

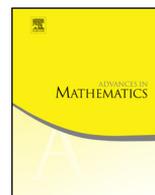


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Algebraic K-theory of group rings and the cyclotomic trace map

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ABSTRACT

We prove that the Farrell–Jones assembly map for connective algebraic K-theory is rationally injective, under mild homological finiteness conditions on the group and assuming that a weak version of the Leopoldt–Schneider conjecture holds for cyclotomic fields. This generalizes a result of Bökstedt, Hsiang, and Madsen, and leads to a concrete description of a large direct summand of $K_n(\mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Q}$ in terms of group homology. In many cases the number theoretic conjectures are true, so we obtain rational injectivity results about assembly maps, in particular for Whitehead groups, under homological finiteness assumptions on the group only. The proof uses the cyclotomic trace map to topological cyclic homology, Bökstedt–Hsiang–Madsen’s functor C , and new general isomorphism and injectivity results about the assembly maps for topological Hochschild homology and C .

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1. Introduction

The algebraic K -theory groups $K_n(\mathbb{Z}[G])$ of the integral group ring of a discrete group G have far-reaching geometric applications to the study of manifolds with fundamental group G and of sufficiently high dimension. One of the most famous and important manifestations of this phenomenon is given by the Whitehead group $Wh(G)$, which is the quotient of $K_1(\mathbb{Z}[G]) = GL(\mathbb{Z}[G])_{\text{ab}}$ by the subgroup generated by the units in $\mathbb{Z}[G]$ of the form $\pm g$ with $g \in G$. By the celebrated s -Cobordism Theorem, $Wh(G)$ completely classifies the isomorphism classes of h -cobordisms over any closed, connected manifold M with $\dim(M) \geq 5$ and $\pi_1(M) \cong G$.

A well-known and still open conjecture, which we review below, predicts that the Whitehead group of any torsion-free group vanishes. However, if a group G has torsion, then in general $Wh(G)$ is not trivial. For example, the structure of the Whitehead groups of finite groups G is well understood (see [87]), and if C is any finite cyclic group of order $c \notin \{1, 2, 3, 4, 6\}$, then $Wh(C)$ is not zero, not even after tensoring with the rational numbers \mathbb{Q} .

One of the main consequences of this work is the following theorem about Whitehead groups of infinite groups with torsion. Its conclusion says that rationally the Whitehead groups $Wh(H)$ of all finite subgroups H of G contribute to $Wh(G)$, and only the obvious relations between these contributions hold. Its assumption is a very weak and natural homological finiteness condition only up to dimension two. As we explain in Section 2, it is satisfied by many geometrically interesting groups, such as outer automorphism groups of free groups and Thompson’s group T , thus yielding the first known results about the

Whitehead groups of these groups. But, in contrast to other known results in this area, the assumption is purely homological, and no geometric input is required.

Theorem 1.1. *Assume that, for every finite cyclic subgroup C of a group G , the first and second integral group homology $H_1(BZ_G C; \mathbb{Z})$ and $H_2(BZ_G C; \mathbb{Z})$ of the centralizer $Z_G C$ of C in G are finitely generated abelian groups. Then the map*

$$\operatorname{colim}_{H \in \operatorname{obj} \operatorname{Sub}G(\mathcal{F}in)} \operatorname{Wh}(H) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \operatorname{Wh}(G) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{1.2}$$

is injective.

Here the colimit is taken over the finite subgroup category $\operatorname{Sub}G(\mathcal{F}in)$, whose objects are the finite subgroups H of G and whose morphisms are induced by subconjugation in G . In the special case when G is abelian, then $\operatorname{Sub}G(\mathcal{F}in)$ is just the poset of finite subgroups of G ordered by inclusion. The general definition is reviewed at the beginning of Section 17.

The injectivity of the map (1.2), as well as the vanishing of the Whitehead groups of torsion-free groups, are in fact consequences of much more general conjectures about assembly maps. Similarly, Theorem 1.1 is a consequence of more general results, namely Main Theorem 1.13 and Main Technical Theorem 1.16 below. In the rest of this section we introduce assembly maps, recall these conjectures, and state our main results.

We denote by $\mathbf{K}(R)$ the non-connective algebraic K -theory spectrum of a ring R , with homotopy groups $\pi_n(\mathbf{K}(R)) = K_n(R)$ for all $n \in \mathbb{Z}$, and we let BG be the classifying space of the group G . The classical assembly map is a map of spectra

$$BG_+ \wedge \mathbf{K}(\mathbb{Z}) \longrightarrow \mathbf{K}(\mathbb{Z}[G]), \tag{1.3}$$

which was first introduced by Loday [65, Proposition 4.1.1 on page 356]. This map is conjectured to be a π_* -isomorphism if the group G has no torsion, as we explain below. After taking homotopy groups and rationalizing, the left-hand side of (1.3) can be explicitly computed: for every $n \in \mathbb{Z}$ there is an isomorphism

$$\bigoplus_{\substack{s+t=n \\ s,t \geq 0}} H_s(BG; \mathbb{Q}) \otimes_{\mathbb{Q}} \left(K_t(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \right) \xrightarrow{\cong} \pi_n(BG_+ \wedge \mathbf{K}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{1.4}$$

We call the map

$$\bigoplus_{\substack{s+t=n \\ s,t \geq 0}} H_s(BG; \mathbb{Q}) \otimes_{\mathbb{Q}} \left(K_t(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \right) \longrightarrow K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{1.5}$$

induced by the composition of (1.4) and (1.3) the *rationalized classical assembly map*. Hence we obtain the following conjecture, which was formulated by Hsiang in his plenary

address at the 1983 ICM under the additional assumption that G has a finite classifying space BG ; see [55, Conjecture 4 on page 114].

Conjecture 1.6 (*The Hsiang Conjecture*). *For every torsion-free group G and every $n \in \mathbb{Z}$ the rationalized classical assembly map (1.5) is an isomorphism.*

There are analogous versions of the classical assembly map (1.3) and Conjecture 1.6 for L -theory instead of algebraic K -theory. The rational injectivity of the L -theory version of (1.3) is equivalent to the famous Novikov Conjecture about the homotopy invariance of higher signatures.

In [17] Bökstedt, Hsiang, and Madsen invented topological cyclic homology and the cyclotomic trace map from algebraic K -theory to topological cyclic homology, and used them to prove the following theorem, which is often referred to as the algebraic K -theory Novikov Conjecture; see [17, Theorem 9.13 on page 535] and [78, Theorem 4.5.4 on page 284].

Theorem 1.7 (*Bökstedt–Hsiang–Madsen*). *Let G be a not necessarily torsion-free group. Assume that the following condition holds.*

[A₁] *For every $s \geq 1$ the integral group homology $H_s(BG; \mathbb{Z})$ is a finitely generated abelian group.*

Then the rationalized classical assembly map (1.5) is injective for all $n \geq 0$.

For groups with torsion the classical assembly map is not surjective in general. For example, the rationalized Whitehead group $Wh(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the cokernel of (1.5) for $n = 1$, and it is usually not zero when G has torsion, as we recalled before Theorem 1.1. Notice that, in particular, Theorem 1.7 does not give any information about Whitehead groups. In contrast, as a consequence of our main result, we detect in Theorem 1.1 a large direct summand of $Wh(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ under extremely mild homological finiteness assumptions on G .

The Farrell–Jones Conjecture [39] predicts the structure of $K_n(\mathbb{Z}G)$ for arbitrary groups G , and reduces to the above-mentioned conjecture about the map (1.3) in the case of torsion-free groups. In the equivalent reformulation due to J. Davis and the first author [26], the Farrell–Jones Conjecture says that the Farrell–Jones assembly map

$$EG(\mathcal{VCyc})_+ \wedge_{\text{Or}G} \mathbf{K}(\mathbb{Z}[Gf-]) \longrightarrow \mathbf{K}(\mathbb{Z}[G]), \tag{1.8}$$

a certain generalization of the classical assembly map (1.3), is a π_* -isomorphism. Here $EG(\mathcal{VCyc})$ is the universal space for the family of virtually cyclic subgroups of G reviewed at the beginning of Section 2, $\text{Or}G$ is the orbit category, $Gf-$ is the action groupoid functor from Subsection 4C, and the map (1.8) is defined in Subsection 4H. We refer to the end of this introduction for some additional remarks about the Farrell–Jones

Conjecture. Now we concentrate on computational consequences for the rationalized homotopy groups and describe the analogue of the isomorphism (1.4).

In order to give a precise statement we need more notation. Given a group G and a subgroup H , we denote by $N_G H$ the normalizer and by $Z_G H$ the centralizer of H in G , and we define the Weyl group as the quotient $W_G H = N_G H / (Z_G H \cdot H)$. Notice that the Weyl group $W_G H$ of a finite subgroup H is always finite, since it embeds into the outer automorphism group of H . We write $\mathcal{F}Cyc$ for the family of finite cyclic subgroups of G , and $(\mathcal{F}Cyc)$ for the set of conjugacy classes of finite cyclic subgroups.

Using results of the first author in [68], Grunewald established in [48, Corollary on page 165] an isomorphism

$$\begin{aligned} \bigoplus_{(C) \in (\mathcal{F}Cyc)} \bigoplus_{\substack{s+t=n \\ s \geq 0, t \geq -1}} H_s(BZ_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \Theta_C \left(K_t(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q} \right) \\ \downarrow \cong \\ \pi_n \left(EG(\mathcal{V}Cyc)_+ \wedge_{\text{Or}G} \mathbf{K}(\mathbb{Z}[Gf-]) \right) \otimes_{\mathbb{Z}} \mathbb{Q} \end{aligned} \tag{1.9}$$

for every $n \in \mathbb{Z}$; see also [75, Theorem 0.3 on page 2]. Here Θ_C is an idempotent endomorphism of $K_t(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q}$, which corresponds to a specific idempotent in the rationalized Burnside ring of C , and whose image is a direct summand of $K_t(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q}$ isomorphic to

$$\text{coker} \left(\bigoplus_{D \not\leq C} \text{ind}_D^C : \bigoplus_{D \not\leq C} K_t(\mathbb{Z}[D]) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow K_t(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q} \right). \tag{1.10}$$

See Section 12 for more details. The Weyl group acts via conjugation on C and hence on $\Theta_C(K_t(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q})$. The Weyl group action on the homology comes from the fact that $EN_G C / Z_G C$ is a model for $BZ_G C$. We remark that the dimensions of the \mathbb{Q} -vector spaces in (1.10) are explicitly computed in [90, Theorem on page 9] for any t and any finite cyclic group C . Notice that if $t < -1$ then $K_t(\mathbb{Z}[C])$ vanishes for any virtually cyclic group C by [40].

We call the map

$$\bigoplus_{(C) \in (\mathcal{F}Cyc)} \bigoplus_{\substack{s+t=n \\ s \geq 0, t \geq -1}} H_s(BZ_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \Theta_C \left(K_t(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q} \right) \longrightarrow K_n(\mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{1.11}$$

induced by the composition of (1.9) and (1.8) the *rationalized Farrell–Jones assembly map*. Hence we obtain the following conjecture, which is a generalization of the Hsiang Conjecture 1.6 for groups that are not necessarily torsion-free.

Conjecture 1.12 (The rationalized Farrell–Jones conjecture). *For every group G and every $n \in \mathbb{Z}$ the rationalized Farrell–Jones assembly map (1.11) is an isomorphism.*

The restriction of the map (1.11) to the summand indexed by the trivial subgroup of G is the map (1.5). So for a torsion-free group the rationalized Farrell–Jones Conjecture 1.12 specializes to the Hsiang Conjecture 1.6.

Our main result generalizes Bökstedt–Hsiang–Madsen’s Theorem 1.7 to the rationalized Farrell–Jones Conjecture 1.12.

Main Theorem 1.13. *Let G be a group. Assume that the following two conditions hold.*

[$A_{\mathcal{FCyc}}$] *For every finite cyclic subgroup C of G and for every $s \geq 1$, the integral group homology $H_s(BZ_G C; \mathbb{Z})$ of the centralizer of C in G is a finitely generated abelian group.*

[$B_{\mathcal{FCyc}}$] *For every finite cyclic subgroup C of G and for every $t \geq 0$, the natural homomorphism*

$$K_t(\mathbb{Z}[\zeta_c]) \longrightarrow \prod_{p \text{ prime}} K_t\left(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_c]; \mathbb{Z}_p\right)$$

is \mathbb{Q} -injective, i.e., injective after applying $- \otimes_{\mathbb{Z}} \mathbb{Q}$. Here c is the order of C and ζ_c is any primitive c -th root of unity.

Then the restriction

$$\bigoplus_{(C) \in (\mathcal{FCyc})} \bigoplus_{\substack{s+t=n \\ s,t \geq 0}} H_s(BZ_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \Theta_C \left(K_t(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q} \right) \longrightarrow K_n(\mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Q} \quad (1.14)$$

of the rationalized Farrell–Jones assembly map (1.11) to the summands satisfying $t \geq 0$ is injective for all $n \in \mathbb{Z}$.

In Assumption [$B_{\mathcal{FCyc}}$] we write \mathbb{Z}_p for the ring of p -adic integers and we use the notation $K_t(R; \mathbb{Z}_p)$ for $\pi_t(\mathbf{K}(R)_p^\wedge)$, the t -th homotopy group of the p -completed algebraic K -theory spectrum of the ring R ; see Subsection 4N. The map to the factor indexed by p is induced by the natural map $K_t(R) \rightarrow K_t(R; \mathbb{Z}_p)$ and the ring homomorphism $\mathbb{Z}[\zeta_c] \rightarrow \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_c]$.

Assumption [$A_{\mathcal{FCyc}}$] implies and is the obvious generalization of Assumption [A_1].

In the next section we thoroughly discuss Assumptions [$A_{\mathcal{FCyc}}$] and [$B_{\mathcal{FCyc}}$]. The discussion can be summarized as follows.

[$A_{\mathcal{FCyc}}$] is satisfied by a very large class of groups.

[$B_{\mathcal{FCyc}}$] is conjecturally always true: it is automatically satisfied if a weak version of the Leopoldt–Schneider Conjecture holds for cyclotomic fields.

Moreover, Assumption $[B_{\mathcal{F}Cyc}]$ is satisfied for all c when $t = 0$ or 1 , and it is satisfied for all t if $c = 1$; for any fixed c , it is satisfied for almost all t . Because of this, only finiteness assumptions appear in [Theorems 1.1 and 1.7](#) and in the following result.

Theorem 1.15 (*Eventual injectivity*). *Assume that there is a finite universal space for proper actions $EG(\mathcal{F}in)$. Then there exists an integer $L > 0$ such that the rationalized Farrell–Jones assembly map (1.11) is injective for all $n \geq L$. The bound L only depends on the dimension of $EG(\mathcal{F}in)$ and on the orders of the finite cyclic subgroups of G .*

Universal spaces are reviewed at the beginning of the next section, where we also give many interesting examples of groups having finite $EG(\mathcal{F}in)$. Here we only remark that, for example, the assumption of [Theorem 1.15](#) is satisfied by outer automorphism groups of free groups, for which there seems to be no other known result about the Farrell–Jones Conjecture. [Theorem 1.15](#) is proved at the end of [Section 18](#).

We now formulate a stronger but more technical version of [Main Theorem 1.13](#), which in particular is needed to deduce [Theorem 1.1](#). We first need some algebraic definitions from [\[74\]](#). An abelian group M is called almost trivial if there exists an $r \in \mathbb{Z} - \{0\}$ such that $rm = 0$ for all $m \in M$. We say that M is almost finitely generated if its torsion part $\text{tors } M$ is almost trivial and if $M/\text{tors } M$ is a finitely generated abelian group. Every finitely generated abelian group M is in particular almost finitely generated. However, notice that $M \otimes_{\mathbb{Z}} \mathbb{Q}$ being finitely generated as a rational vector space does not necessarily imply that M is almost finitely generated. Almost finitely generated abelian groups form a Serre class.

Main Technical Theorem 1.16. *Let G be a group and let $\mathcal{F} \subseteq \mathcal{F}Cyc$ be a family of finite cyclic subgroups of G . Let $N \geq 0$ be an integer and let T be a subset of the non-negative integers $\mathbb{N} = \{0, 1, 2, \dots\}$. Assume that the following two conditions hold.*

$[A'_{\mathcal{F}, \leq N+1}]$ *For every $C \in \mathcal{F}$ and for every $1 \leq s \leq N + 1$, the integral group homology $H_s(BZ_G C; \mathbb{Z})$ of the centralizer of C in G is an almost finitely generated abelian group.*

$[B_{\mathcal{F}, T}]$ *For every $C \in \mathcal{F}$ and for every $t \in T$, the natural homomorphism*

$$K_t(\mathbb{Z}[\zeta_c]) \longrightarrow \prod_{p \text{ prime}} K_t\left(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_c]; \mathbb{Z}_p\right)$$

is \mathbb{Q} -injective, where c is the order of C and ζ_c is any primitive c -th root of unity.

Then the restriction

$$\bigoplus_{(C) \in (\mathcal{F})} \bigoplus_{\substack{s+t=n \\ s \geq 0, t \in T}} H_s(BZ_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \Theta_C \left(K_t(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q} \right) \longrightarrow K_n(\mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Q} \quad (1.17)$$

of the rationalized Farrell–Jones assembly map (1.11) to the summands satisfying (C) \in (F) and $t \in T$ is injective for all $0 \leq n \leq N$.

It is clear that Theorem 1.13 is a special case of Theorem 1.16, since Assumption $[A_{\mathcal{F}}]$ implies $[A'_{\mathcal{F}, \leq N}]$ for all N , and $[B_{\mathcal{F}}]$ is the same as $[B_{\mathcal{F}, \mathbb{N}}]$. Notice that also Theorem 1.7 is a special case of Theorem 1.16. This is because for the trivial family $\mathcal{F} = 1$ the maps (1.5) and (1.17) coincide, and Assumption $[B_{1, \mathbb{N}}]$ holds by Proposition 2.9. Moreover, for torsion-free groups Theorems 1.7 and 1.13 coincide.

Proposition 2.4 says that Assumption $[B_{\mathcal{F}C_{yc}, \{0,1\}}]$ is always satisfied. This allows us to deduce Theorem 1.1 about the rationalized assembly map for Whitehead groups from Theorem 1.16. The proof is given in Section 17. In fact, the following strengthening of Theorem 1.1 is also true.

Addendum 1.18. *The map (1.2) is injective if, for every finite cyclic subgroup C of G , the abelian groups $H_1(BZ_G C; \mathbb{Z})$ and $H_2(BZ_G C; \mathbb{Z})$ are almost finitely generated.*

In Section 3 we outline the strategy of the proof of Theorems 1.13 and 1.16, and explain in Remark 3.7 why with this strategy the homological finiteness assumptions cannot be removed. The main ingredient in the proof is, as in [17], the cyclotomic trace map from algebraic K -theory to topological cyclic homology. A key role is played by a generalization of the functor C defined by Bökstedt, Hsiang, and Madsen in [17, (5.14) on page 497]; see Section 9. In Theorems 6.1 and 9.5, we prove the following general results about the assembly maps for topological Hochschild homology and for C , which we believe are of independent interest. Here we say that a map of spectra $f: \mathbf{X} \rightarrow \mathbf{Y}$ is split injective if there is a map $g: \mathbf{Y} \rightarrow \mathbf{Z}$ to some other spectrum \mathbf{Z} such that $g \circ f$ is a π_* -isomorphism. We say that f is $\pi_n^{\mathbb{Q}}$ -injective if $\pi_n(f) \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective. The conditions very well pointed and connective⁺ for symmetric ring spectra are introduced in Subsection 4J and are very mild. Connective⁺ is a minor strengthening of connectivity; the sphere spectrum and suitable models for all Eilenberg–Mac Lane ring spectra are connective⁺.

Theorem 1.19. *Let G be a group and let \mathcal{F} be a family of subgroups of G . Let \mathbb{A} be a very well pointed symmetric ring spectrum.*

(i) *The assembly map for topological Hochschild homology*

$$EG(\mathcal{F})_+ \wedge_{\text{Or}G} \mathbf{THH}(\mathbb{A}[Gf-]) \longrightarrow \mathbf{THH}(\mathbb{A}[G])$$

is always split injective, and it is a π_ -isomorphism if $C_{yc} \subseteq \mathcal{F}$, i.e., if \mathcal{F} contains all finite and infinite cyclic subgroups of G .*

(ii) *Assume that \mathbb{A} is connective⁺. If $\mathcal{F} \subseteq \mathcal{F}_{in}$ and if Assumption $[A'_{\mathcal{F}, \leq N+1}]$ in Main Technical Theorem 1.16 holds, then the assembly map for Bökstedt–Hsiang–Madsen’s functor C*

$$EG(\mathcal{F})_+ \wedge_{\text{Or}G} \mathbf{C}(\mathbb{A}[Gf-]; p) \longrightarrow \mathbf{C}(\mathbb{A}[G]; p)$$

is $\pi_n^{\mathbb{Q}}$ -injective for all $n \leq N$ and all primes p .

In a separate paper [73] we use the techniques developed here to show that, under stronger finiteness assumptions on G , one also obtains injectivity statements for the assembly map for topological cyclic homology. In particular, we show that if there is a universal space for proper actions $EG(\mathcal{F}in)$ of finite type, then the assembly map for topological cyclic homology

$$EG(\mathcal{F}in)_+ \wedge_{\text{Or}G} \mathbf{TC}(\mathbb{A}[Gf-]; p) \longrightarrow \mathbf{TC}(\mathbb{A}[G]; p)$$

is split injective for any connective⁺ symmetric ring spectrum \mathbb{A} . These results are not needed here and do not lead to stronger statements in algebraic K -theory.

We conclude this introduction with some more information about the Farrell–Jones Conjecture in general, not just its rationalized version [Conjecture 1.12](#). Recall that the conjecture predicts that the Farrell–Jones assembly map (1.8) is a π_* -isomorphism. There is a completely analogous conjecture for L -theory. Both conjectures were stated by Farrell and Jones in [39, 1.6 on page 257], using a different but equivalent formulation; see [49]. Among many others, the results in [34,36–38] provided evidence and paved the way for these conjectures. The Farrell–Jones Conjectures have many applications; most notably, they imply the Borel Conjecture about the topological rigidity of aspherical manifolds in dimensions greater than or equal to five. We refer to [71,9,70] for comprehensive surveys about these conjectures and their applications.

Besides [Theorem 1.7](#), there are many other important injectivity results, including [35, 24,23,101,13,4,58,94]. Furthermore, in recent years there has been tremendous progress in proving isomorphisms results, even for the more general version of the Farrell–Jones Conjectures with coefficients in additive categories and with finite wreath products [12,6]. This more general version has good inheritance properties: it passes to subgroups, overgroups of finite index, finite direct products, free products, and colimits over arbitrary directed systems; see for instance [2,5]. Even this more general version has now been verified for hyperbolic groups, CAT(0)-groups, arithmetic groups, lattices in virtually connected locally compact second countable Hausdorff groups, virtually solvable groups, fundamental groups of 3-manifolds; see for example [3,7,8,10,11,41,57,102,103,122,123].

In this work we only consider the algebraic K -theory version of the conjecture and only with coefficients in the ring of integers \mathbb{Z} or the sphere spectrum \mathbb{S} . For more information about why the method used in [17] and here does not work for L -theory, we refer to [78, [Remark 4.5.5 on pages 285–286](#)]. Notice, however, that all the results mentioned above are about groups satisfying some geometric condition, typically some version of non-positive curvature, and the proofs use geometric tools like controlled topology and flows. Our results only need homological finiteness conditions, and the methods of proof are completely different, coming mostly from equivariant stable homotopy theory. We

cover prominent groups, such as outer automorphism groups of free groups and Thompson’s groups, for which no previous results were known. In particular, [Theorem 1.1](#) about Whitehead groups gives a deep result under very mild assumptions.

2. Discussion of the assumptions

In this section we discuss the assumptions of Main [Theorem 1.13](#). We describe many groups that satisfy Assumption [\[A \$\mathcal{F}Cyc\$ \]](#) and explain why Assumption [\[B \$\mathcal{F}Cyc\$ \]](#) is conjecturally always true.

Let \mathcal{F} be a family of subgroups of G , i.e., a collection of subgroups that is closed under passage to subgroups and conjugates. Examples of families are

$$\mathcal{F} = 1, \mathcal{F}Cyc, \mathcal{F}in, Cyc, \mathcal{V}Cyc,$$

the families consisting respectively of the trivial subgroup only, of all finite cyclic, finite, cyclic, or virtually cyclic subgroups of G . Recall that a group is called virtually cyclic if it contains a cyclic subgroup of finite index. A *universal space for \mathcal{F}* is a G -CW complex $EG(\mathcal{F})$ such that, for all subgroups $H \leq G$, the H -fixed point space

$$(EG(\mathcal{F}))^H \text{ is } \begin{cases} \text{empty} & \text{if } H \notin \mathcal{F}; \\ \text{contractible} & \text{if } H \in \mathcal{F}. \end{cases}$$

Such a space is unique up to G -homotopy equivalence, and it receives a G -map that is unique up to G -homotopy from every G -CW complex all of whose isotropy groups are in \mathcal{F} .

When \mathcal{F} is the trivial family 1, then $EG(1)$ is the universal cover $EG = \widetilde{BG}$ of the classifying space of G . When \mathcal{F} is the family $\mathcal{F}in$ of finite subgroups, $EG(\mathcal{F}in)$ is often denoted \underline{EG} and called the universal space for proper actions.

The following result can be used to verify Assumption [\[A \$\mathcal{F}Cyc\$ \]](#) for a large class of examples, as illustrated below. We say that a G -CW-complex has finite type if it has only finitely many G -cells in each dimension. A group G has type F_∞ if there is a classifying space BG of finite type. Clearly, if G has type F_∞ , then $H_s(BG; \mathbb{Z})$ is a finitely generated abelian group for every s .

Proposition 2.1. *Let $\mathcal{F} \subseteq \mathcal{F}in$ be a family of finite subgroups of G . Then there is a universal space $EG(\mathcal{F})$ of finite type if and only if there are only finitely many conjugacy classes of subgroups in \mathcal{F} and $Z_G H$ has type F_∞ for each $H \in \mathcal{F}$.*

Proof. Since H is finite and $Z_G H$ has finite index in $N_G H$, notice that $Z_G H$ has type F_∞ if and only if $N_G H$ has type F_∞ if and only if $N_G H/H$ has type F_∞ ; see for example [\[43, Corollary 7.2.4 on page 170, Theorem 7.2.21 and Exercise 7.2.1 on pages 175–176\]](#). In the case when $\mathcal{F} = \mathcal{F}in$, [\[67, Theorem 4.2 on page 195\]](#) states that there is an $EG(\mathcal{F})$

of finite type if and only if there are only finitely many conjugacy classes of subgroups in \mathcal{F} and $N_G H/H$ has type F_∞ for each $H \in \mathcal{F}$. By inspection, the proof of this result in [67] goes through for arbitrary $\mathcal{F} \subseteq \mathcal{F}in$. Notice that [67, Lemma 1.3 on page 180], which is used in the proof, in this general case shows that if $H \in \mathcal{F}$ then $(EG(\mathcal{F}))^H$ is a model for $E(N_G H/H)(q_*(\mathcal{F}))$, where $q: N_G H \rightarrow N_G H/H$ is the quotient map and $q_*(\mathcal{F}) = \{ F \leq N_G H/H \mid q^{-1}(F) \in \mathcal{F} \}$. The rest of the proof works verbatim. \square

Corollary 2.2. *If there is a universal space $EG(\mathcal{F}in)$ of finite type then Assumption $[A_{\mathcal{F}Cyc}]$ is satisfied.*

The existence of a universal space $EG(\mathcal{F}in)$ of finite type is already a fairly mild condition. For example, all the following groups have even finite (i.e., finite type and finite dimensional) models for $EG(\mathcal{F}in)$, and therefore satisfy Assumption $[A_{\mathcal{F}Cyc}]$ by Corollary 2.2. Theorem 1.15 about eventual injectivity applies to all these groups.

- Word-hyperbolic groups.
- Groups acting properly, cocompactly, and by isometries on a CAT(0)-space.
- Arithmetic groups in semisimple connected linear \mathbb{Q} -algebraic groups.
- Mapping class groups.
- Outer automorphism groups of free groups.

For more information we refer to [69, Section 4] and also to [85] in the case of mapping class groups.

Moreover, Assumption $[A_{\mathcal{F}Cyc}]$ is much weaker than $EG(\mathcal{F}in)$ having finite type. For example, Thompson’s group T of orientation preserving, piecewise linear, dyadic homeomorphisms of the circle contains finite cyclic subgroups of any given order, and hence there can be no $ET(\mathcal{F}in)$ of finite type by Proposition 2.1. However, in [44, Corollary 1.5] it is proved that Thompson’s group T satisfies Assumption $[A_{\mathcal{F}Cyc}]$.

For Thompson’s group T and for outer automorphism groups of free groups there seem to be no previously known results about the Farrell–Jones Conjecture.

We now turn to Assumption $[B_{\mathcal{F}Cyc}]$. First of all, notice that this assumption depends on the group G only through the set of numbers c that appear as orders of finite cyclic subgroups of G . Fix integers $c \geq 1, t \geq 0$, and a prime number p . The map in Assumption $[B_{\mathcal{F}Cyc}]$, followed by the projection to the factor indexed by p , is the diagonal β of the following commutative diagram.

$$\begin{array}{ccc}
 K_t(\mathbb{Z}[\zeta_c]) & \longrightarrow & K_t(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_c]) \\
 \downarrow & \searrow \beta & \downarrow \\
 K_t(\mathbb{Z}[\zeta_c]; \mathbb{Z}_p) & \longrightarrow & K_t(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_c]; \mathbb{Z}_p)
 \end{array} \tag{2.3}$$

The vertical maps are induced by the natural map $\mathbf{K}(R) \rightarrow \mathbf{K}(R)_p^\wedge$ from the algebraic K -theory spectrum to its p -completion. The horizontal maps are induced by the ring homomorphism $\mathbb{Z}[\zeta_c] \rightarrow \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_c]$. Notice that $\mathbb{Z}[\zeta_c]$ is the ring of integers in the number field $\mathbb{Q}(\zeta_c)$, and that there is a decomposition

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_c] = \prod_{\mathfrak{p}|p} \mathcal{O}_{\mathfrak{p}},$$

where the product runs over all primes \mathfrak{p} in $\mathbb{Z}[\zeta_c]$ over p , and $\mathcal{O}_{\mathfrak{p}}$ is the completion of $\mathbb{Z}[\zeta_c]$ with respect to \mathfrak{p} . Since $K_t(-)$ and $K_t(-; \mathbb{Z}_p)$ respect finite products, one can rewrite diagram (2.3) with suitable products on the right-hand side.

We first consider the cases $t = 0$ and $t = 1$.

Proposition 2.4. *If $t = 0$ or $t = 1$, then the map β in diagram (2.3) is \mathbb{Q} -injective for any prime p and for any $c \geq 1$. In particular, in the notation of [Main Technical Theorem 1.16](#), Assumption $[B_{\mathcal{F}Cyc, \{0,1\}}]$ is always satisfied.*

Proof. The second statement clearly follows from the first.

If $t = 0$, the map β is easily seen to be \mathbb{Q} -injective, because $K_0(\mathbb{Z}[\zeta_c])$ is the sum of \mathbb{Z} and a finite group, the ideal class group of $\mathbb{Q}(\zeta_c)$.

If $t = 1$, the determinant identifies $K_1(R)$ with the group of units R^\times , provided that R is a ring of integers in a number field [[14, Corollary 4.3 on page 95](#)], or a local ring [[125, Lemma III.1.4 on page 202](#)]. The abelian group $\mathbb{Z}[\zeta_c]^\times$ is finitely generated, $\mathcal{O}_{\mathfrak{p}}^\times$ is the sum of a finite group and a finitely generated \mathbb{Z}_p -module, and both $K_0(\mathbb{Z}[\zeta_c])$ and $K_0(\prod_{\mathfrak{p}|p} \mathcal{O}_{\mathfrak{p}})$ are finitely generated abelian groups. Hence all these groups have bounded p -torsion and diagram (2.3) can be identified with the following diagram; compare Subsection 4N.

$$\begin{array}{ccc} \mathbb{Z}[\zeta_c]^\times & \longrightarrow & \prod_{\mathfrak{p}|p} \mathcal{O}_{\mathfrak{p}}^\times \\ \downarrow & & \downarrow \\ (\mathbb{Z}[\zeta_c]^\times)_p^\wedge & \longrightarrow & \prod_{\mathfrak{p}|p} (\mathcal{O}_{\mathfrak{p}}^\times)_p^\wedge \end{array} \tag{2.5}$$

Here A_p^\wedge is the p -completion of the abelian group A , defined as the inverse limit $\lim_i A/p^i$. The natural map $A \rightarrow A_p^\wedge$ is \mathbb{Q} -injective provided that A is a finitely generated \mathbb{Z} - or \mathbb{Z}_p -module. Therefore the vertical maps in (2.5) are \mathbb{Q} -injective. Since the upper horizontal map is clearly injective, too, the result is proved. \square

We remark that also the lower horizontal map in diagram (2.5) is \mathbb{Q} -injective. However, this is highly non-trivial. It follows from the fact that the following conjecture was proved by Brumer [[22, Theorem 2 on page 121](#)] for abelian number fields, and so in particular for $F = \mathbb{Q}(\zeta_c)$.

Conjecture 2.6 (*The Leopoldt conjecture*). *Let \mathcal{O}_F be the ring of integers in a number field F , and let p be a prime. Then the map*

$$(\mathcal{O}_F^\times)_{\widehat{p}} \longrightarrow ((\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_F)^\times)_{\widehat{p}}$$

induced by the natural inclusion $\mathcal{O}_F \longrightarrow \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_F$ is injective.

For more information about the Leopoldt [Conjecture 2.6](#) we refer to [[86, Section X.3](#)] and [[121, Section 5.5](#)]. The conjecture is open for arbitrary number fields.

Next we discuss diagram [\(2.3\)](#) for $t \geq 2$. We can restrict to the case when t is odd, because if $t \geq 2$ is even then the rationalized upper left corner $K_t(\mathbb{Z}[\zeta_c]) \otimes_{\mathbb{Z}} \mathbb{Q}$ vanishes by [[19, Proposition 12.2 on page 271](#)].

The p -completed K -theory groups of \mathcal{O}_p and hence of $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_c]$ are completely known; see for example [[124, Theorem 61 on page 165](#)]. However, the non-completed K -theory groups $K_t(\mathcal{O}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$ seem to be rather intractable; compare [[124, Warning 60 on pages 164–165](#)]. Hence the strategy of going first right and then down in diagram [\(2.3\)](#), which was successful without any effort in the case $t = 1$, does not seem very promising for $t \geq 2$.

However, since the abelian groups $K_t(\mathbb{Z}[\zeta_c])$ are finitely generated by [[93](#)], the left-hand vertical map in diagram [\(2.3\)](#) is always \mathbb{Q} -injective.

The following conjecture concerns the lower horizontal map in diagram [\(2.3\)](#). It is the rationalized analogue for higher algebraic K -theory of the Leopoldt [Conjecture 2.6](#). It appeared in [[104, page 192](#)], and we refer to it as the Schneider Conjecture. A related question was asked by Wagoner in [[118, page 242](#)]. In [Section 18](#) we briefly discuss some equivalent formulations, including Schneider’s original one.

Conjecture 2.7 (*The Schneider conjecture*). *Let \mathcal{O}_F be the ring of integers in a number field F , and let p be a prime. Then for any $t \geq 2$ the map*

$$K_t(\mathcal{O}_F; \mathbb{Z}_p) \longrightarrow K_t(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_F; \mathbb{Z}_p)$$

induced by the natural inclusion $\mathcal{O}_F \longrightarrow \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_F$ is \mathbb{Q} -injective.

The discussion above can then be summarized as follows.

Proposition 2.8. *If the Schneider [Conjecture 2.7](#) holds for $F = \mathbb{Q}(\zeta_c)$, $t \geq 2$, and p , then the map β in diagram [\(2.3\)](#) is \mathbb{Q} -injective. If for each $c \geq 1$ and each $t \geq 2$ there is a prime p such that the Schneider [Conjecture 2.7](#) holds for $F = \mathbb{Q}(\zeta_c)$ and t , then Assumption [[B_{FCyc}](#)] is always satisfied.*

By [Theorem 18.3](#) we know that for a fixed number field F and a fixed odd prime p there can only be finitely many values of t for which the Schneider [Conjecture 2.7](#) is false.

In light of [Propositions 2.4 and 2.8](#), we view our Assumption [[B_{FCyc}](#)] as a weak version of the conjectures of Leopoldt and Schneider.

We conclude this section with the following well-known result, which explains why there is no Assumption $[B_1]$ in [Theorem 1.7](#). In fact, a version of this result enters in the original proof of [Theorem 1.7](#) by Bökstedt–Hsiang–Madsen; compare [\[17, Remark 9.6\(ii\) on page 533\]](#). Recall that a prime p is called regular if p does not divide the order of the ideal class group of $\mathbb{Q}(\zeta_p)$.

Proposition 2.9. *If $c = 1$ and p is a regular prime, then the map*

$$\beta: K_t(\mathbb{Z}) \longrightarrow K_t(\mathbb{Z}_p; \mathbb{Z}_p)$$

in diagram (2.3) is \mathbb{Q} -injective for any $t \geq 0$. In the notation of [Main Technical Theorem 1.16](#), Assumption $[B_{1, \mathbb{N}}]$ is satisfied.

Proof. If p is an odd regular prime, by [Lemma 18.4\(i\)](#) and [Theorem 18.3](#) it suffices to verify statement (vi) in [Theorem 18.3](#) for $F = \mathbb{Q}(\zeta_p)$. This follows from the fact that in this case $Z = 0$; compare [\[104, Beispiele ii\) on page 201\]](#).

If $p = 2$, this is proved in [\[97, Theorem 7.7 on page 318\]](#). \square

3. Strategy of the proof and outline

In this section we describe the strategy of the proof of [Main Technical Theorem 1.16](#) and the outline of the paper. Recall that, as explained in the introduction, [Theorems 1.13 and 1.7](#) are special cases of [Theorem 1.16](#). The strategy can be summarized by the following commutative diagram, which we explain below and refer to as the *main diagram*.

$$\begin{array}{ccc}
 H_n^G(EG(\mathcal{V}\mathcal{C}yc); \mathbf{K}^{\geq 0}(\mathbb{Z}[Gf-])) & \xrightarrow{\alpha^1} & K_n(\mathbb{Z}[G]) \\
 \uparrow H_n^G(\text{incl}) & & \parallel \\
 H_n^G(EG(\mathcal{F}); \mathbf{K}^{\geq 0}(\mathbb{Z}[Gf-])) & \xrightarrow{\alpha^2} & \pi_n \mathbf{K}^{\geq 0}(\mathbb{Z}[G]) \\
 \uparrow \ell\% & & \uparrow \ell \\
 H_n^G(EG(\mathcal{F}); \mathbf{K}^{\geq 0}(\mathbb{S}[Gf-])) & \xrightarrow{\alpha^3} & \pi_n \mathbf{K}^{\geq 0}(\mathbb{S}[G]) \\
 \downarrow \tau\% & & \downarrow \tau \\
 H_n^G\left(EG(\mathcal{F}); \prod_p \mathbf{TC}(\mathbb{S}[Gf-]; p)\right) & \xrightarrow{\alpha^4} & \pi_n\left(\prod_p \mathbf{TC}(\mathbb{S}[G]; p)\right) \\
 \downarrow \sigma\% & & \downarrow \sigma \\
 H_n^G\left(EG(\mathcal{F}); \mathbf{THH}(\mathbb{S}[Gf-]) \times \prod_p \mathbf{C}(\mathbb{S}[Gf-]; p)\right) & \xrightarrow{\alpha^5} & \pi_n\left(\mathbf{THH}(\mathbb{S}[G]) \times \prod_p \mathbf{C}(\mathbb{S}[G]; p)\right)
 \end{array} \tag{3.1}$$

The products are indexed over all prime numbers p .

Here \mathcal{F} is a family of subgroups of G and $EG(\mathcal{F})$ is the corresponding universal space; see the beginning of the previous section. We assume that $\mathcal{F} \subseteq \mathcal{FCyc}$, i.e., that all subgroups in \mathcal{F} are finite cyclic.

The notation $\mathbf{K}^{\geq 0}$ stands for the connective algebraic K -theory spectrum, and \mathbf{THH} , \mathbf{TC} , and \mathbf{C} denote topological Hochschild homology, topological cyclic homology, and Bökstedt–Hsiang–Madsen’s functor C ; see Sections 6, 10, and 9. We construct functors

$$\mathbf{K}^{\geq 0}(\mathbb{A}[Gf-]), \quad \mathbf{THH}(\mathbb{A}[Gf-]), \quad \mathbf{C}(\mathbb{A}[Gf-]; p), \quad \mathbf{TC}(\mathbb{A}[Gf-]; p), \quad (3.2)$$

for any coefficient ring or symmetric ring spectrum \mathbb{A} ; compare Section 15. These are all functors from the orbit category $\text{Or}G$ of G to the category of naive spectra $\mathbb{N}\text{Sp}$. Given any such functor $\mathbf{E}: \text{Or}G \rightarrow \mathbb{N}\text{Sp}$, the assembly map for \mathbf{E} with respect to a family \mathcal{F} of subgroups of G is a map of spectra

$$\text{asbl}: EG(\mathcal{F})_+ \wedge_{\text{Or}G} \mathbf{E} \rightarrow \mathbf{E}(G/G) \quad (3.3)$$

defined in Subsection 4H. There are isomorphisms

$$\mathbf{K}^{\geq 0}(\mathbb{A}[Gf(G/G)]) \cong \mathbf{K}^{\geq 0}(\mathbb{A}[G]),$$

and similarly for the other functors in (3.2).

Taking homotopy groups of $X_+ \wedge_{\text{Or}G} \mathbf{E}$ defines a G -equivariant homology theory $H_*^G(X; \mathbf{E})$. The assembly map (3.3) then gives a group homomorphism

$$\alpha = \pi_n(\text{asbl}): H_n^G(EG(\mathcal{F}); \mathbf{E}) \rightarrow \pi_n \mathbf{E}(G/G).$$

All horizontal maps in diagram (3.1) are given by assembly maps. The top horizontal map α^1 is the Farrell–Jones assembly map for connective algebraic K -theory.

The vertical map $H_n^G(\text{incl})$ is induced by the inclusions $\mathcal{F} \subseteq \mathcal{FCyc} \subseteq \mathcal{VCyc}$. If $\mathcal{F} = \mathcal{FCyc}$ then $H_n^G(\text{incl})$ is a \mathbb{Q} -isomorphism by Example 12.12 and [75, Theorem 0.3 on page 2].

Theorem 12.7 and Example 12.11 identify the source of α^2 rationally with

$$\bigoplus_{(C) \in (\mathcal{F})} \bigoplus_{\substack{s+t=n \\ s,t \geq 0}} H_s(BZ_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \Theta_C \left(K_t(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q} \right). \quad (3.4)$$

Under this identification, the map (1.17) in Main Technical Theorem 1.16 is the restriction of $\alpha^2 \otimes_{\mathbb{Z}} \mathbb{Q}$ to the summands satisfying $t \in T$. In the case $\mathcal{F} = 1$, it is the rationalized classical assembly map (1.5). The map (1.14) in Main Theorem 1.13 is identified with $\alpha^1 \otimes_{\mathbb{Z}} \mathbb{Q}$.

All other vertical maps are induced by natural transformations of functors from $\text{Or}G$ to $\mathbb{N}\text{Sp}$; compare Section 15.

The map ℓ is the linearization map given by changing the group ring coefficients from the sphere spectrum \mathbb{S} to the integers \mathbb{Z} . The spectrum $\mathbf{K}^{\geq 0}(\mathbb{S}[G])$ is a model for $\mathbf{A}(BG)$, Waldhausen’s algebraic K -theory of the classifying space of G . By a result of Waldhausen [119, Proposition 2.2 on page 43], ℓ is a \mathbb{Q} -isomorphism for any group G . The linearization map extends to a natural transformation $\mathbf{K}^{\geq 0}(\mathbb{S}[Gf-]) \rightarrow \mathbf{K}^{\geq 0}(\mathbb{Z}[Gf-])$ of functors $\text{Or}G \rightarrow \text{NSp}$, and therefore it induces the map $\ell_{\%}$. Using Theorem 12.7 and Addendum 12.13 we conclude that also $\ell_{\%}$ is a \mathbb{Q} -isomorphism.

The map τ , followed by the projection to the factor indexed by the prime p , is Bökstedt–Hsiang–Madsen’s cyclotomic trace map trc_p from algebraic K -theory to topological cyclic homology at p . Using the model of Dundas–Goodwillie–McCarthy [30] recalled in Section 15, this map also extends to a natural transformation and induces the map $\tau_{\%}$.

For any coefficient ring or symmetric ring spectrum \mathbb{A} there is a projection map $\mathbf{TC}(\mathbb{A}[G]; p) \rightarrow \mathbf{THH}(\mathbb{A}[G])$. In the special case of spherical coefficients $\mathbb{A} = \mathbb{S}$, there is also a map $\mathbf{TC}(\mathbb{S}[G]; p) \rightarrow \mathbf{C}(\mathbb{S}[G]; p)$ from topological cyclic homology to Bökstedt–Hsiang–Madsen’s functor \mathbf{C} ; compare Section 11. The map σ is then defined as the product of the following maps. For $p = 11$ take the induced map $\mathbf{TC}(\mathbb{S}[G]; 11) \rightarrow \mathbf{THH}(\mathbb{S}[G]) \times \mathbf{C}(\mathbb{S}[G]; 11)$. For $p \neq 11$ take $\mathbf{TC}(\mathbb{S}[G]; p) \rightarrow \mathbf{C}(\mathbb{S}[G]; p)$. The choice of the prime 11 is arbitrary; we could fix any other prime. Also σ extends to a natural transformation and induces the map $\sigma_{\%}$.

This completes the explanation of the main diagram (3.1). The proof of Main Technical Theorem 1.16 is then an immediate consequence of the following two results. Recall that our goal is to show that the restriction of $\alpha^2 \otimes_{\mathbb{Z}} \mathbb{Q}$ to the summands satisfying $t \in T$ in (3.4) is injective. We already know that $\ell_{\%}$ and ℓ are \mathbb{Q} -isomorphisms.

Theorem 3.5 (*Splitting Theorem*). *If Assumption $[A'_{\mathcal{F}, \leq N+1}]$ holds, then for every $n \leq N$ the map $\alpha^5 \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective.*

Theorem 3.6 (*Detection Theorem*). *If Assumption $[B_{\mathcal{F}, T}]$ holds, then for every n the restriction of $(\sigma_{\%} \circ \tau_{\%}) \otimes_{\mathbb{Z}} \mathbb{Q}$ to the summands satisfying $t \in T$ is injective.*

The main ingredient in the proof of the Detection Theorem 3.6 is the theory of equivariant Chern characters due to the first author, which we explain in Section 12. To show in Section 15 that the theory can be applied to our examples, we develop a general framework in Section 14, which uses the formalism of enriched categories of modules that we introduce in Section 13. The proof of the Detection Theorem 3.6 is then completed in Section 16. In this step we also use a deep result of Hesselholt and Madsen [51], building on a theorem of McCarthy [82], about the cyclotomic trace map.

The proof of the Splitting Theorem 3.5 occupies the first half of the paper, Sections 5 to 9. The proof proceeds in stages, starting with general results about the assembly map for the cyclic nerve CN , and leading to Theorem 9.5; see also Theorem 1.19 mentioned at the end of the introduction. The most delicate step occurs when passing from topo-

logical Hochschild homology to Bökstedt–Hsiang–Madsen’s functor C , and is based on the results we developed in [74] and [95].

We summarize the steps of the proof in the following roadmap. Every line represents a result about the assembly map for the theory indicated on the left. Next to the vertical implication symbols, we indicate which assumptions are needed, and some of the important results that are used.

	For the \mathcal{F} -assembly map is ...
Prop. 5.12	$ CN_*(G) $	split inj for all \mathcal{F} , π_* -iso if $Cyc \subseteq \mathcal{F}$ \Downarrow uses [51,106]
Thm. 6.1	$\mathbf{THH}(\mathbb{A}[G])$	split inj for all \mathcal{F} , π_* -iso if $Cyc \subseteq \mathcal{F}$ \Downarrow
Cor. 6.20	$\mathbf{THH}(\mathbb{A}[G])_{hC_{p^n}}$	split inj for all \mathcal{F} , π_* -iso if $Cyc \subseteq \mathcal{F}$ if \mathbb{A} is connective ⁺ \Downarrow uses [51,15,95]
Cor. 8.4	$\text{hofib}(\mathbf{THH}(\mathbb{A}[G])^{C_{p^{n+1}}} \xrightarrow{R} \mathbf{THH}(\mathbb{A}[G])^{C_{p^n}})$	split inj for all \mathcal{F} , π_* -iso if $Cyc \subseteq \mathcal{F}$ \mathcal{E} if G fulfills $[A'_{\mathcal{F}, \leq N+1}] \Downarrow$ uses [74]
Thm. 9.5	$\mathbf{C}(\mathbb{A}[G]; p) \ \& \ \mathbf{THH}(\mathbb{A}[G]) \times \prod_p \mathbf{C}(\mathbb{A}[G]; p)$	$\pi_n^{\mathbb{Q}}$ -injective if $\mathcal{F} \subseteq \mathcal{F}Cyc$, $n \leq N$ \mathcal{E} if G fulfills $[B_{\mathcal{F}, T}] \Downarrow$ uses [17,51,68,30]
Thm. 1.16	$\mathbf{K}^{\geq 0}(\mathbb{S}[G]) \ \& \ \mathbf{K}^{\geq 0}(\mathbb{Z}[G])$	$\pi_n^{\mathbb{Q}}$ -injective if $\mathcal{F} \subseteq \mathcal{F}Cyc$, $n \leq N$, $t \in T$

We conclude this section with two important remarks.

Remark 3.7 (*Limitation of trace methods*). The strategy used both here and in [17] to prove rational injectivity of the assembly map for algebraic K -theory breaks down if the homological finiteness assumptions $[A_1]$, $[A_{\mathcal{F}Cyc}]$, or $[A'_{\mathcal{F}, \leq N+1}]$ are dropped.

For example, let G be the additive group of the rational numbers \mathbb{Q} . Since \mathbb{Q} is torsion-free, $\mathcal{F}Cyc = 1$ and Theorems 1.7 and 1.13 would coincide. Since $H_1(B\mathbb{Q}; \mathbb{Z}) \cong \mathbb{Q}$, assumption $[A_1]$ is not satisfied by $G = \mathbb{Q}$. Recall that $B\mathbb{Q}$ is a Moore space $M(\mathbb{Q}, 1)$, and therefore for any spectrum \mathbf{E} we have that $B\mathbb{Q} \wedge \mathbf{E} \simeq \Sigma \mathbf{E}_{\mathbb{Q}}$, the suspension of the rationalization \mathbf{E} .

Fixing a prime p , the assembly map for $\mathbf{TC}(\mathbb{S}[\mathbb{Q}]; p)$ factors as

$$\begin{array}{ccc}
 B\mathbb{Q}_+ \wedge \mathbf{TC}(\mathbb{S}; p) & \xrightarrow{\alpha} & \mathbf{TC}(\mathbb{S}[\mathbb{Q}]; p) \\
 & \searrow t & \nearrow \\
 & \text{holim}_{\mathcal{R}\mathcal{F}} B\mathbb{Q}_+ \wedge (\mathbf{THH}(\mathbb{S})^{C_{p^m}}) &
 \end{array}$$

where t interchanges smash product and homotopy limit. The assembly map for the trivial group 1 is a π_* -isomorphism and splits off from the assembly map for $\mathbf{TC}(\mathbb{S}[G]; p)$ for any group G . So the interesting part of α factors through

$$\operatorname{holim}_{\mathcal{RF}} B\mathbb{Q} \wedge (\mathbf{THH}(\mathbb{S})^{C_{p^m}}), \tag{3.8}$$

since $B\mathbb{Q} = (B\mathbb{Q}_+)/ (B1_+)$. Using that $\mathbf{THH}(\mathbb{S}) \simeq \mathbb{S}$ (Example 6.16) and the fundamental fibration sequence (Theorem 8.1), we conclude that the homotopy groups of the spectrum in (3.8) vanish for every $n \geq 2$. A similar argument applies to $\prod_p \mathbf{TC}(-; p)$. Therefore, we see that diagram (3.1) cannot be used to prove rational injectivity of the assembly map in algebraic K -theory in this case.

Remark 3.9. With only Assumption $[A_{\mathcal{FCyc}}]$ we are not able to prove that the assembly map α^4 for topological cyclic homology in diagram (3.1) is \mathbb{Q} -injective, not even in the case of the trivial family $\mathcal{F} = 1$. Even treating one prime at the time, our method of proof breaks down without further assumptions on G . The reason for this is subtle, and is explained in detail in Warning 7.13, Remark 10.2, and Warning 11.2.

4. Conventions

4A. Spaces

We denote by \mathbf{Top} the category of compactly generated and weak Hausdorff spaces (e.g., see [28, Section 7.9 on pages 186–195] and [110]), which from now on we simply call *spaces*. A space homeomorphic to a (finite or countable) CW-complex is called a (finite or countable, respectively) *CW-space*. We denote by \mathcal{T} the category of pointed spaces, and we let \mathcal{W} be the full subcategory of \mathcal{T} whose objects are the pointed countable CW-spaces.

4B. Equivariant spaces

Given a discrete group G , we denote by \mathbf{Top}^G the category of left G -spaces and G -equivariant maps. Given G -spaces X and Y , the group G acts by conjugation on the space of maps from X to Y , and the fixed points are the G -equivariant maps. The pointed version of this category is denoted \mathcal{T}^G .

We denote by \mathbf{Sets}^G the category of left G -sets and G -equivariant functions, and we consider \mathbf{Sets}^G as a full subcategory of \mathbf{Top}^G . The *orbit category* $\mathbf{Or}G$ is the full subcategory of \mathbf{Sets}^G with objects G/H for all subgroups H of G .

Given a group homomorphism $\alpha: H \rightarrow G$, the restriction functor

$$\operatorname{res}_\alpha: \mathbf{Top}^G \rightarrow \mathbf{Top}^H \quad \text{has a left adjoint} \quad \operatorname{ind}_\alpha: \mathbf{Top}^H \rightarrow \mathbf{Top}^G$$

which sends an H -space X to the G -space $G \times_H X$ defined as the quotient of $G \times X$ by the H -action $h(g, x) = (g\alpha(h^{-1}), hx)$.

4C. Action groupoids

Given a group G and a G -set S , we denote by GfS the groupoid with $\text{obj } GfS = S$ and $\text{mor}_{GfS}(s, s') = \{g \in G \mid gs = s'\}$.

Identifying the group G with the category $Gfpt$ with only one object, and viewing the G -set S as a functor from G to Sets , then GfS is the Grothendieck construction of this functor; compare [114, Definition 1.1 on page 92]. This definition is functorial in S and thus yields a functor

$$Gf- : \text{Sets}^G \longrightarrow \text{Groupoids}.$$

4D. Enriched categories

We refer to [77, Section VII.7 on pages 180–181] and [59] for the theory of symmetric monoidal categories $(\mathcal{V}, \odot, \mathbf{1})$ and categories and functors enriched over \mathcal{V} , or simply \mathcal{V} -categories and \mathcal{V} -functors. A monoid in a monoidal category \mathcal{V} is the same as a \mathcal{V} -category with only one object. The important examples for us are summarized in the following table.

$(\mathcal{V}, \odot, \mathbf{1})$	\mathcal{V} -categories	Monoids in \mathcal{V}
$(\text{Sets}, \times, \text{pt})$	ordinary categories	ordinary monoids
$(\text{Cat}, \times, \text{pt})$	strict 2-categories	strict monoidal categories
$(\text{Ab}, \otimes, \mathbb{Z})$	pre-additive categories	rings
$(\mathcal{T}, \wedge, S^0)$	topological categories	pointed topological monoids
$(\Sigma\text{Sp}, \wedge, \mathbb{S})$	symmetric spectral categories	symmetric ring spectra

Given a lax monoidal functor $F: \mathcal{V} \rightarrow \mathcal{V}'$, any \mathcal{V} -category \mathcal{X} has an associated \mathcal{V}' -category $F\mathcal{X}$ with the same objects and $(F\mathcal{X})(x, y) = F(\mathcal{X}(x, y))$. We use the term base change for this passage from \mathcal{V} - to \mathcal{V}' -categories.

As indicated in the table, we refer to \mathcal{T} -categories, i.e., categories enriched over pointed spaces, as topological categories; we refer to \mathcal{T} -enriched functors as continuous functors. Given a skeletally small topological category \mathcal{C} , we denote by \mathcal{CT} the category of continuous functors $\mathcal{C} \rightarrow \mathcal{T}$ and continuous natural transformations. Of great importance in this paper is the category \mathcal{WT} of continuous functors to \mathcal{T} from the full subcategory \mathcal{W} of pointed countable CW-spaces defined in 4A. The objects of \mathcal{WT} are called \mathcal{W} -spaces and are models for spectra. In fact, given $\mathbf{X} \in \text{obj } \mathcal{WT}$, for every $A, B \in \text{obj } \mathcal{W}$ there is a map $A \wedge \mathbf{X}(B) \rightarrow \mathbf{X}(A \wedge B)$, and so in particular we get a symmetric spectrum by evaluating on spheres. The category \mathcal{WT} is itself a closed symmetric monoidal category, with respect to the smash product of \mathcal{W} -spaces \mathbf{X} and \mathbf{Y} defined as the continuous left Kan extension along $-\wedge- : \mathcal{W} \times \mathcal{W} \rightarrow \mathcal{W}$ of the composition

$$\mathcal{W} \times \mathcal{W} \xrightarrow{\mathbf{X} \times \mathbf{Y}} \mathcal{T} \times \mathcal{T} \xrightarrow{- \wedge -} \mathcal{T}.$$

Compare Subsections 4E and 4I.

4E. *Coends and left Kan extensions*

Given a small topological category \mathcal{C} and continuous functors $W: \mathcal{C}^{\text{op}} \rightarrow \mathcal{T}$ and $X: \mathcal{C} \rightarrow \mathcal{T}$, we let

$$W \wedge_{\mathcal{C}} X = \text{coend} \left(\mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{W \times X} \mathcal{T} \times \mathcal{T} \xrightarrow{- \wedge -} \mathcal{T} \right).$$

Given a continuous functor $\alpha: \mathcal{C} \rightarrow \mathcal{D}$, the continuous left Kan extension of X along α is the continuous functor

$$\text{Lan}_{\alpha} X: \mathcal{D} \rightarrow \mathcal{T}, \quad d \mapsto \mathcal{D}(\alpha(-), d) \wedge_{\mathcal{C}} X.$$

If we replace \mathcal{T} by \mathcal{WT} or by one of the other topological categories of spectra in Subsection 4I, then coends and Kan extensions are defined levelwise.

4F. *Homotopy colimits*

Given an ordinary small category \mathcal{C} , consider the functor

$$EC: \mathcal{C}^{\text{op}} \rightarrow \text{Top}, \quad c \mapsto |N_{\bullet}(c \downarrow \mathcal{C})|$$

that sends an object c to the classifying space of the under category $c \downarrow \mathcal{C}$. The homotopy colimit of a functor $X: \mathcal{C} \rightarrow \mathcal{T}$ is defined by

$$\text{hocolim}_{\mathcal{C}} X = EC_{+} \wedge_{\mathcal{C}} X.$$

If $W: \mathcal{C}^{\text{op}} \rightarrow \text{Top}$ is a contravariant free \mathcal{C} -CW-complex in the sense of [26, Definition 3.2 on page 221] that is objectwise contractible, then $W \wedge_{\mathcal{C}} X$ is homotopy equivalent to $\text{hocolim}_{\mathcal{C}} X$.

If we replace \mathcal{T} by \mathcal{WT} or by one of the other topological categories of spectra in Subsection 4I, then homotopy colimits are defined levelwise.

4G. *Natural modules*

The category $\text{Mod}_{\mathcal{T}}$ of *natural modules* in \mathcal{T} is defined as follows; compare [30, Section A.10.4.1 on page 404]. The objects are pairs $(\mathcal{C}, \mathbb{X})$, with \mathcal{C} an ordinary small category and $\mathbb{X}: \mathcal{C} \rightarrow \mathcal{T}$ a functor. A morphism in $\text{Mod}_{\mathcal{T}}$ from $(\mathcal{C}, \mathbb{X})$ to $(\mathcal{D}, \mathbb{Y})$ is a pair (f, ν) , with $f: \mathcal{C} \rightarrow \mathcal{D}$ a functor and $\nu: \mathbb{X} \rightarrow f^* \mathbb{Y}$ a natural transformation. Homotopy colimits define a functor

$$\text{hocolim} : \text{Mod}_{\mathcal{T}} \longrightarrow \mathcal{T}, \quad (\mathcal{C}, \mathbb{X}) \longmapsto \text{hocolim}_{\mathcal{C}} \mathbb{X} = EC_+ \wedge_{\mathcal{C}} \mathbb{X}.$$

Analogously we define $\text{Mod}_{\mathcal{WT}}$ and get a functor

$$\text{hocolim} : \text{Mod}_{\mathcal{WT}} \longrightarrow \mathcal{WT}. \tag{4.1}$$

4H. Assembly maps

As a special case of 4E and 4F, take $\mathcal{C} = \text{Or}G$ and $\mathcal{D} = \text{Top}^G$ for a discrete group G , and consider them as topological categories by base change along the strong symmetric monoidal functor $(-)_+ : \text{Top} \longrightarrow \mathcal{T}$. Take $\alpha : \text{Or}G \longrightarrow \text{Top}^G$ to be the inclusion functor. Let NSp be the topological category of naive spectra from 4I, and let $\mathbf{E} : \text{Or}G \longrightarrow \text{NSp}$ be a functor. Define $\mathbf{E}_{\%} = \text{Lan}_{\alpha} \mathbf{E}$.

$$\begin{array}{ccc} \text{Or}G & \xrightarrow{\mathbf{E}} & \text{NSp} \\ \alpha \downarrow & \nearrow \mathbf{E}_{\%} & \\ \text{Top}^G & & \end{array}$$

By the definition in 4E, for any G -space X the spectrum $\mathbf{E}_{\%}(X)$ is the coend of the functor $(\text{Or}G)^{\text{op}} \times \text{Or}G \longrightarrow \text{NSp}$ given by

$$(G/H, G/K) \longmapsto \text{map}(G/H, X)_+^G \wedge \mathbf{E}(G/K) \cong X_+^H \wedge \mathbf{E}(G/K).$$

We write $\mathbf{E}_{\%}(X) = X_+ \wedge_{\text{Or}G} \mathbf{E}$. When $X = G/H$ there is a natural isomorphism

$$G/H_+ \wedge_{\text{Or}G} \mathbf{E} \cong \mathbf{E}(G/H). \tag{4.2}$$

The *assembly map* for \mathbf{E} with respect to a family \mathcal{F} of subgroups of G is the map of spectra

$$\text{asbl} : EG(\mathcal{F})_+ \wedge_{\text{Or}G} \mathbf{E} \longrightarrow \mathbf{E}(G/G) \tag{4.3}$$

induced by the projection $EG(\mathcal{F}) \longrightarrow \text{pt} = G/G$ and by the isomorphism (4.2). Here $EG(\mathcal{F})$ is the universal space for \mathcal{F} ; see the beginning of Section 2. Moreover,

$$EG(\mathcal{F})_+ \wedge_{\text{Or}G} \mathbf{E} \cong EG(\mathcal{F})_+ \wedge_{\text{Or}G(\mathcal{F})} \mathbf{E}|_{\text{Or}G(\mathcal{F})},$$

where $\text{Or}G(\mathcal{F})$ is the full subcategory of $\text{Or}G$ with objects G/H for $H \in \mathcal{F}$. Since the functor $\text{map}(-, EG(\mathcal{F}))^G : \text{Or}G(\mathcal{F})^{\text{op}} \longrightarrow \text{Top}$ is an objectwise contractible contravariant free $\text{Or}G(\mathcal{F})$ -CW-complex, the source of (4.3) is homotopy equivalent to

$$\text{hocolim}(\text{Or}G(\mathcal{F}) \hookrightarrow \text{Or}G \xrightarrow{\mathbf{E}} \text{NSp}).$$

4I. Spectra

As motivated in [Remark 4.6](#) below, we need to use several different model categories of spectra:

- $\mathbb{N}\mathbf{Sp}$ naive spectra (of topological spaces),
- $\Sigma\mathbf{Sp}$ symmetric spectra (of topological spaces),
- $\mathbf{Th}\mathcal{T}$ orthogonal spectra,
- \mathcal{WT} \mathcal{W} -(pointed) spaces,
- $\Gamma\mathcal{S}_*$ Γ -(pointed) simplicial sets.

We refer to [\[80\]](#) for definitions, details, and historical remarks. Two comments are in order.

Our category \mathcal{W} , defined in [4A](#), differs from [\[80, Example 4.6 on page 454\]](#), where only finite CW-spaces are considered. However, it is remarked in [\[80\]](#) that the theory works equally well for countable rather than finite CW-spaces. In [Theorem 8.2](#) we use a result of Blumberg [\[15\]](#) for finite CW-spaces, and there we make that restriction explicit.

Moreover, our notation is slightly different from the one in [\[80\]](#). In particular, we write \mathbf{Sp} instead of \mathcal{S} , Γ instead of \mathcal{F} for Segal’s category Γ^{op} and, following [\[95\]](#), \mathbf{Th} instead of \mathcal{S} for the topological category of real finite dimensional inner product spaces and Thom spaces of orthogonal complement bundles.

These categories are connected by forgetful functors

$$\mathcal{WT} \xrightarrow{\mathbb{U}} \mathbf{Th}\mathcal{T} \xrightarrow{\mathbb{U}} \Sigma\mathbf{Sp} \xrightarrow{\mathbb{U}} \mathbb{N}\mathbf{Sp}; \tag{4.4}$$

compare [\[80, Main Diagram on page 442\]](#). With the sole exception of $\mathbb{N}\mathbf{Sp}$, all other categories in [\(4.4\)](#) are closed symmetric monoidal with respect to the smash product, and the two functors on the left are lax symmetric monoidal.

For all categories in [\(4.4\)](#), homotopy groups and hence π_* -isomorphisms are defined after forgetting down to $\mathbb{N}\mathbf{Sp}$. All categories are Quillen model categories with respect to the stable model structure from [\[80, Theorem 9.2 on page 471\]](#). With the only exception of $\Sigma\mathbf{Sp}$, the weak equivalences are the π_* -isomorphisms. In all cases, the fibrant objects are the ones whose underlying naive spectrum is an Ω -spectrum in the classical sense [\[80, Proposition 9.9\(ii\) on page 472\]](#). All categories in [\(4.4\)](#) are Quillen equivalent [\[80, Theorem 0.1 on page 443\]](#).

In all cases, limits, colimits, and homotopy colimits are defined levelwise. However, homotopy limits have to be defined by first applying a fibrant replacement functor.

Finally, in [Section 15](#) we need to compare the topological categories of spectra that we use with the category $\Gamma\mathcal{S}_*$, which is used in [\[30\]](#). There are strong symmetric monoidal functors

$$\Gamma\mathcal{S}_* \xrightarrow{\mathbb{T}} \Gamma\mathcal{T} \xrightarrow{\mathbb{P}} \mathcal{WT}; \tag{4.5}$$

see [\[80, Section 19 on page 499 and Proposition 3.2 on page 450\]](#). We introduce the notation $\|-\|: \Gamma\mathcal{S}_* \rightarrow \Sigma\mathbf{Sp}$ for the composition of [\(4.5\)](#) and the forgetful func-

tor $\mathbb{U}: \mathcal{WT} \rightarrow \Sigma\mathbf{Sp}$, because we need it explicitly in Section 15. Notice that the choice of using countable instead of finite CW-spaces in the definition of \mathcal{W} does not affect the composition $\mathbb{U} \circ \mathbb{P}: \Gamma\mathcal{T} \rightarrow \Sigma\mathbf{Sp}$.

Remark 4.6. Notice that assembly maps cannot be defined in the stable homotopy category; the category \mathbf{NSp} of naive spectra is sufficient to define and study them. Symmetric spectra have the additional structure that is needed for the input of topological Hochschild homology and its variants. The output naturally comes equipped with more structure, and \mathbf{THH} takes values in \mathcal{WT} and actually even in \mathcal{WT}^{S^1} ; see 4L and Section 6. Striving to use the same category for both input and output is irrelevant for our purposes and would lead to superfluous complications. Working in \mathcal{WT}^{S^1} enables us to consider equivariant shifts as in Definition 4.10 and 4L below, which we use in Section 7 to obtain a convenient point-set level model for the homotopy fiber of the restriction map between the fixed points of \mathbf{THH} . This is crucial for the proof of our splitting theorems; see Remark 10.2. In Section 8 we use (equivariant) orthogonal spectra for the natural model of the Adams isomorphism between homotopy orbits and fixed points developed in [95]. Finally, in Section 15 we use constructions of [30] that are performed using Γ -spaces in the simplicial setting, and so we need to compare their setup with ours.

4J. Properties of symmetric spectra and symmetric spectral categories

As indicated in 4D, we refer to $\Sigma\mathbf{Sp}$ -categories as symmetric spectral categories. Symmetric ring spectra are the monoids in $\Sigma\mathbf{Sp}$, i.e, symmetric spectral categories with only one object.

Definition 4.7. Let \mathbb{E} be a symmetric spectrum. We say that \mathbb{E} is:

- (i) *Well pointed* if for every $x \in \mathbb{N}$ the space \mathbb{E}_x is well pointed.
- (ii) *Cofibrant* if it is cofibrant in the stable model category structure on $\Sigma\mathbf{Sp}$.
- (iii) *Strictly connective* if for every $x \geq 0$ the space \mathbb{E}_x is $(x - 1)$ -connected; compare [80, Definition 17.17 on page 496].
- (iv) *Convergent* if there exists a non-decreasing function $\lambda: \mathbb{N} \rightarrow \mathbb{Z}$ such that $\lim_{x \rightarrow \infty} \lambda(x) = \infty$ and the adjoint structure map $\mathbb{E}_x \rightarrow \Omega\mathbb{E}_{x+1}$ is $(x + \lambda(x))$ -connected for every $x \geq 0$.

Definition 4.8. Let \mathbb{D} be a symmetric spectral category. We say that \mathbb{D} is:

- (i) *Objectwise well pointed* if for every $c, d \in \text{obj } \mathbb{D}$ the symmetric spectrum $\mathbb{D}(c, d)$ is well pointed. Similarly we define objectwise cofibrant, objectwise strictly connective, and objectwise convergent.
- (ii) *Very well pointed* if it is objectwise well pointed and for every $d \in \text{obj } \mathbb{D}$ the identity induces a cofibration $S^0 \rightarrow \mathbb{D}(d, d)(S^0)$ of the underlying unpointed spaces. We say

that a symmetric ring spectrum \mathbb{A} is very well pointed if it is so when considered as a symmetric spectral category with only one object.

Definition 4.9 (*Connective⁺*). We say that a symmetric ring spectrum \mathbb{A} is *connective⁺* if it is strictly connective, convergent, and very well pointed.

We emphasize that the last two conditions are very mild. The last one simply means that $S^0 \rightarrow \mathbb{A}_0$ and $\text{pt} \rightarrow \mathbb{A}_n$ are cofibrations for all $n > 0$. Any (-1) -connected Ω -spectrum is strictly connective and convergent. The sphere spectrum \mathbb{S} and suitable models for all Eilenberg–Mac Lane ring spectra $\mathbb{H}R$ are connective⁺.

4K. *Equivariant spectra*

Given a finite group G , we consider naive G -spectra, i.e., objects of the categories in (4.4) equipped with a G -action in the categorical sense, and G -equivariant maps. We write $\mathcal{W}\mathcal{T}^G$, $\text{Th}\mathcal{T}^G$, ΣSp^G for the corresponding subcategories. Notice that, as the notation suggests, $\mathcal{W}\mathcal{T}^G$ and $\text{Th}\mathcal{T}^G$ are the same as the categories of continuous functors from \mathcal{W} or Th to \mathcal{T}^G . A G -map $f: \mathbf{X} \rightarrow \mathbf{Y}$ in any of these categories is called a π_*^1 -isomorphism if it is a π_* -isomorphism of the underlying non-equivariant spectra.

When working with orthogonal spectra, though, naive is not naive, in the precise sense that the categories $\text{Th}\mathcal{T}^G$ and $\text{Th}_G\mathcal{T}_G$ are equivalent; see [53, Proposition A.19 on page 147], [79, Theorem 1.5 on pages 75–76], [95, Theorem 4.12 on pages 1507–1508]. Here \mathcal{T}_G is the category with $\text{obj } \mathcal{T}_G = \text{obj } \mathcal{T}^G$ and morphism G -spaces of not necessarily equivariant maps equipped with the conjugation G -action. This is a topological G -category, i.e., a \mathcal{T}^G -enriched category. The topological G -category Th_G is defined analogously, and we denote by $\text{Th}_G\mathcal{T}_G$ the corresponding category of continuous G -functors. This is the category of non-naive orthogonal G -spectra from [79]; see also [95, Section 4, pages 1504–1507].

Analogously, we write $\mathcal{W}_G\mathcal{T}_G$ for the category of non-naive G -equivariant \mathcal{W} -spaces, studied in detail in [15]. Here \mathcal{W}_G is the full subcategory of \mathcal{T}_G whose objects are the pointed countable G -CW-spaces, where a (countable) G -CW-space is a G -space G -homeomorphic to a G -CW-complex (with only countably many G -cells). The following definition plays an important role in our work.

Definition 4.10 (*Equivariant shift*). Given $E \in \text{obj } \mathcal{W}_G$ and $\mathbf{X} \in \text{obj } \mathcal{W}\mathcal{T}^G$, we define $\text{sh}^E\mathbf{X} \in \text{obj } \mathcal{W}\mathcal{T}^G$ as follows. Let $\iota: \mathcal{W} \rightarrow \mathcal{W}_G$ be the inclusion of the trivial G -spaces. Restriction along ι gives a functor $\iota^*: \mathcal{W}_G\mathcal{T}_G \rightarrow \mathcal{W}\mathcal{T}^G$. In the other direction, given $\mathbf{X} \in \text{obj } \mathcal{W}\mathcal{T}^G$, define its diagonal extension $\text{ext } \mathbf{X} \in \text{obj } \mathcal{W}_G\mathcal{T}_G$ by $(\text{ext } \mathbf{X})(B) = \mathbf{X}(\text{res } B)$, where $\text{res} = \text{res}_{1 \rightarrow G}$ as in 4B. If $\ell_g^B: B \rightarrow B$ denotes the action of $g \in G$ on B , then the action of g on $(\text{ext } \mathbf{X})(B)$ is defined as

$$\ell_g^{(\text{ext } \mathbf{X})(B)} = \ell_g^{\mathbf{X}(\text{res } B)} \circ \mathbf{X}(\ell_g^B) = \mathbf{X}(\ell_g^B) \circ \ell_g^{\mathbf{X}(\text{res } B)}.$$

For $\mathbf{Y} \in \text{obj } \mathcal{W}_G \mathcal{T}_G$ and $E \in \text{obj } \mathcal{W}_G$, define $\text{sh}^E \mathbf{Y} \in \text{obj } \mathcal{W}_G \mathcal{T}_G$ by $(\text{sh}^E \mathbf{Y})(B) = \mathbf{Y}(E \wedge B)$. Finally, we put

$$\text{sh}^E \mathbf{X} = \iota^* \text{sh}^E \text{ext } \mathbf{X}.$$

In words: extend diagonally to non-naive G -equivariant \mathcal{W} -spaces, shift, and then restrict back to naive G -equivariant \mathcal{W} -spaces.

4L. S^1 -equivariant spaces and spectra

The circle group S^1 acts on the geometric realization of cyclic spaces and spectra; compare 4M. For actions of the non-discrete group S^1 , we keep the notations and conventions introduced in 4B and 4K, e.g., we write \mathcal{T}^{S^1} , \mathcal{WT}^{S^1} , ΣSp^{S^1} , etc.

Now let $\mathbf{X} \in \text{obj } \mathcal{WT}^{S^1}$ and let C be a finite subgroup of S^1 . The fixed points \mathbf{X}^C are defined levelwise: $\mathbf{X}^C(A) = \mathbf{X}(A)^C$ for every $A \in \text{obj } \mathcal{W}$. Given a pointed S^1 -space E such that $\text{res}_{C \rightarrow S^1} E \in \text{obj } \mathcal{W}_C$, using Definition 4.10 we put

$$(\text{sh}^E \mathbf{X})^C = (\text{sh}^{\text{res}_{C \rightarrow S^1} E} \text{res}_{C \rightarrow S^1} \mathbf{X})^C \in \text{obj } \mathcal{WT}.$$

Notice that the assumption on $\text{res}_{C \rightarrow S^1} E$ is satisfied for any subgroup C of S^1 if $E = ES^1_+ = \mathbb{C}P^\infty_+$.

4M. Simplicial and cyclic objects

Given a category \mathcal{C} , we write $\Delta^{\text{op}} \mathcal{C}$ and $\Lambda^{\text{op}} \mathcal{C}$ for the categories of simplicial and cyclic objects in \mathcal{C} , respectively. We use the abbreviation $\mathcal{S}_* = \Delta^{\text{op}} \text{Sets}_*$ for the category of pointed simplicial sets.

Recall that the geometric realization of a cyclic space carries a natural S^1 -action, and so we get a functor $|-|: \Lambda^{\text{op}} \text{Top} \rightarrow \text{Top}^{S^1}$; e.g., see [33], [56, Section 3 on pages 411–414], [29].

The following condition, introduced by Segal [105, Definition A.4 on page 309], becomes important when realizing simplicial spaces.

Definition 4.11. We say that a simplicial unpointed space $X_\bullet \in \text{obj } \Delta^{\text{op}} \text{Top}$ is *Segal good* if for every $n \in \mathbb{N}$ and every $i = 0, \dots, n - 1$ the degeneracy map $s_i: X_{n-1} \rightarrow X_n$ is a cofibration. We call a simplicial or a cyclic pointed space Segal good if the underlying simplicial unpointed space is Segal good.

Lemma 4.12. Let $X_\bullet, Y_\bullet \in \text{obj } \Delta^{\text{op}} \text{Top}$ be simplicial spaces and let $f_\bullet: X_\bullet \rightarrow Y_\bullet$ be a simplicial map. Assume that X_\bullet and Y_\bullet are both Segal good.

- (i) There is a natural weak equivalence $\text{hocolim}_{\Delta^{\text{op}}} X_\bullet \rightarrow |X_\bullet|$.

- (ii) If f_\bullet is a weak equivalence in each simplicial degree, then $|f_\bullet|$ is a weak equivalence.
- (iii) If f_\bullet is a cofibration in each simplicial degree, then $|f_\bullet|$ is a cofibration.

Proof. (ii) follows at once from (i).

(i) and (iii) follow from more general model-theoretic statements, which we now briefly recall. A simplicial space X_\bullet is called Reedy cofibrant if $L_n X_\bullet \subseteq X_n$ is a cofibration for each n , where $L_n X_\bullet = \bigcup_{0 \leq i \leq n-1} s_i(X_{n-1})$. (Slightly stronger versions of the Reedy cofibrant and Segal good conditions are called proper and strictly proper in [81, Definition 11.2 on page 102].) A simplicial map f_\bullet is called a Reedy cofibration if $X_n \cup_{L_n X_\bullet} L_n Y_\bullet \rightarrow Y_n$ is a cofibration for each n . By [63, proof of Corollary 2.4(b) on page 153], Segal good implies Reedy cofibrant. If X_\bullet is Reedy cofibrant, then (i) is proved in [54, Theorem 19.8.7(i) on page 427]. Furthermore, if X_\bullet and Y_\bullet are Reedy cofibrant and f_\bullet is a cofibration in each simplicial degree, then f_\bullet is a Reedy cofibration. This follows from Lillig’s union theorem for cofibrations [28, Theorem 5.4.5 on page 114] and the fact that a pushout square in Top along a closed embedding is a pullback square [110, Proposition 2.35 on page 9]. Then (iii) is a special case of the fact that geometric realization sends Reedy cofibrations between Reedy cofibrant objects to cofibrations; see [45, Proposition VII.3.6 on page 374]. \square

For simplicial or cyclic objects in one of our categories of spectra we define Segal goodness levelwise. Since homotopy colimits are defined levelwise, we have the following immediate consequence of Lemma 4.12. We formulate it explicitly for \mathcal{WT} , but it holds verbatim for all other categories of spectra in (4.4).

Corollary 4.13. *If $X_\bullet \in \text{obj } \Delta^{\text{op}}(\mathcal{WT})$ is levelwise Segal good then there is a natural level equivalence in \mathcal{WT}*

$$\text{hocolim}_{\Delta^{\text{op}}} X_\bullet \longrightarrow |X_\bullet|.$$

If $X_\bullet, Y_\bullet \in \text{obj } \Delta^{\text{op}}(\mathcal{WT})$ are both levelwise Segal good and $f_\bullet: X_\bullet \rightarrow Y_\bullet$ is a π_ -isomorphism in each simplicial degree, then $|f_\bullet|$ is a π_* -isomorphism.*

Lemma 4.14. *Let*

$$X_\bullet \xrightarrow{f_\bullet} Y_\bullet \xrightarrow{g_\bullet} Z_\bullet \tag{4.15}$$

be a sequence of maps in $\Delta^{\text{op}}(\mathcal{WT})$. Assume that X_\bullet, Y_\bullet , and Z_\bullet are levelwise Segal good, and assume that (4.15) is in every simplicial degree a stable homotopy fibration sequence in \mathcal{WT} . Then

$$|X_\bullet| \xrightarrow{|f_\bullet|} |Y_\bullet| \xrightarrow{|g_\bullet|} |Z_\bullet|$$

is a stable homotopy fibration sequence in \mathcal{WT} .

Proof. This follows from [Corollary 4.13](#), using that stable homotopy fibration sequences and stable homotopy cofibration sequences are the same in \mathcal{WT} , that hoco-fib f_\bullet is levelwise Segal good if \mathbf{X}_\bullet and \mathbf{Y}_\bullet are levelwise Segal good, and that geometric realization commutes with homotopy cofibers. \square

4N. *p*-completion of spectra

For a prime p , we define the p -completion $\widehat{\mathbf{X}}_p$ of a naive spectrum \mathbf{X} as the Bousfield localization of \mathbf{X} with respect to the Moore spectrum of $\mathbb{Z}/p\mathbb{Z}$, in the sense of [\[20\]](#). This construction is functorial in the stable homotopy category; it can be made functorial in the symmetric monoidal categories of spectra above, but we do not need it for our purposes. By [\[20, Proposition 2.5 on page 262\]](#), for any $n \in \mathbb{Z}$ there is a splittable short exact sequence

$$0 \longrightarrow \text{Ext}(\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}, \pi_n(\mathbf{X})) \longrightarrow \pi_n(\widehat{\mathbf{X}}_p) \longrightarrow \text{Hom}(\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}, \pi_{n-1}(\mathbf{X})) \longrightarrow 0.$$

If the p -torsion elements in $\pi_n(\mathbf{X})$ and $\pi_{n-1}(\mathbf{X})$ have bounded order, then

$$\begin{aligned} \text{Ext}(\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}, \pi_n(\mathbf{X})) &\cong \pi_n(\widehat{\mathbf{X}}_p) = \lim_{i \in \mathbb{N}} \pi_n(\mathbf{X})/p^i, \\ \text{Hom}(\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}, \pi_{n-1}(\mathbf{X})) &\cong 0; \end{aligned}$$

see [\[21, Section IV.2.1 on pages 166–167\]](#). In particular, if the abelian groups $\pi_*(\mathbf{X})$ are finitely generated, then $\pi_*(\widehat{\mathbf{X}}_p) \cong \pi_*(\mathbf{X}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$. By [\[20, Lemma 1.10 on page 259\]](#), p -completion preserves stable homotopy fibration sequences of spectra.

5. Cyclic nerves

In this section we recall the definition of the cyclic nerve of categories, study the corresponding assembly map, and then prove our first splitting result, [Proposition 5.12](#). This proposition serves as model for all our subsequent splitting theorems.

Definition 5.1 (*Cyclic nerve*). The cyclic nerve of a small category \mathcal{C} is the cyclic set $CN_\bullet(\mathcal{C})$ with q -simplices

$$CN_q(\mathcal{C}) = \coprod_{\substack{c_0, c_1, \dots, c_q \\ \text{in obj } \mathcal{C}}} \mathcal{C}(c_0, c_q) \times \mathcal{C}(c_1, c_0) \times \dots \times \mathcal{C}(c_q, c_{q-1}).$$

The face maps are given by composition (and d_q also uses cyclic permutation), the degeneracies by insertion of identities, and the cyclic structure by cyclic permutations.

In the case of monoids, this construction appears in [\[120, \(2.3\) on page 368\]](#), where the notation N^{cy} is used.

Definition 5.1 produces a functor

$$CN_{\bullet} : \text{Cat} \longrightarrow \Lambda^{\text{op}}\text{Sets}$$

from small categories to cyclic sets. Natural isomorphisms of functors induce cyclic homotopies, and so cyclic nerves of equivalent categories are cyclically homotopy equivalent. Recall from Subsection 4M that the geometric realization of a cyclic set has a natural S^1 -action.

Example 5.2 (*Free loop space*). If M is a monoid, seen as a category with only one object, then there is an S^1 -equivariant map

$$|CN_{\bullet}(M)| \longrightarrow \mathcal{L}BM = \text{map}(S^1, BM), \tag{5.3}$$

where BM is the classifying space of M and $\mathcal{L}BM$ is the free loop space of BM . If M is group-like, i.e., if $\pi_0(M)$ is a group, then this map induces a weak equivalence between the fixed points

$$|CN_{\bullet}(M)|^C \xrightarrow{\simeq} (\mathcal{L}BM)^C$$

of any finite subgroup C of S^1 . However, the induced map on S^1 -fixed points is not an equivalence in general. For more details, see [46, proof of Lemma V.1.3 on pages 209–211], [56, Theorem 6.2 on page 420].

Given a group G , we consider the functor given by the composition

$$\text{Or}G \xrightarrow{Gf-} \text{Groupoids} \xrightarrow{CN_{\bullet}} \Lambda^{\text{op}}\text{Sets} \xrightarrow{|-|_+} \mathcal{T}^{S^1}$$

and the associated assembly map of pointed S^1 -spaces

$$EG(\mathcal{F})_+ \wedge_{\text{Or}G} |CN_{\bullet}(Gf-)|_+ \longrightarrow |CN_{\bullet}(G)|_+.$$

The key to study this assembly map is a certain retraction that we proceed to define.

Let S be a G -set, and consider its action groupoid GfS from Subsection 4C. Let $\text{conj } G$ be the set of conjugacy classes $[g]$ of elements $g \in G$. We denote by $(g_0, g_1, \dots, g_q, s)$ the element in $CN_q(GfS)$ consisting of the $(q + 1)$ -tuple of morphisms (g_0, g_1, \dots, g_q) in the summand indexed by (s_0, s_1, \dots, s_q) , where $s_q = s$ and $s_i = g_{i+1} \cdots g_q s$ for $0 \leq i < q$.

$$\begin{array}{ccccccc} s_0 & \xleftarrow{g_1} & s_1 & \xleftarrow{g_2} & \cdots & \xleftarrow{g_{q-1}} & s_{q-1} & \xleftarrow{g_q} & s_q \\ & \underbrace{\hspace{10em}}_{g_0} & & & & & & & \end{array}$$

Sending (g_0, \dots, g_q, s) to $[g_0 g_1 \cdots g_q] \in \text{conj } G$ defines a map

$$CN_{\bullet}(GfS) \longrightarrow \text{conj } G. \tag{5.4}$$

If we consider $\text{conj } G$ as a constant cyclic set, then (5.4) is a map of cyclic sets, and therefore it induces a decomposition

$$CN_{\bullet}(GfS) = \coprod_{[c] \in \text{conj } G} CN_{\bullet, [c]}(GfS), \tag{5.5}$$

where $CN_{\bullet, [c]}(GfS)$ denotes the preimage of $[c]$ under the map (5.4). Similarly, given a family \mathcal{F} of subgroups of G , we denote by $CN_{\bullet, \mathcal{F}}(GfS)$ the preimage of

$$\text{conj}_{\mathcal{F}} G = \{ [c] \mid \langle c \rangle \in \mathcal{F} \} \subseteq \text{conj } G,$$

and we call $CN_{\bullet, \mathcal{F}}(GfS)$ the \mathcal{F} -part of $CN_{\bullet}(GfS)$. Here and elsewhere $\langle c \rangle$ denotes the cyclic subgroup of G generated by c .

There are cyclic maps

$$CN_{\bullet}(GfS)_+ \begin{matrix} \xrightarrow{\text{pr}_{\mathcal{F}}} \\ \xleftarrow{\text{in}_{\mathcal{F}}} \end{matrix} CN_{\bullet, \mathcal{F}}(GfS)_+, \tag{5.6}$$

where $\text{in}_{\mathcal{F}}$ is the inclusion, and the projection $\text{pr}_{\mathcal{F}}$ sends the components with $\langle c \rangle \notin \mathcal{F}$ to the disjoint basepoint and is the identity on the components with $\langle c \rangle \in \mathcal{F}$. They clearly satisfy $\text{pr}_{\mathcal{F}} \circ \text{in}_{\mathcal{F}} = \text{id}$.

Notice that the decomposition (5.5) and the maps (5.6) are natural in S .

The following lemma gives an explicit computation of the cyclic nerve of action groupoids, and is the basis for all our splitting results for assembly maps. We need to introduce some notation. We put $E_{\bullet}G = N_{\bullet}(Gf(G/1))$, the nerve of the category with set of objects G and precisely one morphism between any two objects. The geometric realization $EG = |E_{\bullet}G|$ is a functorial model for the universal contractible free G -space. Recall that $Z_G H$ denotes the centralizer of H in G .

Lemma 5.7. *Choose a representative c in each conjugacy class $[c] \in \text{conj } G$. Then for any G -set S there is a simplicial homotopy equivalence*

$$\coprod_{[c] \in \text{conj } G} E_{\bullet}Z_G \langle c \rangle \times_{Z_G \langle c \rangle} \text{map}(G/\langle c \rangle, S)^G \longrightarrow CN_{\bullet}(GfS). \tag{5.8}$$

The map (5.8) is natural in S , and it is compatible with the decomposition (5.5) of the target.

Proof. See [72, Proposition 9.9 on page 627]. \square

Remark 5.9. Notice that the source of the map (5.8) does not admit any obvious cyclic structure.

Proposition 5.10. *For every G -space X there is a natural commutative diagram*

$$\begin{CD}
 \bigvee_{[c] \in \text{conj } G} \left(EZ_G \langle c \rangle \times_{Z_G \langle c \rangle} X^{(c)} \right)_+ @>>> X_+ \wedge_{\text{Or } G} |CN_\bullet(Gf-)|_+ \\
 @VVV @VV \text{id} \wedge \text{pr}_{\mathcal{F}} V \\
 \bigvee_{[c] \in \text{conj}_{\mathcal{F}} G} \left(EZ_G \langle c \rangle \times_{Z_G \langle c \rangle} X^{(c)} \right)_+ @>>> X_+ \wedge_{\text{Or } G} |CN_{\bullet, \mathcal{F}}(Gf-)|_+
 \end{CD} \tag{5.11}$$

where the left-hand vertical map is the projection to the wedge summands with $\langle c \rangle \in \mathcal{F}$, and both horizontal maps are homotopy equivalences.

Proof. The simplicial homotopy equivalence from Lemma 5.7 induces the horizontal homotopy equivalence in the following display.

$$\begin{aligned}
 X \times_{\text{Or } G} \left| \coprod_{[c] \in \text{conj } G} E_\bullet Z_G \langle c \rangle \times_{Z_G \langle c \rangle} \text{map}(G/\langle c \rangle, -)^G \right| &\xrightarrow{\cong} X \times_{\text{Or } G} |CN_\bullet(Gf-)| \\
 \text{\textcircled{1}} \uparrow \cong & \\
 X \times_{\text{Or } G} \coprod_{[c] \in \text{conj } G} \left(EZ_G \langle c \rangle \times_{Z_G \langle c \rangle} \text{map}(G/\langle c \rangle, -)^G \right) & \\
 \text{\textcircled{2}} \uparrow \cong & \\
 \coprod_{[c] \in \text{conj } G} EZ_G \langle c \rangle \times_{Z_G \langle c \rangle} \left(X \times_{\text{Or } G} \text{map}(G/\langle c \rangle, -)^G \right) &\stackrel{\text{\textcircled{3}}}{\cong} \coprod_{[c] \in \text{conj } G} EZ_G \langle c \rangle \times_{Z_G \langle c \rangle} X^{(c)}
 \end{aligned}$$

The isomorphism ① comes from the fact that geometric realization commutes with colimits and finite products, ② comes from associativity and distributivity properties, and ③ comes from the Yoneda lemma. Adding disjoint basepoints gives the top horizontal homotopy equivalence in diagram (5.11). Since the vertical maps in (5.11) are retractions, also the bottom horizontal map is a homotopy equivalence. \square

We are now ready to prove the following result, which is the model case of all our splitting and isomorphisms results for assembly maps. Notice that, using the equivalence (5.3), this result also gives a computation of the free loop space $\mathcal{L}BG$.

Proposition 5.12. *Let G be any group. Consider the following commutative diagram of pointed S^1 -spaces.*

$$\begin{CD}
 EG(\mathcal{F})_+ \wedge_{\text{Or } G} |CN_\bullet(Gf-)|_+ @>\text{asbl}>> |CN_\bullet(G)|_+ \\
 @V \text{id} \wedge \text{pr}_{\mathcal{F}} VV @VV \text{pr}_{\mathcal{F}} V \\
 EG(\mathcal{F})_+ \wedge_{\text{Or } G} |CN_{\bullet, \mathcal{F}}(Gf-)|_+ @>\text{asbl}>> |CN_{\bullet, \mathcal{F}}(G)|_+
 \end{CD} \tag{5.13}$$

After forgetting the S^1 -action, the left-hand map and the bottom map are homotopy equivalences. If \mathcal{F} contains all cyclic groups, then the right-hand map is an isomorphism.

Proof. Apply Proposition 5.10 to $X = EG(\mathcal{F})$. Since $EG(\mathcal{F})^{(c)} = \emptyset$ if and only if $\langle c \rangle \notin \mathcal{F}$, the vertical map on the left in diagram (5.13) corresponds under the homotopy equivalences of Proposition 5.10 to the identity.

Recall that the horizontal assembly maps are the maps induced by the projection $EG(\mathcal{F}) \rightarrow \text{pt}$ and the isomorphism $\text{pt}_+ \wedge_{\text{Or}_G} |CN_{\bullet, \mathcal{F}}(Gf-)|_+ \cong |CN_{\bullet, \mathcal{F}}(G)|_+$. If $\langle c \rangle \in \mathcal{F}$, the equivariant map $EZ_G\langle c \rangle \times EG(\mathcal{F})^{(c)} \rightarrow EZ_G\langle c \rangle \times \text{pt}$ is a non-equivariant homotopy equivalence of free $Z_G\langle c \rangle$ -CW-spaces, and hence it remains a homotopy equivalence after taking quotients of the $Z_G\langle c \rangle$ -actions. Using Proposition 5.10, this shows that the horizontal map at the bottom of diagram (5.13) is a homotopy equivalence. Finally, the last statement is clear from the definition of the projection $\text{pr}_{\mathcal{F}}$. \square

6. Topological Hochschild homology

In this section we prove the splitting theorem for assembly maps for topological Hochschild homology, Theorem 1.19(i). More precisely, we establish the following result.

Theorem 6.1. *Let G be any group and \mathbb{A} any very well pointed symmetric ring spectrum. Consider the following commutative diagram in \mathcal{WT}^{S^1} .*

$$\begin{array}{ccc}
 EG(\mathcal{F})_+ \wedge_{\text{Or}_G} \mathbf{THH}(\mathbb{A}[Gf-]) & \xrightarrow{\text{asbl}} & \mathbf{THH}(\mathbb{A}[G]) \\
 \text{id} \wedge \text{pr}_{\mathcal{F}} \downarrow & & \downarrow \text{pr}_{\mathcal{F}} \\
 EG(\mathcal{F})_+ \wedge_{\text{Or}_G} \mathbf{THH}_{\mathcal{F}}(\mathbb{A}[Gf-]) & \xrightarrow{\text{asbl}} & \mathbf{THH}_{\mathcal{F}}(\mathbb{A}[G])
 \end{array}$$

The left-hand map and the bottom map are π_*^1 -isomorphisms. If \mathcal{F} contains all cyclic groups, then the right-hand map is an isomorphism.

We begin by recalling the definition of topological Hochschild homology, following [16] and [31].

Let \mathcal{I} be the category with objects the finite sets $x = \{1, 2, \dots, x\}$ for all $x \geq 0$, and morphisms all the injective functions. Ordered concatenation $(x, y) \mapsto x \sqcup y \cong x + y$ defines a strict cocartesian monoidal structure on \mathcal{I} , symmetric up to natural isomorphism.

Let \mathbb{D} be a symmetric spectral category; compare Subsection 4J. Let $q \in \mathbb{N}$. In order to simplify the notation below, we write $\mathcal{I}^{[q]}$ for \mathcal{I}^{q+1} . Define $\text{cn}_{[q]} \mathbb{D}$ to be the functor

$$\begin{aligned}
 cn_{[q]}\mathbb{D}: \mathcal{I}^{[q]} &\longrightarrow \mathcal{T} \\
 (x_0, \dots, x_q) &\longmapsto \bigvee_{\substack{d_0, \dots, d_q \\ \text{in obj } \mathbb{D}}} \mathbb{D}(d_0, d_q)_{x_0} \wedge \mathbb{D}(d_1, d_0)_{x_1} \wedge \dots \wedge \mathbb{D}(d_q, d_{q-1})_{x_q}.
 \end{aligned}$$

In the special case when \mathbb{D} is the sphere spectrum \mathbb{S} , notice that $cn_{[q]}\mathbb{S}(x_0, \dots, x_q) = S^{x_0} \wedge \dots \wedge S^{x_q}$. Then define

$$\begin{aligned}
 M_{[q]}\mathbb{D}: \mathcal{I}^{[q]} &\longrightarrow \mathcal{WT} \\
 \vec{x} = (x_0, \dots, x_q) &\longmapsto \text{map}(cn_{[q]}\mathbb{S}(\vec{x}), - \wedge cn_{[q]}\mathbb{D}(\vec{x})) \\
 &= \text{map}(S^{x_0} \wedge \dots \wedge S^{x_q}, - \wedge cn_{[q]}\mathbb{D}(x_0, \dots, x_q)),
 \end{aligned}$$

where $-$ denotes the variable in \mathcal{W} .

The key observation for the definition of topological Hochschild homology is that the functors $M_{[q]}\mathbb{D}$ assemble to a cyclic object in the category of natural modules in \mathcal{WT} defined in Subsection 4G:

$$(\mathcal{I}^{[\bullet]}, M_{[\bullet]}\mathbb{D}): \Lambda^{\text{op}} \longrightarrow \text{Mod}_{\mathcal{WT}}, \quad [q] \longmapsto (\mathcal{I}^{[q]}, M_{[q]}\mathbb{D}). \tag{6.2}$$

This is explained in detail in [30, Sections 4.2.2.3–4 on pages 151–153].

Definition 6.3 (*Topological Hochschild homology*). Let \mathbb{D} be a symmetric spectral category. Then $THH_{\bullet}(\mathbb{D}) \in \text{obj } \Lambda^{\text{op}}(\mathcal{WT})$ is defined as the composition of (6.2) and the functor $\text{hocolim}: \text{Mod}_{\mathcal{WT}} \longrightarrow \mathcal{WT}$ in (4.1):

$$THH_q(\mathbb{D}) = \text{hocolim}_{\mathcal{I}^{[q]}} M_{[q]}\mathbb{D}.$$

This defines a functor

$$THH_{\bullet}(-): \Sigma\text{Sp-Cat} \longrightarrow \Lambda^{\text{op}}(\mathcal{WT}),$$

and its composition with the geometric realization functor $|-|: \Lambda^{\text{op}}\mathcal{T} \longrightarrow \mathcal{T}^{S^1}$ gives

$$\mathbf{THH}(-) = |THH_{\bullet}(-)|: \Sigma\text{Sp-Cat} \longrightarrow \mathcal{WT}^{S^1}.$$

Now let \mathbb{A} be a symmetric ring spectrum. Any ordinary category \mathcal{C} defines a symmetric spectral category $\mathbb{A}[\mathcal{C}]$ by base change along the lax symmetric monoidal functor

$$\mathbb{A} \wedge -_+: \text{Sets} \longrightarrow \Sigma\text{Sp}, \quad X \longmapsto \mathbb{A} \wedge X_+;$$

compare Subsection 4D.

Lemma 6.4. *Let \mathbb{A} be a symmetric ring spectrum, and let \mathcal{C} be a small category. Then there is a natural isomorphism of functors $\mathcal{I}^{[q]} \rightarrow \mathcal{T}$*

$$cn_{[q]}(\mathbb{A}[\mathcal{C}])(?) \cong cn_{[q]}\mathbb{A}(?) \wedge CN_q(\mathcal{C})_+ \tag{6.5}$$

and a natural isomorphism of functors $\Lambda^{\text{op}} \rightarrow \text{Mod}_{\mathcal{WT}}$

$$(\mathcal{I}^{[\bullet]}, M_{[\bullet]}(\mathbb{A}[\mathcal{C}])) \cong (\mathcal{I}^{[\bullet]}, \text{map}(cn_{[\bullet]}\mathbb{S}(?), - \wedge cn_{[\bullet]}\mathbb{A}(?) \wedge CN_{\bullet}(\mathcal{C})_+)), \tag{6.6}$$

where $?$ denotes the variable in $\mathcal{I}^{[\bullet]}$ and $-$ the variable in \mathcal{W} .

Proof. Evaluated at $? = (x_0, \dots, x_q) \in \text{obj } \mathcal{I}^{[q]}$, the isomorphism (6.5) is given by

$$\begin{aligned} & \bigvee_{c_0, \dots, c_q} \mathbb{A}_{x_0} \wedge \mathcal{C}(c_0, c_q)_+ \wedge \mathbb{A}_{x_1} \wedge \mathcal{C}(c_1, c_0)_+ \wedge \dots \wedge \mathbb{A}_{x_q} \wedge \mathcal{C}(c_q, c_{q-1})_+ \cong \\ & \cong (\mathbb{A}_{x_0} \wedge \mathbb{A}_{x_1} \wedge \dots \wedge \mathbb{A}_{x_q}) \wedge \left(\prod_{c_0, \dots, c_q} \mathcal{C}(c_0, c_q) \times \mathcal{C}(c_1, c_0) \times \dots \times \mathcal{C}(c_q, c_{q-1}) \right)_+ . \end{aligned}$$

The rest is then clear. \square

Now we specialize to $\mathcal{C} = GfS$ for a group G and a G -set S . Using the decomposition (5.5) of the cyclic nerve of action groupoids, from (6.5) we obtain a decomposition

$$cn_{[\bullet]}(\mathbb{A}[GfS]) \cong \bigvee_{[c] \in \text{conj } G} cn_{[\bullet], [c]}(\mathbb{A}[GfS]),$$

where we define

$$cn_{[\bullet], [c]}(\mathbb{A}[GfS]) = (cn_{[\bullet]}\mathbb{A}) \wedge CN_{\bullet, [c]}(GfS)_+ .$$

Similarly, given a family \mathcal{F} of subgroups of G , we define $cn_{[\bullet], \mathcal{F}}(\mathbb{A}[GfS])$. There are maps

$$cn_{[\bullet]}(\mathbb{A}[GfS]) \begin{matrix} \xrightarrow{\text{pr}_{\mathcal{F}}} \\ \xleftarrow{\text{in}_{\mathcal{F}}} \end{matrix} cn_{[\bullet], \mathcal{F}}(\mathbb{A}[GfS]) \tag{6.7}$$

satisfying $\text{pr}_{\mathcal{F}} \circ \text{in}_{\mathcal{F}} = \text{id}$. Then we define

$$\begin{aligned} M_{[q], \mathcal{F}}(\mathbb{A}[GfS]): \mathcal{I}^{[q]} & \longrightarrow \mathcal{WT} \\ \vec{x} & \longmapsto \text{map}(cn_{[q]}\mathbb{S}(\vec{x}), - \wedge cn_{[q]}\mathbb{A}(\vec{x}) \wedge CN_{q, \mathcal{F}}(GfS)_+) . \end{aligned}$$

As $[q]$ varies, these functors assemble to a cyclic object

$$(\mathcal{I}^{[\bullet]}, M_{[\bullet], \mathcal{F}}(\mathbb{A}[GfS])): \Lambda^{\text{op}} \longrightarrow \text{Mod}_{\mathcal{WT}} .$$

Definition 6.8. The \mathcal{F} -parts of topological Hochschild homology are defined by

$$THH_{q,\mathcal{F}}(\mathbb{A}[GfS]) = \operatorname{hocolim}_{\mathcal{I}^{[q]}} M_{[q],\mathcal{F}}(\mathbb{A}[GfS])$$

and

$$\mathbf{THH}_{\mathcal{F}}(\mathbb{A}[GfS]) = |THH_{\bullet,\mathcal{F}}(\mathbb{A}[GfS])|.$$

The maps (6.7) induce maps

$$\mathbf{THH}(\mathbb{A}[GfS]) \begin{matrix} \xrightarrow{\operatorname{pr}_{\mathcal{F}}} \\ \xleftarrow{\operatorname{in}_{\mathcal{F}}} \end{matrix} \mathbf{THH}_{\mathcal{F}}(\mathbb{A}[GfS]) \tag{6.9}$$

satisfying $\operatorname{pr}_{\mathcal{F}} \circ \operatorname{in}_{\mathcal{F}} = \operatorname{id}$.

Theorem 6.1 follows from Proposition 5.12 using the following result, which in the case of discrete rings is due to Hesselholt and Madsen [51, Theorem 7.1 on page 81]; see also [50, Proposition 3 on page 81]. In fact, they additionally prove that the map φ in (6.11) is a π_*^C -isomorphism for every finite cyclic subgroup C of S^1 . Even though this stronger equivariant statement can also be generalized here, we restrict to the case $C = 1$, since this is all we need.

Theorem 6.10. *Let \mathbb{A} be a very well pointed symmetric ring spectrum, and let \mathcal{C} be a small category. Then there is a natural π_*^1 -isomorphism in \mathcal{WT}^{S^1}*

$$\varphi: \mathbf{THH}(\mathbb{A}) \wedge |CN_{\bullet}(\mathcal{C})|_+ \longrightarrow \mathbf{THH}(\mathbb{A}[\mathcal{C}]). \tag{6.11}$$

If $\mathcal{C} = GfS$, then φ restricts to a π_*^1 -isomorphism

$$\varphi_{\mathcal{F}}: \mathbf{THH}(\mathbb{A}) \wedge |CN_{\bullet,\mathcal{F}}(GfS)|_+ \longrightarrow \mathbf{THH}_{\mathcal{F}}(\mathbb{A}[GfS]) \tag{6.12}$$

which commutes with the projections and inclusions in (5.6) and (6.9).

Proof. First, we define the map φ in (6.11). Given pointed spaces S , A , and C , consider the natural map

$$\operatorname{map}(S, A) \wedge C \longrightarrow \operatorname{map}(S, A \wedge C) \tag{6.13}$$

that is adjoint to $f \mapsto f \wedge \operatorname{id}_C$. Since we need it later, we observe that if $S \cong S^x$, A is a -connected, and C is discrete, then the map (6.13) is $(2(a - x) + 1)$ -connected. The map (6.13) induces a natural transformation of functors $\Lambda^{\operatorname{op}} \longrightarrow \operatorname{Mod}_{\mathcal{WT}}$

$$\begin{aligned} \left(\mathcal{I}^{[\bullet]}, M_{[\bullet]}(\mathbb{A}) \wedge CN_{\bullet}(\mathcal{C})_+ \right) &= \left(\mathcal{I}^{[\bullet]}, \operatorname{map}(cn_{[\bullet]} \mathbb{S}(\?), - \wedge cn_{[\bullet]} \mathbb{A}(\?) \wedge CN_{\bullet}(\mathcal{C})_+) \right) \\ &\quad \downarrow \\ &= \left(\mathcal{I}^{[\bullet]}, \operatorname{map}(cn_{[\bullet]} \mathbb{S}(\?), - \wedge cn_{[\bullet]} \mathbb{A}(\?) \wedge CN_{\bullet}(\mathcal{C})_+) \right), \end{aligned}$$

where \bullet denotes the variable in $\mathcal{I}^{[\bullet]}$ and $-$ the variable in \mathcal{W} . Using the isomorphism (6.6) from Lemma 6.4 and applying the functor $\text{hocolim}: \text{Mod}_{\mathcal{WT}} \rightarrow \mathcal{WT}$ from (4.1), we obtain a map in $\Lambda^{\text{op}}(\mathcal{WT})$

$$\varphi_{\bullet}: THH_{\bullet}(\mathbb{A}) \wedge CN_{\bullet}(\mathcal{C})_+ \cong \text{hocolim}_{\mathcal{I}^{[\bullet]}}(M_{[\bullet]}(\mathbb{A}) \wedge CN_{\bullet}(\mathcal{C})_+) \longrightarrow THH_{\bullet}(\mathbb{A}[\mathcal{C}]), \quad (6.14)$$

where the first isomorphism comes from the fact that $CN_{\bullet}(\mathcal{C})$ is constant on $\mathcal{I}^{[\bullet]}$. Taking geometric realizations and using the fact that geometric realization and smash products commute, we obtain the map φ in (6.11).

If $\mathcal{C} = GfS$, it is clear from the definitions that φ commutes with the projections and inclusions in (5.6) and (6.9).

Next, we show that φ is a π_*^1 -isomorphism. Since retracts of π_*^1 -isomorphisms are π_*^1 -isomorphisms, this then implies that also $\varphi_{\mathcal{F}}$ is a π_*^1 -isomorphism in the case when $\mathcal{C} = GfS$.

The assumption that \mathbb{A} is very well pointed implies that the same is true for $\mathbb{A}[\mathcal{C}]$. Then, by Lemma 6.17(i) below, both the source and the target of φ_{\bullet} in (6.14) are levelwise Segal good. For the source, we also use Fact 6.18(i). Therefore, by Corollary 4.13, it is enough to show that φ_{\bullet} is a π_* -isomorphism in each simplicial degree. For a fixed $q \geq 0$, both the source and the target of φ_q are defined for arbitrary symmetric spectra, not just for symmetric ring spectra, and we proceed to prove that φ_q is a π_* -isomorphism in \mathcal{WT} for every symmetric spectrum \mathbb{A} .

We claim that it is enough to show that φ_q is a π_* -isomorphism for cofibrant symmetric spectra. To prove this claim, take a cofibrant replacement $\gamma: \mathbb{A}_c \rightarrow \mathbb{A}$ in the stable model category of symmetric spectra, with γ a stable fibration and a π_* -isomorphism. From [80, Proposition 9.9(iii) on page 472] it follows that γ is a levelwise weak equivalence. Inspecting the definitions, and using the fact that homotopy colimits preserve weak equivalences, we conclude that γ induces levelwise weak equivalences $THH_q(\mathbb{A}_c) \rightarrow THH_q(\mathbb{A})$ and $THH_q(\mathbb{A}_c[\mathcal{C}]) \rightarrow THH_q(\mathbb{A}[\mathcal{C}])$. Therefore, if φ_q is a π_* -isomorphism for \mathbb{A}_c , then the same is true for \mathbb{A} , establishing the claim.

So it remains to show that φ_q is a π_* -isomorphism for cofibrant symmetric spectra. By [80, Theorem 6.5(iii) on page 461], cofibrant symmetric spectra are retracts of FI -cell complexes in the sense of [80, Definition 5.4 on page 457]. Therefore it is enough to show that φ_q is a π_* -isomorphism for FI -cell complexes. We argue by cellular induction. The base case is treated below. The induction steps are then carried out like in [106, Lemma 4.3.2 and proof of Proposition 4.2.3 on pages 179–180], thus completing the proof.

We show that φ_q is a π_* -isomorphism for a free symmetric spectrum $\mathbb{A} = F_m Z$, where $Z = S_+^{d-1}$ or D_+^d and $m, d \geq 0$. By definition

$$(F_m Z)_x = \begin{cases} \Sigma_{x+} \wedge_{\Sigma_{x-m}} S^{x-m} \wedge Z & \text{if } x \geq m, \\ \text{pt} & \text{if } 0 \leq x < m, \end{cases}$$

and therefore $(F_m Z)_x$ is at least $(x - m - 1)$ -connected.

Recall from (6.14) that $\varphi_q(B)$ is defined by taking $\text{hocolim}_{\mathcal{I}^{[q]}}$ of the natural map

$$\begin{aligned} &\text{map}(cn_{[q]}\mathbb{S}(\vec{x}), B \wedge cn_{[q]}F_m Z(\vec{x})) \wedge CN_q(\mathcal{C})_+ \\ &\quad \downarrow \\ &\text{map}(cn_{[q]}\mathbb{S}(\vec{x}), B \wedge cn_{[q]}F_m Z(\vec{x}) \wedge CN_q(\mathcal{C})_+) \end{aligned} \tag{6.15}$$

as in (6.13), where $\vec{x} = (x_0, \dots, x_q) \in \text{obj } \mathcal{I}^{[q]}$ and $B \in \text{obj } \mathcal{W}$. Let $x = x_0 + \dots + x_q$. Then, evaluated at $B = S^n$, the map (6.15) is at least $(2n - 2(q + 1)m - 1)$ -connected, as observed after (6.13). Since homotopy colimits preserve connectivity of maps (e.g., see [30, Lemma A.7.3.1 on page 373]), the same is true for $\varphi_q(S^n)$, and this then implies that φ_q is a π_* -isomorphism. Here we use the fact that a map $\psi: \mathbf{X} \rightarrow \mathbf{Y}$ in \mathcal{WT} is a π_* -isomorphism provided that there exists a non-decreasing function $\lambda: \mathbb{N} \rightarrow \mathbb{Z}$ such that $\lim_{n \rightarrow \infty} \lambda(n) = \infty$ and $\psi(S^n)$ is $(n + \lambda(n))$ -connected for each $n \geq 0$.

Now for the base case of the induction one needs to allow different free spectra in the different smash factors. More precisely, one replaces $cn_{[q]}F_m Z(\vec{x})$ by $F_{m_0}Z_0(x_0) \wedge F_{m_1}Z_1(x_1) \wedge \dots \wedge F_{m_q}Z_q(x_q)$ and $(q + 1)m$ by $m_0 + m_1 + \dots + m_q$ throughout. The argument above then works unchanged. \square

Example 6.16. Combining Theorem 6.10 with Lemma 5.7, we obtain an isomorphism in the non-equivariant stable homotopy category

$$\mathbf{THH}_{\mathcal{F}}(\mathbb{A}[G]) \simeq \bigvee_{[c] \in \text{conj}_{\mathcal{F}} G} \mathbf{THH}(\mathbb{A}) \wedge BZ_G\langle c \rangle_+.$$

In particular, if G is finite, then $\pi_n(\mathbf{THH}(\mathbb{S}[G])) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for all $n > 0$, since $\mathbf{THH}(\mathbb{S}) \simeq \mathbb{S}$.

In the proof above, as well as in the proof of Theorem 7.2, we use the following lemma. Some versions of it are certainly known to the experts. In particular, part (i) is stated and used in [51, proof of Proposition 2.4 on page 40]. For the sake of completeness, we provide here some details. The edgewise subdivision construction sd_c used in part (ii) is recalled before Lemma 7.6.

Lemma 6.17. *Assume that the symmetric spectral category \mathbb{D} is very well pointed. Then:*

- (i) *The cyclic \mathcal{W} -space $\mathbf{THH}_{\bullet}(\mathbb{D})$ is levelwise Segal good.*
- (ii) *For any finite cyclic subgroup C of S^1 and any $E \in \text{obj } \mathcal{W}^C$, the simplicial \mathcal{W} -space $(\text{sh}^E \text{sd}_c \mathbf{THH}_{\bullet}(\mathbb{D}))^C$ is levelwise Segal good.*

If \mathbb{A} is a very well pointed symmetric ring spectrum and $\mathbb{D} = \mathbb{A}[GfS]$, then the statements in (i) and (ii) also hold for the \mathcal{F} -parts.

Proof. (i) For every $A \in \text{obj } \mathcal{W}$, $i \in \{0, \dots, q\}$, and $\vec{x} \in \text{obj } \mathcal{I}^{[q]}$ the map

$$s_i: A \wedge \text{cn}_{[q]} \mathbb{D}(\vec{x}) \longrightarrow A \wedge \text{cn}_{[q+1]} \mathbb{D}(s_i(\vec{x}))$$

is a cofibration. This uses the assumption that \mathbb{D} is very well pointed, that the object-repeating map $s_i: \text{obj } \mathbb{D}^{[q]} \longrightarrow \text{obj } \mathbb{D}^{[q+1]}$ is injective, and **Facts 6.18(i), (ii)** and **(v)** below. **Fact 6.18(iii)** implies that also the map $s_i: M_{[q]} \mathbb{D}(\vec{x})(A) \longrightarrow M_{[q+1]} \mathbb{D}(s_i(\vec{x}))(A)$ is a cofibration. Since $s_i: \mathcal{I}^{[q]} \longrightarrow \mathcal{I}^{[q+1]}$ only inserts a copy of $0 = \emptyset$, the assumptions of **Fact 6.18(vii)** are verified and we conclude that

$$s_i: \text{hocolim}_{\mathcal{I}^{[q]}} M_{[q]} \mathbb{D}(-)(A) \longrightarrow \text{hocolim}_{\mathcal{I}^{[q+1]}} M_{[q+1]} \mathbb{D}(s_i(-))(A)$$

is a cofibration.

(ii) Observe that

$$\text{cn}_{c[q]} \mathbb{D}(c\vec{x}) = \bigvee_{\substack{(d^1, d^2, \dots, d^c) \\ \text{in } \text{obj } \mathbb{D}^{c[q]}}} \bigwedge_{g \in C} \mathbb{B}_{g(d^1, d^2, \dots, d^c)}(\vec{x}),$$

where the cyclic group C operates by cyclic block permutations on

$$(d^1, d^2, \dots, d^c) = ((d_0^1, d_1^1, \dots, d_q^1), (d_0^2, d_1^2, \dots, d_q^2), \dots, (d_0^c, d_1^c, \dots, d_q^c)) \in \text{obj } \mathbb{D}^{c[q]}$$

and

$$\mathbb{B}_{(d^1, d^2, \dots, d^c)}(\vec{x}) = \mathbb{D}(d_0^1, d_q^1)_{x_0} \wedge \mathbb{D}(d_1^1, d_0^1)_{x_1} \wedge \dots \wedge \mathbb{D}(d_q^1, d_{q-1}^1)_{x_q}.$$

The assumption together with **Fact 6.18(i), (ii), (vi),** and **(iv)** yield that

$$s_i: (\text{sh}^E M_{c[q]} \mathbb{D}(\vec{x})(A))^C \longrightarrow (\text{sh}^E M_{c[q+1]} \mathbb{D}(s_i(\vec{x}))(A))^C$$

is a cofibration. The claim now follows from **(7.8)** and again **Fact 6.18(vii)**. \square

The proof of **Lemma 6.17** used the following basic facts. We emphasize that, even though these facts are concerned with constructions in pointed spaces, the term cofibration always refers to the unpointed version.

Facts 6.18. Suppose $A \longrightarrow X$ is a pointed map in \mathcal{T} that is a cofibration.

- (i) For every $Z \in \text{obj } \mathcal{T}$ the induced map $A \wedge Z \longrightarrow X \wedge Z$ is a cofibration.
- (ii) If X and Z are well pointed, then so is $X \wedge Z$.
- (iii) If $Z \in \text{obj } \mathcal{T}$ is compact, then the induced map $\text{map}(Z, A) \longrightarrow \text{map}(Z, X)$ is a cofibration.

- (iv) If in (i) or (iii) we additionally assume that $Z \in \text{obj } \mathcal{T}^G$ is a G -space and $A \rightarrow X$ is a G -equivariant cofibration, then the induced map is a G -equivariant cofibration.

Suppose $A_s \rightarrow X_s$, $s \in S$ is a family of pointed maps in \mathcal{T} that are cofibrations.

- (v) If all spaces A_s and X_s are well pointed, then the induced map

$$\bigvee_{s \in S} A_s \rightarrow \bigvee_{s \in S} X_s$$

is a cofibration.

- (vi) If S is a G -set and all spaces A_s and X_s are well pointed, then the induced map

$$\bigvee_{s \in S} \bigwedge_{g \in G} A_{gs} \rightarrow \bigvee_{s \in S} \bigwedge_{g \in G} X_{gs}$$

is a G -equivariant cofibration.

Let $(f, \nu): (\mathcal{C}, A) \rightarrow (\mathcal{D}, X)$ be a morphism in $\text{Mod}_{\mathcal{T}}$; see Subsection 4G.

- (vii) Suppose the functor $f: \mathcal{C} \rightarrow \mathcal{D}$ induces an injective function on objects and for all $c, c' \in \text{obj } \mathcal{C}$ the function $\mathcal{C}(c, c') \rightarrow \mathcal{D}(f(c), f(c'))$ is injective. If for all $c \in \text{obj } \mathcal{C}$ the spaces $A(c)$ and $X(c)$ are well-pointed and the natural transformation $\nu_c: A(c) \rightarrow X(f(c))$ is a cofibration, then the induced map

$$\text{hocolim}_{\mathcal{C}} A \rightarrow \text{hocolim}_{\mathcal{D}} X$$

is a cofibration.

We can now prove the main result of this section.

Proof of Theorem 6.1. Apply the functor $\mathbf{THH}(\mathbb{A}) \wedge -$ to the diagram in Proposition 5.12 to obtain the following diagram.

$$\begin{array}{ccc} \mathbf{THH}(\mathbb{A}) \wedge \left(EG(\mathcal{F})_+ \wedge_{\text{Or}G} |CN_{\bullet}(Gf-)|_+ \right) & \xrightarrow{\text{id} \wedge \text{asbl}} & \mathbf{THH}(\mathbb{A}) \wedge |CN_{\bullet}(G)|_+ \\ \text{id} \wedge \text{id} \wedge \text{pr}_{\mathcal{F}} \downarrow & & \downarrow \text{id} \wedge \text{pr}_{\mathcal{F}} \\ \mathbf{THH}(\mathbb{A}) \wedge \left(EG(\mathcal{F})_+ \wedge_{\text{Or}G} |CN_{\bullet, \mathcal{F}}(Gf-)|_+ \right) & \xrightarrow{\text{id} \wedge \text{asbl}} & \mathbf{THH}(\mathbb{A}) \wedge |CN_{\bullet, \mathcal{F}}(G)|_+ \end{array}$$

Then consider the maps

$$\begin{aligned} & \mathbf{THH}(\mathbb{A}) \wedge \left(EG(\mathcal{F})_+ \wedge_{\text{Or}G} |CN_{\bullet, \mathcal{F}}(Gf-)|_+ \right) \\ & \cong \downarrow \\ & EG(\mathcal{F})_+ \wedge_{\text{Or}G} \left(\mathbf{THH}(\mathbb{A}) \wedge |CN_{\bullet, \mathcal{F}}(Gf-)|_+ \right) \xrightarrow{\text{id} \wedge \varphi_{\mathcal{F}}} EG(\mathcal{F})_+ \wedge_{\text{Or}G} \mathbf{THH}_{\mathcal{F}}(\mathbb{A}[Gf-]), \end{aligned}$$

where the first isomorphism uses the associativity and commutativity of smash products, and the second map is induced by (6.12), and similarly at the other corners of diagram (6.19). By Theorem 6.10 these maps induce an objectwise π_*^1 -isomorphism between the commutative square (6.19) and the one in Theorem 6.1. The result then follows from Proposition 5.12. \square

Corollary 6.20. *Let C be any finite cyclic subgroup of S^1 . Consider the following commutative diagram in \mathcal{WT} .*

$$\begin{array}{ccc} EG(\mathcal{F})_+ \wedge_{\text{Or}G} \mathbf{THH}(\mathbb{A}[Gf-])_{hC} & \xrightarrow{\text{asbl}} & \mathbf{THH}(\mathbb{A}[G])_{hC} \\ \text{id} \wedge \text{pr}_{\mathcal{F}} \downarrow & & \downarrow \text{pr}_{\mathcal{F}} \\ EG(\mathcal{F})_+ \wedge_{\text{Or}G} \mathbf{THH}_{\mathcal{F}}(\mathbb{A}[Gf-])_{hC} & \xrightarrow{\text{asbl}} & \mathbf{THH}_{\mathcal{F}}(\mathbb{A}[G])_{hC} \end{array}$$

The left-hand map and the bottom map are π_* -isomorphisms. If \mathcal{F} contains all cyclic groups, then the right-hand map is an isomorphism.

Proof. Apply C -homotopy orbits to the diagram in Theorem 6.1. Since C -homotopy orbits send π_*^1 -isomorphisms to π_* -isomorphisms, and C -homotopy orbits commute up to isomorphisms with smash products over the orbit category, the result follows. \square

7. Homotopy fiber of the restriction map

Fix a prime p . As $n \geq 1$ varies, the C_{p^n} -fixed points of $\mathbf{THH}(\mathbb{D})$ are related by maps

$$R, F: \mathbf{THH}(\mathbb{D})^{C_{p^n}} \longrightarrow \mathbf{THH}(\mathbb{D})^{C_{p^{n-1}}}, \tag{7.1}$$

called restriction and Frobenius. The Frobenius map F is the inclusion of fixed points. The definition of the restriction map R is more involved, and is reviewed in detail below.

The goal of this section is to establish the following result, which gives a point-set level description of the homotopy fiber of the restriction map. This plays a key role in the proof of our splitting theorems; compare Remark 10.2.

Recall that, given $\mathbf{X} \in \text{obj } \mathcal{WT}^{S^1}$ and a finite subgroup C of S^1 , the fixed points $\mathbf{X}^C \in \text{obj } \mathcal{WT}$ are defined levelwise: $\mathbf{X}^C(A) = \mathbf{X}(A)^C$ for every $A \in \text{obj } \mathcal{W}$. The \mathcal{W} -space

$(\text{sh}^{ES^1_+ \mathbf{X}})^C$ is defined in Subsection 4L. The projection $\text{pr}: ES^1 \rightarrow \text{pt}$ induces a map $\text{pr}_*: (\text{sh}^{ES^1_+ \mathbf{X}})^C \rightarrow (\text{sh}^{S^0 \mathbf{X}})^C \cong \mathbf{X}^C$.

Theorem 7.2. *If \mathbb{D} is very well pointed, then for each $n \geq 1$*

$$(\text{sh}^{ES^1_+ \mathbf{T HH}(\mathbb{D})})^{C_{p^n}} \xrightarrow{\text{pr}_*} \mathbf{T HH}(\mathbb{D})^{C_{p^n}} \xrightarrow{R} \mathbf{T HH}(\mathbb{D})^{C_{p^{n-1}}}$$

is a stable homotopy fibration sequence in \mathcal{WT} .

We proceed to explain the definition of R ; compare [17, page 495], [31, Definition 1.5.2 on page 255]. It begins with the following observation, whose proof is straightforward. Recall the definition of natural modules in \mathcal{WT} from Subsection 4G and the functor $\text{hocolim}: \text{Mod}_{\mathcal{WT}} \rightarrow \mathcal{WT}$ in (4.1).

Lemma 7.3. *Let G be a discrete group acting on a natural module $(\mathcal{C}, \mathbb{X}) \in \text{obj Mod}_{\mathcal{WT}}$. Then there is a natural isomorphism in \mathcal{WT}*

$$(\text{hocolim}(\mathcal{C}, \mathbb{X}))^G \cong \text{hocolim}((\mathcal{C}, \mathbb{X})^G). \tag{7.4}$$

where

$$(\mathcal{C}, \mathbb{X})^G = (\mathcal{C}^G, (-)^G \circ \mathbb{X}|_{\mathcal{C}^G}). \tag{7.5}$$

Now let C be a finite cyclic subgroup of S^1 and let c be its order. We denote by $c[\bullet] = [\bullet] \sqcup \dots \sqcup [\bullet]$ the concatenation of $[\bullet]$ with itself c times, and for every $\vec{x} \in \text{obj } \mathcal{I}^{c[\bullet]}$ we let $c\vec{x} = (\vec{x}, \vec{x}, \dots, \vec{x}) \in \text{obj } \mathcal{I}^{c[\bullet]}$. Recall the edgewise subdivision construction sd_c from [17, Section 1, pages 467–471].

Lemma 7.6. *For any symmetric spectral category \mathbb{D} , there is a natural isomorphism in \mathcal{WT}*

$$\mathbf{T HH}(\mathbb{D})^C \cong \left| (\text{sd}_c \mathbf{T HH}_\bullet(\mathbb{D}))^C \right| \tag{7.7}$$

and a natural isomorphism in $\Delta^{\text{op}}(\mathcal{WT})$

$$(\text{sd}_c \mathbf{T HH}_\bullet(\mathbb{D}))^C \cong \text{hocolim}(\mathcal{I}^{c[\bullet]}, (M_{c[\bullet]} \mathbb{D}(c-))^C). \tag{7.8}$$

Proof. The first isomorphism is true for any cyclic space; see [17, Section 1, pages 467–471]. For the second, we have

$$\begin{aligned} (\text{sd}_c \mathbf{T HH}_\bullet(\mathbb{D}))^C &= \left(\text{hocolim}(\mathcal{I}^{c[\bullet]}, M_{c[\bullet]} \mathbb{D}) \right)^C && \text{by definition} \\ &\cong \text{hocolim}(\mathcal{I}^{c[\bullet]}, M_{c[\bullet]} \mathbb{D})^C && \text{by (7.4)} \end{aligned}$$

$$\begin{aligned}
 &= \operatorname{hocolim} \left((\mathcal{I}^{c[\bullet]})^C, (M_{c[\bullet]} \mathbb{D}_{(\mathcal{I}^{c[\bullet]})^C})^C \right) && \text{by (7.5)} \\
 &\cong \operatorname{hocolim} \left(\mathcal{I}^{[\bullet]}, (M_{c[\bullet]} \mathbb{D}(c-)) \right)^C,
 \end{aligned}$$

where in the last isomorphism we use the identification

$$c- : \mathcal{I}^{[\bullet]} \xrightarrow{\cong} (\mathcal{I}^{c[\bullet]})^C, \quad \vec{x} \mapsto c\vec{x}. \quad \square$$

Now we apply Lemma 7.6 for $C = C_{p^n}$ and $C = C_{p^{n-1}}$. In order to define the map R in (7.1) the key ingredient is then a map in \mathcal{WT}

$$\begin{array}{ccc}
 M_{p^n[q]} \mathbb{D}(p^n \vec{x})^{C_{p^n}} & = & \operatorname{map} \left(cn_{p^n[q]} \mathbb{S}(p^n \vec{x}), - \wedge cn_{p^n[q]} \mathbb{D}(p^n \vec{x}) \right)^{C_{p^n}} \\
 \downarrow r & & \downarrow r \\
 M_{p^{n-1}[q]} \mathbb{D}(p^{n-1} \vec{x})^{C_{p^{n-1}}} & = & \operatorname{map} \left(cn_{p^{n-1}[q]} \mathbb{S}(p^{n-1} \vec{x}), - \wedge cn_{p^{n-1}[q]} \mathbb{D}(p^{n-1} \vec{x}) \right)^{C_{p^{n-1}}}
 \end{array} \tag{7.9}$$

for any $\vec{x} \in \operatorname{obj} \mathcal{I}^{[q]}$. For any $A \in \operatorname{obj} \mathcal{W}$ the map r in (7.9) is defined by considering the following maps.

$$\begin{array}{ccc}
 \operatorname{map} \left(cn_{p^n[q]} \mathbb{S}(p^n \vec{x}), A \wedge cn_{p^n[q]} \mathbb{D}(p^n \vec{x}) \right)^{C_{p^n}} & & \\
 \downarrow f \mapsto f^{C_p} & & \\
 \operatorname{map} \left((cn_{p^n[q]} \mathbb{S}(p^n \vec{x}))^{C_p}, A \wedge (cn_{p^n[q]} \mathbb{D}(p^n \vec{x}))^{C_p} \right)^{C_{p^n}/C_p} & & (7.10) \\
 \cong \uparrow \Delta & & \\
 \operatorname{map} \left(cn_{p^{n-1}[q]} \mathbb{S}(p^{n-1} \vec{x}), A \wedge cn_{p^{n-1}[q]} \mathbb{D}(p^{n-1} \vec{x}) \right)^{C_{p^{n-1}}} & &
 \end{array}$$

The first map is given by restricting to the C_p -fixed points, using the fact that the action on A is trivial. For the second map, notice that there is a C_{p^n}/C_p -equivariant homeomorphism

$$\Delta_{\mathbb{D}} : cn_{p^{n-1}[q]} \mathbb{D}(p^{n-1} \vec{x}) \xrightarrow{\cong} (cn_{p^n[q]} \mathbb{D}(p^n \vec{x}))^{C_p}. \tag{7.11}$$

For $\mathbb{D} = \mathbb{S}$, this specializes to the C_{p^n}/C_p -equivariant homeomorphism

$$\Delta_{\mathbb{S}} : (S^{x_0} \wedge \dots \wedge S^{x_q})^{\wedge p^{n-1}} \xrightarrow{\cong} ((S^{x_0} \wedge \dots \wedge S^{x_q})^{\wedge p^n})^{C_p}.$$

The homeomorphisms $\Delta_{\mathbb{S}}$ and $\Delta_{\mathbb{D}}$, together with the identification $C_{p^n}/C_p \cong C_{p^{n-1}}$ that takes every element of order p^n on the circle to its p -th power, induce the homeomorphism Δ in (7.10). Finally, the key map r in (7.9) is defined as the composition of the top map with the inverse of the bottom map in (7.10).

It is not hard to verify that, as $[q]$ varies, the maps r assemble to a cyclic map

$$\left(\mathcal{I}^{[\bullet]}, (M_{p^n[\bullet]}\mathbb{D}(p^n -))^{C_{p^n}}\right) \longrightarrow \left(\mathcal{I}^{[\bullet]}, (M_{p^{n-1}[\bullet]}\mathbb{D}(p^{n-1} -))^{C_{p^{n-1}}}\right)$$

in $\Lambda^{\text{op}}\text{Mod}_{\mathcal{WT}}$. Using (7.8) and (7.7), we thus get the desired map R in (7.1).

One of the key properties of the R and F maps is that they commute:

$$\begin{array}{ccc} \mathbf{THH}(\mathbb{D})^{C_{p^{n+1}}} & \xrightarrow{R} & \mathbf{THH}(\mathbb{D})^{C_{p^n}} \\ F \downarrow & & \downarrow F \\ \mathbf{THH}(\mathbb{D})^{C_{p^n}} & \xrightarrow{R} & \mathbf{THH}(\mathbb{D})^{C_{p^{n-1}}} \end{array} \tag{7.12}$$

e.g., see [30, Section 6.2.3 on pages 237–240].

Warning 7.13. In general, for $\mathbb{D} = \mathbb{A}[GfS]$ there is no map

$$R_{\mathcal{F}}: \mathbf{THH}_{\mathcal{F}}(\mathbb{A}[GfS])^{C_{p^n}} \cdots \xrightarrow{\diamond} \mathbf{THH}_{\mathcal{F}}(\mathbb{A}[GfS])^{C_{p^{n-1}}}.$$

The problem is with the wrong-way homeomorphism Δ in (7.10). The homeomorphism $\Delta_{\mathbb{A}[GfS]}$ in (7.11) does restrict to a map between the \mathcal{F} -parts. But this restricted map is not surjective without further conditions on G and \mathcal{F} . For example, take $\mathbb{A} = \mathbb{S}$, $S = \text{pt}$, $n = 1$, and $q = 0$. Identify $Gf\text{pt} = G$. Then the map $\Delta_{\mathbb{S}[G]}$ restricts to

$$cn_{[0],\mathcal{F}}\mathbb{S}[G](0) \longrightarrow (cn_{p[0],\mathcal{F}}\mathbb{S}[G](0, \dots, 0))^{C_p}. \tag{7.14}$$

Under the identification

$$cn_{p[0],\mathcal{F}}\mathbb{S}[G](0, \dots, 0) \cong \{(g_1, \dots, g_p) \in G \times \cdots \times G \mid \langle g_1 \cdots g_p \rangle \in \mathcal{F}\}_+,$$

in the case $\mathcal{F} = 1$ the map (7.14) corresponds to the inclusion

$$S^0 \longrightarrow \{g \in G \mid g^p = 1\}_+.$$

Next we identify the homotopy fiber of R . The first ingredient is the following.

Lemma 7.15. *Let E be a model for EC_{p^n} , i.e., a contractible free C_{p^n} -CW-space. Let X be a pointed C_{p^n} -CW-space and Y a pointed C_{p^n} -space. Then there is a natural homotopy fibration sequence*

$$\text{map}(X, E_+ \wedge Y)^{C_{p^n}} \xrightarrow{\text{pr}_*} \text{map}(X, Y)^{C_{p^n}} \xrightarrow{r} \text{map}(X^{C_p}, Y^{C_p})^{C_{p^n}/C_p},$$

where pr_* is induced by the projection $\text{pr}: E_+ \wedge Y \longrightarrow Y$, and r takes each C_{p^n} -equivariant map $f: X \longrightarrow Y$ to its restriction $f^{C_p}: X^{C_p} \longrightarrow Y^{C_p}$ to the C_p -fixed parts.

Proof. By mapping the C_{p^n} -equivariant cofibration sequence $X^{C_p} \rightarrow X \rightarrow X/X^{C_p}$ into the C_{p^n} -equivariant map $\text{pr}: E_+ \wedge Y \rightarrow Y$ we get the following map of vertical fibration sequences.

$$\begin{array}{ccc}
 \text{map}(X/X^{C_p}, E_+ \wedge Y)^{C_{p^n}} & \xrightarrow{\simeq} & \text{map}(X/X^{C_p}, Y)^{C_{p^n}} \\
 \cong \downarrow & & \downarrow \\
 \text{map}(X, E_+ \wedge Y)^{C_{p^n}} & \xrightarrow{\text{pr}} & \text{map}(X, Y)^{C_{p^n}} \\
 \downarrow & & \downarrow \\
 \text{map}(X^{C_p}, E_+ \wedge Y)^{C_{p^n}} & \longrightarrow & \text{map}(X^{C_p}, Y)^{C_{p^n}}
 \end{array}$$

The top map is a weak equivalence because the pointed X_{p^n} -CW-space X/X^{C_p} is C_{p^n} -free and $E_+ \wedge Y \rightarrow Y$ is a non-equivariant equivalence. The upper left-hand vertical map is an isomorphism because the lower left-hand space is a single point.

The desired homotopy fibration sequence appears as the middle horizontal map, followed by the lower right-hand vertical map composed with the identification $\text{map}(X^{C_p}, Y)^{C_{p^n}} \cong \text{map}(X^{C_p}, Y^{C_p})^{C_{p^n}/C_p}$. \square

Recall the definition of $\text{sh}^E \mathbf{X} \in \text{obj } \mathcal{WT}^C$ for $E \in \text{obj } \mathcal{W}_C$ and $\mathbf{X} \in \text{obj } \mathcal{WT}^C$; compare Definition 4.10.

Corollary 7.16. *Let E be a model for EC_{p^n} , X a pointed C_{p^n} -CW-space, and Y a pointed C_{p^n} -space. Consider the continuous functor $\mathbb{M} = \text{map}(X, - \wedge Y): \mathcal{W} \rightarrow \mathcal{T}^{C_{p^n}}$. Then there is a levelwise homotopy fibration sequence in \mathcal{WT}*

$$(\text{sh}^{E_+} \mathbb{M})^{C_{p^n}} \xrightarrow{\text{pr}_*} \mathbb{M}^{C_{p^n}} \xrightarrow{r} \text{map}(X^{C_p}, - \wedge Y^{C_p})^{C_{p^n}/C_p}.$$

Proof. This follows at once from Lemma 7.15. \square

Corollary 7.17. *Let $n \geq 1$ and let E be a model for EC_{p^n} . Then for all $[q] \in \text{obj } \Lambda^{\text{op}}$ and all $\vec{x} \in \text{obj } \mathcal{I}^{[q]}$ there is a levelwise homotopy fibration sequence in \mathcal{WT}*

$$(\text{sh}^{E_+} M_{p^n[q]} \mathbb{D}(p^n \vec{x}))^{C_{p^n}} \xrightarrow{\text{pr}_*} (M_{p^n[q]} \mathbb{D}(p^n \vec{x}))^{C_{p^n}} \xrightarrow{r} (M_{p^{n-1}[q]} \mathbb{D}(p^{n-1} \vec{x}))^{C_{p^{n-1}}}.$$

Proof. This follows from the previous Corollary 7.16, identifying the term on the right using the homeomorphism Δ in (7.10). \square

We are now ready to prove the main result of this section.

Proof of Theorem 7.2. Consider ES^1 as a C_{p^n} -space by restricting the action, and notice that it is a model for EC_{p^n} . Recall that in \mathcal{WT} stable homotopy fibration sequences and stable homotopy cofibration sequences are the same, and therefore they are preserved

by homotopy colimits. So by applying $\text{hocolim}_{\mathcal{I}^{[q]}}$ to the (levelwise and hence) stable homotopy fibration sequence from [Corollary 7.17](#) we get a sequence in $\Delta^{\text{op}}(\mathcal{WT})$

$$(\text{sh}^{ES^1_+} \text{sd}_{p^n} \text{THH}_\bullet(\mathbb{D}))^{C_{p^n}} \xrightarrow{\text{Pr}_*} (\text{sd}_{p^n} \text{THH}_\bullet(\mathbb{D}))^{C_{p^n}} \xrightarrow{R} (\text{sd}_{p^{n-1}} \text{THH}_\bullet(\mathbb{D}))^{C_{p^{n-1}}}$$

which in every simplicial degree is a stable homotopy fibration sequence in \mathcal{WT} . To identify the term on the left here we used the fact that [\(7.8\)](#) generalizes to

$$(\text{sh}^E \text{sd}_c \text{THH}_\bullet(\mathbb{D}))^C \cong \text{hocolim} \left(\mathcal{I}^{[\bullet]}, (\text{sh}^E M_{c[\bullet]} \mathbb{D}(c-)) \right)^C.$$

Since we are assuming that \mathbb{D} is very well pointed, [Lemma 6.17\(ii\)](#) ensures that we can apply [Lemma 4.14](#). The fact that [\(7.7\)](#) generalizes to

$$(\text{sh}^E \mathbf{THH}(\mathbb{D}))^C \cong \left| (\text{sh}^E \text{sd}_c \text{THH}_\bullet(\mathbb{D}))^C \right|$$

finishes the proof. \square

8. Adams isomorphism and the fundamental fibration sequence

The main result of this section is the following theorem. Together with [Theorem 7.2](#), it identifies up to natural π_* -isomorphisms the homotopy fiber of the restriction map. We emphasize that this identification is natural before passing to the stable homotopy category. This subtlety is essential when dealing with assembly maps.

Denote by $\mathbb{U}: \mathcal{WT}^C \rightarrow \text{Th}\mathcal{F}^C$ the forgetful functor from C -equivariant \mathcal{W} -spaces to orthogonal C -spectra.

Theorem 8.1. *Let C be a finite cyclic subgroup of S^1 .*

- (i) *If \mathbb{D} is an objectwise strictly connective, objectwise convergent, and very well pointed symmetric spectral category, then there is a zig-zag of natural π_*^1 -isomorphisms of orthogonal spectra between*

$$(\mathbf{UTHH}(\mathbb{D}))_{hC} \quad \text{and} \quad (\mathbb{U}\text{sh}^{ES^1_+} \mathbf{THH}(\mathbb{D}))^C.$$

For each $n \geq 1$, in the stable homotopy category there is an exact triangle

$$(\mathbf{UTHH}(\mathbb{D}))_{hC_{p^n}} \rightarrow \mathbf{UTHH}(\mathbb{D})^{C_{p^n}} \xrightarrow{R} \mathbf{UTHH}(\mathbb{D})^{C_{p^{n-1}}},$$

which is referred to as the fundamental fibration sequence.

- (ii) *Let $\mathbb{D} = \mathbb{A}[GfS]$. Assume that the symmetric ring spectrum \mathbb{A} is connective⁺. Then there is a zig-zag of π_*^1 -isomorphisms of orthogonal spectra between*

$$(\mathbf{UTHH}_{\mathcal{F}}(\mathbb{A}[GfS]))_{hC} \quad \text{and} \quad (\mathbb{U}\text{sh}^{ES^1_+} \mathbf{THH}_{\mathcal{F}}(\mathbb{A}[GfS]))^C$$

that is natural in S and commutes with the projection and inclusion maps induced by (6.9).

The proof of Theorem 8.1 is based on the following key result, which in turn uses two main ingredients. The first is the natural model for the Adams isomorphism developed by the second and last authors in [95], and explained in the course of the proof below. The second is a result by Blumberg [15], giving an equivariant generalization of an unpublished result of Lydakis [76, Proposition 11.7 on page 36]; see also [80, Proposition 17.6 on page 494]. Notice that the assumption on $\mathbb{U}(\text{sh}^{E+}\mathbf{X})$ below is satisfied whenever \mathbf{X} is fibrant in the absolute stable model structure on $\mathcal{W}_C\mathcal{T}_C$ by [15, Theorem 1.3(6) on page 2263].

Theorem 8.2. *Let C be any finite group. Let $\mathbf{X} \in \text{obj } \mathcal{W}\mathcal{T}^C$ and let E be a finite free C - CW -space. Assume that $\mathbb{U}\text{sh}^{E+}\mathbf{X}$ is an orthogonal C - Ω -spectrum. Then there is a zig-zag of natural π_* -isomorphisms of orthogonal spectra between*

$$E_+ \wedge_C \mathbb{U}\mathbf{X} \quad \text{and} \quad (\mathbb{U}\text{sh}^{E+}\mathbf{X})^C.$$

Proof. Choose a complete C -universe \mathcal{U} . In the category $\text{Th}\mathcal{T}^C$ of orthogonal C -spectra, consider the bifunctorial replacement construction $Q^{\mathcal{U}}$ and the natural map $r: \mathbb{Y} \rightarrow Q^{\mathcal{U}}(\mathbb{Y})$ from [95, Theorem 1.1 on pages 1494–1495]. When \mathbb{Y} is good in the sense of [95, Definition 5.5 on pages 1511–1512] (compare also [51, Appendix A on pages 96–98]), then $Q^{\mathcal{U}}(\mathbb{Y})$ is a C - Ω -spectrum and r is a $\underline{\pi}_*$ -isomorphism. In particular, if $f: \mathbb{Y} \rightarrow \mathbb{Y}'$ is a $\underline{\pi}_*$ -isomorphism of good orthogonal C -spectra, then $Q^{\mathcal{U}}(f)$ is a $\underline{\pi}_*$ -isomorphism and therefore a level equivalence of C - Ω -spectra. Recall also that $\underline{\pi}_*$ -isomorphisms of C - Ω -spectra induce π_* -isomorphisms between the C -fixed points; see for example [95, Lemma 6.20(iii) on page 1522]. Here a $\underline{\pi}_*$ -isomorphism is a map $f: \mathbb{Y} \rightarrow \mathbb{Y}'$ such that $\pi_n^H(f): \pi_n^H(\mathbb{Y}) \rightarrow \pi_n^H(\mathbb{Y}')$ is an isomorphism for all $n \in \mathbb{Z}$ and all subgroups H of C .

Consider also a cofibrant replacement functor Γ in the stable model category of orthogonal C -spectra (compare [79, Section III.4 on pages 47–51]), and let $\gamma: \Gamma\mathbb{Y} \rightarrow \mathbb{Y}$ be the corresponding $\underline{\pi}_*$ -isomorphism and fibration. Notice that cofibrant spectra are good by [95, Lemma 5.10(i) on page 1514]. Recall that the fibrant objects are the C - Ω -spectra. In particular, if \mathbb{Y} is a C - Ω -spectrum, then so is $\Gamma\mathbb{Y}$.

We have now the following zig-zag of natural maps of orthogonal spectra, which we explain below.

$$\begin{array}{ccccccc}
 E_+ \wedge_C \mathbb{U}\mathbf{X} & \xleftarrow{\textcircled{1}} & E_+ \wedge_C \Gamma\mathbb{U}\mathbf{X} & \xleftarrow{\textcircled{2}} & EC_+ \wedge_C (E_+ \wedge \Gamma\mathbb{U}\mathbf{X}) & \xrightarrow{\textcircled{3}} & Q^{\mathcal{U}}(E_+ \wedge \Gamma\mathbb{U}\mathbf{X})^C \\
 & & & & & & \uparrow \textcircled{4} \\
 & & & & & & Q^{\mathcal{U}}(\Gamma(E_+ \wedge \Gamma\mathbb{U}\mathbf{X}))^C \\
 & & & & & & \downarrow \textcircled{5} \\
 (\mathbb{U}\text{sh}^{E+}\mathbf{X})^C & \xleftarrow{\textcircled{8}} & (\Gamma\mathbb{U}\text{sh}^{E+}\mathbf{X})^C & \xrightarrow{\textcircled{7}} & Q^{\mathcal{U}}(\Gamma\mathbb{U}\text{sh}^{E+}\mathbf{X})^C & \xleftarrow{\textcircled{6}} & Q^{\mathcal{U}}(\Gamma(E_+ \wedge \mathbb{U}\mathbf{X}))^C
 \end{array}$$

The maps labeled ①, ④, ⑤, and ⑧ are all induced by γ , and the map ⑦ by r . The map ② comes from the isomorphism $EC_+ \wedge_C (E_+ \wedge \Gamma\mathbf{U}\mathbf{X}) \cong (EC \times E)_+ \wedge_C \Gamma\mathbf{U}\mathbf{X}$ and the projection $EC \rightarrow \text{pt}$. Since E is free, ② is a homotopy equivalence and $E_+ \wedge \Gamma\mathbf{U}\mathbf{X}$ is C -free in the sense of [95, (1.6) on page 1495]. Moreover, since $\Gamma\mathbf{U}\mathbf{X}$ is good, also $E_+ \wedge \Gamma\mathbf{U}\mathbf{X}$ is good by [95, Lemma 5.10(v) on page 1515].

Using the facts recalled in the first two paragraphs of the proof, we see that ④, ⑤, ⑦, and ⑧ are all π_* -isomorphisms. Since E is free, ① is a π_* -isomorphism by [95, Lemma 5.13(iv) on page 1517]. It remains to describe the maps ③ and ⑥, which are the key steps in the argument.

By [95, Main Theorem 1.7 on page 1496], for every good and C -free orthogonal C -spectrum \mathbb{Y} there is natural π_* -isomorphism

$$A: EC_+ \wedge_C \mathbb{Y} \rightarrow Q^{\mathcal{U}}(\mathbb{Y})^C.$$

The map A induces in the stable homotopy category the Adams isomorphism of Lewis, May, and Steinberger [64, Theorem II.7.1 on page 97], which in the special case of suspension spectra was proved by Adams [1, Theorem 5.4 on page 500]. We emphasize that the map A is natural before passing to the homotopy category. This explains the natural π_* -isomorphism ③.

By [15, Proposition 3.6 on page 2275], the natural assembly map

$$E_+ \wedge \mathbf{X} \rightarrow \text{sh}^{E_+} \mathbf{X}$$

is a π_* -isomorphism for every $\mathbf{X} \in \text{obj } \mathcal{WT}^C$ and for every finite C -CW-space E . This result in fact holds for any pointed finite C -CW-space B instead of E_+ . This gives a π_* -isomorphism $E_+ \wedge \mathbf{U}\mathbf{X} \cong \mathbb{U}(E_+ \wedge \mathbf{X}) \rightarrow \mathbb{U}\text{sh}^{E_+} \mathbf{X}$. Using the facts above, this induces the natural π_* -isomorphism ⑥, thus completing the proof. \square

In order to apply Theorem 8.2 to the case $\mathbf{X} = \mathbf{THH}(\mathbb{D})$ we need the following result proven by Hesselholt and Madsen [51].

Proposition 8.3. *Let C be a finite cyclic subgroup of S^1 , and let $E \in \text{obj } \mathcal{W}_C$.*

- (i) *Assume that the symmetric spectral category \mathbb{D} is objectwise strictly connective, objectwise convergent, and very well pointed. Then $\mathbb{U}\text{sh}^E \mathbf{THH}(\mathbb{D})$ is an orthogonal C - Ω -spectrum.*
- (ii) *The same is true for $\mathbb{U}\text{sh}^E \mathbf{THH}_{\mathcal{F}}(\mathbb{A}[GfS])$ provided that the symmetric ring spectrum \mathbb{A} is connective⁺.*

Proof. A version of part (i) is proved in [51, proof of Proposition 2.4 on page 40] in the setting of functors with smash products, or FSPs. An inspection of their argument, together with Lemma 6.17(ii), shows that the proof works also under the stated assumptions on \mathbb{D} .

If \mathbb{A} is connective⁺, then $\mathbb{A}[\mathcal{C}]$ is objectwise strictly connective, objectwise convergent, and very well pointed for any small category \mathcal{C} . To verify the second property, notice that if \mathbb{A}_x is $(x - 1)$ -connected and $\mathbb{A}_x \rightarrow \Omega\mathbb{A}_{x+1}$ is $(x + \lambda(x))$ -connected, then $\bigvee \mathbb{A}_x \rightarrow \Omega\bigvee \mathbb{A}_{x+1}$ is roughly $\min(x + \lambda(x), 2x)$ -connected. Here the wedge sum is indexed by any morphism set in \mathcal{C} . The other properties are clear.

Therefore part (i) applies to $\mathbb{D} = \mathbb{A}[GfS]$, and since retracts of Ω -spectra are again Ω -spectra, part (ii) follows. \square

We can now conclude the proof of [Theorem 8.1](#).

Proof of Theorem 8.1. We first prove part (i). Notice that the second claim in (i) follows immediately from the first claim and [Theorem 7.2](#). Consider ES^1 as a C -space by restricting the action. Decompose $ES^1 \cong \text{colim}_{i \in \mathbb{N}} E_i$ as a colimit along closed embeddings of finite free C -CW-complexes E_i . For each i take the zig-zag of natural π_* -isomorphisms given by [Theorem 8.2](#) and [Proposition 8.3](#), and then pass to the homotopy colimit to obtain a zig-zag of natural π_* -isomorphisms between

$$\text{hocolim}_{i \in \mathbb{N}} \left(E_{i+} \wedge_C \mathbf{U}\mathbf{T}\mathbf{H}\mathbf{H}(\mathbb{D}) \right) \quad \text{and} \quad \text{hocolim}_{i \in \mathbb{N}} \left((\mathbf{U}\text{sh}^{E_{i+}} \mathbf{T}\mathbf{H}\mathbf{H}(\mathbb{D}))^C \right).$$

Using well known properties of closed embeddings (e.g., see [\[95, Facts 5.7 on pages 1512–1513\]](#)), we see that on both sides the homotopy colimit is taken levelwise along closed embeddings. Therefore the natural projection $\text{hocolim} \rightarrow \text{colim}$ is on both sides a π_* -isomorphism. Finally, notice that we have isomorphisms

$$\begin{aligned} \text{colim}_{i \in \mathbb{N}} \left(E_{i+} \wedge_C \mathbf{U}\mathbf{T}\mathbf{H}\mathbf{H}(\mathbb{D}) \right) &\cong ES^1_+ \wedge_C \mathbf{U}\mathbf{T}\mathbf{H}\mathbf{H}(\mathbb{D}), \\ \text{colim}_{i \in \mathbb{N}} \left((\mathbf{U}\text{sh}^{E_{i+}} \mathbf{T}\mathbf{H}\mathbf{H}(\mathbb{D}))^C \right) &\cong (\mathbf{U}\text{sh}^{ES^1_+} \mathbf{T}\mathbf{H}\mathbf{H}(\mathbb{D}))^C. \end{aligned}$$

This proves part (i). As explained in the proof of [Proposition 8.3\(ii\)](#), the assumption on \mathbb{A} implies that part (i) applies to $\mathbb{D} = \mathbb{A}[GfS]$. Since retracts of π_*^1 -isomorphisms are π_*^1 -isomorphisms, and using naturality of the zig-zag in [Theorem 8.2](#), part (ii) follows. \square

We close this section with the following result, which is an immediate consequence of [Corollary 6.20](#) and [Theorem 8.1\(ii\)](#).

Corollary 8.4. *Assume that the symmetric ring spectrum \mathbb{A} is connective⁺. Let C be a finite cyclic subgroup of S^1 . Consider the following commutative diagram in \mathcal{WT} .*

$$\begin{array}{ccc} EG(\mathcal{F})_+ \wedge_{\text{Or}G} (\text{sh}^{ES^1_+} \mathbf{T}\mathbf{H}\mathbf{H}(\mathbb{A}[Gf-]))^C & \xrightarrow{\text{asbl}} & (\text{sh}^{ES^1_+} \mathbf{T}\mathbf{H}\mathbf{H}(\mathbb{A}[G]))^C \\ \text{id} \wedge \text{pr}_{\mathcal{F}} \downarrow & & \downarrow \text{pr}_{\mathcal{F}} \\ EG(\mathcal{F})_+ \wedge_{\text{Or}G} (\text{sh}^{ES^1_+} \mathbf{T}\mathbf{H}\mathbf{H}_{\mathcal{F}}(\mathbb{A}[Gf-]))^C & \xrightarrow{\text{asbl}} & (\text{sh}^{ES^1_+} \mathbf{T}\mathbf{H}\mathbf{H}_{\mathcal{F}}(\mathbb{A}[G]))^C \end{array}$$

The left-hand map and the bottom map are π_* -isomorphisms. If \mathcal{F} contains all cyclic groups, then the right-hand map is an isomorphism.

9. Bökstedt–Hsiang–Madsen’s functor C

Bökstedt, Hsiang, and Madsen define in [17, (5.14) on page 497] an infinite loop space $C(X; p)$ for any space X and any prime p . In this section we define a \mathcal{W} -space $\mathbf{C}(\mathbb{D}; p)$ for any symmetric spectral category \mathbb{D} , which generalizes their original construction in the following sense: if $X = BG$ for a discrete group G and $\mathbb{D} = \mathbb{S}[G]$, then $C(BG; p)$ and $\mathbf{C}(\mathbb{S}[G]; p)$ are isomorphic in the stable homotopy category.

Consider an S^1 -equivariant \mathcal{W} -space $\mathbf{X} \in \text{obj } \mathcal{WT}^{S^1}$. We denote by $\mathbf{F}(\mathbf{X}; p)$ the \mathcal{W} -space defined as the homotopy limit in \mathcal{WT} of the C_{p^n} -fixed points of \mathbf{X} along the inclusions F :

$$\mathbf{F}(\mathbf{X}; p) = \text{holim}_{n \in \mathbb{N}} (\dots \rightarrow \mathbf{X}^{C_{p^n}} \xrightarrow{F} \mathbf{X}^{C_{p^{n-1}}} \rightarrow \dots \rightarrow \mathbf{X}^{C_p} \xrightarrow{F} \mathbf{X}).$$

Recall from Subsection 4I that homotopy limits in \mathcal{WT} are defined by first applying a fibrant replacement functor and then taking levelwise homotopy limits.

Definition 9.1 (*Bökstedt–Hsiang–Madsen’s functor C*). Given a symmetric spectral category \mathbb{D} and a prime p , we define

$$\mathbf{C}(\mathbb{D}; p) = \mathbf{F}(\text{sh}^{ES^1_+} \mathbf{THH}(\mathbb{D}); p).$$

If $\mathbb{D} = \mathbb{A}[GfS]$ we define

$$\mathbf{C}_{\mathcal{F}}(\mathbb{A}[GfS]; p) = \mathbf{F}(\text{sh}^{ES^1_+} \mathbf{THH}_{\mathcal{F}}(\mathbb{A}[GfS]); p).$$

Thus we obtain from (6.9) maps

$$\mathbf{C}(\mathbb{A}[GfS]; p) \begin{matrix} \xrightarrow{\text{pr}_{\mathcal{F}}} \\ \xleftarrow{\text{in}_{\mathcal{F}}} \end{matrix} \mathbf{C}_{\mathcal{F}}(\mathbb{A}[GfS]; p) \tag{9.2}$$

satisfying $\text{pr}_{\mathcal{F}} \circ \text{in}_{\mathcal{F}} = \text{id}$.

We also use the \mathbf{F} -construction to define

$$\mathbf{TF}(\mathbb{D}; p) = \mathbf{F}(\mathbf{THH}(\mathbb{D}); p); \tag{9.3}$$

compare [51, (20) on page 56]. The projection $\text{pr}: ES^1 \rightarrow \text{pt}$ induces a map

$$\mathbf{C}(\mathbb{D}; p) \rightarrow \mathbf{TF}(\mathbb{D}; p).$$

Remark 9.4. By Proposition 8.3(ii), $\text{Ush}^E \mathbf{THH}_{\mathcal{F}}(\mathbb{A}[GfS])$ is an orthogonal C_{p^n} - Ω -spectrum for each n , provided that \mathbb{A} is connective⁺. Therefore its C_{p^n} -fixed points are orthogonal Ω -spectra; e.g., see [95, Lemma 6.20(ii) on page 1522]. It follows that also $(\text{sh}^E \mathbf{THH}(\mathbb{A}[GfS]))^{C_{p^n}}$ is a fibrant \mathcal{W} -space by [15, Theorem 1.3(2) on page 2263], and therefore no fibrant replacement would be needed in defining the homotopy limit.

We are now ready to prove our main splitting result, Theorem 1.19(ii) and Theorem 3.5. The deduction of this result from Corollary 8.4 is based on the main theorem of [74].

Theorem 9.5. *Assume that the symmetric ring spectrum \mathbb{A} is connective⁺. Let $\mathcal{F} \subseteq \text{Fin}$ be a family of finite subgroups of G and let $N \geq 0$. Assume that the following condition holds.*

[A'_{\mathcal{F}, \leq N+1}] *For every $C \in \mathcal{F}$ and for every $1 \leq s \leq N + 1$, the integral group homology $H_s(BZ_G C; \mathbb{Z})$ of the centralizer of C in G is an almost finitely generated abelian group.*

Then the assembly maps

$$EG(\mathcal{F})_+ \wedge_{\text{Or}G} \mathbf{C}(\mathbb{A}[Gf-]; p) \xrightarrow{\text{asbl}} \mathbf{C}(\mathbb{A}[G]; p)$$

for all primes p and

$$\begin{aligned} EG(\mathcal{F})_+ \wedge_{\text{Or}G} \left(\mathbf{THH}(\mathbb{A}[Gf-]) \times \prod_{p \text{ prime}} \mathbf{C}(\mathbb{A}[Gf-]; p) \right) \\ \xrightarrow{\text{asbl}} \mathbf{THH}(\mathbb{A}[G]) \times \prod_{p \text{ prime}} \mathbf{C}(\mathbb{A}[G]; p) \end{aligned}$$

are $\pi_n^{\mathbb{Q}}$ -injective for all $n \leq N$.

Proof. We first prove the theorem under the additional assumption that \mathcal{F} contains only finitely many conjugacy classes. We begin with $\mathbf{C}(\mathbb{A}[G]; p)$. Consider the following commutative diagram in \mathcal{WT} .

$$\begin{CD} EG(\mathcal{F})_+ \wedge_{\text{Or}G} \mathbf{C}(\mathbb{A}[Gf-]; p) @>\text{asbl}>> \mathbf{C}(\mathbb{A}[G]; p) \\ @VtVV @VV\parallel V \\ \text{holim}_{m \in \mathbb{N}} \left(EG(\mathcal{F})_+ \wedge_{\text{Or}G} (\text{sh}^{ES^1_+} \mathbf{THH}(\mathbb{A}[Gf-]))^{C_{p^m}} \right) @>\text{holim}(\text{asbl})>> \mathbf{C}(\mathbb{A}[G]; p) \\ @V\text{holim}(\text{id} \wedge \text{pr}_{\mathcal{F}}) \downarrow \textcircled{1}VV @VV\text{pr}_{\mathcal{F}}V \\ \text{holim}_{m \in \mathbb{N}} \left(EG(\mathcal{F})_+ \wedge_{\text{Or}G} (\text{sh}^{ES^1_+} \mathbf{THH}_{\mathcal{F}}(\mathbb{A}[Gf-]))^{C_{p^m}} \right) @>\text{holim}(\text{asbl})>> \mathbf{C}_{\mathcal{F}}(\mathbb{A}[G]; p) \end{CD}$$

\textcircled{2}

The bottom square is obtained by taking the homotopy limit of the diagrams from [Corollary 8.4](#). Therefore the maps ① and ② are π_* -isomorphisms, and so ③ is π_* -injective.

The map t is the natural map that interchanges the order of smashing over $\text{Or}G$ and taking $\text{holim}_{\mathbb{N}}$. We proceed to verify the assumptions (A) to (D) of [\[74, Addendum 1.3 on page 140\]](#), which imply that $\pi_n(t)$ is an almost isomorphism, and so in particular a rational isomorphism, for all $n \leq N$. For assumption (A), the category $\mathcal{C} = \mathbb{N}$ has an obvious 1-dimensional model for $E\mathbb{N}$; compare [\[74, Example 7.1 on page 162\]](#). For (B), [Theorem 8.1](#) and the fact that homotopy orbits preserve connectivity imply that $(\text{sh}^{ES^1_+} \mathbf{THH}(\mathbb{A}[Gf-]))^{C_{p^m}}$ is always (-1) -connected. For $X = EG(\mathcal{F})$, (C) is implied by our assumptions on \mathcal{F} . And finally (D) follows from our assumptions on $H_s(BZ_G H; \mathbb{Z})$ combined with [\[74, Proposition 1.7 on page 142\]](#).

A similar argument applies to

$$t: EG(\mathcal{F})_+ \wedge_{\text{Or}G} \prod_p \mathbf{C}(\mathbb{A}[Gf-]; p) \longrightarrow \prod_p \left(EG(\mathcal{F})_+ \wedge_{\text{Or}G} \mathbf{C}(\mathbb{A}[G]; p) \right).$$

This time the category \mathcal{C} is the set of all prime numbers considered as a discrete category, and the constant functor pt clearly satisfies assumption (A).

Next we explain how to drop the assumption that \mathcal{F} contains only finitely many conjugacy classes. Consider the set \mathcal{J} of subfamilies $\mathcal{E} \subset \mathcal{F}$ that are finite up to conjugacy. Notice that \mathcal{J} is a directed poset with respect to inclusion, and that $\mathcal{F} = \bigcup_{\mathcal{J}} \mathcal{E}$. Appropriate models for $EG(\mathcal{E})$ yield a functor from \mathcal{J} to G -spaces and a G -equivalence

$$\text{hocolim}_{\mathcal{E} \in \mathcal{J}} EG(\mathcal{E}) \xrightarrow{\cong} EG(\mathcal{F}). \tag{9.6}$$

Now there is the following commutative diagram, as well as a corresponding diagram for $\mathbf{THH}(\mathbb{A}[G]) \times \prod_p \mathbf{C}(\mathbb{A}[G]; p)$.

$$\begin{array}{ccc}
 & \text{hocolim}_{\mathcal{J}} \left(EG(\mathcal{E})_+ \wedge_{\text{Or}G} \mathbf{C}(\mathbb{A}[Gf-]; p) \right) & \\
 \textcircled{4} \swarrow & & \searrow \text{hocolim}(\text{asbl}_{\mathcal{E}}) \\
 EG(\mathcal{F})_+ \wedge_{\text{Or}G} \mathbf{C}(\mathbb{A}[Gf-]; p) & \xrightarrow{\text{asbl}_{\mathcal{F}}} & \mathbf{C}(\mathbb{A}[G]; p)
 \end{array}$$

Here ④ is induced by the equivalence in (9.6) and the fact that homotopy colimits commute up to isomorphism with smash products over the orbit category, and therefore ④ is a π_* -isomorphism. Since the functors $\pi_*(-)$ and $-\otimes_{\mathbb{Z}}\mathbb{Q}$ both commute with directed (homotopy) colimits, and directed colimits are exact, it follows from the argument above that $\text{hocolim}(\text{asbl}_{\mathcal{E}})$ is injective on $\pi_n(-) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $n \leq N$. Therefore the same is true for $\text{asbl}_{\mathcal{F}}$, as claimed. \square

The proof of the Splitting [Theorem 3.5](#) is now complete. In fact, [Theorems 6.1 and 9.5](#) establish more general splitting results than what is needed for the [Main Technical](#)

Theorem 1.16. In [Remark 10.2](#) below we explain why our method of proof cannot be used to obtain similar splitting results for topological cyclic homology without additional assumptions on the group and the family.

10. Topological cyclic homology

Topological cyclic homology of a symmetric spectral category \mathbb{D} at a prime p is defined as

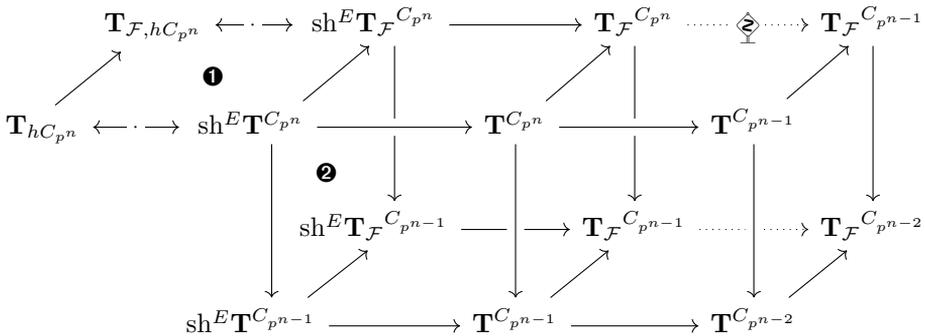
$$\mathbf{TC}(\mathbb{D}; p) = \operatorname{holim}_{n \in \operatorname{obj} \mathcal{RF}} \mathbf{THH}(\mathbb{D})^{C_{p^n}}.$$

Here \mathcal{RF} is the category with set of objects \mathbb{N} and where the morphisms from m to n are the pairs $(i, j) \in \mathbb{N} \times \mathbb{N}$ with $i + j = m - n$. The functoriality in \mathcal{RF} comes from the fact that the maps $R, F: \mathbf{THH}(\mathbb{D})^{C_{p^n}} \rightarrow \mathbf{THH}(\mathbb{D})^{C_{p^{n-1}}}$ commute; see [diagram \(7.12\)](#). So we get a functor by sending the morphisms $(0, 1)$ to the R -maps, and the morphisms $(1, 0)$ to the F -maps. By forgetting the R -maps, we get a natural map

$$\mathbf{TC}(\mathbb{D}; p) \rightarrow \mathbf{TF}(\mathbb{D}; p), \tag{10.1}$$

whose target is defined in [\(9.3\)](#).

Remark 10.2. In general, for $\mathbb{D} = \mathbb{A}[GfS]$, there is no R -map between the fixed points of $\mathbf{THH}_{\mathcal{F}}(\mathbb{A}[GfS])$; see [Warning 7.13](#). Therefore, without additional assumptions on G and \mathcal{F} , we cannot define $\mathbf{TC}_{\mathcal{F}}(\mathbb{A}[GfS]; p)$ and retraction maps as in [\(9.2\)](#), and prove a splitting result for \mathbf{TC} analogous to [Theorem 9.5](#). To see why our method works for \mathbf{C} but not for \mathbf{TC} , consider the following diagram, where we use the abbreviations $\mathbf{T} = \mathbf{THH}(\mathbb{A}[Gf-])$ and $E = ES_+^1$.



The diagonal maps are the projections to the \mathcal{F} -parts. The vertical maps are the inclusions of fixed points F . The commutative diagram **1** comes from [Theorem 8.1](#), where the symbol $\leftarrow \cdot \rightarrow$ indicates a zig-zag of natural π_* -isomorphisms, and it allows us to deduce splitting results for $(\operatorname{sh}^E \mathbf{T})^{C_{p^n}}$ from the ones for \mathbf{T} ; see [Corollary 8.4](#).

On the front face, the other horizontal maps are the homotopy fibration sequences from [Theorem 7.2](#). However, notice that in the back, for the \mathcal{F} -parts, the R -maps are missing, and we do not have homotopy fibration sequences. Therefore we cannot argue by induction and obtain splitting results for the fixed points $\mathbf{T}^{C_{p^n}}$, and then pass to the homotopy limit to get results for \mathbf{TC} .

On the other hand, since $\textcircled{2}$ obviously commutes, we can pass to the homotopy limit over n with respect to the F -maps, and (using the results from [\[74\]](#)) deduce splitting results for \mathbf{C} from the ones for $(\text{sh}^E \mathbf{T})^{C_{p^n}}$; see [Theorem 9.5](#). Notice that we do not need any compatibility between the squares $\textcircled{1}$ and $\textcircled{2}$, as they enter the proof at separate stages. The missing cube on the left and its commutativity are not needed in the proof.

Notice also that if we simply took $\text{hofib } R$ instead of $(\text{sh}^E \mathbf{T})^{C_{p^n}}$, we would not even obtain the existence, let alone the commutativity, of the corresponding diagram $\textcircled{2}$, since there are no R -maps for the \mathcal{F} -parts. This shows the importance of the point-set model for $\text{hofib } R$ given in [Theorem 7.2](#).

In the case of spherical coefficients $\mathbb{A} = \mathbb{S}$, the R -map splits and the splitting induces a map between the \mathcal{F} -parts, but other difficulties arise, as explained in [Warning 11.2](#).

Finally, we mention that in a separate paper [\[73\]](#) we show that under additional assumptions on G and \mathcal{F} it is possible to fill in the missing R -maps in the diagram above, thus obtaining splitting results for \mathbf{TC} , too.

This remark represents years of discussions among the authors.

11. Spherical coefficients

We now analyze the special case $\mathbb{D} = \mathbb{S}[\mathcal{C}]$ for a small category \mathcal{C} . In this case, the key observation is that the R -maps split. This result is well known; compare [\[17, proof of Theorem 5.17, pages 500–501\]](#) and [\[30, Lemma 6.2.5.1 on page 245\]](#). We give a proof below to highlight that the splitting is natural in \mathcal{C} . We then use this fact to construct a zig-zag of natural transformations from $\mathbf{TC}(\mathbb{S}[\mathcal{C}]; p)$ to $\mathbf{C}(\mathbb{S}[\mathcal{C}]; p)$ where the wrong-way maps are all π_* -isomorphisms.

Proposition 11.1. *If $\mathbb{D} = \mathbb{S}[\mathcal{C}]$ for a small category \mathcal{C} , then the R -map splits, i.e., there is a map of \mathcal{W} -spaces*

$$S: \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^{n-1}}} \longrightarrow \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^n}}$$

such that $R \circ S = \text{id}$. The map S is natural in \mathcal{C} . The maps S and F commute.

Proof. We split the map r in [\(7.10\)](#) by a map s that is natural in $A \in \text{obj } \mathcal{W}$ and $\vec{x} \in \text{obj } \mathcal{I}^{[q]}$. As $[q]$ varies, these maps s assemble to a cyclic map that after homotopy colimits and realizations gives rise to S .

For the tuple of finite sets $\vec{x} = (x_0, \dots, x_q) \in \text{obj } \mathcal{I}^{[q]}$ let $|\vec{x}| = x_0 \sqcup \dots \sqcup x_q$ denote the concatenation. There is an orthogonal decomposition of C_{p^n} -representations

$$\mathbb{R}[C_{p^n}] = U \oplus V,$$

where $\mathbb{R}[C_{p^n}]$ denotes the regular C_{p^n} -representation, $V = \mathbb{R}[C_{p^n}]^{C_p}$ its C_p -fixed points and U the orthogonal complement. Since there is an isomorphism of C_{p^n} -sets $|p^n \vec{x}| \cong C_{p^n} \times |\vec{x}|$, the obvious isomorphism

$$cn_{[q]} \mathbb{S}(\vec{x}) = S^{\mathbb{R}[x_0]} \wedge \dots \wedge S^{\mathbb{R}[x_q]} \cong S^{\mathbb{R}[|\vec{x}|]}$$

generalizes to a C_{p^n} -equivariant isomorphism

$$cn_{p^n[q]} \mathbb{S}(p^n \vec{x}) \cong S^{U \otimes \mathbb{R}[|\vec{x}|]} \wedge S^{V \otimes \mathbb{R}[|\vec{x}|]}.$$

Using (6.5) the inclusion of C_p -fixed points

$$\begin{aligned} (cn_{p^n[q]} \mathbb{S}(p^n \vec{x}))^{C_p} &\longrightarrow cn_{p^n[q]} \mathbb{S}(p^n \vec{x}) \\ (A \wedge cn_{p^n[q]} \mathbb{S}[\mathcal{C}](p^n \vec{x}))^{C_p} &\longrightarrow A \wedge cn_{p^n[q]} \mathbb{S}[\mathcal{C}](p^n \vec{x}) \end{aligned}$$

can be identified with the inclusions

$$\begin{aligned} S^{V \otimes \mathbb{R}[|\vec{x}|]} &\longrightarrow S^{U \otimes \mathbb{R}[|\vec{x}|]} \wedge S^{V \otimes \mathbb{R}[|\vec{x}|]} \\ S^{V \otimes \mathbb{R}[|\vec{x}|]} \wedge CN_{p^n q}(\mathcal{C})_+^{C_p} \wedge A &\longrightarrow S^{U \otimes \mathbb{R}[|\vec{x}|]} \wedge S^{V \otimes \mathbb{R}[|\vec{x}|]} \wedge CN_{p^n q}(\mathcal{C})_+ \wedge A. \end{aligned}$$

Now in order to split the upper map in (7.10) we extend each C_{p^n}/C_p -equivariant map

$$S^{V \otimes \mathbb{R}[|\vec{x}|]} \longrightarrow S^{V \otimes \mathbb{R}[|\vec{x}|]} \wedge CN_{p^n q}(\mathcal{C})_+^{C_p} \wedge A$$

to the C_{p^n} -equivariant map

$$S^{U \otimes \mathbb{R}[|\vec{x}|]} \wedge S^{V \otimes \mathbb{R}[|\vec{x}|]} \longrightarrow S^{U \otimes \mathbb{R}[|\vec{x}|]} \wedge S^{V \otimes \mathbb{R}[|\vec{x}|]} \wedge CN_{p^n q}(\mathcal{C})_+ \wedge A$$

obtained by smashing with the identity on $S^{U \otimes \mathbb{R}[|\vec{x}|]}$ and composing with the map induced from the inclusion $CN_{p^n q}(\mathcal{C})^{C_p} \longrightarrow CN_{p^n q}(\mathcal{C})$. Composing with the homeomorphism Δ in (7.10) we obtain the desired map s . \square

Warning 11.2. As opposed to the case of the R -maps explained in Warning 7.13, it is in fact possible to define a map

$$S_{\mathcal{F}} : \mathbf{THH}_{\mathcal{F}}(\mathbb{S}[GfS])^{C_{p^{n-1}}} \longrightarrow \mathbf{THH}_{\mathcal{F}}(\mathbb{S}[GfS])^{C_{p^n}},$$

and we have $S_{\mathcal{F}} \circ F = F \circ S_{\mathcal{F}}$. However, the diagrams

$$\begin{array}{ccc}
 \mathbf{THH}(\mathbb{S}[GfS])^{C_{p^{n-1}}} & \xrightarrow{S} & \mathbf{THH}(\mathbb{S}[GfS])^{C_{p^n}} \\
 \text{pr}_{\mathcal{F}} \downarrow & & \downarrow \text{pr}_{\mathcal{F}} \\
 \mathbf{THH}_{\mathcal{F}}(\mathbb{S}[GfS])^{C_{p^{n-1}}} & \xrightarrow{S_{\mathcal{F}}} & \mathbf{THH}_{\mathcal{F}}(\mathbb{S}[GfS])^{C_{p^n}}
 \end{array}$$

and

$$\begin{array}{ccc}
 \mathbf{THH}(\mathbb{S}[GfS]) & \xrightarrow{F \circ S} & \mathbf{THH}(\mathbb{S}[GfS]) \\
 \text{pr}_{\mathcal{F}} \downarrow & & \downarrow \text{pr}_{\mathcal{F}} \\
 \mathbf{THH}_{\mathcal{F}}(\mathbb{S}[GfS]) & \xrightarrow{F \circ S_{\mathcal{F}}} & \mathbf{THH}_{\mathcal{F}}(\mathbb{S}[GfS])
 \end{array}$$

do not commute without additional assumptions on G and \mathcal{F} , not even in the stable homotopy category. It seems to us that the commutativity of the latter diagram in the case $\mathcal{F} = 1$ is used implicitly in the survey [78, proof of Theorem 4.5.2 on pages 283–284]. The original proof of that theorem [17, Theorem 9.13 on page 535] avoids this problem.

Now consider the following diagram for $n \geq 2$.

$$\begin{array}{ccccc}
 (\text{sh}^{ES^1_+} \mathbf{THH}(\mathbb{S}[\mathcal{C}]))^{C_{p^n}} & & \xrightarrow{\text{pr}_*} & \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^n}} & \xrightarrow{R_n} & \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^{n-1}}} \\
 \downarrow F_n & \swarrow \text{pr}_* & & \downarrow F_n & \swarrow S_n & \downarrow F_{n-1} \\
 & \text{hocofib}(S_n) & \xleftarrow{q_n} & & & \\
 (\text{sh}^{ES^1_+} \mathbf{THH}(\mathbb{S}[\mathcal{C}]))^{C_{p^{n-1}}} & \downarrow f_n & \xrightarrow{\text{pr}_*} & \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^{n-1}}} & \xrightarrow{R_{n-1}} & \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^{n-2}}} \\
 & \downarrow f_n & & \downarrow F_{n-1} & \swarrow S_{n-1} & \\
 & \text{hocofib}(S_{n-1}) & \xleftarrow{q_{n-1}} & & &
 \end{array}$$

Here q_n is the natural map to the homotopy cofiber. By Theorem 7.2, the sequence with pr_* and R_n is a stable homotopy fibration sequence. So Proposition 11.1 implies that the composition $q_n \circ \text{pr}_*$ is a π_* -isomorphism. As n varies, the maps R_n and S_n commute with the inclusions F_n . Therefore there are induced vertical maps f_n and all the squares in the diagram above commute. Now let

$$\mathbf{C}'(\mathbb{S}[\mathcal{C}]; p) = \text{holim}_{n \geq 2} (\cdots \longrightarrow \text{hocofib}(S_n) \xrightarrow{f_n} \text{hocofib}(S_{n-1}) \longrightarrow \cdots).$$

Then the maps q_n induce a map $\mathbf{TF}(\mathbb{S}[\mathcal{C}]; p) \longrightarrow \mathbf{C}'(\mathbb{S}[\mathcal{C}]; p)$ such that the composition

$$\mathbf{C}(\mathbb{S}[\mathcal{C}]; p) \longrightarrow \mathbf{TF}(\mathbb{S}[\mathcal{C}]; p) \longrightarrow \mathbf{C}'(\mathbb{S}[\mathcal{C}]; p) \tag{11.3}$$

is a natural π_* -isomorphism.

Hence we get a zig-zag of natural transformations of functors $\mathbf{Cat} \rightarrow \mathcal{WT}$

$$\mathbf{TC}(\mathbb{S}[\mathcal{C}]; p) \rightarrow \mathbf{TF}(\mathbb{S}[\mathcal{C}]; p) \rightarrow \mathbf{C}'(\mathbb{S}[\mathcal{C}]; p) \xleftarrow{\simeq} \mathbf{C}(\mathbb{S}[\mathcal{C}]; p), \tag{11.4}$$

where the first map comes from (10.1) and the last map is the natural π_* -isomorphism (11.3) above. The natural map induced in the stable homotopy category by the zig-zag (11.4) is denoted α .

The rest of this section is devoted to the following result, which generalizes from groups to arbitrary small categories a fundamental theorem due to [17, Theorem 5.17 on page 500]; see also [98, Theorem 1.16 on page 885] and [30, Section 6.4.2.3 on pages 259–261].

Theorem 11.5. *For any small category \mathcal{C} and any prime p , in the stable homotopy category there is a homotopy cartesian square*

$$\begin{array}{ccc} \mathbf{TC}(\mathbb{S}[\mathcal{C}]; p) & \xrightarrow{\alpha} & \mathbf{C}(\mathbb{S}[\mathcal{C}]; p) \\ \text{pr} \downarrow & & \downarrow \text{pr} \\ \mathbf{THH}(\mathbb{S}[\mathcal{C}]) & \xrightarrow{1-F_1 \circ S_1} & \mathbf{THH}(\mathbb{S}[\mathcal{C}]). \end{array} \tag{11.6}$$

Proof. The proof in [17, Section 5] can be made more precise with our point-set models for all the relevant spectra and maps. In detail, from Proposition 11.1 and Theorem 7.2 we get inductively π_* -isomorphisms

$$\rho_n: (\text{sh}^E \mathbf{T})^{C_{p^n}} \vee (\text{sh}^E \mathbf{T})^{C_{p^{n-1}}} \vee \dots \vee (\text{sh}^E \mathbf{T})^{C_p} \vee \mathbf{THH}(\mathbb{S}[\mathcal{C}]) \rightarrow \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^n}},$$

where we use the abbreviation $\text{sh}^E \mathbf{T} = \text{sh}^{ES^1} \mathbf{THH}(\mathbb{S}[\mathcal{C}])$. Here ρ_n is the sum of the maps $S^{n-m} \circ \text{pr}_* : (\text{sh}^E \mathbf{T})^{C_{p^m}} \rightarrow \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^n}}$ for $1 \leq m \leq n$ and $S^n : \mathbf{THH}(\mathbb{S}[\mathcal{C}]) \rightarrow \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^n}}$.

Moreover, the following diagrams commute.

$$\begin{array}{ccc} (\text{sh}^E \mathbf{T})^{C_{p^n}} \vee (\text{sh}^E \mathbf{T})^{C_{p^{n-1}}} \vee \dots \vee (\text{sh}^E \mathbf{T})^{C_p} \vee \mathbf{THH}(\mathbb{S}[\mathcal{C}]) & \xrightarrow{\rho_n} & \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^n}} \\ \swarrow F_n & \dots & \downarrow F_n \\ (\text{sh}^E \mathbf{T})^{C_{p^{n-1}}} \vee \dots \vee (\text{sh}^E \mathbf{T})^{C_p} \vee \mathbf{THH}(\mathbb{S}[\mathcal{C}]) & \xrightarrow{\rho_{n-1}} & \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^{n-1}}} \end{array}$$

$\begin{array}{ccc} & \downarrow \text{pr}_* \circ F_1 & \downarrow F_1 \circ S_1 \\ & \searrow & \downarrow \end{array}$

$$\begin{array}{ccc} (\text{sh}^E \mathbf{T})^{C_{p^n}} \vee (\text{sh}^E \mathbf{T})^{C_{p^{n-1}}} \vee \dots \vee (\text{sh}^E \mathbf{T})^{C_p} \vee \mathbf{THH}(\mathbb{S}[\mathcal{C}]) & \xrightarrow{\rho_n} & \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^n}} \\ \downarrow \text{id} \swarrow \kappa & \vdots & \downarrow \text{id} \swarrow \kappa \\ (\text{sh}^E \mathbf{T})^{C_{p^{n-1}}} \vee \dots \vee (\text{sh}^E \mathbf{T})^{C_p} \vee \mathbf{THH}(\mathbb{S}[\mathcal{C}]) & \xrightarrow{\rho_{n-1}} & \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^{n-1}}} \end{array}$$

$\begin{array}{ccc} \downarrow \text{id} \swarrow \kappa & \downarrow \text{id} \swarrow \kappa & \downarrow \text{id} \swarrow \kappa \\ \downarrow \text{id} \swarrow \kappa & \downarrow \text{id} \swarrow \kappa & \downarrow \text{id} \swarrow \kappa \end{array}$

The downward vertical map on the left in the second diagram projects the first summand to the base point.

For each $n \geq 1$ consider the following commutative diagram in \mathcal{WT} , with horizontal stable homotopy (co)fibration sequences.

$$\begin{array}{ccccc}
 \mathbf{THH}(\mathbb{S}[\mathcal{C}]) & \xrightarrow{S^{n+1}} & \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^{n+1}}} & \longrightarrow & \text{hocofib}(S^{n+1}) \\
 \downarrow F_1 \circ S_1 & \searrow & \downarrow F_{n+1} & \searrow R_{n+1} & \downarrow \bar{F}_{n+1} \searrow \bar{R}_{n+1} \\
 \mathbf{THH}(\mathbb{S}[\mathcal{C}]) & \xrightarrow{S^n} & \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^n}} & \longrightarrow & \text{hocofib}(S^n) \\
 \downarrow F_1 \circ S_1 & \searrow & \downarrow F_n & \searrow R_n & \downarrow \bar{F}_n \searrow \bar{R}_n \\
 \mathbf{THH}(\mathbb{S}[\mathcal{C}]) & \xrightarrow{S^{n-1}} & \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^{n-1}}} & \longrightarrow & \text{hocofib}(S^{n-1})
 \end{array}$$

Passing to homotopy limits over the R -maps, we get another commutative diagram with horizontal stable homotopy fibration sequences.

$$\begin{array}{ccccc}
 \mathbf{THH}(\mathbb{S}[\mathcal{C}]) & \xrightarrow{S^\infty} & \mathbf{TR}(\mathbb{S}[\mathcal{C}]; p) & \longrightarrow & \text{hocofib}(S^\infty) \\
 F_1 \circ S_1 \downarrow & & F \downarrow & & \bar{F} \downarrow \\
 \mathbf{THH}(\mathbb{S}[\mathcal{C}]) & \xrightarrow{S^\infty} & \mathbf{TR}(\mathbb{S}[\mathcal{C}]; p) & \longrightarrow & \text{hocofib}(S^\infty)
 \end{array}$$

Here

$$\mathbf{TR}(\mathbb{S}[\mathcal{C}]; p) = \text{holim}_{n \in \mathbb{N}} (\dots \longrightarrow \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^n}} \xrightarrow{R_n} \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_{p^{n-1}}} \longrightarrow \dots)$$

is defined analogously to $\mathbf{TF}(\mathbb{S}[\mathcal{C}]; p)$, replacing the Frobenius F -maps with the restriction R -maps; compare (9.3) and [51, (20) on page 56]. Taking homotopy fixed points for the vertical self-maps, or equivalently their homotopy equalizers with the identity maps, gives the stable homotopy fibration sequence

$$\text{hoeq}(\text{id}, F_1 \circ S_1) \xrightarrow{S^\infty} \mathbf{TC}(\mathbb{S}[\mathcal{C}]; p) \xrightarrow{\alpha} \text{hoeq}(\text{id}, \bar{F}).$$

Here we use the well known equivalent ways to compute homotopy limits over \mathcal{RF} ; e.g., see [30, Section 6.4.2.1 on pages 255–256].

The π_* -isomorphisms ρ_n induce compatible π_* -isomorphisms

$$\bar{\rho}_n : (\text{sh}^E \mathbf{T})^{C_{p^n}} \vee \dots \vee (\text{sh}^E \mathbf{T})^{C_p} \longrightarrow \text{hocofib}(S^n)$$

with homotopy limit

$$\bar{\rho}_\infty : \text{holim}_{n \in \mathbb{N}} ((\text{sh}^E \mathbf{T})^{C_{p^n}} \vee \dots \vee (\text{sh}^E \mathbf{T})^{C_p}) \longrightarrow \text{hocofib}(S^\infty),$$

and the inclusion of wedge sums into products induces a π_* -isomorphism

$$\operatorname{holim}_{n \in \mathbb{N}} ((\operatorname{sh}^E \mathbf{T})^{C_{p^n}} \vee \dots \vee (\operatorname{sh}^E \mathbf{T})^{C_p}) \longrightarrow \prod_{n=1}^{\infty} (\operatorname{sh}^E \mathbf{T})^{C_{p^n}},$$

such that \overline{F} corresponds to the product of the $F_n: (\operatorname{sh}^E \mathbf{T})^{C_{p^n}} \rightarrow (\operatorname{sh}^E \mathbf{T})^{C_{p^{n-1}}}$. Hence we have a natural zig-zag of π_* -isomorphisms between the target $\operatorname{hoeq}(\operatorname{id}, \overline{F})$ of α and

$$\operatorname{hoeq}(\operatorname{id}, \prod_n F_n) = \operatorname{holim}_{n \geq 1} (\operatorname{sh}^E \mathbf{T})^{C_{p^n}} \simeq \mathbf{C}(\mathbb{S}[\mathcal{C}]; p),$$

by the usual model for a sequential homotopy limit.

So far, this argument follows [17, Section 5], and establishes the left-hand vertical homotopy fiber sequence of (5.19) on page 502 in that paper. To construct the homotopy cartesian square (11.6), an additional argument is needed, which we now describe.

Restricting the homotopy limit defining $\mathbf{TC}(\mathbb{S}[\mathcal{C}]; p)$ to the full subcategory of \mathcal{RF} generated by the objects $\{0, 1\} \subset \mathbb{N}$, we get the middle vertical map in the following commutative diagram in \mathcal{WT} .

$$\begin{array}{ccccc} \operatorname{hoeq}(\operatorname{id}, F_1 \circ S_1) & \xrightarrow{S^\infty} & \mathbf{TC}(\mathbb{S}[\mathcal{C}]; p) & \xrightarrow{\alpha} & \operatorname{hoeq}(\operatorname{id}, \overline{F}) \simeq \mathbf{C}(\mathbb{S}[\mathcal{C}]; p) \\ \parallel & & \operatorname{res} \downarrow & & \downarrow \\ \operatorname{hoeq}(\operatorname{id}, F_1 \circ S_1) & \xrightarrow{S_1} & \operatorname{hoeq}(R_1, F_1) & \xrightarrow{q_1} & \operatorname{hocofib}(S_1) \simeq (\operatorname{sh}^E \mathbf{T})^{C_p} \end{array} \tag{11.7}$$

The right-hand vertical map factors in the canonical way through $\operatorname{hocofib}(S^\infty)$. The left-hand lower map is induced by $S_1: \mathbf{THH}(\mathbb{S}[\mathcal{C}]) \rightarrow \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_p}$ and the identity on $\mathbf{THH}(\mathbb{S}[\mathcal{C}])$. The right-hand lower map is induced by $q_1: \mathbf{THH}(\mathbb{S}[\mathcal{C}])^{C_p} \rightarrow \operatorname{hocofib}(S_1)$ and $\mathbf{THH}(\mathbb{S}[\mathcal{C}]) \rightarrow \operatorname{pt}$. Hence the lower row is a stable homotopy fibration sequence, and the right-hand square is homotopy cartesian in \mathcal{WT} .

In the stable homotopy category, $\operatorname{hoeq}(R_1, F_1)$ is isomorphic to $\operatorname{hofib}(F_1 - R_1)$, and $F_1 - R_1$ corresponds under ρ_1 and ρ_0 to the difference of $\operatorname{pr}_* \circ F_1 + F_1 \circ S_1$ and $0 + \operatorname{id}$, which equals the difference of $\operatorname{pr}_* \circ F_1 + 0$ and $0 + (\operatorname{id} - F_1 \circ S_1)$, as maps $(\operatorname{sh}^E \mathbf{T})^{C_p} \vee \mathbf{THH}(\mathbb{S}[\mathcal{C}]) \rightarrow \mathbf{THH}(\mathbb{S}[\mathcal{C}])$. Hence $\operatorname{hofib}(F_1 - R_1)$ is also isomorphic to the homotopy pullback of these two maps, and we have a homotopy cartesian square

$$\begin{array}{ccc} \operatorname{hoeq}(R_1, F_1) & \xrightarrow{q_1} & (\operatorname{sh}^E \mathbf{T})^{C_p} \\ R_1 \downarrow & & \downarrow \operatorname{pr}_* \circ F_1 \\ \mathbf{THH}(\mathbb{S}[\mathcal{C}]) & \xrightarrow{\operatorname{id} - F_1 \circ S_1} & \mathbf{THH}(\mathbb{S}[\mathcal{C}]) \end{array} \tag{11.8}$$

in the stable homotopy category.

Now combine the homotopy cartesian squares in (11.7) and (11.8). The composition $R_1 \circ \operatorname{res}$ is the canonical projection $\operatorname{pr}: \mathbf{TC}(\mathbb{S}[\mathcal{C}]; p) \rightarrow \mathbf{THH}(\mathbb{S}[\mathcal{C}])$, and the composition $\operatorname{hoeq}(\operatorname{id}, \overline{F}) \rightarrow \operatorname{hocofib}(S_1) \simeq (\operatorname{sh}^E \mathbf{T})^{C_p} \rightarrow \mathbf{THH}(\mathbb{S}[\mathcal{C}])$ corresponds to the canonical

projection $\text{pr}: \mathbf{C}(\mathbb{S}[\mathcal{C}]; p) \longrightarrow \mathbf{THH}(\mathbb{S}[\mathcal{C}])$. So we obtain the asserted homotopy cartesian square (11.6). \square

The following easy consequence of [Theorem 11.5](#) is very important for the proof of the Detection [Theorem 3.6](#).

Corollary 11.9. *If G is a finite group, then the map*

$$(\text{pr}, \alpha): \mathbf{TC}(\mathbb{S}[G]; p) \longrightarrow \mathbf{THH}(\mathbb{S}[G]) \times \mathbf{C}(\mathbb{S}[G]; p) \tag{11.10}$$

is $\pi_0^{\mathbb{Q}}$ -injective and a $\pi_n^{\mathbb{Q}}$ -isomorphism for all $n \geq 1$.

Proof. Apply [Theorem 11.5](#) to $\mathcal{C} = G$ and consider the Mayer–Vietoris sequence coming from the homotopy cartesian square (11.6). From [Example 6.16](#) we know that $\pi_n(\mathbf{THH}(\mathbb{S}[G])) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for all $n \geq 1$ if G is finite, and the claim follows. \square

Remark 11.11. In fact, it can be shown that the map (11.10) is not $\pi_{-1}^{\mathbb{Q}}$ -injective.

12. Equivariant homology theories and Chern characters

In this section we review the notion of an equivariant homology theory and a very general way of producing examples; see [Theorem 12.2](#). Next we recall under which conditions equivariant Chern characters can be used to compute rationalized equivariant homology theories; see [Theorem 12.7](#) and [Addendum 12.13](#). These results are key to the proof of the Detection [Theorem 3.6](#) in [Section 16](#).

Given a group G , a G -homology theory \mathcal{H}_*^G is a collection of functors \mathcal{H}_n^G , one for every $n \in \mathbb{Z}$, from the category of G -CW-complex pairs to abelian groups, satisfying the obvious G -equivariant analogues of the familiar axioms in the non-equivariant setting; see [[68, Section 1, pages 197–198](#)]. An *equivariant homology theory* $\mathcal{H}_*^?$ is a collection of G -homology theories \mathcal{H}_*^G , one for every group G , together with induction isomorphisms

$$\text{ind}_{\alpha}: \mathcal{H}_*^H(X, A) \xrightarrow{\cong} \mathcal{H}_*^G(\text{ind}_{\alpha} X, \text{ind}_{\alpha} A) \tag{12.1}$$

for every group homomorphism $\alpha: H \longrightarrow G$ and every H -CW-complex pair (X, A) on which $\ker \alpha$ acts freely. The induction isomorphisms are required to be functorial in α and to be compatible with conjugation and boundary homomorphisms; see [[68, Section 1, pages 198–199](#)].

The following result provides a framework for constructing equivariant homology theories. Recall the action groupoid functor $Gf-: \mathbf{Sets}^G \longrightarrow \mathbf{Groupoids}$ from [Subsection 4C](#). Recall also that, given a functor $\mathbf{E}: \text{Or}G \longrightarrow \mathbf{NSp}$, its continuous left Kan extension $\mathbf{E}_{\%}: \mathbf{Top}^G \longrightarrow \mathbf{NSp}$ along the inclusion $\text{Or}G \hookrightarrow \mathbf{Top}^G$ is defined as follows. For any G -space X let $\mathbf{E}_{\%}(X) = X_+ \wedge_{\text{Or}G} \mathbf{E}$ be the coend of the functor

$$(\text{Or}G)^{\text{op}} \times \text{Or}G \longrightarrow \text{NSp}, \quad (G/H, G/K) \longmapsto X_+^H \wedge \mathbf{E}(G/K);$$

compare Subsection 4E.

Theorem 12.2. *Let $\mathbf{E}: \text{Groupoids} \rightarrow \text{NSp}$ be a functor. Assume that \mathbf{E} preserves equivalences, in the sense that \mathbf{E} sends equivalences of groupoids to π_* -isomorphisms. For any group G , consider the composition*

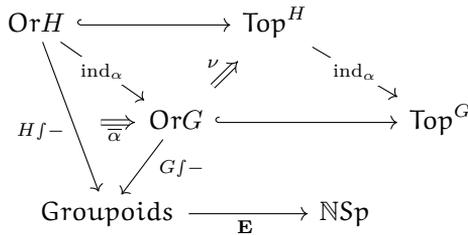
$$\text{Or}G \xrightarrow{Gf-} \text{Groupoids} \xrightarrow{\mathbf{E}} \text{NSp},$$

and for any G -CW-pair (X, A) define

$$H_*^G(X, A; \mathbf{E}) = \pi_* (\mathbf{E}(Gf-)\% (X/A)) = \pi_* \left(X/A_+ \wedge_{\text{Or}G} \mathbf{E}(Gf-) \right).$$

Then $H_*^?(-; \mathbf{E})$ is an equivariant homology theory.

Proof. This is proved in [71, Proposition 157 on page 796]. We explain here in a more conceptual way how the induction homomorphisms (12.1) are defined. Let $\alpha: H \rightarrow G$ be a group homomorphism. Define a functor $\text{ind}_\alpha: \text{Or}H \rightarrow \text{Or}G$ by $H/K \mapsto G/\alpha(K)$. Consider the following diagram.



Here $\bar{\alpha}$ is the natural transformation defined in (14.9), and ν is the natural isomorphisms $G \times_H H/K \cong G/\alpha(K)$. Then we get the following natural transformations of functors $\text{Top}^H \rightarrow \text{NSp}$

$$\mathbf{E}(Hf-)\% \longrightarrow \mathbf{E}(Gf\text{ind}_\alpha-)\% \longrightarrow \mathbf{E}(Gf-)\% \circ \text{ind}_\alpha .$$

The first natural transformation is induced by $\bar{\alpha}$, the second one by the version of Fact 13.5 for topological categories. Passing to homotopy groups, the composition of these natural transformations defines the induction homomorphisms (12.1). \square

Next we want to recall some of the main results of [68] about equivariant Chern characters. We first introduce some more notation.

Let $\mathcal{H}_*^?$ be an equivariant homology theory. Given a group G , consider the category $\text{Sub}G(\mathcal{F}in)$ defined right after Theorem 1.1. Then for any $n \in \mathbb{Z}$ we get a functor

$$\mathcal{H}_n^G(G/-): \text{Sub}G(\mathcal{F}in) \longrightarrow \text{Ab}, \quad H \longmapsto \mathcal{H}_n^G(G/H); \tag{12.3}$$

compare [68, paragraph after Lemma 3.7 on page 207]. If $\mathcal{H}_*^? = H_*^?(-; \mathbf{E})$ for some functor $\mathbf{E}: \text{Groupoids} \rightarrow \text{NSp}$ as in Theorem 12.2, then this is also proved in [74, Lemma 3.11 on page 152]. Notice that in this case we have $H_*^G(G/H; \mathbf{E}) \cong \pi_* \mathbf{E}(H)$, where we identify any group H with the groupoid H/pt .

We denote by FinGroupsInj the category of finite groups and injective group homomorphisms. Given any equivariant homology theory $\mathcal{H}_*^?$ and any $n \in \mathbb{Z}$ we get a functor

$$M_* : \text{FinGroupsInj} \rightarrow \text{Ab}, \quad G \mapsto \mathcal{H}_n^G(\text{pt}), \tag{12.4}$$

where for every injective group homomorphism $\alpha: H \rightarrow G$ we define $M(\alpha)$ as the composition

$$M_*(H) = \mathcal{H}_n^H(\text{pt}) \xrightarrow{\text{ind}_\alpha} \mathcal{H}_n^G(G/\alpha(H)) \xrightarrow{\mathcal{H}_n^G(\text{pr})} \mathcal{H}_n^G(\text{pt}) = M_*(G). \tag{12.5}$$

Notice that, if $\mathcal{H}_*^? = H_*^?(-; \mathbf{E})$ for some functor $\mathbf{E}: \text{Groupoids} \rightarrow \text{NSp}$ as in Theorem 12.2, then the functor M_* coincides with the composition

$$\text{FinGroupsInj} \hookrightarrow \text{Groupoids} \xrightarrow{\mathbf{E}} \text{NSp} \xrightarrow{\pi_n} \text{Ab}, \quad G \mapsto \pi_n \mathbf{E}(G). \tag{12.6}$$

A Mackey functor consists of a pair of functors

$$M_* : \text{FinGroupsInj} \rightarrow \text{Ab} \quad \text{and} \quad M^* : (\text{FinGroupsInj})^{\text{op}} \rightarrow \text{Ab}$$

that agree on objects, i.e., $M_*(G) = M^*(G)$ for every finite group G , and that satisfy the following axioms.

- (i) If $\alpha: H \rightarrow G$ is an isomorphism, then $M^*(\alpha)$ is the inverse of $M_*(\alpha)$.
- (ii) If $\alpha = \text{conj}_g: G \rightarrow G, x \mapsto gxg^{-1}$, for some $g \in G$, then

$$M_*(\text{conj}_g) = \text{id}_{M(G)}.$$

- (iii) (Double coset formula). If H and K are subgroups of G , let $\mathcal{J} = K \backslash G / H$. For every $j \in \mathcal{J}$, choose a representative $g_j \in j = Kg_jH$, and define $L_j = H \cap g_j^{-1}Kg_j$ and $c_j = \text{conj}_{g_j}: L_j \rightarrow K$. Then

$$M^*(K \hookrightarrow G) \circ M_*(H \hookrightarrow G) = \sum_{j \in \mathcal{J}} M_*(c_j) \circ M^*(L_j \hookrightarrow H).$$

We say that an equivariant homology theory $\mathcal{H}_*^?$ has a Mackey structure if, for every $n \in \mathbb{Z}$, the functor defined in (12.4) and (12.5) extends to a Mackey functor.

A fundamental example of a Mackey functor is the Burnside ring A ; see for example [27, (2.18) on pages 18–19]. For every finite group G , $A(G)$ is the Grothendieck ring

of the set of isomorphism classes of finite G -sets, with addition given by the disjoint union and multiplication given by the cartesian product. As an abelian group, $A(G)$ is the free abelian group with basis given by the isomorphism classes of the orbits G/H . There is an injective ring homomorphism

$$\varphi_G: A(G) \longrightarrow \prod_{(H)} \mathbb{Z},$$

where the product is indexed over the conjugacy classes of subgroups of G . The composition of φ_G with the projection to the factor indexed by the conjugacy class of a subgroup H maps S to $\#(S^H)$, the number of elements in the H -fixed point set. Moreover, $\varphi_G \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism. Now let $e_G \in \prod_{(H)} \mathbb{Z}$ be the idempotent that corresponds to the factor indexed by the conjugacy class (G) . We then define the idempotent

$$\Theta_G = \left(\varphi_G \otimes_{\mathbb{Z}} \mathbb{Q}\right)^{-1} (e_G \otimes 1) \in A(G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

in the rationalized Burnside ring of G .

The Burnside ring A has a natural action on any other Mackey functor M , in the precise sense of [68, Section 7, pages 222–223]. In particular, the idempotent $\Theta_G \in A(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ acts on $M(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ for every G .

We are now ready to formulate the following result.

Theorem 12.7. *Keep the notation and assumption of Theorem 12.2. Assume that the equivariant homology theory $H_*^?(-; \mathbf{E})$ has a Mackey structure, i.e., for every $n \in \mathbb{Z}$ the composition (12.6) extends to a Mackey functor. Then for every proper G -CW-complex X there are isomorphisms*

$$\begin{array}{ccc} \bigoplus_{s+t=n} \bigoplus_{(H) \in (\mathcal{F}in)} H_s(X^H/Z_G H; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G H]} S_t(H; \mathbf{E}) & & \\ \downarrow h & & \\ \bigoplus_{s+t=n} H_s(X^-/Z_G -; \mathbb{Q}) \otimes_{\mathbb{Q}[\text{Sub}G(\mathcal{F}in)]} \left(\pi_t \mathbf{E}(-) \otimes_{\mathbb{Z}} \mathbb{Q}\right) & & (12.8) \\ \downarrow c & & \\ H_n^G(X; \mathbf{E}) \otimes_{\mathbb{Z}} \mathbb{Q}. & & \end{array}$$

Their composition $ch = c \circ h$ is called Chern character. Both h and c are natural in X , and h respects the outer sum decompositions. Here, for every finite subgroup H of G ,

$$S_t(H; \mathbf{E}) = \text{coker} \left(\bigoplus_{K \lesssim H} \pi_t \mathbf{E}(K) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \pi_t \mathbf{E}(H) \otimes_{\mathbb{Z}} \mathbb{Q} \right). \tag{12.9}$$

Moreover, for every finite cyclic subgroup C of G there is a natural isomorphism

$$S_t(C; \mathbf{E}) \cong \Theta_C \left(\pi_t \mathbf{E}(C) \otimes_{\mathbb{Z}} \mathbb{Q} \right). \tag{12.10}$$

Proof. The existence of a Mackey structure implies that $\pi_t \mathbf{E}(-) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a projective and hence flat $\mathbb{Q}[\text{Sub}G(\mathcal{F}in)]$ -module by [68, Theorem 5.2 on page 216]. The same theorem together with [68, (2.12) on page 205] yields the isomorphism h . We remark that the definition of h depends on choices [68, (2.10) on page 204], and so h is a priori not natural in \mathbf{E} .

The map c is constructed in [68, Section 4, in particular Lemma 4.3(b) on page 211]. It is defined for any equivariant homology theory, and its definition only depends on the induction structure (no choices). The flatness condition above allows to interchange homology and tensor product in the source of c . Then [68, Theorem 4.4 on page 213], together with the isomorphism at the beginning of that proof, implies that c is an isomorphism.

Finally, the isomorphism (12.10) is proved in [72, Lemma 7.4 on page 622]. \square

Example 12.11. If $X = EG(\mathcal{F})$ for a family \mathcal{F} of finite subgroups of G , then the maps

$$EG(\mathcal{F})^H / Z_G H \longleftarrow EZ_G H \times_{Z_G H} EG(\mathcal{F})^H \longrightarrow EZ_G H / Z_G H$$

induce an isomorphism

$$H_s(EG(\mathcal{F})^H / Z_G H; \mathbb{Q}) \cong H_s(BZ_G H; \mathbb{Q});$$

compare [74, proof of Lemma 8.1 on page 164]. Under the Chern character isomorphisms (12.8), an inclusion $\mathcal{F} \subseteq \mathcal{F}'$ of families of finite subgroups induces the inclusion of the summands indexed by $(H) \in \mathcal{F}$, which in particular is always injective.

Example 12.12. If R is an arbitrary discrete ring and $\mathbf{E} = \mathbf{K}^{\geq 0}(R[Gf-])$, then $S_t(H; \mathbf{K}^{\geq 0}(R[Gf-])) = 0$ for any $t \geq 0$ and any non-cyclic finite subgroup H of G by [72, Lemma 7.2 on page 621]. In particular, the inclusion $\mathcal{F}Cyc \subseteq \mathcal{F}in$ induces an isomorphism

$$H_n^G(EG(\mathcal{F}Cyc); \mathbf{K}^{\geq 0}(R[Gf-])) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} H_n^G(EG(\mathcal{F}in); \mathbf{K}^{\geq 0}(R[Gf-])) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Next we investigate the naturality of Theorem 12.7 in the functor \mathbf{E} . Because of the subtleties explained in Remark 15.4, we also need to consider natural transformations that do not necessarily respect the Mackey structures.

Addendum 12.13. Keep the notation and assumption of Theorem 12.2. Consider another equivalence-preserving functor $\mathbf{E}' : \text{Groupoids} \rightarrow \text{NSp}$ and a natural transformation $\tau : \mathbf{E} \rightarrow \mathbf{E}'$. Assume that both $H_*^?(-; \mathbf{E})$ and $H_*^?(-; \mathbf{E}')$ have Mackey structures.

(i) The induced homomorphism

$$H_n^G(X; \tau) \otimes_{\mathbb{Z}} \mathbb{Q} : H_n^G(X; \mathbf{E}) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H_n^G(X; \mathbf{E}') \otimes_{\mathbb{Z}} \mathbb{Q}$$

is compatible with the isomorphisms c in (12.8), even when τ does not respect the Mackey structures.

- (ii) If τ induces a natural transformation of Mackey functors, then there are choices of the maps h in (12.8) for which $H_n^G(X; \tau) \otimes_{\mathbb{Z}} \mathbb{Q}$ is compatible with the isomorphisms $ch = c \circ h$ for any \mathcal{FCyc} - G - CW -complex X .
- (iii) If τ induces a natural transformation of Mackey functors, and the homomorphism $\pi_t \mathbf{E}(C) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \pi_t \mathbf{E}'(C) \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective for all finite cyclic subgroups C of G and all $t \in \mathbb{Z}$, then $H_n^G(X; \tau) \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective for any \mathcal{FCyc} - G - CW -complex X .

Proof. First of all, notice that τ induces a natural transformation of equivariant homology theories $H^G(-; \mathbf{E}) \longrightarrow H^G(-; \mathbf{E}')$ which is compatible with the induction structures. Therefore it induces a natural transformation between the functors $\text{Sub}G(\mathcal{Fin}) \longrightarrow \text{Ab}$ defined in (12.3). Since the definition of the map h only depends on the induction structure, (i) follows.

Furthermore, since also definition (12.9) only depends on the induction structure, τ always induces a map $S_t(H; \mathbf{E}) \longrightarrow S_t(H; \mathbf{E}')$. Now assume that X is a \mathcal{FCyc} - G - CW -complex X . Then in the source of h we only have to consider finite cyclic subgroups $H = C$, and in this case we have the isomorphism (12.10).

If we additionally assume that τ induces a natural transformation of Mackey functors, then we get a commutative diagram

$$\begin{array}{ccccc}
 \Theta_C \left(\pi_t \mathbf{E}(C) \otimes_{\mathbb{Z}} \mathbb{Q} \right) & \hookrightarrow & \pi_t \mathbf{E}(C) \otimes_{\mathbb{Z}} \mathbb{Q} & \twoheadrightarrow & S_t(C, \mathbf{E}) \\
 \downarrow & & \downarrow & & \downarrow \\
 \Theta_C \left(\pi_t \mathbf{E}'(C) \otimes_{\mathbb{Z}} \mathbb{Q} \right) & \hookrightarrow & \pi_t \mathbf{E}'(C) \otimes_{\mathbb{Z}} \mathbb{Q} & \twoheadrightarrow & S_t(C, \mathbf{E}')
 \end{array} \tag{12.14}$$

for any finite cyclic subgroup C . The composition of the horizontal maps is an isomorphism. The inverse of this isomorphism, followed by the inclusion of the image of Θ_C , then gives compatible choices of the sections needed in [68, (2.10) on page 204] to define the map h . This proves (ii).

Diagram (12.14) also shows that the two vertical maps on the outside are injective provided that the one in the middle is injective. Then (iii) follows because the Weyl group $W_G H$ is finite for every finite group H , hence $\mathbb{Q}[W_G H]$ is semisimple, and the functor $M \otimes_{\mathbb{Q}[W_G H]} -$ preserves injectivity for every M . \square

13. Enriched categories of modules over a category

The goal of this technical section is to show that the formalism of modules over a category (for example, see [66, Section 9, page 162]) extends to the enriched setting. For our main applications in the following sections, it would be enough to consider spectral categories only. But since we actually need two different models of spectral categories (with enrichments in either $\Sigma\mathcal{S}p$ or $\Gamma\mathcal{S}_*$), and since the proofs are not any easier in these special cases, we proceed in complete generality. In the case of pre-additive categories, the constructions in this section can be simplified, because products and coproducts are the same. Since this is not the case in categories of spectra, we resort to machinery from enriched category theory, which we now recall. The standard modern reference for this material (and much more) is [59].

Fix a closed symmetric monoidal category $(\mathcal{V}, \odot, \mathbf{1})$; compare Subsection 4D. We always assume that \mathcal{V} is both complete and cocomplete, as in all the examples in 4D. We denote by $\mathcal{V}\text{-Cat}$ the 2-category of small \mathcal{V} -categories, \mathcal{V} -functors, and \mathcal{V} -natural transformations. To distinguish between \mathcal{V} -categories and **Sets**-categories, we call the latter ordinary categories and denote them $\mathcal{C}, \mathcal{D}, \dots$, whereas we use the symbols $\mathcal{X}, \mathcal{Y}, \dots$ to denote \mathcal{V} -categories.

Let

$$V = \mathcal{V}(\mathbf{1}, -): \mathcal{V} \longrightarrow \mathbf{Sets}$$

be the underlying set functor. Notice that in some of our examples ($\mathbf{Cat}, \Sigma\mathcal{S}p, \Gamma\mathcal{S}_*$) the functor V is neither faithful nor conservative. Since we are assuming that \mathcal{V} is cocomplete, V has a left adjoint

$$\mathbf{1}[-]: \mathbf{Sets} \longrightarrow \mathcal{V}, \quad S \longmapsto \mathbf{1}[S] = \coprod_S \mathbf{1}.$$

Both functors $\mathbf{1}$ and V are lax monoidal, and base change along them (see Subsection 4D) induces a 2-adjunction

$$\mathcal{V}\text{-Cat} \begin{array}{c} \xrightarrow{V} \\ \xleftarrow{\mathbf{1}[-]} \end{array} \mathbf{Cat};$$

compare [59, Section 2.5 on page 35].

We make extensive use of the \mathcal{V} -categories of functors $\mathcal{V}\text{-fun}(\mathbf{1}[\mathcal{C}], \mathcal{X})$, where \mathcal{C} is an ordinary small category, and \mathcal{X} is a \mathcal{V} -category. To simplify the notation we simply write $\mathcal{V}\text{-fun}(\mathcal{C}, \mathcal{X})$ in this case, and this \mathcal{V} -category can be explicitly described as follows. The objects are precisely the functors from \mathcal{C} to the underlying category $V\mathcal{X}$, i.e., $\text{obj } \mathcal{V}\text{-fun}(\mathcal{C}, \mathcal{X}) = \text{obj } \text{fun}(\mathcal{C}, V\mathcal{X})$. Given two such functors X and Y , the \mathcal{V} -object of morphisms from X to Y is defined as the end in $V\mathcal{V}$ of the functor

$$\mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{X^{\text{op}} \times Y} V\mathcal{X}^{\text{op}} \times V\mathcal{X} \xrightarrow{\mathcal{X}(-, -)} V\mathcal{V}.$$

Notice that

$$V(\mathcal{V}\text{-fun}(\mathcal{C}, \mathcal{X})) \cong \text{fun}(\mathcal{C}, V\mathcal{X}). \tag{13.1}$$

Given a functor $\alpha: \mathcal{C} \rightarrow \mathcal{D}$, the functor $\text{res}_\alpha: \text{fun}(\mathcal{D}, V\mathcal{X}) \rightarrow \text{fun}(\mathcal{C}, V\mathcal{X})$ given by restriction along α is clearly a \mathcal{V} -functor

$$\text{res}_\alpha: \mathcal{V}\text{-fun}(\mathcal{D}, \mathcal{X}) \rightarrow \mathcal{V}\text{-fun}(\mathcal{C}, \mathcal{X}). \tag{13.2}$$

Now assume that \mathcal{X} is \mathcal{V} -complete and \mathcal{V} -cocomplete, which is equivalent to assuming that \mathcal{X} is \mathcal{V} -tensorable and \mathcal{V} -cotensorable and that the underlying category $V\mathcal{X}$ is complete and cocomplete; see [59, Theorem 3.73 on page 54]. Then the assumptions on \mathcal{X} imply that the left Kan extension along α

$$\text{Lan}_\alpha: \text{fun}(\mathcal{C}, V\mathcal{X}) \rightarrow \text{fun}(\mathcal{D}, V\mathcal{X}), \quad X \mapsto \left(d \mapsto \mathcal{D}(\alpha(-), d) \otimes_{\mathcal{C}} X(-) \right)$$

exists and agrees with the \mathcal{V} -enriched left Kan extension, and therefore defines a \mathcal{V} -functor

$$\text{Lan}_\alpha: \mathcal{V}\text{-fun}(\mathcal{C}, \mathcal{X}) \rightarrow \mathcal{V}\text{-fun}(\mathcal{D}, \mathcal{X})$$

which is left 2-adjoint to the \mathcal{V} -functor res_α in (13.2); compare [59, second paragraph on page 50].

Next, we recall some basic facts about Kan extensions that are needed later. The first two are well-known, and the third is also used in [95, proof of Proposition 8.3 on page 1528].

Fact 13.3. Given functors $\alpha: \mathcal{B} \rightarrow \mathcal{C}$ and $\beta: \mathcal{C} \rightarrow \mathcal{D}$, then there is a natural isomorphism $\text{Lan}_{\beta \circ \alpha} \cong \text{Lan}_\beta \circ \text{Lan}_\alpha$.

Fact 13.4. If α is an equivalence of categories, then there are natural isomorphisms $\text{res}_\alpha \text{Lan}_\alpha \cong \text{id}$ and $\text{Lan}_\alpha \text{res}_\alpha \cong \text{id}$.

Fact 13.5. Consider a diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\alpha} & \mathcal{B} \\ \gamma \downarrow & \nearrow \nu & \downarrow \beta \\ \mathcal{C} & \xrightarrow{\delta} & \mathcal{D} \\ X \downarrow & & \\ V\mathcal{X} & & \end{array}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and \mathcal{D} are ordinary small categories, and let ν be a natural transformation from $\delta\gamma$ to $\beta\alpha$. Then ν induces a natural transformation

$$\nu_* : \text{Lan}_\alpha \text{res}_\gamma X \longrightarrow \text{res}_\beta \text{Lan}_\delta X$$

as we proceed to explain. For a functor $\delta : \mathcal{C} \longrightarrow \mathcal{D}$ define

$$\begin{aligned} \mathcal{D}_\delta : \mathcal{C}^{\text{op}} \times \mathcal{D} &\longrightarrow \text{Sets}, & (c, d) &\mapsto \mathcal{D}(\delta(c), d), \\ \delta\mathcal{D} : \mathcal{D}^{\text{op}} \times \mathcal{C} &\longrightarrow \text{Sets}, & (d, c) &\mapsto \mathcal{D}(d, \delta(c)). \end{aligned}$$

Notice that by Yoneda Lemma, for any functor $Y : \mathcal{D} \longrightarrow V\mathcal{X}$, we have $\text{res}_\delta Y \cong \delta\mathcal{D} \otimes_{\mathcal{D}} Y$. Recall that $\text{Lan}_\delta X = \mathcal{D}_\delta \otimes_{\mathcal{C}} X$.

Therefore we have

$$\begin{aligned} \text{Lan}_\alpha \text{res}_\gamma X &\cong \mathcal{B}_\alpha \otimes_{\mathcal{A}} (\gamma\mathcal{C} \otimes_{\mathcal{C}} X) \cong (\mathcal{B}_\alpha \otimes_{\mathcal{A}} \gamma\mathcal{C}) \otimes_{\mathcal{C}} X, & \text{and} \\ \text{res}_\beta \text{Lan}_\delta X &\cong \beta\mathcal{D} \otimes_{\mathcal{D}} (\mathcal{D}_\delta \otimes_{\mathcal{C}} X) \cong (\beta\mathcal{D} \otimes_{\mathcal{D}} \mathcal{D}_\delta) \otimes_{\mathcal{C}} X \cong \beta\mathcal{D}_\delta \otimes_{\mathcal{C}} X. \end{aligned}$$

Now let $\bar{\nu}$ be the composition

$$\bar{\nu} : \mathcal{B}_\alpha \otimes_{\mathcal{A}} \gamma\mathcal{C} \longrightarrow \beta\mathcal{D}_{\beta\alpha} \otimes_{\mathcal{A}} \beta\alpha\mathcal{D}_\delta \longrightarrow \beta\mathcal{D} \otimes_{\mathcal{D}} \mathcal{D}_\delta \cong \beta\mathcal{D}_\delta$$

where the first map is induced by the functors β and δ together with the natural transformation ν . Then define ν_* by applying $- \otimes_{\mathcal{C}} X$ to $\bar{\nu}$ and using the isomorphisms above. Furthermore:

- (i) If $\bar{\nu}$ is a natural isomorphism of functors $\mathcal{C}^{\text{op}} \times \mathcal{B} \longrightarrow \text{Sets}$, then ν_* is a natural isomorphism of functors $\mathcal{B} \longrightarrow V\mathcal{X}$, natural in X .
- (ii) If α and δ are equivalences of categories and ν is a natural isomorphism, then ν_* is an isomorphism.

Proof. (i) follows directly from the construction. (ii) follows from (i) by observing that the assumptions imply that $\bar{\nu}$ is an isomorphism. Notice also that [Fact 13.4](#) follows from (ii). \square

Now fix also a monoid \mathbb{A} in \mathcal{V} , or in other words, a \mathcal{V} -category with exactly one object. Let $\mathbb{A}\text{-Mod}$ be the \mathcal{V} -category of modules over \mathbb{A} , and notice that $\mathbb{A}\text{-Mod} \cong \mathcal{V}\text{-fun}(\mathbb{A}, \mathcal{V})$. Notice that $\mathbb{A}\text{-Mod}$ is \mathcal{V} -complete and \mathcal{V} -cocomplete; see for example [\[59, Proposition 3.75 on page 54\]](#). Therefore the discussion above about left Kan extensions applies to functors with values in $\mathbb{A}\text{-Mod}$.

Definition 13.6 (*\mathbb{A} -modules over \mathcal{C}*). Given an ordinary small category \mathcal{C} , define

$$\mathcal{M}_{\mathbb{A}}(\mathcal{C}) = \mathcal{V}\text{-fun}(\mathcal{C}, \mathbb{A}\text{-Mod})$$

to be the \mathcal{V} -category of \mathcal{V} -functors from \mathcal{C} to the \mathcal{V} -category of \mathbb{A} -modules. The objects of this category are called *\mathbb{A} -modules over \mathcal{C}* .

Next, we want to define what it means for an \mathbb{A} -module over \mathcal{C} to be free and finitely generated. Notice that for the underlying category of $\mathcal{M}_{\mathbb{A}}(\mathcal{C})$ we have

$$V\mathcal{M}_{\mathbb{A}}(\mathcal{C}) \cong \text{fun}(\mathcal{C}, V(\mathbb{A}\text{-Mod}))$$

by (13.1). There are two pairs of adjoint functors

$$V\mathbb{A}\text{-Mod} \begin{matrix} \xleftarrow{u} \\ \mathbb{A}\odot- \\ \xrightarrow{V} \end{matrix} V\mathcal{V} \begin{matrix} \xleftarrow{V} \\ \mathbb{1}[-] \\ \xrightarrow{\text{Sets}} \end{matrix}$$

where the left adjoints are displayed at the bottom, and they induce adjoint functors between the ordinary functor categories

$$\text{fun}(\mathcal{C}, V\mathbb{A}\text{-Mod}) \begin{matrix} \xleftarrow{u_*} \\ (\mathbb{A}\odot-)_* \\ \xrightarrow{\text{fun}(\mathcal{C}, V\mathcal{V})} \end{matrix} \text{fun}(\mathcal{C}, V\mathcal{V}) \begin{matrix} \xleftarrow{V_*} \\ (\mathbb{1}[-])_* \\ \xrightarrow{\text{fun}(\mathcal{C}, \text{Sets})} \end{matrix}$$

We use the abbreviation $\mathbb{A}[X] = \mathbb{A} \odot \mathbb{1}[X]$. Furthermore, the inclusion ι of the discrete subcategory $\text{obj } \mathcal{C}$ into \mathcal{C} induces another pair of adjoint functors

$$\text{fun}(\mathcal{C}, \text{Sets}) \begin{matrix} \xleftarrow{\iota^*} \\ L \\ \xrightarrow{\text{fun}(\text{obj } \mathcal{C}, \text{Sets})} \end{matrix}$$

where the left adjoint L is defined as follows. Any $X \in \text{obj } \text{fun}(\text{obj } \mathcal{C}, \text{Sets})$ is just a collection $\{X(c)\}_{c \in \text{obj } \mathcal{C}}$ of sets indexed by the objects of \mathcal{C} . The functor LX is then defined as

$$LX = \coprod_{c \in \text{obj } \mathcal{C}} \coprod_{X(c)} \mathcal{C}(c, -).$$

Composing these functors gives an adjunction

$$\text{fun}(\mathcal{C}, V\mathbb{A}\text{-Mod}) \begin{matrix} \xleftarrow{U} \\ F \\ \xrightarrow{\text{fun}(\text{obj } \mathcal{C}, \text{Sets})} \end{matrix}$$

Notice that, given X as above, then FX is isomorphic to

$$\coprod_{c \in \text{obj } \mathcal{C}} \coprod_{X(c)} \mathbb{A}[\mathcal{C}(c, -)]. \tag{13.7}$$

Definition 13.8 (*Finitely generated and free \mathbb{A} -modules over \mathcal{C}*). If $M \in \text{obj } \mathcal{M}_{\mathbb{A}}(\mathcal{C})$ is an \mathbb{A} -module over \mathcal{C} , we say that M is *free* if it is isomorphic in $V\mathcal{M}_{\mathbb{A}}(\mathcal{C}) = \text{fun}(\mathcal{C}, V\mathbb{A}\text{-Mod})$ to FX for some $X \in \text{obj } \text{fun}(\text{obj } \mathcal{C}, \text{Sets})$, and it is *finitely generated free* if $X(c)$ is finite for all $c \in \text{obj } \mathcal{C}$ and nonempty only for finitely many $c \in \text{obj } \mathcal{C}$. We let

$$\mathcal{F}_{\mathbb{A}}(\mathcal{C})$$

be the full \mathcal{V} -subcategory of $\mathcal{M}_{\mathbb{A}}(\mathcal{C})$ whose objects are the finitely generated free modules.

Example 13.9. There is a \mathcal{V} -functor $\mathbb{A}[\mathcal{C}] \rightarrow \mathcal{F}_{\mathbb{A}}(\mathcal{C})$, $c \mapsto \mathbb{A}[\mathcal{C}(c, -)]$.

Since \mathcal{V} has all coproducts, the same is true in $\mathbb{A}\text{-Mod}$, and therefore also in $\mathcal{M}_{\mathbb{A}}(\mathcal{C})$, as coproducts are computed objectwise in functor categories. The coproduct of two finitely generated free modules is again finitely generated free by inspection, and therefore also $\mathcal{F}_{\mathbb{A}}(\mathcal{C})$ has all finite coproducts. Notice that if \mathcal{C} is small, then $\mathcal{F}_{\mathbb{A}}(\mathcal{C})$ has a small skeleton.

Now let $\alpha: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. The left Kan extension along α gives a \mathcal{V} -functor $\text{Lan}_{\alpha}: \mathcal{M}_{\mathbb{A}}(\mathcal{C}) \rightarrow \mathcal{M}_{\mathbb{A}}(\mathcal{D})$. By inspection, one verifies that Lan_{α} restricts to a functor

$$\mathcal{F}_{\mathbb{A}}(\alpha): \mathcal{F}_{\mathbb{A}}(\mathcal{C}) \rightarrow \mathcal{F}_{\mathbb{A}}(\mathcal{D}), \tag{13.10}$$

and that it preserves coproducts. In fact, if $M \in \text{obj } \mathcal{F}_{\mathbb{A}}(\mathcal{C})$ is isomorphic to (13.7) for some $X \in \text{obj fun}(\text{obj } \mathcal{C}, \text{Sets})$, then

$$\mathcal{F}_{\mathbb{A}}(\alpha)(M) \cong \coprod_{d \in \text{obj } \mathcal{D}} \coprod_{c \in \alpha^{-1}(d)} \coprod_{X(c)} \mathbb{A}[\mathcal{D}(d, -)]. \tag{13.11}$$

Since Kan extension are functorial up to natural isomorphisms, in the sense of Fact 13.3, we get a pseudo-functor

$$\mathcal{F}_{\mathbb{A}}: \text{Cat} \rightarrow \mathcal{V}\text{-CAT}^{\text{copr}},$$

where we denote by $\mathcal{V}\text{-CAT}^{\text{copr}}$ the 2-category of skeletally small \mathcal{V} -categories with finite coproducts, coproduct preserving \mathcal{V} -functors, and \mathcal{V} -natural transformations. For the definition of pseudo-functor, see for example [18, Definition 7.5.1 on pages 296–297].

For the applications in the next section, it is necessary to work with small categories, not just skeletally small ones, and with functors, not just pseudo-functors. This can be achieved as follows. First, choose a functorial coproduct in $\mathbb{A}\text{-Mod}$. Then, let $\mathcal{F}'_{\mathbb{A}}(\mathcal{C})$ be the small full subcategory of $\mathcal{F}_{\mathbb{A}}(\mathcal{C})$ whose objects are precisely the ones in (13.7). Finally, using (13.11), it is clear how to define $\mathcal{F}'_{\mathbb{A}}(\alpha)$ in order to get a strict functor

$$\mathcal{F}'_{\mathbb{A}}: \text{Cat} \rightarrow \mathcal{V}\text{-Cat}^{\text{copr}},$$

where $\mathcal{V}\text{-Cat}^{\text{copr}}$ is the full subcategory of the small \mathcal{V} -categories with finite coproducts. In order to simplify the exposition, we do not distinguish between the equivalent constructions $\mathcal{F}_{\mathbb{A}}$ and $\mathcal{F}'_{\mathbb{A}}$ in the next section.

14. Constructing homology theories with Mackey structures

The main result of this section, Theorem 14.2, gives a very general procedure to construct equivariant homology theories with Mackey structures. This is used in Section 15

to construct the main diagram (3.1) and show that the results of Section 12 apply. A special case of Theorem 14.2 for additive categories was proved in [72, Proposition 6.5 on page 620].

As in the previous section, we fix a closed symmetric monoidal category $(\mathcal{V}, \odot, \mathbb{1})$ and a monoid \mathbb{A} in \mathcal{V} . We assume that \mathcal{V} is complete and cocomplete. We denote by $\mathcal{V}\text{-Cat}^{\text{copr}}$ the category of small \mathcal{V} -categories with finite coproducts.

Let $\mathbf{T}: \mathcal{V}\text{-Cat}^{\text{copr}} \rightarrow \mathbb{N}\text{Sp}$ be a functor; see Example 14.3 below. For any group G , we consider the composition

$$\text{Or}G \hookrightarrow \text{Sets}^G \xrightarrow{Gf-} \text{Groupoids} \xrightarrow{\mathcal{F}_{\mathbb{A}}} \mathcal{V}\text{-Cat}^{\text{copr}} \xrightarrow{\mathbf{T}} \mathbb{N}\text{Sp} \tag{14.1}$$

and the corresponding G -homology theory $H_*^G(-; \mathbf{T}\mathcal{F}_{\mathbb{A}})$ from Theorem 12.2. Here $\mathcal{F}_{\mathbb{A}}$ is the restriction to Groupoids of the functor constructed at the end of the last section, which sends an ordinary small category to the \mathcal{V} -category of finitely generated and free \mathbb{A} -modules over it. We can now state the main result of this section.

Theorem 14.2. *Let $\mathbf{T}: \mathcal{V}\text{-Cat}^{\text{copr}} \rightarrow \mathbb{N}\text{Sp}$ be a functor. Assume that \mathbf{T} preserves equivalences and products, in the sense that \mathbf{T} sends equivalences to π_* -isomorphisms and the natural map $\mathbf{T}(\mathcal{X} \times \mathcal{Y}) \rightarrow \mathbf{T}(\mathcal{X}) \times \mathbf{T}(\mathcal{Y})$ is a π_* -isomorphism for all \mathcal{X} and \mathcal{Y} . Then the equivariant homology theory $H_*^?(-; \mathbf{T}\mathcal{F}_{\mathbb{A}})$ has a Mackey structure.*

Example 14.3. Examples of functors \mathbf{T} that satisfy the assumptions of Theorem 14.2 are given by the composition

$$\Gamma\mathcal{S}_*\text{-Cat}^{\text{copr}} \rightarrow \text{SymMon}\Gamma\mathcal{S}_*\text{-Cat} \xrightarrow{\mathbf{K}} \mathbb{N}\text{Sp},$$

where the first functor is the evident inclusion and the second is the algebraic K -theory functor from (15.2), and by

$$\Sigma\text{Sp}_*\text{-Cat}^{\text{copr}} \rightarrow \Sigma\text{Sp}_*\text{-Cat} \xrightarrow{\mathbf{TC}(-;p)} \mathbb{N}\text{Sp},$$

where the first functor is the forgetful one.

In order to prove Theorem 14.2, we analyze the composition

$$\text{Sets}^G \xrightarrow{Gf-} \text{Groupoids} \xrightarrow{\mathcal{F}_{\mathbb{A}}} \mathcal{V}\text{-Cat}^{\text{copr}}$$

through which (14.1) factors. The following examples illustrate this construction and guide our intuition.

Example 14.4. For any subgroup H of G , $\mathcal{F}_{\mathbb{A}}(Gf(G/H))$ is equivalent to the \mathcal{V} -category of finitely generated free $\mathbb{A}[H]$ -modules.

Example 14.5. Consider the special case $(\mathcal{V}, \odot, \mathbb{1}) = (\mathbf{Sets}, \times, \text{pt})$ and $\mathbb{A} = \text{pt}$, and notice that then $\text{pt-Mod} = \mathbf{Sets}$. Then for any G -set S the categories

$$\mathcal{F}_{\text{pt}}(GfS) \quad \text{and} \quad \mathbf{Sets}_{\text{cof,free}}^G \downarrow S$$

are equivalent, where $\mathbf{Sets}_{\text{cof,free}}^G \downarrow S$ is the comma category of cofinite and free G -sets over S .

To see this, consider first the category $\mathcal{M}_{\text{pt}}(GfS)$. There is an equivalence of categories

$$\mathcal{M}_{\text{pt}}(GfS) \longrightarrow \mathbf{Sets}^G \downarrow S, \tag{14.6}$$

sending a functor $X: GfS \rightarrow \mathbf{Sets}$ to the G -set $\coprod_{s \in S} X(s)$ together with the obvious G -map to S sending any $x \in X(s)$ to s . In the opposite direction, a G -map $p: T \rightarrow S$ is sent to the functor $s \mapsto p^{-1}(\{s\})$.

It is then easy to see that the equivalence of categories given by (14.6) induces an equivalence between $\mathcal{F}_{\text{pt}}(GfS)$ and $\mathbf{Sets}_{\text{cof,free}}^G$.

We now establish a couple of preliminary results. The first one follows immediately from the definitions.

Lemma 14.7. *For any group G and any pair of cofinite G -sets S_1 and S_2 there is a natural isomorphism of \mathcal{V} -categories with coproducts*

$$\varphi: \mathcal{F}_{\mathbb{A}}(Gf(S_1 \amalg S_2)) \longrightarrow \mathcal{F}_{\mathbb{A}}(GfS_1) \times \mathcal{F}_{\mathbb{A}}(GfS_2).$$

Lemma 14.8. *Under the assumptions of Theorem 14.2, for any group G and any cofinite G -set S there is a natural isomorphism*

$$H_*^G(S; \mathbf{T}\mathcal{F}_{\mathbb{A}}) \cong \pi_*(\mathbf{T}\mathcal{F}_{\mathbb{A}}(GfS)).$$

Proof. Recall that, by construction, $H_*^G(S; \mathbf{T}\mathcal{F}_{\mathbb{A}}) = \pi_*((\mathbf{T}\mathcal{F}_{\mathbb{A}}(Gf-))_{\%}(S))$, where $(\mathbf{T}\mathcal{F}_{\mathbb{A}}(Gf-))_{\%}$ is the continuous left Kan extension of (14.1) along the inclusion $\text{Or}G \hookrightarrow \text{Top}^G$. Choose an isomorphism $f: S \rightarrow G/H_1 \amalg \cdots \amalg G/H_\ell$. We then get the following maps.

$$\begin{array}{ccc}
 (\mathbf{T}\mathcal{F}_{\mathbb{A}}(Gf-))_{\%}(S) & \xrightarrow{\cong} & (\mathbf{T}\mathcal{F}_{\mathbb{A}}(Gf-))_{\%}(G/H_1 \amalg \cdots \amalg G/H_{\ell}) \\
 & & \cong \downarrow \textcircled{1} \\
 & & \mathbf{T}\mathcal{F}_{\mathbb{A}}(Gf(G/H_1)) \vee \cdots \vee \mathbf{T}\mathcal{F}_{\mathbb{A}}(Gf(G/H_{\ell})) \\
 & & \downarrow \textcircled{2} \\
 \mathbf{T}\mathcal{F}_{\mathbb{A}}(GfS) & \xrightarrow{\cong} & \mathbf{T}\mathcal{F}_{\mathbb{A}}(Gf(G/H_1 \amalg \cdots \amalg G/H_{\ell})) \\
 & & \cong \downarrow \textcircled{3} \\
 & & \mathbf{T}(\mathcal{F}_{\mathbb{A}}(Gf(G/H_1)) \times \cdots \times \mathcal{F}_{\mathbb{A}}(Gf(G/H_{\ell}))) \\
 & & \downarrow \textcircled{4} \\
 & & \mathbf{T}\mathcal{F}_{\mathbb{A}}(Gf(G/H_1)) \times \cdots \times \mathbf{T}\mathcal{F}_{\mathbb{A}}(Gf(G/H_{\ell}))
 \end{array}$$

The horizontal isomorphisms are induced by f . The isomorphism $\textcircled{1}$ follows from the fact that the discrete category with ℓ objects given by the inclusions $G/H_i \rightarrow G/H_1 \amalg \cdots \amalg G/H_{\ell}$ is cofinal in $\text{Or}G \downarrow (G/H_1 \amalg \cdots \amalg G/H_{\ell})$. The maps $\textcircled{2}$ and $\textcircled{4}$ are the natural ones induced by functoriality, and $\textcircled{3}$ is induced by the isomorphism φ from Lemma 14.7. We are assuming that $\textcircled{4}$ is a π_* -isomorphism. The composition $\textcircled{4} \circ \textcircled{3} \circ \textcircled{2}$ is the natural map from the finite coproduct to the product in $\mathbb{N}\text{Sp}$, which is also a π_* -isomorphism. So we conclude that $\textcircled{2}$ is a π_* -isomorphism, proving the result. \square

Now consider a group homomorphism $\alpha: H \rightarrow G$. For any H -set S there is a functor

$$\bar{\alpha}: HfS \rightarrow Gf\text{ind}_{\alpha} S, \tag{14.9}$$

defined on objects by $s \mapsto [1, s]$ and on morphisms by $h \mapsto \alpha(h)$. On the other hand, for any G -set T there is a functor

$$\underline{\alpha}: Hf\text{res}_{\alpha} T \rightarrow GfT,$$

which is the identity on objects and α on morphisms.

Using $\bar{\alpha}$ we define

$$\text{Ind}_{\alpha} = \text{Lan}_{\bar{\alpha}}: \mathcal{F}_{\mathbb{A}}(HfS) \rightarrow \mathcal{F}_{\mathbb{A}}(Gf\text{ind}_{\alpha} S) \tag{14.10}$$

as in (13.10). On the other hand, restriction along $\underline{\alpha}$ defines a functor

$$\text{Res}_{\alpha} = \text{res}_{\underline{\alpha}}: \mathcal{M}_{\mathbb{A}}(GfT) \rightarrow \mathcal{M}_{\mathbb{A}}(Hf\text{res}_{\alpha} T).$$

If we assume that α is injective and $\#(G/\alpha(H)) < \infty$, then Res_{α} yields a functor

$$\text{Res}_{\alpha} = \text{res}_{\underline{\alpha}}: \mathcal{F}_{\mathbb{A}}(GfT) \rightarrow \mathcal{F}_{\mathbb{A}}(Hf\text{res}_{\alpha} T). \tag{14.11}$$

These induction and restriction functors (14.10) and (14.11) for cofinite equivariant sets satisfy the following double coset formula.

Lemma 14.12 (Double coset formula). *Let H and K be subgroups of G with $\#(G/K) < \infty$, and let $a: H \rightarrow G$ and $b: K \rightarrow G$ be the inclusions. Let $\mathcal{J} = K \backslash G/H$. For every $j \in \mathcal{J}$, choose a representative $g_j \in j = Kg_jH$, define $L_j = H \cap g_j^{-1}Kg_j$, and let $d_j: L_j \rightarrow H$ be the inclusion and $c_j = \text{conj}_{g_j}: L_j \rightarrow K$. There is a commutative diagram*

$$\begin{array}{ccc}
 \prod_{j \in \mathcal{J}} G/L_j & \xrightarrow{(c_j)} & G/K \\
 \cong \searrow & & \downarrow b \\
 G/H \times G/K & \longrightarrow & G/K \\
 \downarrow & & \downarrow \\
 G/H & \xrightarrow{a} & G/G
 \end{array}$$

(d_j) is the arrow from $\prod_{j \in \mathcal{J}} G/L_j$ to G/H .

whose inner and outer squares are pullbacks, where we write a, b, d_j also for the maps of orbits induced by the inclusions, and $c_j: G/L_j \rightarrow G/K$ sends xL_j to $xg_j^{-1}K$. Then, for any cofinite H -set S , the diagram

$$\begin{array}{ccc}
 \prod_{j \in \mathcal{J}} \mathcal{F}_{\mathbb{A}}(L_j \text{fres}_{d_j} S) & \xrightarrow{\prod \text{Ind}_{c_j}} & \prod_{j \in \mathcal{J}} \mathcal{F}_{\mathbb{A}}(K \text{find}_{c_j} \text{res}_{d_j} S) \\
 \uparrow (\text{Res}_{d_j}) & & \cong \uparrow \varphi \\
 & & \mathcal{F}_{\mathbb{A}}\left(K \text{f} \prod_{j \in \mathcal{J}} \text{ind}_{c_j} \text{res}_{d_j} S\right) \\
 & & \cong \uparrow \mathcal{F}_{\mathbb{A}}^K(K \text{f} w^{-1}) \\
 & & \mathcal{F}_{\mathbb{A}}(K \text{fres}_b \text{ind}_a S) \\
 & & \uparrow \text{Res}_b \\
 \mathcal{F}_{\mathbb{A}}(H \text{f} S) & \xrightarrow{\text{Ind}_a} & \mathcal{F}_{\mathbb{A}}(G \text{f} \text{ind}_a S)
 \end{array} \tag{14.13}$$

commutes up to natural isomorphism, where

$$w: \prod_{j \in \mathcal{J}} \text{ind}_{c_j} \text{res}_{d_j} S \xrightarrow{\cong} \text{res}_b \text{ind}_a S$$

is the natural K -isomorphism sending $[k, s] \in \text{ind}_{c_j} \text{res}_{d_j} S = K \times_{L_j} \text{res}_{d_j} S$ to $[kg_j, s] \in \text{res}_b \text{ind}_a S = \text{res}_b(G \times_H S)$, and φ is the natural isomorphism from Lemma 14.7.

Proof. First of all, notice that \mathcal{J} is finite since $\#(G/K) < \infty$. Notice also that L_j and hence c_j and d_j depend on the choice of representative. It is enough to prove that diagram (14.13) commutes up to isomorphism after composing with the projection from the top right corner to each factor $\mathcal{F}_{\mathbb{A}}(K \text{f} \text{ind}_{c_j} \text{res}_{d_j} S)$. Consider the following diagram.

$$\begin{array}{ccc}
 L_j \text{fres}_{d_j} S & \xrightarrow{\bar{c}_j} & K \text{find}_{c_j} \text{res}_{d_j} S \\
 \downarrow d_j & \nearrow \nu & \downarrow K \text{fincl}_j \\
 & & K \text{f} \coprod_{j \in \mathcal{J}} \text{ind}_{c_j} \text{res}_{d_j} S \\
 & & \cong \downarrow K \text{fw} \\
 & & K \text{fres}_b \text{ind}_a S \\
 & & \downarrow b \\
 H \text{f} S & \xrightarrow{\bar{a}} & G \text{find}_a S
 \end{array} \tag{14.14}$$

Diagram (14.14) does not commute. Given $s \in \text{res}_{d_j} S$, the composition through the left bottom corner sends s to $[1, s]$, the composition through the top right corner sends s to $[g_j, s]$. So there is a natural isomorphism $\nu = g_j$ in the indicated direction. Notice that the functors \bar{a} and \bar{c}_j are equivalences of categories. Now apply Fact 13.5 and the definition of the functors in diagram (14.13) to obtain the result. \square

We are now ready to prove the main result of this section.

Proof of Theorem 14.2. The map (12.5) corresponds to the one obtained by applying $\pi_n \mathbf{T}(-)$ to

$$\mathcal{F}_{\mathbb{A}}(H \text{fpt}) \xrightarrow{\text{Ind}_{\alpha}} \mathcal{F}_{\mathbb{A}}(G \text{find}_{\alpha} \text{pt}) \xrightarrow{\mathcal{F}_{\mathbb{A}}(G \text{fpr})} \mathcal{F}_{\mathbb{A}}(G \text{fpt}).$$

We have to show that this extends to a Mackey functor. The contravariant functoriality is induced by

$$\mathcal{F}_{\mathbb{A}}(G \text{fpt}) \xrightarrow{\text{Res}_{\alpha}} \mathcal{F}_{\mathbb{A}}(H \text{fpt}).$$

To verify axiom (i), notice that if $\alpha: H \rightarrow G$ is an isomorphism, then $\bar{\alpha} = \alpha: H \text{fpt} \rightarrow G \text{fpt}$ is an isomorphism. Then apply Fact 13.4.

(ii) follows from Fact 13.5 and the fact that g is a natural isomorphism from $\text{conj}_g: G \text{fpt} \rightarrow G \text{fpt}$ to the identity.

(iii) follows by applying $\pi_n \mathbf{T}(-)$ to the diagram (14.13) in Lemma 14.12 for $S = \text{pt}$ combined with the following commutative diagram.

$$\begin{array}{ccc}
 \prod_{j \in \mathcal{J}} \mathcal{F}_{\mathbb{A}}(K \text{find}_{c_j} \text{pt}) & \longrightarrow & \prod_{j \in \mathcal{J}} \mathcal{F}_{\mathbb{A}}(K \text{fpt}) \\
 \cong \uparrow \varphi & & \cong \uparrow \varphi \\
 \mathcal{F}_{\mathbb{A}}\left(K \text{f} \coprod_{j \in \mathcal{J}} \text{ind}_{c_j} \text{pt}\right) & \longrightarrow & \mathcal{F}_{\mathbb{A}}\left(K \text{f} \coprod_{j \in \mathcal{J}} \text{pt}\right) \\
 \cong \downarrow \mathcal{F}_{\mathbb{A}}^K(K \text{fw}) & & \cong \downarrow \\
 \mathcal{F}_{\mathbb{A}}(K \text{fres}_b \text{ind}_a \text{pt}) & \longrightarrow & \mathcal{F}_{\mathbb{A}}(K \text{fpt}) \\
 \uparrow \text{Res}_b & & \uparrow \text{Res}_b \\
 \mathcal{F}_{\mathbb{A}}(G \text{find}_a \text{pt}) & \longrightarrow & \mathcal{F}_{\mathbb{A}}(G \text{fpt})
 \end{array}$$

Here the unlabeled functors are induced by the projections to pt . One also needs to use [Lemma 14.8](#) and the assumption on \mathbf{T} . \square

Using the constructions and results in this section, we can also prove that $H_*^?(-; \mathbf{T}\mathcal{F}_{\mathbb{A}})$ is an equivariant homology theory with restriction structure in the sense of [\[68, Section 6, pages 217–218\]](#). This is not needed here, and details will appear elsewhere.

15. The cyclotomic trace map and the main diagram

In this section we explain the construction of the main diagram [\(3.1\)](#). An essential ingredient is the natural model for the cyclotomic trace map constructed by Dundas, Goodwillie, and McCarthy in [\[30\]](#). Using the results from [Section 14](#), it follows that all the equivariant homology theories in [\(3.1\)](#) have Mackey structures, and therefore the theory of Chern characters from [Section 12](#) becomes available. The proof of the [Detection Theorem 3.6](#) in [Section 16](#) is based on this.

In essence, diagram [\(3.1\)](#) is induced by natural transformations

$$\mathbf{K}^{\geq 0}(\mathbb{Z}[-]) \xleftarrow{\ell} \mathbf{K}^{\geq 0}(\mathbb{S}[-]) \xrightarrow{\tau} \prod_p \mathbf{TC}(\mathbb{S}[-]; p) \xrightarrow{\sigma} \mathbf{THH}(\mathbb{S}[-]) \times \prod_p \mathbf{C}(\mathbb{S}[-]; p)$$

of functors $\text{Groupoids} \rightarrow \mathbb{N}\text{Sp}$. In detail, though, both τ and σ are zig-zags of natural transformations whose wrong-way maps are all natural π_* -isomorphisms. Since natural π_* -isomorphisms induce isomorphisms of the associated homology theories, we use the same symbols for the zig-zags of natural transformations and for the induced one-way maps in homology.

We proceed to give technical details about ℓ , τ , and σ .

The category of spectra used in [\[30\]](#) is $\Gamma\mathcal{S}_*$; see [Subsection 4I](#). Given a $\Gamma\mathcal{S}_*$ -category \mathcal{D} , let $\mathbf{TC}(\mathcal{D}; p)$ be the geometric realization of the spectrum denoted $\underline{TC}(\mathcal{D}, S^0; p)$ in [\[30, Definition 6.4.1.1 on pages 253–254\]](#).

In order to compare this definition to the one used in this paper, consider the lax monoidal functor $\|-\|: \Gamma\mathcal{S}_* \rightarrow \Sigma\text{Sp}$ from [Subsection 4I](#). It induces a change of base functor

$$\|-\|: \Gamma\mathcal{S}_*\text{-Cat} \rightarrow \Sigma\text{Sp-Cat},$$

which we denote with the same symbol; compare [Subsection 4D](#). Notice that, if \mathcal{A} is a monoid in $\Gamma\mathcal{S}_*$ (i.e., an \mathbf{S} -algebra in the sense of [\[30\]](#)) and \mathcal{C} is an ordinary small category, then $\|\mathcal{A}[\mathcal{C}]\| \cong \|\mathcal{A}\|[\mathcal{C}]$. The following result follows easily by inspecting the definitions.

Lemma 15.1. *For any $\Gamma\mathcal{S}_*$ -category \mathcal{D} there is a natural π_* -isomorphism in $\mathbb{N}\text{Sp}$*

$$\mathbf{TC}(\mathcal{D}; p) \rightarrow \mathbf{UTC}(\|\mathcal{D}\|; p).$$

Denote by $\text{SymMon}\Gamma\mathcal{S}_*\text{-Cat}$ the category of small symmetric monoidal $\Gamma\mathcal{S}_*$ -categories. The algebraic K -theory spectrum of a symmetric monoidal $\Gamma\mathcal{S}_*$ -category is defined in [30, Subsection 5.2.1.4 on pages 192–193]. By taking geometric realization, we get a functor

$$\mathbf{K}: \text{SymMon}\Gamma\mathcal{S}_*\text{-Cat} \longrightarrow \mathbb{N}\text{Sp}. \tag{15.2}$$

The cyclotomic trace map is then defined in [30, Definition 7.1.1.2 and the rest of Section 7.1.1 on page 285] as a zig-zag of natural transformations

$$\text{trc}_p: \mathbf{K}(\mathcal{D}) \xleftarrow{\simeq} \dots \longrightarrow \dots \xleftarrow{\simeq} \mathbf{TC}(\mathcal{D}; p) \tag{15.3}$$

of functors $\text{SymMon}\Gamma\mathcal{S}_*\text{-Cat} \longrightarrow \mathbb{N}\text{Sp}$, where the wrong-way maps are all natural π_* -isomorphisms.

We are now ready to describe ℓ and τ . Fix models in the category of monoids in $\Gamma\mathcal{S}_*$ for the Eilenberg–Mac Lane ring spectrum $\mathbb{H}\mathbb{Z}$, the sphere spectrum \mathbb{S} , and the Hurewicz map $\mathbb{S} \longrightarrow \mathbb{H}\mathbb{Z}$. We then obtain a natural transformation ℓ from the bottom to the top composition in

$$\text{Groupoids} \begin{array}{c} \xrightarrow{\mathcal{F}_{\mathbb{H}\mathbb{Z}}} \\ \xrightarrow{\mathcal{F}_{\mathbb{S}}} \end{array} \Gamma\mathcal{S}_*\text{-Cat}^{\text{copr}} \longleftarrow \text{SymMon}\Gamma\mathcal{S}_*\text{-Cat} \xrightarrow{\mathbf{K}} \mathbb{N}\text{Sp},$$

where \mathbf{K} is the algebraic K -theory functor from (15.2); compare also Example 14.3.

Next, using (15.3) and taking the product over all primes p , we obtain a zig-zag of natural transformations

$$\tau = (\text{trc}_p): \mathbf{K}(\mathcal{D}) \xleftarrow{\simeq} \dots \longrightarrow \dots \xleftarrow{\simeq} \prod_p \mathbf{TC}(\mathcal{D}; p)$$

of functors $\text{SymMon}\Gamma\mathcal{S}_*\text{-Cat} \longrightarrow \mathbb{N}\text{Sp}$, where the wrong-way maps are all natural π_* -isomorphisms.

In order to describe σ , we first switch to enrichments in ΣSp , and then we need to use Morita invariance, Lemma 15.5 below, because the zig-zag (11.4) between \mathbf{TC} and \mathbf{C} is not defined for categories of modules.

In more detail, Lemma 15.1 gives a natural π_* -isomorphism from the top to the bottom composition in the diagram

$$\text{Groupoids} \begin{array}{c} \xrightarrow{\mathcal{F}_{\mathbb{S}}} \\ \xrightarrow{\mathcal{F}_{\mathbb{S}}} \end{array} \Gamma\mathcal{S}_*\text{-Cat} \begin{array}{c} \xrightarrow{\Pi\mathbf{TC}(-; p)} \\ \xrightarrow{\Pi\mathbf{TC}(-; p)} \end{array} \mathbb{N}\text{Sp}.$$

Lemma 15.5(ii) gives a natural π_* -isomorphism from the bottom to the top composition in

$$\text{OrG} \xrightarrow{Gf-} \text{Groupoids} \begin{array}{c} \xrightarrow{\mathcal{F}_{\mathbb{S}}} \\ \xrightarrow{\mathbb{S}[-]} \end{array} \Sigma\text{Sp-Cat} \xrightarrow{\Pi\mathbf{TC}(-; p)} \mathbb{N}\text{Sp}.$$

Finally, using (11.4) and taking the product over all primes p , and for $p = 11$ using also the projection $\mathbf{TC}(-; 11) \rightarrow \mathbf{THH}(-)$, we obtain a zig-zag of natural transformations

$$\sigma: \prod_p \mathbf{TC}(\mathbb{S}[\mathcal{C}]; p) \rightarrow \cdots \xleftarrow{\cong} \mathbf{THH}(\mathbb{S}[\mathcal{C}]) \times \prod_p \mathbf{C}(\mathbb{S}[\mathcal{C}]; p)$$

of functors $\mathbf{Groupoids} \rightarrow \mathbf{NSp}$.

Remark 15.4. Because of the naturality of Theorem 14.2 in the functor \mathbf{T} , the maps ℓ and τ induce natural transformations of Mackey functors. Arguing as above with Morita invariance and Theorem 14.2, we see that also the homology theory associated with $\mathbf{THH}(\mathbb{S}[-]) \times \prod_p \mathbf{C}(\mathbb{S}[-]; p)$ has a Mackey structure. However, we only know that the natural transformation induced by σ respects the induction structures, but not necessarily the Mackey structures.

We finish this section with the well-known Morita invariance property needed above. For a symmetric ring spectrum \mathbb{A} , consider the symmetric spectral category $\mathcal{F}_{\mathbb{A}}(\mathcal{C})$ of finitely generated free \mathbb{A} -modules over \mathcal{C} from Definition 13.8, and the functor of symmetric spectral categories $\mathbb{A}[\mathcal{C}] \rightarrow \mathcal{F}_{\mathbb{A}}(\mathcal{C})$ from Example 13.9.

Lemma 15.5 (Morita invariance). *Let \mathbb{A} be a symmetric ring spectrum.*

(i) *The natural transformation*

$$\mathbf{THH}(\mathbb{A}[Gf-]) \rightarrow \mathbf{THH}(\mathcal{F}_{\mathbb{A}}(Gf-))$$

of functors $\mathbf{Or}G \rightarrow \mathcal{WT}^{S^1}$ is objectwise a π_^1 -isomorphism.*

(ii) *Assume that \mathbb{A} is connective⁺. Then the natural transformations*

$$\begin{aligned} \mathbf{C}(\mathbb{A}[Gf-]; p) &\rightarrow \mathbf{C}(\mathcal{F}_{\mathbb{A}}(Gf-); p), \\ \mathbf{TC}(\mathbb{A}[Gf-]; p) &\rightarrow \mathbf{TC}(\mathcal{F}_{\mathbb{A}}(Gf-); p) \end{aligned}$$

of functors $\mathbf{Or}G \rightarrow \mathcal{WT}$ are objectwise π_ -isomorphisms.*

Proof. (i) Consider the following commutative diagram of functors of spectral categories.

$$\begin{array}{ccc} \mathbb{A}[Gf(G/H)] & \longrightarrow & \mathcal{F}_{\mathbb{A}}(Gf(G/H)) \\ \uparrow & & \uparrow \\ \mathbb{A}[H] & \longrightarrow & \mathcal{F}_{\mathbb{A}}(H) \cong \mathcal{F}_{\mathbb{A}[H]}(\text{pt}) \end{array} \tag{15.6}$$

The vertical maps induce π_*^1 -isomorphisms on \mathbf{THH} by an argument as in [30, Corollary 4.3.5.2 on page 171]. For any symmetric ring spectrum \mathbb{B} , the functor $\mathbb{B} \rightarrow \mathcal{F}_{\mathbb{B}}(\text{pt})$

induces a π_*^1 -isomorphism on **THH** by an argument similar to that of [30, Lemma 4.3.5.11 on page 176]. So we conclude that also the top horizontal functor in (15.6) induces a π_*^1 -isomorphism on **THH**.

(ii) The assumption on \mathbb{A} implies that $\mathbb{A}[C]$ and $\mathcal{F}_{\mathbb{A}}(C)$ are objectwise strictly connective, objectwise convergent, and very well pointed for any small category \mathcal{C} . The result then follows from part (i), the fundamental fibration sequence from Theorem 8.1(i), and the fact that homotopy limits respect π_* -isomorphisms. \square

16. Proof of the Detection Theorem

The goal of this section is to prove the Detection Theorem 3.6, thus completing the proof of Main Technical Theorem 1.16.

Let $\mathcal{F} \subseteq \mathcal{FCyc}$ be a family of finite cyclic subgroups of G , and let $T \subseteq \mathbb{N}$ be a subset. Recall that Assumption $[B_{\mathcal{F}, T}]$ reads as follows.

$[B_{\mathcal{F}, T}]$ For every $C \in \mathcal{F}$ and for every $t \in T$, the natural homomorphism

$$K_t(\mathbb{Z}[\zeta_c]) \longrightarrow \prod_{p \text{ prime}} K_t\left(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_c]; \mathbb{Z}_p\right)$$

is \mathbb{Q} -injective, where c is the order of C and ζ_c is any primitive c -th root of unity.

Lemma 16.1. *If Assumption $[B_{\mathcal{F}, T}]$ holds then, for every $C \in \mathcal{F}$ and for every $t \in T$, the natural homomorphism*

$$K_t(\mathbb{Z}[C]) \longrightarrow \prod_{p \text{ prime}} K_t(\mathbb{Z}_p[C]; \mathbb{Z}_p)$$

is \mathbb{Q} -injective.

Proof. Let \mathcal{MC} be the maximal \mathbb{Z} -order in $\mathbb{Q}[C]$ containing $\mathbb{Z}[C]$. Since $\mathbb{Q}[C] \cong \prod_{d|c} \mathbb{Q}(\zeta_d)$, we have that $\mathcal{MC} \cong \prod_{d|c} \mathbb{Z}[\zeta_d]$; compare [96, Theorem 10.5 on page 128] or [87, Theorem 1.5(i) on page 25]. Using that algebraic K -theory respects finite products, we get the following commutative diagram.

$$\begin{CD} K_t(\mathbb{Z}[C]) @>>> \prod_p K_t(\mathbb{Z}_p[C]; \mathbb{Z}_p) \\ @VVV @VVV \\ K_t(\mathcal{MC}) \cong \bigoplus_{d|c} K_t(\mathbb{Z}[\zeta_d]) @>>> \prod_p \bigoplus_{d|c} K_t\left(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_d]; \mathbb{Z}_p\right) \end{CD} \tag{16.2}$$

The finite sums here commute with the product and with rationalization. Hence, since the family \mathcal{F} is closed under passage to subgroups, Assumption $[B_{\mathcal{F}, T}]$ implies that the

bottom horizontal map is \mathbb{Q} -injective for every $t \in T$. By [96, Theorem 41.1 on page 379], we have that $c\mathcal{M}C \subseteq \mathbb{Z}[C]$, and so we get the following cartesian square of rings.

$$\begin{array}{ccc} \mathbb{Z}[C] & \longrightarrow & \mathbb{Z}[C]/c\mathcal{M}C \\ \downarrow & & \downarrow \\ \mathcal{M}C & \longrightarrow & \mathcal{M}C/c\mathcal{M}C \end{array}$$

Both rings on the right are finite, and for any finite ring R and any $t \geq 1$ we have $K_t(R) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ by [125, Proposition IV.1.16 on page 296]. Then the Mayer–Vietoris long exact sequence in $K_*(-) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{c}]$ from [25, Corollary 2 on page 6 and examples on page 7] implies that the left-hand vertical map in (16.2) is a \mathbb{Q} -isomorphism for every $t \geq 1$. For $t = 0$, the natural map $K_0(\mathbb{Z}) \rightarrow K_0(\mathbb{Z}[C])$ is a \mathbb{Q} -isomorphism by [113, Proposition 9.1 on page 573]. Using this, we see that $K_0(\mathbb{Z}[C]) \rightarrow K_0(\mathcal{M}C)$ is \mathbb{Q} -injective. So we conclude that the top horizontal map in (16.2) is \mathbb{Q} -injective for every $t \in T$. \square

Combining this lemma with a result of Hesselholt and Madsen [51], which depends on a theorem of McCarthy [82], we obtain the following corollary.

Corollary 16.3. *If Assumption $[B_{\mathcal{F}, T}]$ holds then, for every $C \in \mathcal{F}$ and for every $t \in T$, the map*

$$\tau = (\text{trc}_p) : \mathbf{K}^{\geq 0}(\mathbb{S}[C]) \longrightarrow \prod_{p \text{ prime}} \mathbf{TC}(\mathbb{S}[C]; p)$$

induced by the cyclotomic trace maps is $\pi_t^{\mathbb{Q}}$ -injective.

Proof. Consider the following commutative diagram.

$$\begin{array}{ccccc} \mathbf{K}^{\geq 0}(\mathbb{S}[C]) & \xrightarrow{\ell} & \mathbf{K}^{\geq 0}(\mathbb{Z}[C]) & \xrightarrow{c} & \prod_p \mathbf{K}^{\geq 0}(\mathbb{Z}_p[C])_{\widehat{p}} \\ (\text{trc}_p) \downarrow & & & & \downarrow \prod (\text{trc}_p)_{\widehat{p}} \\ \prod_p \mathbf{TC}(\mathbb{S}[C]; p) & \xrightarrow{\hspace{10em}} & & & \prod_p \mathbf{TC}(\mathbb{Z}_p[C]; p)_{\widehat{p}} \end{array}$$

The linearization map ℓ is a $\pi_*^{\mathbb{Q}}$ isomorphism by [119, Proposition 2.2 on page 43]. Lemma 16.1 implies that $\pi_t(c) \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective for every $t \in T$. Since the \mathbb{Z}_p -algebra $\mathbb{Z}_p[C]$ is finitely generated as a \mathbb{Z}_p -module, the p -completed cyclotomic trace map $(\text{trc}_p)_{\widehat{p}} : \mathbf{K}^{\geq 0}(\mathbb{Z}_p[C])_{\widehat{p}} \rightarrow \mathbf{TC}(\mathbb{Z}_p[C]; p)_{\widehat{p}}$ is a π_t -isomorphism for every $t \geq 0$ and every prime p by [51, Theorem D on page 30]. The result follows. \square

Proof of Theorem 3.6. We want to apply Addendum 12.13 to the composition of the natural transformations

$$\mathbf{K}^{\geq 0}(\mathbb{S}[-]) \xrightarrow{\tau} \coprod_p \mathbf{TC}(\mathbb{S}[-]; p) \xrightarrow{\sigma} \mathbf{THH}(\mathbb{S}[-]) \times \coprod_p \mathbf{C}(\mathbb{S}[-]; p)$$

of functors $\mathbf{Groupoids} \rightarrow \mathbf{NSp}$. Here we are suppressing some wrong-way natural π_* -isomorphisms, as explained in Section 15. Recall from Remark 15.4 that both τ and σ induce natural transformations of equivariant homology theories that respect the induction structures, but we only know that τ also respects the Mackey structures.

By Addendum 12.13(i), both τ and σ are compatible with the isomorphisms c in (12.8). Therefore it is enough to consider one $t \in T$ at a time.

Assume first that $t > 0$. Then Corollary 16.3 and an argument similar to Addendum 12.13(iii) imply that the restriction of $\tau_{\%} \otimes_{\mathbb{Z}} \mathbb{Q}$ to the summand indexed by t is injective. Moreover, for all finite groups C , $\pi_t(\mathbf{THH}(\mathbb{S}[C])) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ by Example 6.16 and $\mathbf{TC}(\mathbb{S}[C]; p) \rightarrow \mathbf{C}(\mathbb{S}[C]; p)$ is a $\pi_t^{\mathbb{Q}}$ -isomorphism for each p by Corollary 11.9. Therefore σ induces a \mathbb{Q} -isomorphism between the corresponding functors $\mathbf{Sub}G(\mathcal{F}in) \rightarrow \mathbf{Ab}$, and an isomorphism between the summands indexed by t . We conclude then that the restriction $(\sigma_{\%} \circ \tau_{\%}) \otimes_{\mathbb{Z}} \mathbb{Q}$ to the summands indexed by t is injective.

Now let $t = 0$. Recall that the linearization map $\ell: \mathbf{K}^{\geq 0}(\mathbb{S}[G]) \rightarrow \mathbf{K}^{\geq 0}(\mathbb{Z}[G])$ is a $\pi_*^{\mathbb{Q}}$ -isomorphism for every group G , and therefore it induces an isomorphism $S_t(C; \mathbf{K}^{\geq 0}(\mathbb{S}[-])) \cong S_t(C; \mathbf{K}^{\geq 0}(\mathbb{Z}[-]))$. By [113, Proposition 9.1 on page 573], the natural map $K_0(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism for any finite group C . Therefore, $S_0(C; \mathbf{K}^{\geq 0}(\mathbb{S}[-])) = 0$ whenever $C \neq 1$. This implies that the map $EG \rightarrow EG(\mathcal{F})$ induces on the summand indexed by $t = 0$ an isomorphism

$$H_n(BG; \mathbb{Q}) \otimes_{\mathbb{Q}} \left(K_0(\mathbb{S}) \otimes_{\mathbb{Z}} \mathbb{Q} \right) \xrightarrow{\cong} \bigoplus_{(C) \in (\mathcal{F}Cyc)} H_n(BZ_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} S_0(C; \mathbf{K}^{\geq 0}(\mathbb{S}[-])).$$

So it is enough to consider the case $\mathcal{F} = 1$; compare Example 12.11. Since $H_n(BG; \mathbb{Q}) \otimes_{\mathbb{Q}} -$ clearly preserves injectivity, it is enough to show that $\pi_0(\sigma \circ \tau)$ in the following commutative diagram is \mathbb{Q} -injective.

$$\begin{array}{ccc} K_0(\mathbb{S}) & \xrightarrow{\pi_0(\sigma \circ \tau)} \pi_0\left(\mathbf{THH}(\mathbb{S}) \times \coprod_p \mathbf{C}(\mathbb{S}; p)\right) & \xrightarrow{\pi_0(\text{pr}_1)} \pi_0(\mathbf{THH}(\mathbb{S})) \\ \pi_0(\ell) \downarrow & & \downarrow \\ K_0(\mathbb{Z}) & \xrightarrow{B} \pi_0(\mathbf{THH}(\mathbb{Z})) & \\ & \searrow D & \downarrow \\ & & HH_0(\mathbb{Z}) \end{array}$$

Here HH is Hochschild homology, D is the Dennis trace, B is the Bökstedt trace, and pr_1 is the projection on the first factor. Since ℓ and D are isomorphisms in dimension 0, it follows that $\pi_0(\sigma \circ \tau)$ is injective, thus finishing the proof. \square

17. Whitehead groups

This section is devoted to the proof of [Addendum 1.18](#) and its special case, [Theorem 1.1](#), about the rationalized assembly map for Whitehead groups. First we need to introduce some notation.

The finite subgroup category $\text{Sub}G(\mathcal{F}in)$ that appears in [\(1.2\)](#) has as objects the finite subgroups H of G ; the morphisms are defined as follows. Given finite subgroups H and K of G , let $\text{conhom}_G(H, K)$ be the set of all group homomorphisms $H \rightarrow K$ given by conjugation by an element of G . The group $\text{inn}(K)$ of inner automorphisms of K acts on $\text{conhom}_G(H, K)$ by post-composition. The set of morphisms in $\text{Sub}G(\mathcal{F}in)$ from H to K is then defined as the quotient $\text{conhom}_G(H, K)/\text{inn}(K)$. For example, in the special case when G is abelian, then $\text{Sub}G(\mathcal{F}in)$ is just the poset of finite subgroups of G ordered by inclusion.

Given a space X , not pointed and not equivariant, we denote by $\Pi(X)$ its fundamental groupoid, and consider the functor $\text{Top} \rightarrow \mathbb{N}\text{Sp}$ given by $X \mapsto \mathbf{K}^{\geq 0}(\mathbb{Z}[\Pi(X)])$. There is the classical assembly map

$$X_+ \wedge \mathbf{K}^{\geq 0}(\mathbb{Z}) \rightarrow \mathbf{K}^{\geq 0}(\mathbb{Z}[\Pi(X)]); \tag{17.1}$$

see [\[126\]](#). (We suppress here an irrelevant wrong-way natural π_* -isomorphism.) We denote by $\mathbf{Wh}_{\mathbb{Z}}^{\geq 0}(X)$ the homotopy cofiber of [\(17.1\)](#).

Now, given a group G , we consider the composition of the functors above with

$$\text{Or}G \xrightarrow{Gf-} \text{Groupoids} \xrightarrow{B} \text{Top},$$

where $B = |N_{\bullet}(-)|$ is the classifying space functor. We obtain the following diagram of functors $\text{Or}G \rightarrow \mathbb{N}\text{Sp}$

$$B(Gf-)_+ \wedge \mathbf{K}^{\geq 0}(\mathbb{Z}) \rightarrow \mathbf{K}^{\geq 0}(\mathbb{Z}[\Pi(B(Gf-))]) \rightarrow \mathbf{Wh}_{\mathbb{Z}}^{\geq 0}(B(Gf-)), \tag{17.2}$$

which, by definition, is objectwise a homotopy cofibration sequence. For any subgroup H of G we have

$$\pi_n \mathbf{Wh}_{\mathbb{Z}}^{\geq 0}(B(Gf(G/H))) \cong \begin{cases} Wh(H) & \text{if } n = 1; \\ \tilde{K}_0(\mathbb{Z}[H]) & \text{if } n = 0; \\ 0 & \text{if } n < 0. \end{cases} \tag{17.3}$$

From this we get the following lemma.

Lemma 17.4. *For every group G there is an isomorphism*

$$H_1^G(EG(\mathcal{F}in); \mathbf{Wh}_{\mathbb{Z}}^{\geq 0}(B(-))) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \text{colim}_{H \in \text{obj } \text{Sub}G(\mathcal{F}in)} Wh(H) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Proof. From [26, Theorem 4.7 on pages 234–235] there is a spectral sequence converging to $H_{i+j}^G(EG(\mathcal{F}in); \mathbf{Wh}_{\mathbb{Z}}^{\geq 0}(B(-)))$, with E_2 -term given by the Bredon homology groups

$$H_i^G(EG(\mathcal{F}in); \pi_j \mathbf{Wh}_{\mathbb{Z}}^{\geq 0}(B(-))).$$

Since $\pi_0 \mathbf{Wh}_{\mathbb{Z}}^{\geq 0}(B(Gf(G/H))) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \tilde{K}_0(\mathbb{Z}[H]) \otimes_{\mathbb{Z}} \mathbb{Q}$ vanishes if H is finite [113, Proposition 9.1 on page 573], the edge homomorphism of the spectral sequence gives an isomorphism

$$H_1^G(EG(\mathcal{F}in); \mathbf{Wh}_{\mathbb{Z}}^{\geq 0}(B(-))) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_0^G(EG(\mathcal{F}in); \pi_1 \mathbf{Wh}_{\mathbb{Z}}^{\geq 0}(B(-))) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Using the definition of Bredon homology, the fact that the cellular chain complex of $EG(\mathcal{F}in)$ is a free resolution of the constant functor on $\text{Or}G(\mathcal{F}in)$ with value \mathbb{Z} , and right exactness of $- \otimes_{\text{Or}G(\mathcal{F}in)} M$, we get an isomorphism

$$H_0^G(EG(\mathcal{F}in); \pi_1 \mathbf{Wh}_{\mathbb{Z}}^{\geq 0}(B(-))) \cong \text{colim}_{G/H \in \text{obj Or}G(\mathcal{F}in)} Wh(H).$$

Finally, using [74, Lemma 3.11 on page 152], we see that

$$\text{colim}_{G/H \in \text{obj Or}G(\mathcal{F}in)} Wh(H) \cong \text{colim}_{H \in \text{obj Sub}G(\mathcal{F}in)} Wh(H). \quad \square$$

Proof of Addendum 1.18. Consider the homotopy cofibration sequence (17.2). The natural homeomorphism $EG \times_G S \cong B(GfS)$ (e.g., see [72, Lemma 9.9(i) on page 628]), identifies the term on the left with

$$\mathbf{E}(-) = (EG \times_G -)_+ \wedge \mathbf{K}^{\geq 0}(\mathbb{Z}).$$

For any groupoid \mathcal{G} there is an equivalence $\mathcal{G} \rightarrow \Pi(B\mathcal{G})$, which induces a π_* -isomorphism between the middle term in (17.2) and $\mathbf{K}^{\geq 0}(\mathbb{Z}[Gf-])$.

The homotopy cofibration sequence (17.2) produces a long exact sequence of the associated G -equivariant homology theories from Theorem 12.2. The maps induced by the projection $EG(\mathcal{F}in) \rightarrow \text{pt}$ give the following commutative diagram with exact columns.

$$\begin{array}{ccc}
 H_n^G(EG(\mathcal{F}in); \mathbf{E}) & \xrightarrow{\textcircled{1}} & H_n^G(\text{pt}; \mathbf{E}) \\
 \downarrow & & \downarrow \\
 H_n^G(EG(\mathcal{F}in); \mathbf{K}^{\geq 0}(\mathbb{Z}[-])) & \xrightarrow{\textcircled{2}} & K_n(\mathbb{Z}[G]) \\
 \downarrow & & \downarrow \\
 H_n^G(EG(\mathcal{F}in); \mathbf{Wh}_{\mathbb{Z}}^{\geq 0}(B(-))) & \xrightarrow{\textcircled{3}} & H_n^G(\text{pt}; \mathbf{Wh}_{\mathbb{Z}}^{\geq 0}(B(-))) \\
 \downarrow & & \downarrow \\
 H_{n-1}^G(EG(\mathcal{F}in); \mathbf{E}) & \xrightarrow{\textcircled{4}} & H_{n-1}^G(\text{pt}; \mathbf{E})
 \end{array}$$

By [26, Lemma 5.4 on page 239], the assembly map for \mathbf{E} is an isomorphism for every group G and every family of subgroups. So the top and the bottom horizontal maps $\textcircled{1}$ and $\textcircled{4}$ are isomorphisms.

In the notation of Main Technical Theorem 1.16, we are assuming that condition $[A'_{\mathcal{F}Cyc, \leq 2}]$ holds, and $[B_{\mathcal{F}Cyc, \{0,1\}}]$ is satisfied by Proposition 2.4. Therefore the map $\textcircled{2}$ is \mathbb{Q} -injective if $n = 1$. An elementary diagram chase, the four lemma, implies that also the map $\textcircled{3}$ is \mathbb{Q} -injective if $n = 1$.

When $n = 1$ the target of $\textcircled{3}$ is isomorphic to $Wh(G)$ by (17.3). The source is computed rationally by Lemma 17.4. \square

18. Schneider conjecture

We briefly review the relation between algebraic K -theory and étale cohomology of rings of integers in number fields, and give equivalent formulations of the Schneider Conjecture 2.7 and some useful reductions. Other equivalent formulations are discussed in [109,60,42].

Let \mathcal{O}_F denote the ring of integers in a finite field extension $F|\mathbb{Q}$. Fix a prime p . For a prime ideal \mathfrak{p} in \mathcal{O}_F with $\mathfrak{p}|p$, let $\mathcal{O}_{\mathfrak{p}}$ denote the completion, $F_{\mathfrak{p}}$ its field of fractions, and $k_{\mathfrak{p}}$ the residue field at \mathfrak{p} . Consider the following commutative square.

$$\begin{array}{ccc}
 \mathcal{O}_F & \longrightarrow & \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_F \cong \prod_{\mathfrak{p}|p} \mathcal{O}_{\mathfrak{p}} \\
 \downarrow & & \downarrow \\
 \mathcal{O}_F[\frac{1}{p}] & \longrightarrow & \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_F[\frac{1}{p}] \cong \prod_{\mathfrak{p}|p} F_{\mathfrak{p}}
 \end{array}$$

These ring homomorphisms, together with the natural map from K -theory to étale K -theory [32, Proposition 4.4 on page 256] for $\mathbb{Z}[\frac{1}{p}]$ -algebras, the spectral sequence from étale cohomology to étale K -theory [32, Proposition 5.1 on page 260], and the identification of étale cohomology with continuous group cohomology [84, Proposition II.2.9

on page 170] induce the following commutative diagram. The top horizontal map is the map in the Schneider [Conjecture 2.7](#).

$$\begin{array}{ccc}
 K_{2i-1}(\mathcal{O}_F; \mathbb{Z}_p) & \longrightarrow & \prod_{\mathfrak{p}|p} K_{2i-1}(\mathcal{O}_{\mathfrak{p}}; \mathbb{Z}_p) \\
 \textcircled{1} \downarrow & & \downarrow \textcircled{1} \\
 K_{2i-1}(\mathcal{O}_F[\frac{1}{p}]; \mathbb{Z}_p) & \longrightarrow & \prod_{\mathfrak{p}|p} K_{2i-1}(F_{\mathfrak{p}}; \mathbb{Z}_p) \\
 \textcircled{2} \downarrow & & \downarrow \textcircled{2} \\
 K_{2i-1}^{\text{ét}}(\mathcal{O}_F[\frac{1}{p}]; \mathbb{Z}_p) & \longrightarrow & \prod_{\mathfrak{p}|p} K_{2i-1}^{\text{ét}}(F_{\mathfrak{p}}; \mathbb{Z}_p) \\
 \textcircled{3} \downarrow & & \downarrow \textcircled{3} \\
 H_{\text{ét}}^1(\mathcal{O}_F[\frac{1}{p}]; \mathbb{Z}_p(i)) & \longrightarrow & \prod_{\mathfrak{p}|p} H_{\text{ét}}^1(F_{\mathfrak{p}}; \mathbb{Z}_p(i)) \\
 \textcircled{4} \downarrow & & \downarrow \textcircled{4} \\
 H_{\text{cont}}^1(G_{\Sigma}; \mathbb{Z}_p(i)) & \longrightarrow & \prod_{\mathfrak{p}|p} H_{\text{cont}}^1(G_{\mathfrak{p}}; \mathbb{Z}_p(i))
 \end{array} \tag{18.1}$$

Here G_{Σ} is the Galois group of the extension $F_{\Sigma}|F$, where F_{Σ} is the union of all finite extensions F' of F for which $\mathcal{O}_F[\frac{1}{p}] \rightarrow \mathcal{O}_{F'}[\frac{1}{p}]$ is unramified [[104, page 182](#)], [[84, Section I.4 on page 48](#)], and $G_{\mathfrak{p}}$ denotes the absolute Galois group of the local field $F_{\mathfrak{p}}$. We write $r_1(F)$ for the number of real embeddings and $r_2(F)$ for the number of pairs of complex embeddings of F . The degree of $F|\mathbb{Q}$ is $r_1(F) + 2r_2(F)$.

Theorem 18.2. *Let F be a number field, and if $p = 2$ assume that F is totally imaginary. Let $i \geq 2$. In diagram (18.1), all vertical maps are isomorphisms, and therefore if one of the horizontal maps is \mathbb{Q} -injective, then all the horizontal maps are \mathbb{Q} -injective. All groups in (18.1) are finitely generated \mathbb{Z}_p -modules, and their \mathbb{Z}_p -ranks are given in the following table.*

	Left column	Right column
i odd	$r_1(F) + r_2(F)$	$r_1(F) + 2r_2(F)$
i even	$r_2(F)$	$r_1(F) + 2r_2(F)$

Proof. The fact that the maps $\textcircled{1}$ and $\textcircled{1}$ are isomorphisms follows from the localization sequence in algebraic K -theory [[92, Section 5, Corollary on page 113](#)] and Quillen’s computation [[91, Theorem 8 on page 583](#)] of the algebraic K -theory groups of finite fields. Notice that p -completion of spectra preserves stable homotopy fibration sequences; see Subsection 4N.

The global and local Lichtenbaum–Quillen conjectures, asserting that the maps $\textcircled{2}$ and $\textcircled{2}$ are isomorphisms, follow from celebrated results of Voevodsky and others. For fields of characteristic zero, the Milnor and Bloch–Kato conjectures were confirmed

in [116, Corollary 7.5 on page 97] and [117, Theorem 6.1 on page 429], and shown to imply the Beilinson–Lichtenbaum conjecture in [112, Theorem 7.4 on page 169]. Using Grayson’s [47,111] or Levine’s [62,115] constructions of the motivic spectral sequence, it follows that algebraic and étale K -theory, both with \mathbb{Z}_p -coefficients, agree in positive degrees for global and local number fields. Using localization techniques, as in [100, Section 6] and [61], it follows that the Dwyer–Friedlander comparison map is also an isomorphism for the rings of p -integers in global number fields. For $p = 2$, the comparison is made for totally imaginary number fields in [99, Theorem 0.1 on page 101], and for real number fields in [88, Theorem 4 on page 201].

Independently, ❷ was shown to be an isomorphism for p odd by Hesselholt and Madsen [52, Theorem 6.1.10 on page 103], using topological cyclic homology and cyclotomic trace methods.

The isomorphisms ❸ and ❹ follow at once from the spectral sequence of [32, Proposition 5.1 on page 260], since $\mathcal{O}_F[\frac{1}{p}]$ and F_p have étale cohomological dimension 2. For $p = 2$ this uses our hypothesis that F is totally imaginary.

For the isomorphisms ❺ and ❻, see [84, Proposition II.2.9 on page 170] and [83, III.1.7 on page 86].

The rank of $K_{2i-1}(\mathcal{O}_F)$ was calculated by Borel [19, Proposition 12.2 on page 271], and agrees with the \mathbb{Z}_p -rank of $K_{2i-1}(\mathcal{O}_F; \mathbb{Z}_p)$ by Quillen’s finite generation theorem [93, Theorem 1 on page 179]. A similar calculation of the \mathbb{Z}_p -rank of $K_{2i-1}(F_p; \mathbb{Z}_p)$ was outlined in [118]; see also [89]. The corresponding results for Galois cohomology were proven by Soulé in [108, Fact b) on page 376 and Remark 1 on page 390], as applications of Iwasawa theory. For the local result, see also [104, §3 Satz 4 on page 188]. □

Theorem 18.3. *Assume that p is odd, and let $i \geq 2$. The following statements are equivalent.*

- (i) *The horizontal maps in diagram (18.1) are \mathbb{Q} -injective.*
- (ii) *The group $H^2(G_\Sigma; \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ is finite.*
- (iii) *The group $H^2(G_\Sigma; \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ is zero.*
- (iv) *The group $H^2_{\text{cont}}(G_\Sigma; \mathbb{Z}_p(1-i))$ is finite.*
- (v) *The kernel of the map $H^1(G_\Sigma; \mathbb{Q}_p/\mathbb{Z}_p(i)) \longrightarrow \prod_{\mathfrak{p}|p} H^1(G_{\mathfrak{p}}; \mathbb{Q}_p/\mathbb{Z}_p(i))$ is finite.*
- (vi) *Let $F(\mu_{p^\infty})$ denote the union of the fields $F(\zeta_{p^n})$. Let L be the maximal abelian pro- p -extension of $F(\mu_{p^\infty})$ for which all primes are completely split. Let $G = \text{Gal}(F(\mu_{p^\infty})|F)$ and let Z be the Pontryagin dual of $\text{Gal}(L|F(\mu_{p^\infty}))$. Then $H^0(G; Z(i))$ is finite.*

Furthermore, for a fixed number field F and a fixed odd prime p , the equivalent conditions above hold for almost all $i \geq 2$.

That $H^2(G_\Sigma; \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ is finite was conjectured by Schneider in [104]. Neukirch, Schmidt, and Wingberg [86, page 641] also refer to this assertion as a conjecture of

Schneider. The notation $H_{\Sigma}^{2,1-i} = H^2(G_{\Sigma}; \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ for arbitrary $i \in \mathbb{Z}$ is introduced in [104, page 189]. For $i = 0$ this group is computed in [104, §4 Lemma 2ii) on page 190]. For $i \neq 0$ the groups are of the form $H_{\Sigma}^{2,1-i} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{i_{1-i}(F)}$ for some finite numbers $i_{1-i}(F)$ by [104, §4 Lemma 2i) on page 190]. On page 192 it is conjectured that the numbers $i_{1-i}(F) = 0$ for all $i \neq 0$.

Proof of Theorem 18.3. This is due to Schneider [104].

We first explain the equivalence of (i) and (ii). The lower horizontal map in (18.1) sits in the Poitou–Tate 9-term exact sequence, where the group to the left is the Pontryagin dual of the group in (ii), and the group further left is finite since $H_{\text{cont}}^0(G_p; \mathbb{Z}_p(i))$ is finite cyclic; compare [124, Lemma 13 on page 148]. Indeed, let Σ denote the finite set of primes over p together with all infinite primes. Then the Poitou–Tate exact sequence as stated for example in [86, (8.6.10) on page 489], with coefficients in $\mathbb{Z}/p^{\nu}\mathbb{Z}(i)$ and ramification allowed over Σ , consists of finite groups by [104, §2 Satz 1i) and Satz 2i) on page 185]. Passing to the limit over the reduction maps $\mathbb{Z}/p^{\nu+1}\mathbb{Z}(i) \rightarrow \mathbb{Z}/p^{\nu}\mathbb{Z}(i)$ yields the desired exact sequence.

Passing instead to the colimit over the inclusions $\mathbb{Z}/p^{\nu}\mathbb{Z}(i) \rightarrow \mathbb{Z}/p^{\nu+1}\mathbb{Z}(i)$, we obtain similarly the equivalence of (iv) and (v); compare [104, §5 proof of Lemma 3 on page 196].

Statement (ii) is equivalent to (iii) since the group is divisible by [104, §4 Lemma 2i) on page 190].

The equivalence of (iii) and (v) is [104, §5 Corollary 4 on page 197], where the notation $R_i(F)$ for the kernel of the map in (v) is introduced on page 196.

Then $R_i(F)$ is identified with $H^0(G; Z(i))$ in [104, §6 Lemma 1 on page 199], proving the equivalence of (v) and (vi).

Finally, the last claim is [104, Satz 3 on page 200]. \square

Lemma 18.4. Fix a prime p and an integer $c \geq 3$. Let $t = 2i - 1$ with $i \geq 2$.

- (i) Suppose $E|F$ is a finite extension. If the Schneider Conjecture 2.7 holds for E and t , then it holds for F and t .
- (ii) Let $\mathbb{Q}(\zeta_c)^+$ denote the maximal real subfield of $\mathbb{Q}(\zeta_c)$. Assume that i is odd. If the Schneider Conjecture 2.7 holds for $\mathbb{Q}(\zeta_c)^+$ and t , then it holds for $\mathbb{Q}(\zeta_c)$ and t .

Proof. (i) The inclusion of F into E induces the following commutative diagram.

$$\begin{array}{ccc}
 K_{2i-1}(\mathcal{O}_F; \mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & K_{2i-1}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_F; \mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q} \\
 \downarrow & & \downarrow \\
 K_{2i-1}(\mathcal{O}_E; \mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & K_{2i-1}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_E; \mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q}
 \end{array}$$

The vertical maps are injective, since composed with restriction both maps are isomorphisms.

(ii) Consider the diagram above for $F = \mathbb{Q}(\zeta_c)^+$ and $E = \mathbb{Q}(\zeta_c)$. If i is odd then the vertical map on the left is an isomorphism. This follows from the statement about ranks in [Theorem 18.2](#) and since $r_1(E) + r_2(E) = \varphi(c)/2 = r_1(F) + r_2(F)$. \square

Remark 18.5. There are partial results about the equivalent statements in [Theorem 18.3](#), but unfortunately they cannot be used to verify Assumption $[B_{\mathcal{F}Cyc}]$. For example, Soule verified in [\[107, Théorème 5 on page 268\]](#) that the condition (iii) of [Theorem 18.3](#) holds for all $i < 0$. The results in [\[104, §6 Satz 6 on page 201\]](#) and [\[42, Lemma 4.3a\) and b\)\]](#) deal with totally real number fields and i even or negative.

We close the paper by showing how the last statement in [Theorem 18.3](#) leads to [Theorem 1.15](#).

Proof of Theorem 1.15. If $EG(\mathcal{F}in)$ is finite, then by [Proposition 2.1](#) there are only finitely many conjugacy classes of finite cyclic subgroup of G . So the last statement in [Theorem 18.3](#) and [Proposition 2.8](#) imply that there exists an $M \in \mathbb{N}$ such that Assumption $[B_{\mathcal{F}Cyc, [M, \infty)}]$ holds. By [Corollary 2.2](#), Assumption $[A_{\mathcal{F}Cyc}]$ holds, and so also $[A'_{\mathcal{F}Cyc, \leq N}]$ holds for every $N \in \mathbb{N}$. Moreover, by [Example 12.11](#), for every finite cyclic subgroup C of G we have $H_s(BZ_G C; \mathbb{Q}) \cong H_s(EG(\mathcal{F}in)^C/Z_G C; \mathbb{Q})$, and these groups vanish if $s > D = \dim EG(\mathcal{F}in)$. Now apply [Main Technical Theorem 1.16](#) to obtain [Theorem 1.15](#) for $L = M + D$. \square

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