A Union Theorem for Cofibrations

By

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1. Results. We work in the category $\mathcal{T}op$ of topological spaces and continuous maps. If A is a subspace of X, we denote the closure of A by $\bar{A}$ and the interior of A by $\overset{\circ}{A}$. Let I be the unit interval. We say that a map $i: A \to X$ has the homotopy extension property (HEP) with respect to Z if, given maps $f: X \to Z$ and $\varphi: A \times I \to Z$ such that $f(x) = \varphi(x, 0)$ for $x \in A$, there exists a map $\Phi: X \times I \to Z$ such that $\Phi | X \times 0 = f$ and $\Phi | A \times I = \varphi$. We call $i: A \to X$ a cofibration if it has the HEP with respect to every space Z and a closed cofibration if in addition $i(A)$ is closed. We consider subspaces A and B of a space X such that the inclusion maps $A \subset X$ and $B \subset X$ have the HEP w.r.t. Z, and investigate under what assumptions $A \cup B \subset X$ has the HEP w.r.t. Z.

Theorem 1. *Assume that $A \subset X$ has the* HEP *w.r.t. Z. Assume there exists a continuous map $u: X \to I$ with $A \subset u^{-1}(0)$ and $(u | B)^{-1}(0) = A \cap B$. If $B \times I \subset X \times I$ and $(A \cap B) \times I \subset B \times I$ have the* HEP *w.r.t. Z, then $A \cup B \subset X$ has the* HEP *w.r.t. Z.*

(Independently, A. Dold has proved the same result.)

If we assume A and B separated, then we obtain a symmetric result. We define an equivalence relation $\sim$ in $X \times I$ by identifying (x, t) and $(x, 0)$ for $t \in I$ and $x \in A \cap B$. Let $[x, t]$ denote the class of (x, t) in $X \times I/\sim$ and let $\mathrm{pr}: X \times I/\sim \to X$ be given by $\mathrm{pr}[x, t] = x$.

Definition. We call two subspaces A and B of X *separated* if there exists a continuous map $j: X \to X \times I/\sim$ such that $\mathrm{pr} \circ j = \mathrm{id}(X)$ and $j(x) = [x, 0]$ for $x \in A$, $j(x) = [x, 1]$ for $x \in B$.

In section 3 we show that closed cofibrations are separated.

Now we can state a symmetrical variant of theorem 1:

Theorem 2. *Assume that $A \subset X$, $B \subset X$ have the* HEP *w.r.t. Z. If $(A \cap B) \times I \subset X \times I$ has the* HEP *w.r.t. Z and A and B are separated, then $A \cup B \subset X$ has the* HEP *w.r.t. Z.*

Reformulated for cofibrations, we obtain the

Union Theorem. *Let $A \subset X$ and $B \subset X$ be cofibrations. Let*
either (a) *$A \cap B \subset B$ be a cofibration and $\bar{A} \cap B = A \cap B$,*
or (b) *$A \cap B \subset X$ be a cofibration and A, B separated.*
Then $A \cup B \subset X$ is a cofibration.

This theorem follows immediately from theorem 1 and theorem 2. In case (a) note that if $A \subset X$ is a cofibration then so is $\bar{A} \subset X$ [2; Cor. 5]. Moreover a closed cofibration is a *Nullstellen*-set (i.e. a subspace A of a space X is a *Nullstellen*-set if there exists a continuous map $u\colon X \to I$ with $u^{-1}(0) = A$) (see [1; Satz 3.26]).

From the union theorem we deduce the product theorem for cofibrations:

Corollary 1. *Let $A \subset X$ and $B \subset Y$ be cofibrations. Let A be closed in X. Then $(X \times B) \cup (A \times Y) \subset X \times Y$ is a cofibration.*

One deduces immediately from the assumptions that $X \times B \subset X \times Y$, $A \times Y \subset X \times Y$ and $(X \times B) \cap (A \times Y) = A \times B \subset X \times Y$ are cofibrations. Since A is closed in X, we have $\overline{(A \times Y)} \cap (X \times B) = (A \times Y) \cap (X \times B)$, and case (a) proves the corollary.

Since closed cofibrations are separated (see section 3), we have

Corollary 2. *If $A \subset X$ and $B \subset X$ are closed cofibrations and if $A \cap B \subset X$ is a cofibration, then $A \cup B \subset X$ is a cofibration.*

By induction we obtain

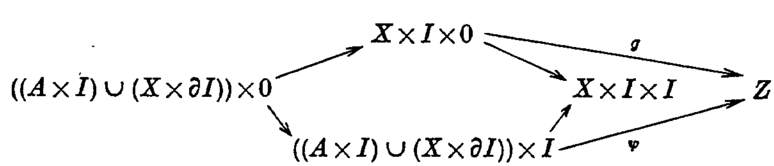
Corollary 3. *Let $A_1 \subset X, \ldots, A_n \subset X$ be closed cofibrations. For all $\sigma \subset \{1, \ldots, n\}$ let $A_\sigma = \bigcap_{l \in \sigma} A_l \subset X$ be a cofibration. Then $\bigcup_{l=1}^{n} A_l \subset X$ is a cofibration.*

Corollary 3 does not hold in general for countably many cofibrations. For let $X = I$, $A_l = \{0, 1/l\}$ for $l = 1, 2, \ldots$ and $A = \bigcup_{l=1}^{\infty} A_l$. The set A is closed in X. The inclusion maps $A_l \subset X$ are obviously closed cofibrations. For all finite σ, $A_\sigma \subset X$ are cofibrations but $A \subset X$ is not a cofibration [1; Beispiel 3.14 (3)].

2. Proof of theorem 1. We will need the lemma (a generalization of a special case of the product theorem for cofibrations):

Lemma 1. *If $i\colon A \times I \subset X \times I$ has the HEP w.r.t. Z, then $(A \times I) \cup (X \times \partial I) \to X \times I$ and $(A \times I) \cup (X \times 0) \to X \times I$ have the HEP w.r.t. Z. Here $(A \times I) \cup (X \times \partial I)$ is not considered as a subspace of $X \times I$, but as a quotient space of the topological sum $(A \times I) \cup (X \times \partial I)$ obtained by identifying $(a, 0)$ with $i(a, 0)$ and $(a, 1)$ with $i(a, 1)$. Similarly for $(A \times I) \cup (X \times 0)$.*

Proof. Assume given a commutative diagram



Let $Q\colon I \times I \to I \times I$ be a homeomorphism with

$$Q((I \times 0) \cup (\partial I \times I)) = I \times 0\,.$$

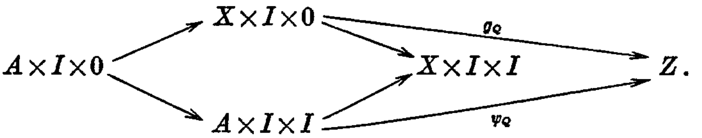
Define maps $g_Q' = g \circ ((\mathrm{id} \times Q^{-1}) \mid X \times I \times 0)$ and

$$\psi_Q' = \psi \circ ((\mathrm{id} \times Q^{-1}) \mid (A \times I \times I) \cup (X \times \partial I \times I)).$$

Let $\psi_Q = \psi_Q' \mid A \times I \times I$ and let $g_Q \colon X \times I \times 0 \to Z$ be given by

$$g_Q \mid (\mathrm{id} \times Q^{-1})(X \times I \times 0) = g_Q' \text{ and}$$
$$g_Q \mid (\mathrm{id} \times Q^{-1})(X \times \partial I \times I) = \psi_Q' \mid (\mathrm{id} \times Q^{-1})(X \times \partial I \times I).$$

We obtain the commutative diagram:



Since $A \times I \subset X \times I$ has the HEP w.r.t. Z, there exists a continuous map $\Phi_Q \colon X \times I \times \times I \to Z$ completing the diagram commutatively. Now we define $\Phi \colon X \times I \times I \to Z$ by $\Phi = \Phi_Q \circ (\mathrm{id} \times Q)$. One checks that Φ is the desired extension of ψ and g. Analogously one proves the second part of the lemma using a homeomorphism $P \colon I \times \times I \to I \times I$ with $P((I \times 0) \cup (0 \times I)) = I \times 0$. ∎

We also need the following lemma:

Lemma 2. *Let $A \subset X$ be a Nullstellen-set. Let $f, g \colon X \to Z$ be continuous maps with $\Phi \colon f \simeq g$ rel A. Then there exists a homotopy $\tilde{\Phi} \colon f \simeq g$ rel A with*

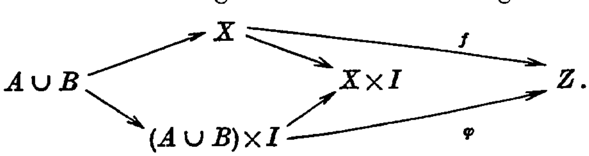
$$\tilde{\Phi}(x, t) = \tilde{\Phi}(x, u(x)) = \Phi(x, 1) \text{ for } x \in X \text{ and } t \geqq u(x).$$

Proof. Let $u \colon X \to I$ be a map with $u^{-1}(0) = A$. We define $\tilde{\Phi} \colon X \times I \to Z$ by

$$\tilde{\Phi}(x, t) = \begin{cases} \Phi(x, 1) & \text{for } t \geqq u(x), \\ \Phi(x, t) & \text{for } u(x) = 0, \\ \Phi\left(x, \dfrac{t}{u(x)}\right) & \text{for } t \leqq u(x) \text{ and } u(x) \neq 0. \end{cases}$$

Obviously $\tilde{\Phi}$ is well-defined; the continuity of $\tilde{\Phi}$ is proved in [1; Satz 3.26]. ∎

Proof of theorem 1. Assume given a commutative diagram



The inclusion map $A \subset X$ has the HEP w.r.t. Z, and consequently there exists an extension $\Phi \colon X \times I \to Z$ of $\varphi \mid A \times I$ and f. Now define maps

$$\Phi \colon X \times I \times 0 \to Z \text{ by } \Phi(x, s, 0) = \Phi(x, s),$$
$$\varphi \colon B \times I \times 1 \to Z \text{ by } \varphi(b, s, 1) = \varphi(b, s),$$
$$F \colon X \times 0 \times I \to Z \text{ by } F(x, 0, t) = f(x) \text{ for every } t \in I \text{ and}$$

$$\psi\colon (A \cap B) \times I \times I \to Z \quad \text{by} \quad \psi(a, s, t) = \psi(a, s, 0) = \varphi(a, s)$$

for $a \in A \cap B$ and $t \in I$. Since $(A \cap B) \times I \subset B \times I$ has the HEP w.r.t. Z, there exists, by lemma 1, a homotopy $\Psi\colon B \times I \times I \to Z$ with

$$\begin{aligned} &\Psi(b, s, 0) = \Phi(b, s), \\ &\Psi(b, s, 1) = \varphi(b, s), \\ &\Psi(b, 0, t) = f(b) \quad \text{for} \quad b \in B \quad \text{and} \\ &\Psi(a, s, t) = \psi(a, s, t) = \varphi(a, s) \quad \text{for} \quad a \in A \cap B \quad \text{and} \quad t \in I. \end{aligned}$$

In view of lemma 2 we can deform Ψ to $\widetilde{\Psi}\colon B \times I \times I \to Z$ with $u'\colon B \times I \to Z$ defined by $u'(b, s) = u(b)$. The map $B \times I \subset X \times I$ has the HEP w.r.t. Z; hence, by lemma 1, there exists a continuous map $\Omega\colon X \times I \times I \to Z$ with

$$\begin{aligned} &\Omega(x, s, 0) = \Phi(x, s, 0), \\ &\Omega(x, 0, t) = F(x, 0, t) \quad \text{and} \\ &\Omega(b, s, t) = \widetilde{\Psi}(b, s, t) \quad \text{for} \quad b \in B. \end{aligned}$$

We now define the desired extension $H\colon X \times I \to Z$ of φ and f by $H(x, s) = \Omega(x, s, u(x))$. It is clear that H is continuous and that $H | X \times 0 = f$ and $H | (A \cup B) \times I = \varphi$. ▮

3. Proof of theorem 2. In this section we shall first give several criteria for the separation of two subspaces of a space X.

Lemma 3. (a) *Given subspaces A and B of X and a continuous map $u\colon X - (A \cap B) \to I$ with $A - (A \cap B) \subset u^{-1}(0)$ and $B - (A \cap B) \subset u^{-1}(1)$, then A and B are separated and $\partial A \cap \partial B \subset A \cap B$. ($\partial A$ is the boundary of A.)*

(b) i) *If A and B are Nullstellen-sets, then a map u exists satisfying the hypothesis in* (a).

In particular: if $A \subset X$ and $B \subset X$ are closed cofibrations, then A and B are separated.

ii) *If $\bar{A}$ and $\bar{B}$ are Nullstellen-sets and if $\partial A \cap \partial B \subset A \cap B$, then a map u exists satisfying the hypothesis in* (a).

In particular: if $A \subset X$ and $B \subset X$ are cofibrations and if $\partial A \cap \partial B \subset A \cap B$, then A and B are separated.

Proof. (a) We define $j\colon X \to X \times I/\!\sim$ by

$$j(x) = \begin{cases} [x, u(x)] & \text{for} \quad x \notin A \cap B, \\ [x, 0] = [x, t] & \text{for} \quad x \in A \cap B \quad \text{and} \quad t \in I. \end{cases}$$

(b) The case i) follows from ii) because $\partial A \cap \partial B \subset A \cap B$. Now let $\lambda, \mu\colon X \to I$ be continuous maps with $\bar{A} = \lambda^{-1}(0)$ and $\bar{B} = \mu^{-1}(0)$. We define a map $u\colon X - (A \cap B) \to I$ by

$$u(x) = \begin{cases} \dfrac{\lambda(x)}{\lambda(x) + \mu(x)} & \text{for} \quad x \notin A \cap B, \\ 1 & \text{for} \quad x \in (\bar{A} - A) \cap \mathring{B}, \\ 0 & \text{for} \quad x \in (\bar{B} - B) \cap \mathring{A}. \end{cases}$$

It is obviously sufficient to show continuity in the points of $\mathring{A} \cap (\bar{B} - B)$. But this is clear because, for an arbitrary $x \in \mathring{A} \cap (\bar{B} - B)$, a neighborhood U of $\mathring{A}$ can be chosen such that $u(x) = 0$ for every $x \in U$. ▮

Remark. If A and B are separated then there need not exist a continuous map $u: X - (A \cap B) \to I$ satisfying lemma 3(a) as following example of D. Puppe shows: Take $X = \{a, b, c\}$ with the open sets $\emptyset$, $\{a\}$, $\{a, c\}$, X. Let $A = \{a, c\}$ and $B = \{b, c\}$. Then $j: X \to X \times I/\sim$ is continuous but there is no continuous $u: X - \{c\} \to I$.

For the proof of theorem 2 we need the following lemma.

Lemma 4. *Let A be a subspace of X such that $A \times I \subset X \times I$ has the* HEP *w.r.t. Z. Let $K, L: X \times I \to Z$ be homotopies with $K_0 = L_0$ and $K \mid A \times I = L \mid A \times I$. Then there exists a homotopy $\Phi: K \simeq L$* rel $(A \times I) \cup (X \times 0)$.

Proof. We define maps $g: (X \times I \times \partial I) \cup (X \times 0 \times I) \to Z$ by

$$g(x, s, t) = \begin{cases} K(x, s) & \text{for} \quad t = 0 \quad \text{or} \quad s = 0, \\ L(x, s) & \text{for} \quad t = 1 \end{cases}$$

and $\psi: A \times I \times I \to Z$ by $\psi(a, s, t) = K(a, s) = L(a, s)$ for every $t \in I$. Since g is defined by continuous maps on closed subspaces, it is continuous. The map $A \times I \subset X \times I$ has the HEP w.r.t. Z, and consequently, by lemma 1, there exists a continuous map $\Phi: X \times I \times I \to Z$ with $\Phi \mid A \times I \times I = \psi$ and $\Phi \mid (X \times I \times \partial I) \cup (X \times 0 \times I) = g$. It is clear that Φ satisfies all the required conditions. ▮

Proof of theorem 2. Assume given a homotopy $\varphi: (A \cup B) \times I \to Z$ with $\varphi_0 = f: X \to Z$. The maps $A \subset X$ and $B \subset X$ have the HEP w.r.t. Z. Hence there exist extensions $\Phi^A: X \times I \to Z$ of $\varphi \mid A \times I$ and $\Phi^B: X \times I \to Z$ of $\varphi \mid B \times I$ with $\Phi_0^A = \Phi_0^B = f$. Then $\Phi^A \mid (A \cap B) \times I = \varphi \mid (A \cap B) \times I = \Phi^B \mid (A \cap B) \times I$. Since $(A \cap B) \times I \subset X \times I$ has the HEP w.r.t. Z, there exists a homotopy $\Psi: \Phi^A \simeq \Phi^B \operatorname{rel}((A \cap B) \times I) \cup (X \times 0)$, by lemma 4. By assumptions, we have a continuous map $j: X \to X \times I/\sim$ with $j(x) = [x, 0]$ for $x \in A$ and $j(x) = [x, 1]$ for $x \in B$. We consider the diagram

$$\begin{array}{ccccc} X \times I \times I & \xrightarrow{\mathrm{id} \times T} & X \times I \times I & \xrightarrow{\Psi} & Z \\ \big\downarrow{\scriptstyle p \times \mathrm{id}} & & & \nearrow{\scriptstyle \Omega} & \\ (X \times I/\sim) \times I & & & & \end{array}$$

where $p: X \times I \to X \times I/\sim$ is the identification map and $T: I \times I \to I \times I$ switches the factors. The composite $\Psi \circ (\mathrm{id} \times T)$ factors through $p \times \mathrm{id}$ and hence induces a map $\Omega: (X \times I/\sim) \times I \to Z$. A required extension of φ and f is given by $\Omega \circ (j \times \mathrm{id})$: $X \times I \to (X \times I/\sim) \times I \to Z$. ▮

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