ and X .

Connected subspaces:

Recall that for $A \subset Y \subset X$ we write $\bar{A}$ for the closure of A in X . The following is Lemma 23.1, reformulated in terms of closures in place of “limit points”.

Lemma: Let $Y \subset X$ be a subspace. A separation of Y is a pair A and B of nonempty subsets with $A \cup B = Y$, $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$.

Proof: If A and B is a separation of Y , then B is closed in Y , so that $B = Y \cap \bar{B}$, which implies

$$A \cap \bar{B} = A \cap Y \cap \bar{B} = A \cap B = \emptyset.$$

Switching the roles of A and B proves that $\bar{A} \cap B = \emptyset$.

Conversely, given A and B as above, $A \cap B \subset A \cap \bar{B} = \emptyset$, so $A \cap B = \emptyset$. Then

$$Y \cap \bar{B} = (A \cup B) \cap \bar{B} = (A \cap \bar{B}) \cup (B \cap \bar{B}) = \emptyset \cup B = B$$

is closed in B , since $\bar{B}$ is closed in X , so that $A = Y - B$ is open in Y . Symmetrically, B is open in Y . Hence A and B is a separation of Y .

Example: Let $Y = [-1, 0) \cup (0, 1]$ as a subspace of $\mathbb{R}$. Then $A = [-1, 0)$ and $B = (0, 1]$ is a separation of Y , because $\bar{A} = [-1, 0]$ does not meet B and $\bar{B} = [0, 1]$ does not meet A .

Example: Let $Y = [-1, 1]$, $A = [-1, 0]$ and $B = (0, 1]$. Then A and B is not a separation of Y , since A meets $\bar{B} = [0, 1]$ in 0 . We will see that Y is connected.

Example: The subspace of rationals $\mathbb{Q} \subset \mathbb{R}$ is disconnected. If $Y \subset \mathbb{Q}$ contains at least two points, say $p < q$, then there is an irrational $r \in (p, q)$ and $A = Y \cap (-\infty, r)$, $B = Y \cap (r, \infty)$ is a separation of Y . Hence the only connected subspaces of $\mathbb{Q}$ are the singleton sets (and the empty set).

Example: Let $Y \subset \mathbb{R}^2$ be the subspace

$$Y = \{(x, 0) \mid x \in \mathbb{R}\} \cup \{(x, y) \mid x > 0, y = 1/x\}.$$

Then $A = \{(x, 0) \mid x \in \mathbb{R}\}$ and $B = \{(x, y) \mid x > 0, y = 1/x\}$ form a separation of Y , since $A = \bar{A}$ and $B = \bar{B}$ are disjoint.

New connected spaces from old:

Continuous images, many unions, and all products of connected spaces are connected.

Theorem 23.5: The continuous image of a connected space is connected.

Proof: Let $f: X \rightarrow Y$ be a surjective map, with X connected. We show that Y is connected. Suppose that U and V is a separation of Y . Then $f^{-1}(U)$ and $f^{-1}(V)$ are a separation of X . ((ETC: Elaborate?)) Since X is connected, this is impossible. Hence no separation of Y exists, and Y is connected.

More generally, if $f: X \rightarrow Y$ is continuous, but not necessarily surjective, then corestriction $g: X \rightarrow f(X)$ is a surjective map, and the argument above shows that $f(X)$ is a connected subspace of Y . QED

Lemma 23.2: If C and D is a separation of X , and $Y \subset X$ is a connected subspace, then $Y \subset C$ or $Y \subset D$.

Proof: Let $A = Y \cap C$ and $B = Y \cap D$. These are open in Y , with union Y and empty intersection. If they are both nonempty, they give a separation of Y . Since Y is connected, no such separation exists, which means that A or B is empty, so that Y is contained in D or C , respectively. QED

Theorem 23.3: If $X = \bigcup_{\alpha \in J} Y_\alpha$ and $p \in \bigcap_{\alpha \in J} Y_\alpha$, with each Y_α connected, then X is connected.

Proof: Suppose that C and D is a separation of X . Then $p \in C$ or $p \in D$, but not both. Suppose the first. Each Y_α is contained in C or D . Since $p \in Y_\alpha$ we cannot have $Y_\alpha \subset D$, so

$Y_\alpha \subset C$. This holds for each α , so $X \subset C$. This implies $C = X$ and $D = \emptyset$, which is not a valid separation. QED

Theorem 23.6+: Any product of connected spaces is connected.

Proof: An empty product is a single point, which is connected.

We prove that a product $X \times Y$ of two connected spaces is (empty or) connected. Choose $p = (a, b) \in X \times Y$. For each $x \in X$ the union

$$T_x = X \times \{b\} \cup \{x\} \times Y$$

is a union of two connected spaces that meet in (x, b) , hence is connected. Moreover

$$X \times Y = \bigcup_{x \in X} T_x$$

is a union of connected spaces that all contain (a, b) . Hence this is also connected.

The case of finite products follows by induction.

The case of infinite products is left to Example 23.7 and Exercise 23.10. QED

Theorem 23.4: Let $Y \subset X$ be a connected subspace. If $Y \subset Z \subset \bar{Y}$ then Z is also connected.

Proof: Suppose that C and D are a separation of Z . Then the connected subspace Y lies in C or D . Suppose the first, so that $Y \subset C$. Recall that $\bar{C} \cap D = \emptyset$. Then $Z \subset \bar{Y} \subset \bar{C}$ does not meet D , which contradicts the assumption that $D \subset Z$ is nonempty. QED

§24 Connected Subspaces of the Real Line: Munkres considers quite general ordered sets, with the order topology from Section 14, which we have omitted. We specialize to the case of $\mathbb{R}$, with the usual metric topology.

[Follow §3.2-§3.7 from `topology.pdf`.]

Matrix groups. Let $n \geq 1$. An $n \times n$ matrix

$$A = (a_{i,j})_{i,j=1}^n$$

with real entries can be viewed as an n -tuple of column vectors

$$A = [v_1 | \dots | v_n]$$

with each $v_j \in \mathbb{R}^n$, for $1 \leq j \leq n$. Let

$$M_n(\mathbb{R}) = \{A \mid A \text{ is a real } n \times n \text{ matrix}\}$$

be the vector space of all such matrices. Then there are linear isomorphisms

$$M_n(\mathbb{R}) \cong (\mathbb{R}^n)^n \cong \mathbb{R}^{n^2}.$$

We topologize $M_n(\mathbb{R})$ so that these are homeomorphisms, where $\mathbb{R}^{n^2}$ has the product topology, which equals the metric topology from any norm on this vector space.

The product of matrices defines a continuous function

$$\cdot: M_n(\mathbb{R}) \times M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R}),$$

since each entry in $A \cdot B = AB$ is a polynomial in the entries of A and B .

The determinant of a matrix defines a continuous function

$$\det: M_n(\mathbb{R}) \longrightarrow \mathbb{R},$$

since $\det(A)$ is a polynomial in the entries of A . Let

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$$

be the subspace of invertible matrices, let

$$GL_n^+(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) > 0\}$$

be the subspace of orientation-preserving matrices, and let

$$SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\}$$

be the subspace of volume-preserving matrices. Each of these is preserved under matrix multiplication, and matrix inverse, hence defines a group. In each case the multiplication pairing, as well as the inverse map sending A to A^{-1} , is continuous, so these are topological groups.

Note that $GL_n(\mathbb{R})$ and $GL_n^+(\mathbb{R})$ are open subsets of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ hence are n^2 -dimensional manifolds. One can prove that $SL_n(\mathbb{R})$ is an $(n^2 - 1)$ -dimensional manifold.

Note that $GL_n(\mathbb{R})$ is not connected, since it maps onto $\mathbb{R} - \{0\}$. We claim that $GL_n^+(\mathbb{R})$ and $SL_n(\mathbb{R})$ are path connected, so that $GL_n(\mathbb{R})$ has precisely two (path) connected components. To prove the first claim, we use Gaussian elimination/row reduction of matrices. For $i \neq j$ let

$$E_{i,j}(r) = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \dots & r & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

be the elementary matrix that differs from the identity matrix I_n by an entry r in the (i, j) -th position. The rule $A \mapsto E_{i,j}(R)A$ is the row operation that adds r times the j -th row to the i -th row. For any invertible matrix A there is a finite sequence of row operations that brings A to the form

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_i & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & \lambda_j & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & \lambda_n \end{pmatrix}$$

with $\lambda_1, \dots, \lambda_{n-1} > 0$. If $\det(A) > 0$ then also $\lambda_n > 0$. This uses that the operation of interchanging two rows and changing the sign of one of them can be realized by three row operations of the kind mentioned above. Hence

$$D = E_{i_k, j_k}(r_k) \cdot \dots \cdot E_{i_1, j_1}(r_1)A$$

for some finite k . Note that $E_{i,j}(r)^{-1} = E_{i,j}(-r)$. Hence

$$A = E_{i_1, j_1}(-r_1) \cdot \dots \cdot E_{i_k, j_k}(-r_k)D.$$

Letting $t \in [0, 1]$ we obtain a path

$$f(t) = E_{i_1, j_1}(-tr_1) \cdots E_{i_k, j_k}(-tr_k)D$$

in $GL_n(\mathbb{R})$ from $f(0) = D$ to $f(1) = A$. If $\lambda_1, \dots, \lambda_n > 0$ then there is also a path

$$g(t) = (1 - t)D + tI_n$$

from $g(0) = D$ to $g(1) = I_n$. Hence each $A \in GL_n^+(\mathbb{R})$ lies in the same path component as I_n , so $GL_n^+(\mathbb{R})$ is path connected. The surjective map

$$GL_n^+(\mathbb{R}) \longrightarrow SL_n(\mathbb{R})$$

sending A to $A/\det(A)^{1/n}$ shows that $SL_n(\mathbb{R})$ is also path connected.

The subspaces $GL_n(\mathbb{R})$ and $GL_n^+(\mathbb{R})$ of $M_n(\mathbb{R})$ are not closed, since the zero matrix lies in the closure of either one, hence these groups are not compact. They are, however, locally compact. Trivially, $SL_1(\mathbb{R}) = \{1\}$ is compact, but $SL_n(\mathbb{R})$ is not compact for $n \geq 2$. Once one proves that $SL_n(\mathbb{R})$ is a manifold, it follows that it is locally compact.

Let

$$O_n = \{A \in M_n(\mathbb{R}) \mid A^t A = I_n\}$$

be the subspace of orthogonal matrices. It is a subspace of $GL_n(\mathbb{R})$, since $\det(A) = \{\pm 1\}$ for each orthogonal A . It is preserved under the group operations, hence is a topological subgroup of $GL_n(\mathbb{R})$. Let

$$SO_n = \{A \in O_n \mid \det(A) = 1\}$$

be the special orthogonal group. It is a topological subgroup of $SL_n(\mathbb{R})$.

We claim that O_n and SO_n are compact. To see this, note that the function

$$f: M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$$

sending A to $A^t A$ is continuous, so $O_n = f^{-1}(I_n)$ is the preimage of a closed point, hence is closed in $M_n(\mathbb{R})$. In particular, for

$$A = [v_1 \mid \dots \mid v_n]$$

orthogonal each v_i is a unit vector, so

$$O_n \subset S^{n-1} \times \dots \times S^{n-1}$$

is a bounded subspace of $(\mathbb{R}^n)^n$. Hence O_n is compact, and so is its closed subspace $SO_n = \det^{-1}(1)$. In fact, O_n is a maximal compact subgroup of $GL_n(\mathbb{R})$, while SO_n is a maximal compact subgroup of $GL_n^+(\mathbb{R})$ and of $SL_n(\mathbb{R})$.

One can show that O_n and SO_n are $n(n-1)/2$ -dimensional manifolds, and are in particular locally path connected. The Gram-Schmidt orthogonalization process defines a surjective map

$$g: GL_n(\mathbb{R}) \longrightarrow O_n$$

that maps $GL_n^+(\mathbb{R})$ onto SO_n . Hence SO_n is (path) connected, and O_n has precisely two (path) components.

For $n = 2$, the group SO_2 consists of the rotations

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

for $0 \leq \theta < 2\pi$, and is homeomorphic to S^1 . The other component of O_2 consists of the reflections in lines through the origin, hence is also homeomorphic to $\mathbb{R}P^1 \cong S^1$.

One often calls SO_n the n -dimensional rotation group, but note that for $n \geq 4$, more than one planar rotation is usually involved. For example,

$$A = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 & 0 \\ \sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & \cos \beta & -\sin \beta \\ 0 & 0 & \sin \beta & \cos \beta \end{pmatrix} \in SO_4$$

acts on $\mathbb{R}^4$ the composite of a rotation by α in the first two coordinates and a rotation by β in the last two coordinates.