Math. Z. 233, 251–263 (2000)

Mathematische Zeitschrift

Two-primary algebraic K-theory of two-regular number fields

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Received August 1, 1998

Abstract. We explicitly calculate all the 2-primary higher algebraic K-groups of the rings of integers of all 2-regular quadratic number fields, cyclotomic number fields, or maximal real subfields of such. Here 2-regular means that (2) does not split in the number field, and its narrow Picard group is of odd order.

Mathematics Subject Classification (1991): 19D50, 11R11, 11R18, 11R37, 11R42, 19F27.

Introduction

As a consequence of Voevodsky's proof [V] of the Milnor Conjecture, the 2-primary algebraic K-groups of rings of integers in arbitrary number fields were computed in [RW], using the Bloch–Lichtenbaum spectral sequence of [BL]. Similar results were obtained by Kahn in [K] for totally imaginary number fields. The results of these computations were expressed in terms of the étale cohomology groups of the number ring. Here we make these calculations completely explicit in the case of a large number of quadratic and cyclotomic number fields, and their maximal real subfields.

A number field F is called 2-*regular* [GJ] if the 2-Sylow subgroup $R_2(F)$ of the kernel in $K_2(F)$ of the regular symbols attached to the non-complex places of F, is trivial. This is equivalent to the combined hypothesis that the rational prime ideal (2) does not split into distinct prime ideals in $\mathcal{O}_F$, and that the narrow Picard group $\mathrm{Pic}_+(R_F)$ of the ring $R_F = \mathcal{O}_F[\frac{1}{2}]$ of 2-integers in F has odd order. (See §2 for definitions and further discussion.)

The 2-regular quadratic number fields were classified by Browkin and Schinzel in [BS]. All their 2-primary algebraic K-groups are determined in Theorems 3.2 and 4.2. More generally, a classification of 2-regular Abelian 2-extensions of $\mathbb{Q}$ was given by Gras in [G, p. 331].

We also consider the cyclotomic number fields $\mathbb{Q}(\zeta_q)$, and their maximal real subfields $\mathbb{Q}(\zeta_q + \bar{\zeta}_q)$. These can only be 2-regular when $q = p^m$ is a prime power. We note in Theorems 3.3 and 4.3 that these are all 2-regular when $p = 2$, and in Examples 3.5, 3.6 and 4.5 that these are all 2-regular when $q \neq 29$ is an odd prime power with $\varphi(q) \leq 66$, or a Sophie Germain prime, such that 2 is a primitive root mod q. In the 2-regular cases all the 2-primary algebraic K-groups of the cyclotomic fields and their maximal real subfields are calculated in Theorems 3.3, 3.4, 4.3 and 4.4.

We remark that in the 2-regular cases $K_4(\mathcal{O}_F) = K_4(R_F) = 0$ modulo odd torsion, so the map $K_4^M(F) \to K_4(F)$ from Milnor K-theory to algebraic K-theory is zero in degree 4, in these cases.

By Wiles' proof [Wi] of the Main Conjecture in Iwasawa theory, and Kolster's appendix in [RW], these calculations of 2-primary algebraic K-groups give 2-primary calculations of the values of the Dedekind zeta functions $\zeta_F(s)$ at negative odd integers, in the cases when F is totally real Abelian. We present these calculations for the 2-regular real quadratic or maximal real subfields of cyclotomic number fields in Theorems 5.2, 5.3 and 5.4.

1. Notations and review

Consider a number field F over $\mathbb{Q}$, with r_1 real and $2r_2$ complex embeddings. Units of F with positive image under each real embedding are called *totally positive*. We say that F is *totally imaginary* when $r_1 = 0$, F is *real* when $r_1 > 0$, and F is *totally real* when $r_2 = 0$. Let $\mathcal{O}_F$ be the ring of integers in F (a *number ring*), and let $R_F = \mathcal{O}_F[\frac{1}{2}]$ be the ring of 2-integers.

Recall that the *ideal class group* $\mathrm{Cl}(F)$ of F is the group of fractional ideals of F modulo the principal fractional ideals, the *narrow ideal class group* $\mathrm{Cl}_+(F)$ is the group of fractional ideals modulo the totally positive principal fractional ideals, the *Picard group* $\mathrm{Pic}(R_F)$ is the quotient group of $\mathrm{Cl}(F)$ by the classes of prime ideals above (2), while the *narrow Picard group* $\mathrm{Pic}_+(R_F)$ is the quotient group of $\mathrm{Cl}_+(F)$ by the (narrow) classes of prime ideals above (2). Also recall that the 2-*rank* $\mathrm{rk}_2\, A$ of an Abelian group A is the dimension of its exponent 2 subgroup ${}_2A$ as a vector space over $\mathbb{Z}/2$.

Let h_F be the class number of F, i.e., the order of $\mathrm{Cl}(F)$. Let s be the number of primes of $\mathcal{O}_F$ over the rational prime (2), let t be the 2-rank of the

Picard group $\mathrm{Pic}(R_F)$, and let u be the 2-rank of the narrow Picard group $\mathrm{Pic}_+(R_F)$.

Let $\zeta_n = \exp(2\pi\sqrt{-1}/n)$ be a primitive nth root of unity. For each integer i, $w_i^{(2)}(F)$ is defined as the maximal power 2^n of 2 such that $\mathrm{Gal}(F(\zeta_{2^n})/F)$ has exponent dividing i. If no maximal power exists, such as for $i = 0$, we write $w_i^{(2)}(F) = 2^\infty$. We say that F is *exceptional* if the Galois group $\mathrm{Gal}(F(\zeta_{2^n})/F)$ is not cyclic for some n. Any real number field is exceptional.

We write $v_2(i)$ for the 2-adic valuation of an integer i, i.e., the greatest number n such that 2^n divides i. Let $\cong_{(2)}$ denote isomorphism after 2-localization, which for finite Abelian groups amounts to isomorphism of 2-Sylow subgroups. Let $H^n_{\text{ét}}(R_F; M)$ be the nth étale cohomology group of R_F with coefficients in the étale sheaf M.

The following results were obtained in [RW] as applications of Voevodsky's proof of the Milnor conjecture [V], and the Bloch–Lichtenbaum spectral sequence [BL]. See also [K] and [W2].

Theorem 1.1 [RW, 0.4(a)]. *Let E be a totally imaginary number field. Then for all $n \geq 2$*

$$K_n(\mathcal{O}_E) \cong_{(2)} K_n(R_E) \cong_{(2)} \begin{cases} H^2_{\text{ét}}(R_E; \mathbb{Z}_2(i+1)) & \text{for } n = 2i, \\ \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i^{(2)}(E) & \text{for } n = 2i-1. \end{cases}$$

Theorem 1.2 [RW, 0.6]. *Let F be a real number field. Then for all $n \geq 2$*

$$K_n(\mathcal{O}_F) \cong_{(2)} K_n(R_F) \cong_{(2)} \begin{cases} H^2_{\text{ét}}(R_F; \mathbb{Z}_2(4k+1)) & \text{for } n = 8k, \\ \mathbb{Z}^{r_1+r_2} \oplus \mathbb{Z}/2 & \text{for } n = 8k+1, \\ H^2_{\text{ét}}(R_F; \mathbb{Z}_2(4k+2)) & \text{for } n = 8k+2, \\ \mathbb{Z}^{r_2} \oplus (\mathbb{Z}/2)^{r_1-1} & \\ \quad \oplus \mathbb{Z}/2w^{(2)}_{4k+2}(F) & \text{for } n = 8k+3, \\ (?) & \text{for } n = 8k+4, \\ \mathbb{Z}^{r_1+r_2} & \text{for } n = 8k+5, \\ \tilde{H}^2_{\text{ét}}(R_F; \mathbb{Z}_2(4k+2)) & \text{for } n = 8k+6, \\ \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w^{(2)}_{4k+4}(F) & \text{for } n = 8k+7. \end{cases}$$

Here $\tilde{H}^2_{\text{ét}}(R_F; \mathbb{Z}_2(4k+2))$ is the kernel of the natural surjective map

$$H^2_{\text{ét}}(R_F; \mathbb{Z}_2(4k+2)) \longrightarrow \bigoplus^{r_1} H^2_{\text{ét}}(\mathbb{R}; \mathbb{Z}_2(4k+2)) \cong (\mathbb{Z}/2)^{r_1}$$

induced by the r_1 real embeddings of F.

The missing group $K_{8k+4}(R_F)$ is isomorphic, modulo odd finite groups, to the maximal finite quotient of $K_{8k+5}(R_F; \mathbb{Q}_2/\mathbb{Z}_2)$. The latter K-group *fits into the short exact sequence*

$$\begin{aligned} 0 \to (\mathbb{Z}/2)^{r_1-1} &\longrightarrow K_{8k+5}(R_F;\mathbb{Q}_2/\mathbb{Z}_2) \\ &\longrightarrow (\mathbb{Q}_2/\mathbb{Z}_2)^{r_1+r_2} \oplus H^2_{\acute{e}t}(R_F;\mathbb{Z}_2(4k+3)) \to 0. \end{aligned}$$

Theorem 1.3 [RW, 0.7]**.** *Let F be a real number field. For $n \geq 2$*

$$\mathrm{rk}_2\, K_n(R) = \begin{cases} s+t-1 & \textit{for } n = 8k, \\ 1 & \textit{for } n = 8k+1, \\ r_1+s+t-1 & \textit{for } n = 8k+2, \\ r_1 & \textit{for } n = 8k+3, \\ s+u-1 & \textit{for } n = 8k+4, \\ 0 & \textit{for } n = 8k+5, \\ s+u-1 & \textit{for } n = 8k+6, \\ 1 & \textit{for } n = 8k+7. \end{cases}$$

Recall from e.g. [RW, 1.8, 4.4, 4.6, 6.11 and 6.12]:

Proposition 1.4. *Let F be a number field and $i \geq 0$ an integer. Then*

$$H^n_{\acute{e}t}(R_F;\mathbb{Q}_2/\mathbb{Z}_2(i)) \cong \begin{cases} \mathbb{Z}/w_i^{(2)}(F) & \textit{for } n = 0, \\ (\mathbb{Q}_2/\mathbb{Z}_2)^r & \\ \quad \oplus H^2_{\acute{e}t}(R_F;\mathbb{Z}_2(i)) & \textit{for } n = 1,\, i \neq 0,1, \\ (\mathbb{Z}/2)^{r_1} & \textit{for } n \geq 2,\, i-n \textit{ odd}, \\ 0 & \textit{for } n \geq 2,\, i-n \textit{ even}. \end{cases}$$

Here $r = r_2$ for i even, and $r = r_1 + r_2$ for i odd. $H^2_{\acute{e}t}(R_F;\mathbb{Z}_2(i))$ is a finite group for $i \neq 0, 1$, with 2-rank

$$\mathrm{rk}_2\, H^2_{\acute{e}t}(R_F;\mathbb{Z}_2(i)) = \begin{cases} r_1+s+t-1 & \textit{for } i \neq 0 \textit{ even}, \\ s+t-1 & \textit{for } i \neq 1 \textit{ odd}. \end{cases}$$

Also recall from [W1, 6.3]:

Proposition 1.5. *Let F be a number field. Let a be maximal such that $F(\sqrt{-1})$ contains a primitive 2^ath root of unity, let i be an integer, and let $b = v_2(i)$ be its 2-adic valuation.*

(a) If $\sqrt{-1} \in F$ then $w_i^{(2)}(F) = 2^{a+b}$.
(b) If $\sqrt{-1} \notin F$ and i is odd then $w_i^{(2)}(F) = 2$.
(c) If $\sqrt{-1} \notin F$, F is exceptional and i is even then $w_i^{(2)}(F) = 2^{a+b}$.
(d) If $\sqrt{-1} \notin F$, F is non-exceptional and i is even then $w_i^{(2)}(F) = 2^{a+b-1}$.

2. Two-regular number fields

Let F be a number field. The common kernel of the regular symbols on $K_2(F)$ associated to the finite places of F is called the *tame kernel*, and is isomorphic to $K_2(\mathcal{O}_F)$. Ignoring odd torsion, $K_2(\mathcal{O}_F) \cong_{(2)} K_2(R_F)$ by the localization sequence. By 1.1 and 1.2, we have a natural isomorphism $K_2(R_F) \cong_{(2)} H^2_{\text{ét}}(R_F; \mathbb{Z}_2(2))$. See also [T] and [S, Lemme 10].

In [G], Gras introduced the *modified tame kernel* $R_2(F)$, as the subgroup of the tame kernel where also the regular symbols associated to the r_1 real embeddings of F vanish. On 2-Sylow subgroups, these symbols are identified with the natural surjection

$$\alpha^2 : H^2_{\text{ét}}(R_F; \mathbb{Z}_2(2)) \to \bigoplus^{r_1} H^2_{\text{ét}}(\mathbb{R}; \mathbb{Z}_2(2)) \cong (\mathbb{Z}/2)^{r_1}$$

and so $R_2(F) \cong \tilde{H}^2_{\text{ét}}(R_F; \mathbb{Z}_2(2))$ in the notation of 1.2.

Definition 2.1. A number field F is 2*-regular* if the 2-Sylow subgroup of the modified tame kernel $R_2(F)$ is trivial.

As noted in 1.3, the 2-rank of $K_2(R_F)$ is $r_1 + s + t - 1$. Fixing r_1, the minimal possible 2-rank is thus r_1. Totally real fields satisfying 2.2(b) were studied by Berger in [B]. Some of the following characterizations were pointed out to us by an anonymous referee, and improve on our initial results.

Proposition 2.2. *For number fields F the following are equivalent:*

(a) F *is* 2*-regular.*
(b) $K_2(\mathcal{O}_F)\{2\} \cong K_2(R_F)\{2\} \cong (\mathbb{Z}/2)^{r_1}$ *is elementary Abelian of minimal* 2*-rank.*
(c) The ideal (2) *does not split in* F *and the narrow Picard group* $\mathrm{Pic}_+(R_F)$ *has odd order, i.e.,* $s = 1$ *and* $t = u = 0$.
(d)

$$H^2_{\text{ét}}(R_F; \mathbb{Z}_2(i)) \cong \begin{cases} (\mathbb{Z}/2)^{r_1} & \text{for } i \neq 0 \text{ even,} \\ 0 & \text{for } i \neq 1 \text{ odd.} \end{cases}$$

Proof. Suppose $i \neq 0$ is even. Then $H^n_{\text{ét}}(\mathbb{R}; \mathbb{Z}_2(i))$ is 0 for n odd and $\mathbb{Z}/2$ for $n > 0$ even. We compare the Bockstein sequences in étale cohomology for R_F and $\mathbb{R}$ under the r_1 real embeddings:

(2.3)

$$\begin{tikzcd}
0 \to H^1_{\text{ét}}(R_F; \mathbb{Z}_2(i))/2 \arrow[r] & H^1_{\text{ét}}(R_F; \mathbb{Z}/2(i)) \arrow[r] \arrow[d, "\alpha^1"] & {}_2H^2_{\text{ét}}(R_F; \mathbb{Z}_2(i)) \to 0 \arrow[d] \\
& \bigoplus^{r_1} H^1_{\text{ét}}(\mathbb{R}; \mathbb{Z}/2(i)) \arrow[r, "\cong"] & \bigoplus^{r_1} {}_2H^2_{\text{ét}}(\mathbb{R}; \mathbb{Z}_2(i))
\end{tikzcd}$$

Similarly:

$$
(2.4)\qquad
\begin{array}{ccc}
H^2_{\text{ét}}(R_F;\mathbb{Z}_2(i))/2 & \xrightarrow{\cong} & H^2_{\text{ét}}(R_F;\mathbb{Z}/2(i)) \\
\downarrow & & \downarrow \\
\bigoplus^{r_1} H^2_{\text{ét}}(\mathbb{R};\mathbb{Z}_2(i))/2 & \xrightarrow{\cong} & \bigoplus^{r_1} H^2_{\text{ét}}(\mathbb{R};\mathbb{Z}/2(i))
\end{array}
$$

Note that cohomology with coefficients in $\mathbb{Z}/2(i)$ is independent of i. By the Kummer sequence $H^1_{\text{ét}}(R_F;\mathbb{Z}/2(1)) \cong R_F^\times/2 \oplus {}_2\operatorname{Pic}(R_F)$ and $H^2_{\text{ét}}(R_F;\mathbb{Z}/2(1)) \cong \operatorname{Pic}(R_F)/2 \oplus {}_2\operatorname{Br}(R_F)$, where $\operatorname{Br}(R_F)$ is the Brauer group. By the Brauer–Hasse–Noether theorem the natural map

$$
{}_2\operatorname{Br}(R_F) \to \bigoplus^{r_1} {}_2\operatorname{Br}(\mathbb{R}) \cong (\mathbb{Z}/2)^{r_1}
$$

is surjective, so the right-hand vertical map in (2.4) is surjective, which explains why the left-hand vertical map is always surjective.

Now suppose that F is 2-regular. Then $R_2(F)\{2\} \cong \tilde{H}^2_{\text{ét}}(R_F;\mathbb{Z}_2(2)) = 0$, so $K_2(R_F)\{2\} \cong H^2_{\text{ét}}(R_F;\mathbb{Z}_2(2)) \cong (\mathbb{Z}/2)^{r_1}$. This proves $(a) \Rightarrow (b)$. The converse $(b) \Rightarrow (a)$ is equally obvious.

Let $R_{F+}^\times \subset R_F^\times$ denote the groups of totally positive units in R_F and units in R_F, respectively. Then there is an exact sequence

$$
(2.5)\qquad 0 \to R_{F+}^\times \to R_F^\times \xrightarrow{\sigma} (\mathbb{Z}/2)^{r_1} \to \operatorname{Pic}_+(R_F) \to \operatorname{Pic}(R_F) \to 0
$$

obtained by applying the snake lemma to the exact sequences defining $\operatorname{Pic}_+(R_F)$ and $\operatorname{Pic}(R_F)$. Here σ is the *signature map*.

Now suppose that $K_2(R_F)\{2\} \cong H^2_{\text{ét}}(R_F;\mathbb{Z}_2(2)) \cong (\mathbb{Z}/2)^{r_1}$. Evaluating 2-ranks we find $r_1+s+t-1 = r_1$, so $s=1$ and $t=0$. Hence (2) does not split, and $\operatorname{Pic}(R_F)$ has odd order. The right-hand vertical map in (2.3) is an isomorphism for $i=2$, so the middle vertical map

$$
\alpha^1 : H^1_{\text{ét}}(R_F;\mathbb{Z}/2(i)) \to \bigoplus^{r_1} H^1_{\text{ét}}(\mathbb{R};\mathbb{Z}/2(i))
$$

is a surjection for all i. Since ${}_2\operatorname{Pic}(R_F) = 0$ we can identify this map with the signature map σ, extended over the natural surjection $R_F^\times \to R_F^\times/2$. Hence σ is surjective and $\operatorname{Pic}_+(R_F) \cong \operatorname{Pic}(R_F)$ has odd order. Thus $u=0$ and $(b) \Rightarrow (c)$.

Suppose $s=1$ and $t=u=0$. Note that the image of the signature map can be identified with that of α^1 since $t=0$, and σ is surjective by (2.5).

The group $H^2_{\text{ét}}(R_F;\mathbb{Z}_2(i))$ is a finitely generated $\mathbb{Z}_2$-module for all integers i, see [RW, 6.12]. Hence the Bockstein exact sequence

$$\begin{aligned} 0 \longrightarrow H^2_{\text{ét}}(R_F;\mathbb{Z}_2(i))/2 &\longrightarrow H^2_{\text{ét}}(R_F;\mathbb{Z}/2(i)) \\ \longrightarrow {}_2H^3_{\text{ét}}(R_F;\mathbb{Z}/2(i)) &\longrightarrow 0 \end{aligned}$$

and 1.4 shows that $H^2_{\text{ét}}(R_F;\mathbb{Z}_2(i)) = 0$ for $i \neq 1$ odd, and that $\operatorname{rk}_2 H^2_{\text{ét}}(R_F;\mathbb{Z}_2(i)) \leq r_1$ for $i \neq 0$ even. Surjectivity of α^1 shows that r_1 is also a lower bound for the 2-rank of $H^2_{\text{ét}}(R_F;\mathbb{Z}_2(i))$ for $i \neq 0$ even. From surjectivity of the right-hand vertical map in (2.3) we find that the surjection $H^2_{\text{ét}}(R_F;\mathbb{Z}_2(i)) \to (\mathbb{Z}/2)^{r_1}$ is in fact an isomorphism. This proves that $(c) \Rightarrow (d)$.

Finally $(d) \Rightarrow (b)$ by specialization, which completes the proof. □

We say that an extension E/F of number fields is *finitely ramified* if a finite (non-Archimedean) prime of F ramifies in the extension.

Proposition 2.6. *Let E/F be an extension of number fields that does not contain any nontrivial Abelian finitely ramified subextension. For example, E can be a quadratic finitely ramified extension. Then the narrow norm maps* $\mathrm{Cl}_+(E) \to \mathrm{Cl}_+(F)$ *and* $\mathrm{Pic}_+(R_E) \to \mathrm{Pic}_+(R_F)$ *are surjective.*

Proof. By class field theory, $\mathrm{Cl}(F)$ is the Galois group of the maximal Abelian totally unramified extension $H(F)$ of F, called the Hilbert class field. Likewise $\mathrm{Cl}_+(F)$ is the Galois group of the maximal Abelian finitely unramified extension $H_+(F)$ of F, called the *narrow class field* of F.

The composite $E \cdot H_+(F)$ is Abelian and finitely unramified over E, because $H_+(F)$ is Abelian and finitely unramified over F. Hence $E \cdot H_+(F)$ is contained in $H_+(E)$. By assumption $E \cap H_+(F) = F$, so we obtain a surjective group homomorphism:

$$\begin{aligned} \mathrm{Cl}_+(E) = \mathrm{Gal}(H_+(E)/E) &\to \mathrm{Gal}(E \cdot H_+(F)/E) \\ &\cong \mathrm{Gal}(H_+(F)/F) = \mathrm{Cl}_+(F) \end{aligned}$$

As in [Wa, p. 340] this equals the map induced by the norm on fractional ideals. The surjection $\mathrm{Cl}_+(E) \to \mathrm{Pic}_+(R_E)$ annihilates the classes of ideals of $\mathcal{O}_E$ lying over (2), and likewise for F. The norm preserves ideals over (2), and so we obtain an induced surjection $\mathrm{Pic}_+(R_E) \to \mathrm{Pic}_+(R_F)$, as claimed. □

We now consider cyclotomic fields. Let $E = \mathbb{Q}(\zeta_q)$ be the qth cyclotomic field. Its maximal real subfield is $F = \mathbb{Q}(\zeta_q + \bar{\zeta}_q)$.

Proposition 2.7. *(a) The qth cyclotomic field $E = \mathbb{Q}(\zeta_q)$ is 2-regular if and only if $q = 2^m$, or if $q = p^m$ where p an odd prime, 2 is a primitive root mod q and $\mathrm{Pic}(R_E)$ has odd order.*

(b) The maximal real subfield $F = \mathbb{Q}(\zeta_q + \bar{\zeta}_q)$ of the qth cyclotomic field is 2-regular if and only if $q = 2^m$, or if $q = p^m$ where p is an odd prime, 2 is a primitive root mod q and $\mathrm{Pic}_+(R_F)$ has odd order.

Proof. In both cases it follows easily from the cyclotomic reciprocity law [Wa, p. 14] that the prime ideal (2) does not split in E, resp. in F, if and only if $q = 2^m$, or if $q = p^m$ with p an odd prime and 2 a primitive root mod q. Hence in these cases the number field is 2-regular if and only if its narrow Picard group has odd order. Of course $\mathrm{Pic}(R_E) = \mathrm{Pic}_+(R_E)$ for E totally imaginary.

When $q = 2^m$ the class number of $E = \mathbb{Q}(\zeta_{2^m})$ is odd [Wa, 10.5]. In this case the quadratic extension E/F is finitely ramified above (2), so $\mathrm{Pic}(R_E)$ and $\mathrm{Pic}_+(R_F)$ have odd order by 2.6. □

3. Totally imaginary number fields

Theorem 3.1. *Let E be a 2-regular totally imaginary number field. Then for $n \geq 2$*

$$K_n(\mathcal{O}_E) \cong_{(2)} K_n(R_E) \cong_{(2)} \begin{cases} 0 & \text{for } n = 2i, \\ \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i^{(2)}(E) & \text{for } n = 2i-1. \end{cases}$$

Proof. This is clear from 1.1 and 2.2. □

Theorem 3.2. *Let $E = \mathbb{Q}(\sqrt{m})$ be a quadratic number field with $m < 0$ square free. Then E is 2-regular if and only if $m = -1$, $m = -2$, $m = -p$ or $m = -2p$ with $p \equiv \pm 3 \bmod 8$ prime. In these cases, for $n \geq 2$*

$$K_n(\mathbb{Z}[\sqrt{-1}, \frac{1}{2}]) \cong_{(2)} \begin{cases} 0 & \text{for } n = 2i, \\ \mathbb{Z} \oplus \mathbb{Z}/2^{2+v_2(i)} & \text{for } n = 2i-1, \end{cases}$$

while

$$K_n(\mathbb{Z}[\sqrt{m}, \frac{1}{2}]) \cong_{(2)} \begin{cases} 0 & \text{for } n = 2i, \\ \mathbb{Z} \oplus \mathbb{Z}/2^{2+v_2(i)} & \text{for } n = 2i-1,\ i \text{ even}, \\ \mathbb{Z} \oplus \mathbb{Z}/2 & \text{for } n = 2i-1,\ i \text{ odd}, \end{cases}$$

for $m \neq 1$.

Proof. The classification of 2-regular imaginary quadratic number fields comes from [BS]. The numbers $w_i^{(2)}(E)$ are obtained from 1.5, by noting that the fields $\mathbb{Q}(\sqrt{m})$ are non-exceptional for $m = -1, -2$, and exceptional otherwise. $\square$

Theorem 3.3. *Let $E = \mathbb{Q}(\zeta_{2^m})$ with $m \geq 2$. Then E is 2-regular. Hence for $n \geq 2$*

$$K_n(\mathbb{Z}[\zeta_{2^m}]) \cong_{(2)} \begin{cases} 0 & \text{for } n = 2i, \\ \mathbb{Z}^{2^{m-2}} \oplus \mathbb{Z}/2^{m+v_2(i)} & \text{for } n = 2i-1. \end{cases}$$

Proof. We use 2.7 and 3.1. The calculation of $w_i^{(2)}(E) = 2^{m+v_2(i)}$ is immediate from 1.5, since $\sqrt{-1} \in E$. $\square$

The Euler φ-function $\varphi(n)$ equals the number of units in the ring $\mathbb{Z}/n$. This is the degree of $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$.

Theorem 3.4. *Let $E = \mathbb{Q}(\zeta_q)$ with q an odd prime power. Suppose 2 is a primitive root modulo q and $\operatorname{Pic}(R_E)$ has odd order. Then for $n \geq 2$*

$$K_n(\mathbb{Z}[\zeta_q]) \cong_{(2)} \begin{cases} 0 & \text{for } n = 2i, \\ \mathbb{Z}^{\varphi(q)/2} \oplus \mathbb{Z}/2^{2+v_2(i)} & \text{for } n = 2i-1,\ i \text{ even}, \\ \mathbb{Z}^{\varphi(q)/2} \oplus \mathbb{Z}/2 & \text{for } n = 2i-1,\ i \text{ odd}. \end{cases}$$

Proof. The discriminant of E is a power of q, so (2) is inert in these cases. We apply 3.1. The numbers $w_i^{(2)}(E)$ are obtained from 1.5 by noting that E is exceptional. $\square$

Example 3.5. Let $E = \mathbb{Q}(\zeta_q)$ with q an odd prime power. By the table in [Wa, pp. 352–353], the class number h_E is odd for all prime powers q with Euler φ-function $\varphi(q) \leq 66$, except $q = 29$. Hence the hypothesis of 3.4 holds for all odd prime powers $q \neq 29$ with $\varphi(q) \leq 66$, such that 2 is a primitive root modulo q.

A *Sophie Germain prime* p is a prime such that $(p-1)/2$ is also prime.

Example 3.6. Let $E = \mathbb{Q}(\zeta_p)$ with p a Sophie Germain prime. By a theorem of Estes [E], h_E is odd for Sophie Germain primes p such that (2) is inert in the maximal real subfield of E. This is certainly the case if (2) is inert in E. Thus the hypothesis of 3.4 holds for all Sophie Germain primes p with 2 a primitive root modulo p. These are precisely the Sophie Germain primes $p \equiv 3 \bmod 8$, and $p = 5$. The first few examples of such primes are $p = 5, 11, 59, 83, 107$ and 179.

4. Real number fields

Theorem 4.1. *Let F be a 2-regular real number field. Then for $n \geq 2$*

$$K_n(\mathcal{O}_F) \cong_{(2)} K_n(R_F) \cong_{(2)} \begin{cases} 0 & \textit{for } n = 8k, \\ \mathbb{Z}^{r_1+r_2} \oplus \mathbb{Z}/2 & \textit{for } n = 8k+1, \\ (\mathbb{Z}/2)^{r_1} & \textit{for } n = 8k+2, \\ \mathbb{Z}^{r_2} \oplus (\mathbb{Z}/2)^{r_1-1} & \\ \quad \oplus \mathbb{Z}/2w_{4k+2}^{(2)}(F) & \textit{for } n = 8k+3, \\ 0 & \textit{for } n = 8k+4, \\ \mathbb{Z}^{r_1+r_2} & \textit{for } n = 8k+5, \\ 0 & \textit{for } n = 8k+6, \\ \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_{4k+4}^{(2)}(F) & \textit{for } n = 8k+7. \end{cases}$$

The torsion subgroups in degrees $n = 8k+1$ and $n = 8k+2$ are detected by the natural maps

$$K_n(R_F) \longrightarrow \bigoplus^{r_1} \pi_n(\mathbb{Z} \times BO) \cong (\mathbb{Z}/2)^{r_1}$$

induced by the r_1 real embeddings.

Proof. The formulas for the algebraic K-groups follow from 1.2, 1.3 and 2.2. Note that there is no extension issue in degree $8k+4$ because the 2-rank there is $s + u - 1 = 0$. The Adams μ-element μ_{8k+1} in the stable $(8k+1)$-stem maps to a class in $K_{8k+1}(R_F)$ that is detected by real topological K-theory [A]. The claims in degree $8k+1$ and $8k+2$ follow from this, and 2.2. $\square$

Theorem 4.2. *Let $F = \mathbb{Q}(\sqrt{m})$ be a quadratic number field with $m > 0$ square free. Then F is 2-regular if and only if $m = 2$, $m = p$ or $p = 2p$ with $p \equiv \pm 3 \bmod 8$ prime. In these cases, for $n \geq 2$*

$$K_n(\mathbb{Z}[\sqrt{m}, \frac{1}{2}]) \cong_{(2)} \begin{cases} 0 & \textit{for } n = 8k, \\ \mathbb{Z}^2 \oplus \mathbb{Z}/2 & \textit{for } n = 8k+1, \\ (\mathbb{Z}/2)^2 & \textit{for } n = 8k+2, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2w_{4k+2}^{(2)}(F) & \textit{for } n = 8k+3, \\ 0 & \textit{for } n = 8k+4, \\ \mathbb{Z}^2 & \textit{for } n = 8k+5, \\ 0 & \textit{for } n = 8k+6, \\ \mathbb{Z}/w_{4k+4}^{(2)}(F) & \textit{for } n = 8k+7. \end{cases}$$

When i is even, $w_i^{(2)}(F) = 2^{3+v_2(i)}$ for $m = 2$, while $w_i^{(2)}(F) = 2^{2+v_2(i)}$ for $m \neq 2$.

Proof. The classification of 2-regular real quadratic number fields also comes from [BS]. To determine the numbers $w_i^{(2)}(F)$, note that F contains ζ_8 only for $m = 2$, and that F is real, thus exceptional. $\square$

Theorem 4.3. *Let $F = \mathbb{Q}(\zeta_{2^m} + \bar\zeta_{2^m})$ with $m \geq 2$. Then F is 2-regular. Hence for $n \geq 2$*

$$K_n(\mathbb{Z}[\zeta_{2^m} + \bar\zeta_{2^m}]) \cong_{(2)} \begin{cases} 0 & \text{for } n = 8k, \\ \mathbb{Z}^{2^{m-2}} \oplus \mathbb{Z}/2 & \text{for } n = 8k+1, \\ (\mathbb{Z}/2)^{2^{m-2}} & \text{for } n = 8k+2, \\ (\mathbb{Z}/2)^{2^{m-2}-1} \oplus \mathbb{Z}/2^{m+2} & \text{for } n = 8k+3, \\ 0 & \text{for } n = 8k+4, \\ \mathbb{Z}^{2^{m-2}} & \text{for } n = 8k+5, \\ 0 & \text{for } n = 8k+6, \\ \mathbb{Z}/2^{m+2+v_2(k+1)} & \text{for } n = 8k+7. \end{cases}$$

Proof. We use 2.7 and 4.1. The numbers $w_i^{(2)}(F)$ are obtained from 1.5. $\square$

Theorem 4.4. *Let $E = \mathbb{Q}(\zeta_q)$ and $F = \mathbb{Q}(\zeta_q + \bar\zeta_q)$ with q an odd prime power such that 2 is a primitive root mod q. Suppose $\mathrm{Pic}(R_E)$ has odd order, or more generally that $\mathrm{Pic}_+(R_F)$ has odd order. Then F is 2-regular, and for $n \geq 2$*

$$K_n(\mathbb{Z}[\zeta_q + \bar\zeta_q]) \cong_{(2)} \begin{cases} 0 & \text{for } n = 8k, \\ \mathbb{Z}^{\varphi(q)/2} \oplus \mathbb{Z}/2 & \text{for } n = 8k+1, \\ (\mathbb{Z}/2)^{\varphi(q)/2} & \text{for } n = 8k+2, \\ (\mathbb{Z}/2)^{\varphi(q)/2-1} \oplus \mathbb{Z}/16 & \text{for } n = 8k+3, \\ 0 & \text{for } n = 8k+4, \\ \mathbb{Z}^{\varphi(q)/2} & \text{for } n = 8k+5, \\ 0 & \text{for } n = 8k+6, \\ \mathbb{Z}/2^{4+v_2(k+1)} & \text{for } n = 8k+7. \end{cases}$$

In degree $n = 8k+3$ the unit homomorphism $\mathbb{Z} \to \mathbb{Z}[\zeta_q + \bar\zeta_q]$ takes $K_n(\mathbb{Z}) \cong_{(2)} \mathbb{Z}/16$ isomorphically to the $\mathbb{Z}/16$-summand in $K_n(\mathbb{Z}[\zeta_q + \bar\zeta_q])$.

Proof. Write $q = p^e$ for an odd prime p. Then (p) is totally ramified in E, hence E/F is finitely ramified at the prime above (p). Thus the assumption that $\mathrm{Pic}(R_E)$ has odd order and 2.6 imply that $\mathrm{Pic}_+(R_F)$ has odd order. We can therefore quote 4.1. To apply 1.5, note that $\mathbb{Q}(\zeta_q + \bar\zeta_q, \sqrt{-1})$ is contained in $\mathbb{Q}(\zeta_q, \sqrt{-1}) = \mathbb{Q}(\zeta_{4q})$, hence contains ζ_4 but not ζ_8.

The map $\mathbb{Z} \to \mathbb{Z}[\zeta_q + \bar\zeta_q]$ induces isomorphisms $H^0_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Q}_2/\mathbb{Z}_2(4k+2)) \to H^0_{\text{ét}}(R_F; \mathbb{Q}_2/\mathbb{Z}_2(4k+2))$, and therefore takes $K_{8k+3}(\mathbb{Z}[\frac{1}{2}]) \cong \mathbb{Z}/16$

split injectively to a $\mathbb{Z}/16$-summand of $K_{8k+3}(R_F)$, by naturality of 1.2. The final claim follows. $\square$

Examples 4.5. As noted in 3.5 the hypothesis of this theorem holds for all odd prime powers $q \neq 29$ with $\varphi(q) \leq 66$ and 2 a primitive root mod q.

Likewise by Estes' theorem [E] the hypothesis holds for all Sophie Germain primes p with 2 a primitive root mod p.

5. Values of zeta functions

Let $\#A$ denote the order of a finite group A, and let $\zeta_F(s)$ be the zeta-function of the number field F. Recall the following theorem of Kolster, Rognes and Weibel, combining the results of [RW] with Wiles' proof [Wi] of the Main Conjecture in Iwasawa theory.

Theorem 5.1 [RW, 0.1]**.** *Let F be a totally real Abelian number field with r_1 real embeddings. Then for all even $i \geq 2$*

$$2^{r_1} \cdot \frac{\#K_{2i-2}(R_F)}{\#K_{2i-1}(R_F)} = \frac{\#H^2_{\acute{e}t}(R_F; \mathbb{Z}_2(i))}{\#H^1_{\acute{e}t}(R_F; \mathbb{Z}_2(i))} = \zeta_F(1-i)$$

up to odd multiples.

Let $v_2(q) = v_2(a) - v_2(b)$ denote the 2-adic valuation of the rational number $q = a/b$.

Theorem 5.2. *Let $F = \mathbb{Q}(\sqrt{m})$. Then for all even $i \geq 2$*

$$v_2(\zeta_F(1-i)) = -1 - v_2(i)$$

when $m = 2$, while

$$v_2(\zeta_F(1-i)) = -v_2(i)$$

when $m = p$ or $m = 2p$ with $p \equiv \pm 3 \bmod 8$ prime.

Theorem 5.3. *Let $F = \mathbb{Q}(\zeta_{2^m} + \bar{\zeta}_{2^m})$ with $m \geq 2$. Then for all even $i \geq 2$*

$$v_2(\zeta_F(1-i)) = 2^{m-2} - (m + v_2(i)).$$

Theorem 5.4. *Let $F = \mathbb{Q}(\zeta_q + \bar{\zeta}_q)$ with q an odd prime power and 2 a primitive root mod q. Suppose $\mathrm{Pic}_+(R_F)$ has odd order. Then for all even $i \geq 2$*

$$v_2(\zeta_F(1-i)) = \varphi(q)/2 - (2 + v_2(i)).$$

This accounts for all 2-regular real quadratic or maximal real subfields of cyclotomic number fields. These results may be reformulated in terms of L-functions by the identity [Wa, 4.3]

$$\zeta_F(s) = \prod_{\chi \in X} L(s, \chi)$$

where X is the group of Dirichlet characters associated to the Abelian field F, and s is complex. In turn the L-function values at negative integers can be expressed in terms of generalized Bernoulli numbers using the formula [Wa, 4.2]

$$L(1-n, \chi) = -\frac{B_{n,\chi}}{n}$$

for integers $n \geq 1$. We omit the easy transcriptions.

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