

INTRODUCTION TO SPECTRAL SEQUENCES, II

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We introduce the Atiyah–Hirzebruch [AH61] and Serre [Ser51] spectral sequences, in their homological and cohomological versions, with some examples.

1. THE ATIYAH–HIRZEBRUCH SPECTRAL SEQUENCE (AHSS)

Let X be a CW complex, with skeleton filtration

$$\emptyset = X^{(-1)} \subset \dots \subset X^{(s-1)} \subset X^{(s)} \subset \dots \subset X,$$

and let $M_*(-)$ be a generalized homology theory. It is represented by a spectrum M , with

$$M_n(X) = \pi_n(M \wedge X_+) = [S^n, M \wedge X_+]$$

for all $n \in \mathbb{Z}$. Its *coefficient groups* M_* are

$$M_n = M_n(*) = \pi_n(M).$$

The associated generalized cohomology theory $M^*(-)$ is given by

$$M^n(X) = \pi_{-n}F(X_+, M) = [X_+, \Sigma^n M].$$

Its coefficient groups M^* are

$$M^n = M^n(*) = \pi_{-n}(M).$$

The skeleton filtration leads to a sequence of spectra

$$* = Y_{-\infty} = Y_{-1} \rightarrow \dots \rightarrow Y_{s-1} \xrightarrow{f_s} Y_s \rightarrow \dots \rightarrow Y_{\infty}$$

with $Y_s = M \wedge X_+^{(s)}$ and $Y_{\infty} \simeq M \wedge X_+$. For each s , we have the exact sequence

$$\dots \rightarrow M_*(X^{(s-1)}) \xrightarrow{\alpha_s} M_*(X^{(s)}) \xrightarrow{\beta_s} M_*(X^{(s)}, X^{(s-1)}) \xrightarrow{\gamma_s} M_{*-1}(X^{(s-1)}) \rightarrow \dots$$

The associated unrolled exact couple

$$\begin{array}{ccccccc} A_{-\infty} & \longrightarrow & \dots & \longrightarrow & A_{s-1} & \xrightarrow{\alpha_s} & A_s & \longrightarrow & \dots & \longrightarrow & A_{\infty} \\ & & & & \swarrow \gamma_s & & \downarrow \beta_s & & & & \\ & & & & & & E_s & & & & \end{array}$$

has $A_s = M_*(X^{(s)})$, $A_{\infty} \cong M_*(X)$, $A_{-\infty} = RA_{-\infty} = 0$ and

$$E_s = \pi_*(Y_s, Y_{s-1}) = M_*(X^{(s)}, X^{(s-1)}).$$

By excision with respect to the pushout square

$$\begin{array}{ccc} \coprod_{\alpha} \partial D^s & \longrightarrow & \coprod_{\alpha} D^s \\ \varphi \downarrow & & \downarrow \Phi \\ X^{(s-1)} & \longrightarrow & X^{(s)} \end{array}$$

we have isomorphisms

$$C_s^{\text{CW}}(X; M_{*-s}) \cong \bigoplus_{\alpha} M_*(D^s, \partial D^s) \cong M_*(\prod_{\alpha} D^s, \prod_{\alpha} \partial D^s) \cong M_*(X^{(s)}, X^{(s-1)}),$$

where C_s^{CW} denotes the *cellular* chain complex, and α ranges over the set of s -cells in X . Hence the associated spectral sequence

$$\mathcal{E}_{s,*-s}^1 = (\mathcal{E}_s^1)_* \implies M_*(X)$$

has $\mathcal{E}^1$ -page

$$\mathcal{E}_{s,*-s}^1 = C_s^{\text{CW}}(X; M_{*-s}).$$

Introducing the complementary degree $t = * - s$, this can be written

$$\mathcal{E}_{s,t}^1 = C_s^{\text{CW}}(X; M_t) \implies M_{s+t}(X).$$

This is the homological Atiyah–Hirzebruch spectral sequence for $M_*(-)$ and X . It is concentrated in the half-plane with $s \geq 0$, hence is a strongly convergent spectral sequence with exiting differentials.

The d^1 -differential

$$d_{s,t}^1: \mathcal{E}_{s,t}^1 = C_s^{\text{CW}}(X; M_t) \longrightarrow C_{s-1}^{\text{CW}}(X; M_t) = \mathcal{E}_{s-1,t}^1$$

is the composite

$$M_{s+t}(X^{(s)}, X^{(s-1)}) \xrightarrow{\partial} M_{s+t-1}(X^{(s-1)}) \longrightarrow M_{s+t-1}(X^{(s-1)}, X^{(s-2)}),$$

hence equals the cellular boundary homomorphism. Passing to homology, we obtain the $\mathcal{E}^2$ -page

$$\mathcal{E}_{s,t}^2 = H_s^{\text{CW}}(X; M_t),$$

where H_*^{CW} denotes cellular homology.

Theorem 1.1. *For each CW complex X and spectrum M , there is a strongly convergent Atiyah–Hirzebruch spectral sequence*

$$\mathcal{E}_{s,t}^2 = H_s(X; M_t) \implies M_{s+t}(X).$$

We can also write this as $\mathcal{E}_{*,*}^2 = H_*^{\text{CW}}(X; M_*) \implies M_*(X)$.

2. UNIQUENESS OF ORDINARY HOMOLOGY

If $M_*(X) = H_*(X; \Gamma)$ is ordinary homology with coefficients in an abelian group Γ , then $M_0 = \Gamma$ and $M_t = 0$ for $t \neq 0$. Conversely, if $M_*(-)$ is a homology theory with $M_* = \Gamma$ concentrated in degree 0, then the Atiyah–Hirzebruch spectral sequence for M and any CW complex X has the form

$$\mathcal{E}_{s,t}^2 = H_s^{\text{CW}}(X; M_t) = \begin{cases} H_s^{\text{CW}}(X; \Gamma) & \text{for } t = 0, \\ 0 & \text{for } t \neq 0. \end{cases}$$

Hence it is concentrated on the s -axis. It follows that $d^r = 0$ for $r \geq 2$, since d^r increases the t -index by $r - 1$, and therefore maps from or to a trivial group in each instance. Thus $\mathcal{E}_{s,t}^2 = \mathcal{E}_{s,t}^{\infty}$. By strong convergence, in each degree n the abutment $M_n(X)$ is exhaustively filtered by the images

$$F_s M_n(X) = \text{im}(M_n(X^{(s)}) \rightarrow M_n(X)),$$

with $F_s M_n(X)/F_{s-1} M_n(X) \cong \mathcal{E}_{s,n-s}^\infty$ and $F_s M_n(X) = 0$ for $s < 0$. Hence $F_{s-1} M_n(X) = F_s M_n(X)$ in all cases, except when $s = n$, in which case $F_{n-1} M_n(X) = 0$ and $F_n M_n(X) \cong \mathcal{E}_{n,0}^\infty \cong H_n^{\text{CW}}(X; \Gamma)$. It follows that

$$M_n(X) \cong \operatorname{colim}_s F_s M_n(X) \cong H_n^{\text{CW}}(X; \Gamma).$$

This proves the Eilenberg–Steenrod characterization [ES45] of ordinary homology: Any homology theory $M_*(-)$ that has coefficient groups $M_* = \Gamma$ concentrated in degree 0 is naturally isomorphic to cellular homology with coefficients in Γ . This applies, in particular, to singular homology with coefficients in Γ , recovering the fact that singular and cellular homology agree for CW complexes. Hence the Atiyah–Hirzebruch spectral sequence can be written

$$\mathcal{E}_{s,t}^2 = H_s(X; M_t) \Longrightarrow M_{s+t}(X)$$

or

$$\mathcal{E}_{*,*}^2 = H_*(X; M_*) \Longrightarrow M_*(X).$$

The construction also extends to the case of pairs (X, A) , to reduced homology theories, and to the case of spectra X . (For spectra X we only consider reduced homology, writing $M_*(X)$ in place of $\tilde{M}_*(X)$.)

3. THE STABLE HUREWICZ THEOREM

Let S be the sphere spectrum, with coefficient groups

$$\pi_n(S) = \operatorname{colim}_k \pi_{n+k}(S^k)$$

the stable homotopy groups of spheres, and associated homology theory

$$S_*(X) = \pi_n(S \wedge X_+) = \operatorname{colim}_k \pi_{n+k}(S^k \wedge X_+)$$

the unreduced stable homotopy groups of X , also denoted $\pi_n^S(X_+)$. For based spaces X , the reduced homology theory is $\tilde{S}_*(X) = \pi_n(S \wedge X) = \pi_n^S(X)$.

By cellular approximation, $\pi_n(S) = 0$ for $n < 0$. Hopf proved that $\pi_0(S) \cong \mathbb{Z}$ via the degree of a map, and Freudenthal used his suspension theorem to prove that $\pi_1(S) \cong \mathbb{Z}/2\{\eta\}$, generated by the stable class of the Hopf fibration $\eta: S^3 \rightarrow S^2$. Suppose X is $(n-1)$ -connected. The Atiyah–Hirzebruch spectral sequence

$$\mathcal{E}_{s,t}^2 = \tilde{H}_s(X; \pi_t(S)) \Longrightarrow \pi_{s+t}^S(X)$$

is then concentrated in the quadrant $s \geq n, t \geq 0$. The bidegrees in the (s, t) -plane contributing to $\pi_*^S(X)$ for $* \leq n+1$ are shown below.

0	?	?	?	?
0	$\tilde{H}_n(X; \mathbb{Z}/2)$	?	?	?
0	$\tilde{H}_n(X; \mathbb{Z})$	$\tilde{H}_{n+1}(X; \mathbb{Z})$	$\tilde{H}_{n+2}(X; \mathbb{Z})$	?
0	0	0	0	0

The only possible differential in this region is $d_{n+2,0}^2$. Hence $\mathcal{E}_{n,1}^\infty = \text{cok}(d_{n+2,0}^2)$. In total degree n , the only nonzero group is $\mathcal{E}_{n,0}^\infty = \mathcal{E}_{n,0}^2 \cong \tilde{H}_n(X; \mathbb{Z})$, recovering the stable Hurewicz theorem

$$h_n: \pi_n^S(X) \cong \tilde{H}_n(X; \mathbb{Z})$$

for an $(n-1)$ -connected space (or spectrum) X . In total degree $(n+1)$, we have $\mathcal{E}_{n,1}^\infty$ as above, and $\mathcal{E}_{n+1,0}^\infty \cong \tilde{H}_{n+1}(X; \mathbb{Z})$. Hence there is a short exact sequence

$$0 \rightarrow \text{cok}(d_{n+2,0}^2) \longrightarrow \pi_{n+1}^S(X) \xrightarrow{h_{n+1}} \tilde{H}_{n+1}(X; \mathbb{Z}) \rightarrow 0.$$

We can expand this to a four-term exact sequence

$$\tilde{H}_{n+2}(X; \mathbb{Z}) \xrightarrow{d^2} \tilde{H}_n(X; \mathbb{Z}/2) \longrightarrow \pi_{n+1}^S(X) \xrightarrow{h_{n+1}} \tilde{H}_{n+1}(X; \mathbb{Z}) \rightarrow 0.$$

In particular, h_{n+1} is always surjective.

((ETC: Serre proved that $\pi_n(S)$ is finite for each $n \geq 1$. Can apply exact functor, such as rationalization or other localization, to a spectral sequence. This might not commute with the limits A_∞ , RA_∞ , $\lim_s F_s G$ and $R \lim_s F_s G$, but this is no concern if $A_s = 0$ for all $s < 0$. Rational equivalence $\pi_n(X) \rightarrow \tilde{H}_n(X)$.)

((In the case $X = MU$, the complex bordism spectrum, it is known that

$$H_*(MU) \cong H_*(BU) \cong B = \mathbb{Z}[b_i \mid i \geq 1]$$

with $|b_i| = 2i$, and that

$$\pi_*(MU) \cong L = \mathbb{Z}[x_i \mid i \geq 1]$$

with $|x_i| = 2i$. Here B is a self-dual Hopf algebra of symmetric polynomials, and L is the Lazard ring corepresenting one-dimensional commutative formal group laws. Kochman [Koc90] ran the Atiyah–Hirzebruch spectral sequence

$$\mathcal{E}_{s,t}^2 = H_s(MU; \pi_t(S)) \implies MU_{s+t}$$

in reverse, to determine $\pi_*(S)$ is a significant range.))

4. REINDEXING FOR COHOMOLOGY (AHSS)

The skeleton filtration also leads to a sequence of spectra

$$* = Z_{-\infty} \rightarrow \cdots \rightarrow Z_{-s-1} \rightarrow Z_{-s} \rightarrow \cdots \rightarrow Z_0 = Z_\infty = F(X_+, M)$$

with $Z_{-s} = F(X/X^{(s-1)}, M)$ for $s \geq 0$. For each s , we have the exact sequence

$$\cdots \rightarrow M^*(X, X^{(s)}) \xrightarrow{\alpha_s} M^*(X, X^{(s-1)}) \xrightarrow{\beta_s} M^*(X^{(s)}, X^{(s-1)}) \xrightarrow{\gamma_s} M^{*+1}(X, X^{(s)}) \rightarrow \cdots$$

The associated unrolled exact couple has $A_{-s} = M^*(X, X^{(s-1)})$, $A_\infty = M^*(X)$, $A_{-\infty} = RA_{-\infty} = 0$ and

$$E_{-s} = \pi_{-*}(Z_{-s}, Z_{-s-1}) = M^*(X^{(s)}, X^{(s-1)}).$$

By excision, we have isomorphisms

$$C_{\text{CW}}^s(X; M^{*-s}) \cong \prod_{\alpha} M^*(D^s, \partial D^s) \cong M^*(X^{(s)}, X^{(s-1)}).$$

Hence the associated spectral sequence has $\mathcal{E}^1$ -page

$$\mathcal{E}_{-s,-t}^1 = (\mathcal{E}_{-s}^1)_{-s-t} = C_{\text{CW}}^s(X; M^t) \implies M^{s+t}(X).$$

We reindex this for cohomology, by setting $\mathcal{E}_r^{s,t} = \mathcal{E}_{-s,-t}^r$ for all $r \geq 1$ and $s, t \in \mathbb{Z}$, with differentials

$$d_r^{s,t}: \mathcal{E}_r^{s,t} \longrightarrow \mathcal{E}_r^{s+r,t-r+1}$$

of cohomological bidegree $(r, -r + 1)$. The increasing filtration

$$F_{-s}M^*(X) = \text{im}(\pi_{-*}(Z_{-s}) \longrightarrow M^*(X))$$

of the abutment $M^*(X)$ is reindexed as the *decreasing* filtration

$$F^s M^*(X) = F_{-s} M^*(X),$$

so that $F^{s+1}M^*(X) \subset F^s M^*(X)$. The spectral sequence is weakly convergent when each comparison map

$$\frac{F^s M^*(X)}{F^{s+1} M^*(X)} \longrightarrow (\mathcal{E}_\infty^s)^*$$

is an isomorphism. The filtration is exhaustive, with $F^0 M^*(X) = M^*(X)$, and it is complete Hausdorff if and only if $\lim_s F^s M^*(X) = \text{Rlim}_s F^s M^*(X) = 0$.

The resulting cohomological Atiyah–Hirzebruch spectral sequence for $M^*(-)$ and X has $\mathcal{E}_2$ -page

$$\mathcal{E}_2^{s,t} = H^s(X; M^t) \Longrightarrow M^{s+t}(X).$$

It is concentrated in the half-plane with $s \geq 0$, and is a conditionally convergent spectral sequence with entering differentials. By Boardman's criterion, it is strongly convergent if and only if $R\mathcal{E}_\infty = 0$.

5. MULTIPLICATIVE STRUCTURE (AHSS)

Now suppose that $M^*(-)$ is a multiplicative cohomology theory, represented by a ring spectrum M . In order to get a multiplicative spectral sequence from the general theory, we need a pairing $\phi_{*,*}: Z_* \otimes Z_* \rightarrow Z_*$ of filtered spectra, i.e., pairings

$$\phi_{p,q}: F(X/X^{(p-1)}, M) \otimes F(X/X^{(q-1)}, M) \longrightarrow F(X/X^{(p+q-1)}, M)$$

for all $p, q \in \mathbb{Z}$. These will be induced by compatible maps

$$D_{p,q}: X/X^{(p+q-1)} \longrightarrow X/X^{(p-1)} \wedge X/X^{(q-1)}$$

for all p and q , and the ring spectrum pairing $\mu: M \otimes M \rightarrow M$, as the composite

$$\begin{aligned} F(X/X^{(p-1)}, M) \otimes F(X/X^{(q-1)}, M) &\xrightarrow{\otimes} F(X/X^{(p-1)} \otimes X/X^{(q-1)}, M \otimes M) \\ &\xrightarrow{F(D_{p,q}, \mu)} F(X/X^{(p+q-1)}, M). \end{aligned}$$

To obtain the $D_{p,q}$, we start with the diagonal map $\Delta: X \rightarrow X \times X$. This is hardly ever cellular, when $X \times X$ has the product CW structure, but we can choose a cellular approximation $D: X \rightarrow X \times X$ that is homotopic to Δ . Then

$$D(X^{(p+q-1)}) \subset (X \times X)^{(p+q-1)} \subset X^{(p-1)} \times X \cup X \times X^{(q-1)}$$

for all p and q , which implies that D induces the required maps $D_{p,q}$. Hence there are pairings

$$\mu_r: \mathcal{E}_r^{*,*} \otimes \mathcal{E}_r^{*,*} \longrightarrow \mathcal{E}_r^{*,*}$$

giving the cohomological Atiyah–Hirzebruch spectral sequence a multiplicative structure. The pairing of $\mathcal{E}_1$ -pages is given by the diagonal approximation, so that the pairing of $\mathcal{E}_2$ -pages

$$\mu_2: H^p(X; M^*) \otimes H^q(X; M^*) \longrightarrow H^{p+q}(X; M^*)$$

is given by the cup product in cohomology, combined with the coefficient ring pairing $M^* \otimes M^* \rightarrow M^*$. The pairing of abutments

$$\mu: M^*(X) \otimes M^*(X) \longrightarrow M^*(X)$$

is induced by the diagonal approximation D and the ring spectrum pairing, but since $D \simeq \Delta$, this agrees with the cup product in the cohomology theory $M^*(-)$.

Theorem 5.1. *For each CW complex X and ring spectrum M , there is a conditionally convergent and multiplicative Atiyah–Hirzebruch spectral sequence*

$$\mathcal{E}_2^{s,t} = H^s(X; M^t) \Longrightarrow M^{s+t}(X).$$

It is strongly convergent if and only if $R\mathcal{E}_\infty = 0$.

6. EVEN COHOMOLOGY THEORIES

The classifying space $\mathbb{C}P^\infty = BU(1)$ of the circle group has homology

$$H_*(\mathbb{C}P^\infty) \cong \mathbb{Z}\{\beta_i \mid i \geq 0\}$$

and cohomology

$$H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[y],$$

with $|y| = 2$ and y^i dual to β_i .

If M_* is concentrated in even degrees, so that $M_t = 0$ for all odd t , then

$$\mathcal{E}_{s,t}^2 = H_s(\mathbb{C}P^\infty; M_t) \cong H_s(\mathbb{C}P^\infty) \otimes M_t$$

is concentrated in (even, even) bidegrees. In particular $\mathcal{E}_{s,t}^2 = 0$ for $s+t$ odd, which implies that $\mathcal{E}_{s,t}^r = 0$ whenever the total degree $s+t$ is odd. Since d^r reduces the total degree by 1, either the source or the target of each $d_{s,t}^r$ is the trivial group. Hence $d^r = 0$ for all $r \geq 2$, and the homological Atiyah–Hirzebruch spectral sequence converging to $M_*(\mathbb{C}P^\infty)$ *collapses* at the $\mathcal{E}^2$ -page.

The condition on M_* is equivalent to M^* being concentrated in even degrees, since $M^t = M_{-t}$. Assuming this, also

$$\mathcal{E}_2^{s,t} = H^s(\mathbb{C}P^\infty; M^t)$$

is concentrated in (even, even) bidegrees, and in even total degrees, so that $d_r = 0$ for all $r \geq 2$ and $\mathcal{E}_2 = \mathcal{E}_\infty$. In particular, $R\mathcal{E}_\infty = 0$, so that also the cohomological Atiyah–Hirzebruch spectral sequence is strongly convergent, in these cases. The algebra structure is given by

$$\mathcal{E}_2^{*,*} = H^*(\mathbb{C}P^\infty; M^*) \cong \mathbb{Z}[y] \otimes M^*.$$

It follows by induction on n , using the short exact sequences

$$0 \rightarrow \mathcal{E}_\infty^{2n,*} \longrightarrow \frac{G}{F^{2n+1}G} \longrightarrow \frac{G}{F^{2n-1}G} \rightarrow 0,$$

that the decreasing filtration $(F^s G)_s$ of $G = M^*(\mathbb{C}P^\infty)$ satisfies

$$M^*(\mathbb{C}P^n) \cong \frac{G}{F^{2n+1}G} \cong \mathbb{Z}[y]/(y^{n+1}) \otimes M^*.$$

Passing to the limit over n , we deduce that

$$M^*(\mathbb{C}P^\infty) \cong \lim_n M^*(\mathbb{C}P^n) \cong M^*[[y]],$$

where $M^*[[y]] = \lim_n M^*[y]/(y^{n+1})$ denotes the ring of formal power series with coefficients in M^* .

Examples of spectra M with $M_* = \pi_*(M)$ concentrated in even degrees include the Eilenberg–MacLane spectra $H\Gamma$, complex K -theory KU , connective complex K -theory ku , the complex bordism spectrum MU , the p -local Brown–Peterson spectrum BP , the Lubin–Tate spectra E_n , the truncated Brown–Peterson spectra $BP\langle n \rangle$, the Johnson–Wilson spectra $E(n)$, and the Morava K -theory spectra $K(n)$

$$\begin{aligned} \pi_*(H\Gamma) &= \Gamma \\ \pi_*(KU) &= \mathbb{Z}[u^{\pm 1}] \\ \pi_*(ku) &= \mathbb{Z}[u] \\ \pi_*(MU) &\cong \mathbb{Z}[x_i \mid i \geq 1] \\ \pi_*(BP) &\cong \mathbb{Z}_{(p)}[v_i \mid i \geq 1] \\ \pi_*(E_n) &\cong \mathbb{W}_{\mathbb{F}_{p^n}}[[u_1, \dots, u_{n-1}]] [u^{\pm 1}] \\ \pi_*(BP\langle n \rangle) &\cong \mathbb{Z}_{(p)}[v_1, \dots, v_n] \\ \pi_*(E(n)) &\cong \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}] \\ \pi_*(K(n)) &\cong \mathbb{F}_p[v_n^{\pm 1}], \end{aligned}$$

with $|u| = 2$, $|x_i| = 2i$, $|v_i| = 2p^i - 2$ and $|u_i| = 0$. If we assume that $\Gamma = \Lambda$ is a ring, then these all admit $\mathbb{E}_1$ -ring structures, or better.

The original Atiyah–Hirzebruch paper considered the case $M = KU$, for which

$$KU^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[u^{\pm 1}][[y]]$$

with u and y in cohomological degrees -2 and 2 , respectively. In particular, $KU^0(\mathbb{C}P^\infty) \cong \mathbb{Z}[[x]]$, with $x = uy$ in degree 0 , is the ring of functions on a formal affine line $\hat{\mathbb{A}}_{\mathbb{Z}}^1$, i.e., a formal neighborhood of the origin 0 in the affine line $\mathbb{A}_{\mathbb{Z}}^1$ over $\text{Spec } \mathbb{Z}$.

((ETC: $K(n)_*(BC_p)$ finite.))

7. SERRE FIBRATIONS

A map $p: E \rightarrow B$ is a Serre fibration if it has the homotopy lifting property with respect to D^n for each $n \geq 0$:

$$\begin{array}{ccc} D^n & \xrightarrow{\quad} & E \\ i_0 \downarrow & \nearrow & \downarrow p \\ D^n \times I & \xrightarrow{\quad} & B. \end{array}$$

We call E the *total space* and B the *base space*. Given a base point $b_0 \in B$, we call $F = p^{-1}(b_0)$ the *fiber* of p . We also refer to $F \rightarrow E \rightarrow B$ as a Serre fiber sequence. Every fiber bundle $p: E \rightarrow B$ is a Serre fibration.

The geometric realization of any Kan fibration is a Serre fibration [Qui68].

The path fibration $p: PX \rightarrow X$ associated to a based space (X, x_0) is a Serre fibration, where $PX = F(I, X)$ is the space of maps $f: (I, 0) \rightarrow (X, x_0)$, i.e., paths

starting at x_0 , and $p = \text{ev}_1$ takes $f \in PX$ to the endpoint $f(1) \in X$. The fiber at x_0 is then the loop space ΩX , and we have a fiber sequence

$$\Omega X \longrightarrow PX \longrightarrow X.$$

Note that PX is contractible, and the connecting maps $\partial: \pi_n(X) \rightarrow \pi_{n-1}(\Omega X)$ in the long exact sequence in homotopy are the standard isomorphisms.

8. THE SERRE SPECTRAL SEQUENCE (SSS)

In place of [Ser51], who uses cubical singular chains, we follow [Kud50], [Kud52] and use CW structures; see also [Whi78].

Let $p: E \rightarrow B$ be a Serre fibration with fiber F over the base point b_0 . Suppose that B is a connected CW complex. The preimages along p of the skeleton filtration

$$\emptyset = B^{(-1)} \subset \dots \subset B^{(s-1)} \subset B^{(s)} \subset \dots \subset B$$

define a filtration of spaces

$$\emptyset = E^{-1} \subset \dots \subset E^{s-1} \subset E^s \subset \dots \subset E,$$

where $E^s = p^{-1}(B^{(s)})$ for each s . (This is typically not the s -skeleton of a CW structure on E .) Applying homology with coefficients in Γ leads to the exact sequences

$$\dots \rightarrow H_*(E^{s-1}; \Gamma) \xrightarrow{\alpha_s} H_*(E^s; \Gamma) \xrightarrow{\beta_s} H_*(E^s, E^{s-1}; \Gamma) \xrightarrow{\gamma_s} H_{*-1}(E^{s-1}; \Gamma) \rightarrow \dots$$

The associated unrolled exact couple $(A_s, E_s, \alpha_s, \beta_s, \gamma_s)$ has $A_s = H_*(E^s; \Gamma)$, $A_\infty = H_*(E; \Gamma)$, $A_{-\infty} = RA_{-\infty} = 0$ and

$$E_s = H_*(E^s, E^{s-1}; \Gamma).$$

For each s -cell of B we have a pullback square of pairs

$$\begin{array}{ccc} (D^s \times_B E, \partial D^s \times_B E) & \longrightarrow & (E^s, E^{s-1}) \\ \downarrow & & \downarrow p \\ (D^s, \partial D^s) & \xrightarrow{\Phi_\alpha} & (B^{(s)}, B^{(s-1)}). \end{array}$$

The Serre fibration hypothesis, together with the fact that D^s is contractible, ensures that there exists a fiber homotopy equivalence $D^s \times_B E \simeq D^s \times F$ over D^s , giving a homotopy equivalence of pairs

$$(D^s \times_B E, \partial D^s \times_B E) \simeq (D^s \times F, \partial D^s \times F).$$

When combined with excision, we get isomorphisms

$$\begin{aligned} C_s^{\text{CW}}(B; H_{*-s}(F; \Gamma)) &\cong \bigoplus_{\alpha} H_*(D^s \times F, \partial D^s \times F; \Gamma) \\ &\cong \bigoplus_{\alpha} H_*(D^s \times_B E, \partial D^s \times_B E; \Gamma) \\ &\cong H_*(E^s, E^{s-1}; \Gamma). \end{aligned}$$

This gives us the homological Serre spectral sequence

$$E_{s,t}^1 = C_s^{\text{CW}}(B; H_t(F; \Gamma)) \implies H_{s+t}(E; \Gamma).$$

It is concentrated in the quadrant where $s \geq 0$ and $t \geq 0$, and is strongly convergent. In each total degree n , the filtration

$$0 = F_{-1}H_n(E; \Gamma) \subset \cdots \subset F_nH_n(E; \Gamma) = H_n(E; \Gamma)$$

has finite length, since $F_sH_n(E; \Gamma) = \text{im}(H_n(E^s; \Gamma) \rightarrow H(E; \Gamma))$ and $E^n \rightarrow E$ is n -connected. Convergence amounts to the isomorphisms

$$\frac{F_sH_n(E; \Gamma)}{F_{s-1}H_n(E; \Gamma)} \cong \mathcal{E}_{s, n-s}^\infty$$

for $0 \leq s \leq n$.

The choice (up to 2-homotopy) of fiber homotopy equivalence depends on a choice of path from b_0 to the s -cell in question, which implies that the d^1 -differential

$$\begin{aligned} d_{s,t}^1: \mathcal{E}_{s,t}^1 &= C_s^{\text{CW}}(B; H_t(F; \Gamma)) \cong H_{s+t}(E^s, E^{s-1}; \Gamma) \\ &\xrightarrow{\partial} H_{s+t-1}(E^{s-1}, E^{s-2}; \Gamma) \cong C_{s-1}^{\text{CW}}(B; H_t(F; \Gamma)) = \mathcal{E}_{s-1,t}^1 \end{aligned}$$

corresponds to the cellular boundary with *twisted coefficients*, for a geometrically defined action by $\pi_1(B, b_0)$ on $H_t(F; \Gamma)$. Hence the $\mathcal{E}^2$ -page of the Serre spectral sequence

$$\mathcal{E}_{s,t}^2 = H_s(B; H_t(F; \Gamma))$$

will in general be given by homology with twisted coefficients. If B is simply-connected, this is the same as homology with ordinary coefficients.

Theorem 8.1. *For each Serre fibration $F \rightarrow E \rightarrow B$ with B path-connected, and coefficient group Γ , there is a strongly convergent Serre spectral sequence*

$$\mathcal{E}_{s,t}^2 = H_s(B; H_t(F; \Gamma)) \implies H_{s+t}(E; \Gamma).$$

We can also write this as $\mathcal{E}_{*,*}^2 = H_*(B; H_*(F; \Gamma)) \implies H_*(E; \Gamma)$.

9. TRANSGRESSIONS

Let $n \geq 2$. If X is $(n-1)$ -connected, then ΩX is $(n-2)$ -connected, and the Serre spectral sequence

$$\mathcal{E}_{s,t}^2 = H_s(X; H_t(\Omega X)) \implies H_{s+t}(PX)$$

satisfies

$$\mathcal{E}_{s,t}^2 \cong \begin{cases} H_s(X) & \text{for } t = 0, \\ H_t(\Omega X) & \text{for } s = 0, \\ 0 & \text{for } 1 \leq t \leq n-2 \text{ or } 1 \leq s \leq n-1. \end{cases}$$

The remaining terms lie in the region $s \geq n$ and $t \geq n-1$, with total degree $s+t \geq 2n-1$. Since $H_*(PX) = \mathbb{Z}$ concentrated in total degree 0, we must have

$\mathcal{E}_{*,*}^\infty = \mathbb{Z}$ concentrated in bidegree $(0, 0)$.

$$\begin{array}{cccccc}
 H_{2n-2}(\Omega X) & 0 & 0 & ? & ? & ? \\
 \swarrow & & & & & \\
 H_n(\Omega X) & 0 & 0 & ? & ? & ? \\
 \swarrow & & & & & \\
 H_{n-1}(\Omega X) & 0 & 0 & ? & ? & ? \\
 \swarrow & & & & & \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 \swarrow & & & & & \\
 \mathbb{Z} & 0 & 0 & H_n(X) & H_{n+1}(X) & H_{2n-1}(X)
 \end{array}$$

$d_{n,n-1}^n$ (arrow from $H_n(\Omega X)$ to $H_{2n-2}(\Omega X)$)
 $d_{2n-1,0}^{2n-1}$ (arrow from $H_{2n-1}(X)$ to $H_{2n-2}(\Omega X)$)

To achieve this, the long (‘transgression’) differentials

$$\tau = d_{s,0}^s: H_s(X) \xrightarrow{\cong} H_{s-1}(\Omega X),$$

which are defined for $n \leq s \leq 2n-2$, must be isomorphisms. Moreover,

$$d_{2n-1,0}^{2n-1}: H_{2n-1}(X) \xrightarrow{\cong} H_{2n-2}(\Omega X) / \text{im}(d_{n,n-1}^n)$$

must also be an isomorphism. It follows that the composite ‘homology suspension’

$$\tilde{H}_{s-1}(\Omega X) \xleftarrow[\cong]{\partial} H_s(PX, \Omega X) \xrightarrow{p_*} \tilde{H}_s(X)$$

is an isomorphism for $s \leq 2n-2$, and a surjection for $s = 2n-1$. This result can be viewed as dual to the Freudenthal suspension theorem concerning $E: \pi_s(X) \rightarrow \pi_{s+1}(\Sigma X)$. The Hurewicz isomorphism also follows by induction on n , since

$$\pi_n(X) \cong \pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X) \cong H_n(X)$$

for $n \geq 2$.

More generally, for a fiber sequence $F \rightarrow E \rightarrow B$ with F being $(m-1)$ -connected and B being $(n-1)$ -connected (for $m \geq 1$ and $n \geq 2$), the convergence of the Serre spectral sequence leads to the (finite length) Serre exact sequence

$$\begin{aligned}
 H_{n+m-1}(F) \rightarrow H_{n+m-1}(E) &\xrightarrow{p_*} H_{n+m-1}(B) \xrightarrow{\tau} H_{n+m-2}(F) \rightarrow \dots \\
 \dots \rightarrow H_2(E) &\xrightarrow{p_*} H_2(B) \xrightarrow{\tau} H_1(F) \rightarrow H_1(E) \rightarrow 0.
 \end{aligned}$$

This is formally dual to the homotopy excision (Blakers–Massey) theorem.

10. REINDEXING FOR COHOMOLOGY (SSS)

Applying cohomology with coefficients in Γ leads to the exact sequences

$$\dots \rightarrow H^*(E, E^s; \Gamma) \xrightarrow{\alpha_s} H^*(E, E^{s-1}; \Gamma) \xrightarrow{\beta_s} H^*(E^s, E^{s-1}; \Gamma) \xrightarrow{\gamma_s} H^{*+1}(E^{s-1}; \Gamma) \rightarrow \dots$$

The associated unrolled exact couple has $A_{-s} = H^*(E, E^{s-1}; \Gamma)$, $A_0 = A_\infty = H^*(E; \Gamma)$, $A_{-\infty} = RA_{-\infty} = 0$ and

$$E_{-s} = H^*(E^s, E^{s-1}; \Gamma).$$

The Serre fibration hypothesis and excision give isomorphisms

$$H^*(E^s, E^{s-1}; \Gamma) \cong C_{\text{CW}}^s(B; H^{*-s}(F; \Gamma))$$

and yield the cohomological Serre spectral sequence

$$\mathcal{E}_1^{s,t} = C_{\text{CW}}^s(B; H^t(F; \Gamma)) \implies H^{s+t}(E; \Gamma).$$

It is concentrated in the quadrant where $s \geq 0$ and $t \geq 0$, and is conditionally convergent. For each (s, t) we have $d_r^{s,t} = 0$ when r is sufficiently large, so $R\mathcal{E}_\infty = 0$, and the spectral sequence is in fact strongly convergent. In each total degree n , the filtration

$$0 = F^{n+1}H^n(E; \Gamma) \subset \cdots \subset F^0H^n(E; \Gamma) = H^n(E; \Gamma)$$

has finite length, since $F^sH^n(E; \Gamma) = \text{im}(H^n(E, E^{s-1}; \Gamma) \rightarrow H^n(E; \Gamma))$ and (E, E^n) is n -connected. We have isomorphisms

$$\frac{F^sH^n(E; \Gamma)}{F^{s+1}H^n(E; \Gamma)} \cong \mathcal{E}_\infty^{s, n-s}$$

for $0 \leq s \leq n$.

Again, there is an action of $\pi_1(B, b_0)$ on $H^*(F; \Gamma)$, and the differential

$$d_1^{s,t}: \mathcal{E}_1^{s,t} \cong C_{\text{CW}}^s(B; H^t(F; \Gamma)) \xrightarrow{\delta} C_{\text{CW}}^{s+1}(B; H^t(F; \Gamma)) \cong \mathcal{E}_1^{s+1,t}$$

is the cellular coboundary with twisted coefficients, so that

$$\mathcal{E}_2^{s,t} = H^s(B; H^t(F; \Gamma))$$

is given by cohomology with twisted coefficients.

Theorem 10.1. *For each Serre fibration $F \rightarrow E \rightarrow B$ with B path-connected, and coefficient group Γ , there is a strongly convergent Serre spectral sequence*

$$\mathcal{E}_2^{s,t} = H^s(B; H^t(F; \Gamma)) \implies H^{s+t}(E; \Gamma).$$

We can also write this as $\mathcal{E}_2^{*,*} = H^*(B; H^*(F; \Gamma)) \implies H^*(E; \Gamma)$.

11. MULTIPLICATIVE STRUCTURE (SSS)

Let Λ be a ring. The diagonal map $\Delta: B \rightarrow B \times B$ admits a cellular approximation $D: B \rightarrow B \times B$. A homotopy $\Delta \simeq D$ gives a homotopy $(p \times p) \circ \tilde{\Delta} = \Delta \circ p \simeq D \circ p$, which lifts to a homotopy from $\tilde{\Delta}: E \rightarrow E \times E$ to a map $\tilde{D}: E \rightarrow E \times E$ taking the filtration $(E^s)_s$ of E to the convolved filtration of $E \times E$. Hence

$$\tilde{D}(E^{p+q-1}) \subset E^{p-1} \times E \cup E \times E^{q-1},$$

so that $\tilde{D}$ induces compatible maps

$$\tilde{D}_{p,q}: E/E^{p+q-1} \longrightarrow E/E^{p-1} \wedge E/E^{q-1}$$

for all p and q . Hence there are pairings

$$\mu_r: \mathcal{E}_r^{*,*} \otimes \mathcal{E}_r^{*,*} \longrightarrow \mathcal{E}_r^{*,*}$$

giving multiplicative structure to the cohomological Serre spectral sequence for $F \rightarrow E \rightarrow B$ and the coefficient ring Λ . The pairing of $\mathcal{E}_2$ -pages

$$\mu_2: H^p(B; H^*(F; \Lambda)) \otimes H^q(B; H^*(F; \Lambda)) \longrightarrow H^{p+q}(B; H^*(F; \Lambda))$$

is given by the cup products in cohomology for B and F . The pairing of abutments

$$\mu: H^*(E; \Lambda) \otimes H^*(E; \Lambda) \longrightarrow H^*(E; \Lambda)$$

is induced by $\tilde{D} \simeq \tilde{\Delta}$, hence is equal to the cohomology cup product for E .

Theorem 11.1. *For each Serre fibration $F \rightarrow E \rightarrow B$ with B path-connected, and coefficient ring Λ , there is a strongly convergent and multiplicative Serre spectral sequence*

$$\mathcal{E}_2^{s,t} = H^s(B; H^t(F; \Lambda)) \implies H^{s+t}(E; \Lambda).$$

12. p -TORSION IN $\pi_{2p}(S^3)$

By the Hurewicz theorem, $\pi_3(S^3) \cong H_3(S^3) \cong \mathbb{Z}$. Let $k^3: S^3 \rightarrow K(\mathbb{Z}, 3)$ be the π_3 -isomorphism representing a generator of $H^3(S^3) \cong \mathbb{Z}$. Let $E = \text{fib}(k^3)$ be its homotopy fiber. Then $S^3 \rightarrow K(\mathbb{Z}, 3)$ is the third Postnikov section of S^3 , and $p: E \rightarrow S^3$ is its 3-connected Whitehead cover. The homotopy fiber $\text{fib}(p)$ is then equivalent to $\Omega K(\mathbb{Z}, 3) \simeq K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$, as in the Puppe fiber sequence. In particular, we have a fiber sequence (up to homotopy equivalences)

$$\mathbb{C}P^\infty \longrightarrow E \xrightarrow{p} S^3$$

with E a 3-connected space. The associated cohomology Serre spectral sequence

$$\mathcal{E}_2^{*,*} = H^*(S^3; H^*(\mathbb{C}P^\infty)) \implies H^*(E)$$

has

$$\mathcal{E}_2^{*,*} \cong H^*(S^3; \mathbb{Z}[y]) \cong \Lambda(z) \otimes \mathbb{Z}[y],$$

where $|y| = 2$ and $|z| = 3$. Here $\Lambda(z) = \mathbb{Z}\{1, z\}$ denotes the exterior algebra, with $z^2 = 0$.

$$\begin{array}{ccccc}
 \mathbb{Z}\{y^4\} & & \cdot & & \cdot & & \vdots \\
 & \searrow & & & & & \\
 & & \cdot & & \cdot & & \cdot \\
 & & & & & & \\
 \mathbb{Z}\{y^3\} & & \cdot & & \cdot & & \mathbb{Z}\{y^3z\} \\
 & \searrow & & & & & \\
 & & \cdot & & \cdot & & \cdot \\
 & & & & & & \\
 \mathbb{Z}\{y^2\} & & \cdot & & \cdot & & \mathbb{Z}\{y^2z\} \\
 & \searrow & & & & & \\
 & & \cdot & & \cdot & & \cdot \\
 & & & & & & \\
 \mathbb{Z}\{y\} & & \cdot & & \cdot & & \mathbb{Z}\{yz\} \\
 & \searrow & & & & & \\
 & & \cdot & & \cdot & & \cdot \\
 & & & & & & \\
 \mathbb{Z}\{1\} & & \cdot & & \cdot & & \mathbb{Z}\{z\}
 \end{array}$$

The dots indicate trivial groups. Since E is 3-connected, the $\mathcal{E}_\infty$ -term must be 0 in total degrees $1 \leq n \leq 3$. It follows that y must support a differential, and z must be hit by a differential. The only possibility is that

$$d_3(y) = \pm z.$$

By possibly replacing z with $-z$, we may assume the sign is $+$. Then, by the Leibniz rule we obtain

$$d_3(y^2) = d_3(y) \cup y + y \cup d_3(y) = zy + yz = 2yz.$$

By induction, it follows that

$$d_3(y^k) = ky^{k-1}z$$

for all $k \geq 1$. All later differentials vanish, since the $\mathcal{E}_2$ -page is concentrated in filtrations $s \in \{0, 3\}$, so $\mathcal{E}_4 = \mathcal{E}_\infty$ appears as follows.

$$\begin{array}{cccc}
 0 & & & \vdots \\
 & \cdot & \cdot & \\
 & & & \\
 & \cdot & \cdot & \cdot \\
 & & & \\
 0 & & & \mathbb{Z}/4\{y^3z\} \\
 & \cdot & \cdot & \cdot \\
 & & & \\
 0 & & & \mathbb{Z}/3\{y^2z\} \\
 & \cdot & \cdot & \cdot \\
 & & & \\
 0 & & & \mathbb{Z}/2\{yz\} \\
 & \cdot & \cdot & \cdot \\
 & & & \\
 \mathbb{Z}\{1\} & & & 0
 \end{array}$$

Since there is at most one nonzero group in each total degree, there are no extension questions. Hence

$$H^n(E) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ \mathbb{Z}/k & \text{for } n = 2k + 1, k \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

The homological Serre spectral sequence shows (also without a precise calculation of its differentials) that each $H_n(E)$ is a finitely generated abelian group, for each n ,

so the universal coefficient theorem and the calculation of $H^*(E)$ tells us that

$$H_n(E) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ \mathbb{Z}/k & \text{for } n = 2k, k \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for any prime p , the p -localized homology

$$H_n(E)_{(p)} = H_n(E; \mathbb{Z}_{(p)})$$

begins with $\mathbb{Z}_{(p)}$ in degree 0 and $\mathbb{Z}/p$ in degree $2p$. By a p -localized version of the Hurewicz theorem, realized either by Bousfield–Kan localization of spaces, or by working with Serre classes of abelian groups, it follows that

$$\pi_n(E)_{(p)} \cong \begin{cases} 0 & \text{for } n < 2p, \\ \mathbb{Z}/p & \text{for } n = 2p. \end{cases}$$

But $E \rightarrow S^3$ induces an isomorphism on π_n for $n \geq 4$. Hence

$$\pi_n(S^3)_{(p)} \cong \begin{cases} 0 & \text{for } n < 3, \\ \mathbb{Z}_{(p)} & \text{for } n = 3, \\ 0 & \text{for } 3 < n < 2p, \\ \mathbb{Z}/p & \text{for } n = 2p. \end{cases}$$

The order p generator of $\pi_{2p}S^3$ can be denoted α_1 , and is represented by a homotopy class of maps $S^{2p} \rightarrow S^3$.

Further work (see e.g. [Spa66]) shows that the iterated stabilization map

$$\pi_{2p}(S^3) \xrightarrow{\cong} \operatorname{colim}_i \pi_{2p+i}(S^{3+i}) = \pi_{2p-3}(S)$$

is an isomorphism, so that the stable class of α_1 is the generator of the lowest-degree p -torsion in the stable homotopy groups of spheres, i.e., $\pi_{2p-3}(S)_{(p)} \cong \mathbb{Z}/p\{\alpha_1\}$. When $p = 2$, the map $\alpha_1: S^4 \rightarrow S^3$ is the suspension of the Hopf fibration $\eta: S^3 \rightarrow S^2$, and $\pi_1(S) \cong \mathbb{Z}/2\{\eta\}$.

13. GROUP (CO-)HOMOLOGY

If $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ is an extension of groups, there are models for their classifying spaces such that $BN \rightarrow BG \rightarrow BQ$ is a Serre fibration. In the Serre spectral sequence

$$\mathcal{E}_{s,t}^2 = H_s(BQ; H_t(BN; \Gamma)) \Longrightarrow H_{s+t}(BG; \Gamma)$$

the action by $Q = \pi_1(BQ)$ on $H_*(BN; \Gamma)$ is induced by the conjugation action of G on N : for each $q \in Q$ we can choose lift $g \in G$, and $n \mapsto gng^{-1}$ defines an action of g on N , BN and $H_*(BN; \Gamma)$. When $g \in N$ the latter turns out to be the trivial action, hence gives a well-defined action of $q \in Q$ on $H_*(BN; \Gamma)$. We can rewrite this in terms of group homology as

$$\mathcal{E}_{s,t}^2 = H_s^{\operatorname{gp}}(Q; H_t^{\operatorname{gp}}(N; \Gamma)) \Longrightarrow H_{s+t}^{\operatorname{gp}}(G; \Gamma),$$

which is known as the (Lyndon–)Hochschild–Serre spectral sequence [HS53]. If N is central in G , the conjugation action is trivial, and then $H_*^{\operatorname{gp}}(N; \Gamma)$ has the trivial Q -action.

The cohomological version takes the form

$$\mathcal{E}_2^{s,t} = H_{\text{gp}}^s(Q; H_{\text{gp}}^t(N; \Gamma)) \implies H_{\text{gp}}^{s+t}(G; \Gamma).$$

This can also be written in terms of Ext for group rings, as the Cartan–Eilenberg spectral sequence

$$\mathcal{E}_2^{s,t} = \text{Ext}_{\mathbb{Z}[Q]}^s(\mathbb{Z}, \text{Ext}_{\mathbb{Z}[N]}^t(\mathbb{Z}, \Gamma)) \implies \text{Ext}_{\mathbb{Z}[G]}^{s,t}(\mathbb{Z}, \Gamma),$$

which is constructed purely algebraically in [CE56], and generalized to extensions of augmented algebras. It is also a special case of Grothendieck’s composite functor spectral sequence [Gro57].

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