

TOPOLOGICAL CYCLIC HOMOLOGY

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We survey the Bökstedt–Hsiang–Madsen definition and Nikolaus–Scholze redefinition of topological cyclic homology, and exhibit a third presentation emphasized by Drinfeld and Bhatt–Lurie.

1. TRACE MAPS

For a ring R and a finitely generated projective R -module P , Hattori [Hat65] and Stallings [Sta65] studied the trace of an endomorphism $f: P \rightarrow P$ an element $\mathrm{tr}(f)$ in $\mathrm{HH}_0(R) = R/[R, R]$. It satisfies the cyclic invariance property that $\mathrm{tr}(f_1 f_2) = \mathrm{tr}(f_2 f_1)$ when $f_1: P_1 \rightarrow P_2$ and $f_2: P_2 \rightarrow P_1$, so that both composites make sense. Applied to the identity automorphism of P it yields the rank of P . The resulting homomorphisms

$$K_0(R) \longrightarrow K_0^{\mathrm{cy}}(R) \longrightarrow \mathrm{HH}_0(R)$$

were extended to homomorphisms

$$K_*(R) \longrightarrow \mathrm{HH}_*(R)$$

for $* \geq 0$ by Dennis (late 1970s), to a spectrum map

$$A(X) = K(\mathbb{S}[G]) \longrightarrow \mathrm{THH}(\mathbb{S}[G]) \simeq \Sigma_+^\infty(\Lambda X)$$

by Waldhausen [Wal79], where X is a based and connected space, $\Omega X \simeq G$ and $\Lambda X = \mathrm{Map}(S^1, X) \simeq B^{\mathrm{cy}}(G)$, and to the trace map

$$\mathrm{tr}: K(B) \longrightarrow \mathrm{THH}(B)$$

for general $\mathbb{E}_1$ -rings B by Bökstedt (ca. 1986). Maybe ‘rank’ would be more appropriate than ‘trace’, but the terminology is fixed by now.

2. THE BÖKSTEDT–HSIANG–MADSEN DEFINITION

Recall that $\mathrm{THH}(B)$ is the realization of a cyclic object, therefore has a natural S^1 -action. In a letter to Waldhausen, Goodwillie (July/August 1987) argued that the trace map extends over cyclic K -theory, see [Sch09], and also that it lifts through the C_m -fixed points of $\mathrm{THH}(B)$ for each finite subgroup $C_m \subset S^1$.

$$\begin{array}{ccc} K(B) & \longrightarrow & \mathrm{THH}(B)^{C_m} \\ \downarrow & \searrow \mathrm{tr} & \downarrow \\ K^{\mathrm{cy}}(B) & \longrightarrow & \mathrm{THH}(B) \end{array}$$

The C_m -action on $\mathrm{THH}(B)$ roughly takes the trace of $f_1 \dots f_{m-1} f_m$ to the trace of $f_m f_1 \dots f_{m-1}$. In particular, it fixes the traces of m -fold iterated endomorphisms of the form f^m , including the case where f is an identity morphism, and this leads to the displayed lift of tr .

We shall concentrate on p -completed constructions, for a fixed prime p , and may then restrict attention to the cyclic p -groups $C_{p^n} \subset S^1$, with $m = p^n$ in the above. Associated to the inclusion $C_{p^{n-1}} \subset C_{p^n}$ there is a natural forgetful map

$$F: \mathrm{THH}(B)^{C_{p^n}} \longrightarrow \mathrm{THH}(B)^{C_{p^{n-1}}},$$

later seen by Hesselholt [HM97] (with some input in 1994 from Pirashvili) to be related to the p -typical Witt vector Frobenius F . Moreover, associated to the projection $C_{p^n} \rightarrow C_{p^n}/C_p \cong C_{p^{n-1}}$, there is a natural restriction map

$$R: \mathrm{THH}(B)^{C_{p^n}} \longrightarrow \mathrm{THH}(B)^{C_{p^{n-1}}},$$

related to the p -typical Witt vector restriction R . This is a particular feature of the spectrum $\mathrm{THH}(B)$, not shared by general spectra with S^1 -action. More precisely, there is an S^1 -equivariant equivalence

$$(1) \quad \mathrm{THH}(B)^{\varphi C_p} \simeq \mathrm{THH}(B),$$

where $X^{\varphi C_p} = \Phi^{C_p}(X)$ denotes the geometric C_p -fixed points of X . (This equivalence was the 20th century formulation of cyclotomic structure on $\mathrm{THH}(B)$.) When combined with the natural map $r: X^{C_p} \rightarrow X^{\varphi C_p}$, the composite

$$\mathrm{THH}(B)^{C_{p^n}} \xrightarrow{r^{C_{p^{n-1}}}} (\mathrm{THH}(B)^{\varphi C_p})^{C_{p^{n-1}}} \simeq \mathrm{THH}(B)^{C_{p^{n-1}}}$$

specifies R .

In the case $B = \mathbb{S}[\Omega X]$, the R -map corresponds to a map

$$(\Sigma_+^\infty(\Lambda X))^{C_{p^n}} \longrightarrow (\Sigma_+^\infty(\Lambda X)^{C_p})^{C_{p^{n-1}}} \cong (\Sigma_+^\infty \Lambda X)^{C_{p^{n-1}}}$$

that first restricts a C_{p^n} -map $S^V \rightarrow (\Lambda X)_+ \wedge S^V$ to its C_p -fixed points $S^W \rightarrow (\Lambda X)_+^{C_p} \wedge S^W$. Here V is some C_{p^n} -representation, and $W = V^{C_p}$ is a $C_{p^n}/C_p \cong C_{p^{n-1}}$ -representation. Next it uses the p -th root homeomorphism $\rho: S^1 \cong S^1/C_p$ to identify $(\Lambda X)^{C_p}$ with ΛX .

The lifts $\mathrm{tr}_n: K(B) \rightarrow \mathrm{THH}(B)^{C_{p^n}}$ were shown by Bökstedt, Hsiang and Madsen [BHM93] to be compatible under both the F - and the R -maps, so that

$$F \circ \mathrm{tr}_n \simeq \mathrm{tr}_{n-1} \simeq R \circ \mathrm{tr}_n: K(B) \longrightarrow \mathrm{THH}(B)^{C_{p^{n-1}}}.$$

$$\begin{array}{ccc} K(B) & \xrightarrow{\quad \mathrm{tr}_{n-1} \quad} & \mathrm{THH}(B)^{C_{p^{n-1}}} \\ & \searrow \mathrm{tr}_n & \uparrow F \\ & \mathrm{THH}(B)^{C_{p^n}} & \xrightarrow[\quad R \quad]{\quad F \quad} \mathrm{THH}(B)^{C_{p^{n-1}}} \end{array}$$

The first compatibility leads to a lift

$$\mathrm{tr}_F: K(B) \longrightarrow \mathrm{TF}(B) := \lim_{n, F} \mathrm{THH}(B)^{C_{p^n}}$$

of the Bökstedt trace tr , where the (homotopy) limit defining the ‘topological Frobenius homology’ $\mathrm{TF}(B)$ is formed over

$$\dots \xrightarrow{F} \mathrm{THH}(B)^{C_{p^n}} \xrightarrow{F} \mathrm{THH}(B)^{C_{p^{n-1}}} \xrightarrow{F} \dots \xrightarrow{F} \mathrm{THH}(B).$$

Moreover, F and R commute, so the R -maps induce a self-map $R: \mathrm{TF}(B) \rightarrow \mathrm{TF}(B)$, and the second compatibility implies that $\mathrm{tr}_F \simeq R \circ \mathrm{tr}_F$. This leads to a lift

$$\mathrm{trc}: K(B) \longrightarrow \mathrm{TC}(B) := \mathrm{eq}(\mathrm{id}, R: \mathrm{TF}(B) \rightarrow \mathrm{TF}(B))$$

where the ‘topological cyclic homology’ $\mathrm{TC}(B)$ is defined to be the (homotopy) equalizer of the two maps

$$\mathrm{TC}(B) \xrightarrow{\pi} \mathrm{TF}(B) \begin{array}{c} \xrightarrow{\mathrm{id}} \\ \xrightarrow{R} \end{array} \mathrm{TF}(B).$$

This lift is called the ‘cyclotomic trace map’, following a suggestion by Soulé. These constructions can be shown to be lax symmetric monoidal in B , so if B is an $\mathbb{E}_k$ -ring for $k \geq 1$ then $\mathrm{TC}(B)$ is an $\mathbb{E}_{k-1}$ -ring and trc is an $\mathbb{E}_{k-1}$ -map. Additively, $\mathrm{TC}(B)$ is also the fiber of $R - \mathrm{id}$.

3. TOPOLOGICAL RESTRICTION HOMOLOGY

Goodwillie (MSRI lectures, 1990) observed that the situation is symmetric in F and R , so one can also define the ‘topological restriction homology’ $\mathrm{TR}(B)$ as the (homotopy) limit over

$$\dots \xrightarrow{R} \mathrm{THH}(B)^{C_{p^n}} \xrightarrow{R} \mathrm{THH}(B)^{C_{p^{n-1}}} \xrightarrow{R} \dots \xrightarrow{R} T\mathrm{HH}(B),$$

obtain a lift

$$\mathrm{tr}_R: K(B) \longrightarrow \mathrm{TR}(B) = \lim_{n, R} \mathrm{THH}(B)^{C_{p^n}}$$

of tr , note that F induces a self-map $F: \mathrm{TR}(B) \rightarrow \mathrm{TR}(B)$, and see that

$$\mathrm{TC}(B) \simeq \mathrm{eq}(\mathrm{id}, F: \mathrm{TR}(B) \rightarrow \mathrm{TR}(B)),$$

with trc lifting tr_R . For connective B with $\pi_0(B)$ commutative, $\pi_0 \mathrm{TR}(B) \cong W(\pi_0(B))$ is given by the p -typical Witt vectors [HM97], and

$$\pi_{-1} \mathrm{TC}(B) \cong W(\pi_0(B))_F$$

is given by its Frobenius coinvariants. For example, if $\pi_0(B) = \mathbb{F}_q$ is a field with p^f elements, then $W(\mathbb{F}_q)$ is an unramified degree f extension of the ring $\mathbb{Z}_p$ of p -adic integers, $\mathbb{Z}_p = W(\mathbb{F}_p) \cong W(\mathbb{F}_q)^F$, and also $W(\mathbb{F}_q)_F \cong \mathbb{Z}_p$.

4. HOMOTOPY FIXED POINTS

In the case $B = \mathbb{S}[\Omega X]$ for a space X , one can use the Segal–tom Dieck splitting formula to analyze the fixed point spectra $\mathrm{THH}(\mathbb{S}[\Omega X])^{C_{p^n}}$, and thereby $\mathrm{TC}(\mathbb{S}[\Omega X])$. This was carried out in [BHM93]. For ordinary rings, and $\mathbb{E}_1$ -rings closer to these, a different strategy was pursued by Bökstedt–Madsen [BM94], [BM95] for $B = \mathbb{Z}_p$ or $W(\mathbb{F}_q)$ at p odd, Hesselholt–Madsen [HM97] for $B = \mathbb{F}_q$, R. [Rog99] for $B = \mathbb{Z}_2$, Ausoni–R. [AR02] for $B = \mathrm{BP}\langle 1 \rangle$ at $p \geq 5$, [AR12] for $B = k(1)$ at $p \geq 5$, and Ausoni [Aus10] for $B = \mathrm{ku}$ at $p \geq 5$ and Angelini–Knoll–Ausoni–Culver–Höning–R. [AKAC⁺25] for $B = \mathrm{BP}\langle 2 \rangle$ at $p \geq 7$.

This approach relies on comparing G -fixed points with G -homotopy fixed points, using the comparison map

$$X^G = F(S^0, G)^G \longrightarrow F(EG_+, X)^G = X^{hG}$$

induced by the collapse map $c: EG_+ \rightarrow *_+ = S^0$. Here X is any G -spectrum. When G is a finite p -group and $X = \mathbb{S}$, the Segal conjecture asserts that this map becomes an equivalence after p -completion:

$$\mathbb{S}^G \xrightarrow{\simeq_p} \mathbb{S}^{hG}.$$

This was proven for $G = C_2$ by Lin, and for $G = C_p$ with p odd by Gunawardena. The extension to general p -groups is due to Carlsson [Car84]. This gives computational access to G -fixed points, since there is a ‘ G -homotopy fixed point spectral sequence’

$$\mathcal{E}_{s,t}^2 = H^{-s}(G; \pi_t(X)) \implies \pi_{s+t}(X^{hG}).$$

Here $H^*(G; -)$ denotes group cohomology. For example, $H^*(C_{p^n}; \mathbb{F}_p) = \Lambda(u_n) \otimes \mathbb{F}_p[t_n]$ for $p^n > 2$, with $u_n \in H^1$ and $t_n \in H^2$. (For $p^n = 2$ we instead have $u_1^2 = t_1$.)

For $X = \mathrm{THH}(B)$, Carlsson’s argument was adapted by Tsalidis [Tsa97], [Tsa98], proving that if

$$\Gamma_1 : \mathrm{THH}(B)^{C_p} \xrightarrow{\sim} \mathrm{THH}(B)^{hC_p}$$

is an equivalence above some degree k , then the same holds for Γ_n for each $n \geq 1$. A moderate generalization, with coefficients, appeared in [BBLNR14]. For example, it is known for $B = \mathbb{F}_q$ that Γ_1 is an equivalence above degree -2 , while for $B = \mathbb{Z}_p$ or $W(\mathbb{F}_q)$ the mod p version of Γ_1 is an equivalence above degree -1 . Somewhat surprisingly, for $B = \mathrm{MU}$ and $B = \mathrm{BP}$ the map Γ_1 is an equivalence in all degrees [LNR11], as in the original Segal conjecture.

Thus, if one can verify that Γ_1 is an equivalence above some degree k , then all C_{p^n} -fixed points in the definition of $\mathrm{TC}(B)$ can be replaced by C_{p^n} -homotopy fixed points, making computations feasible. Passing to the (homotopy) limit, it also follows that

$$\Gamma : \mathrm{TF}(B) = \lim_n \mathrm{THH}(B)^{C_{p^n}} \xrightarrow{\sim} \lim_n \mathrm{THH}(B)^{hC_{p^n}} \simeq_p \mathrm{THH}(B)^{hS^1}$$

is an equivalence above degree k (or so). Here

$$\mathrm{TC}^-(B) := \mathrm{THH}(B)^{hS^1}$$

can be called the ‘topological negative homology’ of B , and should maybe be denoted $\mathrm{TN}(B)$. There is an S^1 -homotopy fixed point spectral sequence

$$(2) \quad \mathcal{E}_{s,t}^2 = H^{-s}(S^1; \pi_* \mathrm{THH}(B)) \implies \pi_{s+t}(\mathrm{TC}^-(B)),$$

with $H^*(S^1; \mathbb{F}_p) = \mathbb{F}_p[t]$ where $t \in H^2$.

5. TATE FIXED POINTS

When $G = C_{p^n}$, the comparison map extends to a comparison of ‘norm–restriction sequences’, or ‘isotropy separation sequences’, i.e., a vertical map

$$\begin{array}{ccccccc} X_{hC_{p^n}} & \xrightarrow{N} & X^{C_{p^n}} & \xrightarrow{R} & (X^{\varphi C_p})^{C_{p^{n-1}}} & \xrightarrow{\partial} & \Sigma X_{hC_{p^n}} \\ \parallel & & \downarrow \Gamma_n & & \downarrow \hat{\Gamma}_n & & \parallel \\ X_{hC_{p^n}} & \xrightarrow{N^h} & X^{hC_{p^n}} & \xrightarrow{\mathrm{can}} & X^{tC_{p^n}} & \xrightarrow{\partial^h} & \Sigma X_{hC_{p^n}} \end{array}$$

of horizontal cofiber sequences. The central square is Cartesian, and is also called the ‘Tate square’. Letting $\widetilde{EG}$ denote the cofiber of $c : EG_+ \rightarrow S^0$, the G -Tate fixed point spectrum, or G -Tate construction, admits the model

$$X^{tG} = [\widetilde{EG} \wedge F(EG_+, X)]^G.$$

The latter vanishes for G -free spectra, such as $X = \mathbb{S}[G]$, and can be viewed as a localization of the G -homotopy fixed points away from these. The ‘canonical map’

from homotopy fixed points to Tate fixed points was previously denoted R^h and referred to as the homotopy restriction map.

The homotopy orbits X_{hG} and Tate fixed points X^{tG} are also computationally accessible, using the ‘ G -homotopy orbit spectral sequence’

$$\mathcal{E}_{s,t}^2 = H_s(G; \pi_t(X)) \implies \pi_{s+t}(X_{hG})$$

and the ‘ G -Tate spectral sequence’

$$\mathcal{E}_{s,t}^2 = \hat{H}^{-s}(G; \pi_t(X)) \implies \pi_{s+t}(X^{tG}).$$

Here $\hat{H}^*(G; -)$ denotes Tate cohomology. For example, $\hat{H}^*(C_{p^n}; \mathbb{F}_p) = \Lambda(u_n) \otimes \mathbb{F}_p[t_n^{\pm 1}]$ for $p^n > 2$, with $u_n \in H^1$ and $t_n \in H^2$. (For $p^n = 2$ we instead have $u_1^2 = t_1$.)

Notice that Γ_n is an equivalence above degree k if and only if $\hat{\Gamma}_n$ is an equivalence above degree k , and by Tsalidis’s theorem, this holds if and only if

$$\hat{\Gamma}_1: X^{\varphi_{C_p}} \xrightarrow{\sim} X^{tC_p}$$

is an equivalence above degree k . When $X = \mathrm{THH}(B)$, the classical cyclotomic equivalence (1) lets us rewrite this as an S^1 -map

$$(3) \quad \varphi_p: \mathrm{THH}(B) \simeq \mathrm{THH}(B)^{\varphi_{C_p}} \xrightarrow{\hat{\Gamma}_1} \mathrm{THH}(B)^{tC_p}.$$

The 21st century Nikolaus–Scholze [NS18] formulation of the cyclotomic structure on $\mathrm{THH}(B)$ revolves only around φ_p as a map of spectra with S^1 -action, called the ‘cyclotomic structure map’, or perhaps the ‘Frobenius map’. If it is an equivalence above degree k , it then follows that

$$\hat{\Gamma}: \mathrm{TF}(B) = \lim_n \mathrm{THH}(B)^{C_{p^{n-1}}} \xrightarrow{\sim} \lim_n \mathrm{THH}(B)^{tC_{p^n}} \simeq_p \mathrm{THH}(B)^{tS^1}$$

is an equivalence above degree k (or so). Here

$$\mathrm{TP}(B) := \mathrm{THH}(B)^{tS^1}$$

can be called the ‘topological periodic homology’ of B . There is an S^1 -Tate spectral sequence

$$(4) \quad \mathcal{E}_{s,t}^2 = \hat{H}^{-s}(S^1; \pi_* \mathrm{THH}(B)) \implies \pi_{s+t}(\mathrm{TP}(B)),$$

with $\hat{H}^*(S^1; \mathbb{F}_p) = \mathbb{F}_p[t^{\pm 1}]$ where $t \in H^2$, see e.g. [HR24].

Remark 5.1. In the original algebraic setting, $\mathrm{HC}(R) \simeq \mathrm{HH}(R)_{hS^1}$ is called cyclic homology, $\mathrm{HC}^-(R) \simeq \mathrm{HH}(R)^{hS^1}$ is called negative cyclic homology, and $\mathrm{HP}(R) \simeq \mathrm{HH}(R)^{tS^1}$ is called periodic cyclic homology. At first, HP was denoted $\mathrm{HC}^{\mathrm{per}}$, but the superscript was cumbersome. For the same reason, some authors switched from HC^- to HN . Unfortunately, the same typographic improvement was not widely adopted in the topological setting.

6. THE NIKOLAUS–SCHOLZE REFORMULATION

The following equivalence was verified by direct calculation in the earlier papers [BM94, Thm. 6.7], [HM97, p. 69], [AR02, Lem. 7.1], but is now known to hold e.g. for all connective B . It is often omitted from the notation, but we retain it for clarity.

Proposition 6.1. *If $\mathrm{THH}(B)$ is bounded below, then the natural map*

$$G: \mathrm{TP}(B) = \mathrm{THH}(B)^{tS^1} = (\mathrm{THH}(B)^{tC_p})^{S^1} \xrightarrow{\simeq_p} (\mathrm{THH}(B)^{tC_p})^{hS^1}$$

from the S^1 -fixed points to the S^1 -homotopy fixed points of $\mathrm{THH}(B)^{tC_p}$ becomes an equivalence after p -completion.

Proof. If $H_*(\mathrm{THH}(B); \mathbb{F}_p)$ is of finite type, then this is [BBLNR14, Prop. 3.8]. This is the case, e.g. if $H_*(B; \mathbb{F}_p)$ is a connected $\mathbb{F}_p$ -algebra of finite type. The finite type hypothesis is removed in [NS18, Lem. II.4.2]. $\square$

To calculate $\mathrm{TC}(B)$ when $\mathrm{THH}(B)$ is bounded below, the traditional method of computation is to replace $\mathrm{TF}(B)$ in the source and target of id and R , via Γ and $G \circ \hat{\Gamma}$, by $\mathrm{TC}^-(B) = \mathrm{THH}(B)^{hS^1}$ and $(\mathrm{THH}(B)^{tC_p})^{hS^1}$, respectively, at least above the degree k mentioned above. Then R is replaced by $G \circ \mathrm{can}$, due to the commutative square

$$\begin{array}{ccc} \mathrm{TF}(B) & \xrightarrow{R} & \mathrm{TF}(B) \\ \Gamma \downarrow & & \downarrow G \circ \hat{\Gamma} \\ \mathrm{TC}^-(B) & \xrightarrow{G \circ \mathrm{can}} & (\mathrm{THH}(B)^{tC_p})^{hS^1} . \end{array}$$

Simultaneously, id is replaced by $\varphi_p^{hS^1}$, due to the commutative square

$$\begin{array}{ccc} \mathrm{TF}(B) & \xrightarrow{\mathrm{id}} & \mathrm{TF}(B) \\ \Gamma \downarrow & & \downarrow G \circ \hat{\Gamma} \\ \mathrm{TC}^-(B) & \xrightarrow{\varphi_p^{hS^1}} & (\mathrm{THH}(B)^{tC_p})^{hS^1} . \end{array}$$

This means that in range of degrees $* > k$, with k as before, $\mathrm{TC}(B)$ is equivalent after p -completion to the (homotopy) equalizer

$$\mathrm{TC}(B) \sim \mathrm{eq}(\varphi_p^{hS^1}, G \circ \mathrm{can}: \mathrm{TC}^-(B) \rightarrow (\mathrm{THH}(B)^{tC_p})^{hS^1})$$

of $\varphi_p^{hS^1}$ and $G \circ \mathrm{can}$. The notation simplifies if we invert G , and instead consider the equalizer of the two maps

$$(5) \quad \mathrm{TC}^-(B) \xrightarrow[\mathrm{can}]{G^{-1} \circ \varphi_p^{hS^1}} \mathrm{TP}(B) .$$

Additively, this is the fiber of $\mathrm{can} - G^{-1} \circ \varphi_p^{hS^1}$. Abbreviating $G^{-1} \circ \varphi_p^{hS^1}$ to φ , we see that $\mathrm{TC}(B)$ agrees after p -completion, above degree k , with the fiber of $\mathrm{can} - \varphi$. So we have an implicitly p -complete cofiber sequence

$$(6) \quad \Sigma^{-1}\mathrm{TP}(B) \xrightarrow{\partial} \mathrm{TC}(B) \xrightarrow{\pi} \mathrm{TC}^-(B) \xrightarrow{\mathrm{can} - \varphi} \mathrm{TP}(B) .$$

The computations then proceeded by analyzing the spectral sequences (2) and (4) to understand can , and a ‘localized S^1 -homotopy fixed point spectral sequence’

$$(7) \quad \mu^{-1}\mathcal{E}_{s,t}^2 = H^{-s}(S^1; \pi_t \mathrm{THH}(B)^{tC_p}) \implies \pi_{s+t}(\mathrm{THH}(B)^{tC_p})^{hS^1}$$

to access $\varphi_p^{hS^1}$ and the equivalence G .

Nikolaus-Scholze turned this calculational strategy into a definition.

Definition 6.2 ([NS18, Def. II.1.1]). A p -cyclotomic spectrum X is a bounded below, p -complete spectrum with S^1 -action, together with an S^1 -equivariant map

$$\varphi_p: X \longrightarrow X^{tC_p}.$$

Here S^1 acts on the target via the isomorphism $\rho: S^1 \cong S^1/C_p$. Let

$$\mathrm{TC}(X) = \mathrm{eq}(\varphi, \mathrm{can}: \mathrm{TC}^-(X) \rightarrow \mathrm{TP}(X))$$

be the p -completed equalizer of $\varphi = G^{-1} \circ \varphi_p^{hS^1}: X^{hS^1} \rightarrow (X^{tC_p})^{hS^1} \simeq X^{tS^1}$ and $\mathrm{can}: X^{hS^1} \rightarrow X^{tS^1}$.

For example, $X = \mathrm{THH}(B)_p^\wedge$ is a p -cyclotomic spectrum for each connective $\mathbb{E}_1$ -ring B , with the cyclotomic structure map $\varphi_p = \hat{\Gamma}_1: \mathrm{THH}(B) \rightarrow \mathrm{THH}(B)^{tC_p}$ (implicitly p -completed). Then $\mathrm{TC}(X)$ is the Nikolaus–Scholze definition of the spectrum $\mathrm{TC}(B)$. These constructions are lax symmetric monoidal in B , so if B is an $\mathbb{E}_k$ -ring for $k \geq 1$ then $\mathrm{TC}(B)$ is an $\mathbb{E}_{k-1}$ -ring and trc is an $\mathbb{E}_{k-1}$ -map.

Theorem 6.3 ([NS18, Thm. II.3.8]). *For each B with $\mathrm{THH}(B)_p^\wedge$ bounded below, the p -completed Bökstedt–Hsiang–Madsen spectrum $\mathrm{TC}(B)$ is equivalent to the p -completed Nikolaus–Scholze spectrum $\mathrm{TC}(B)$.*

In particular, the apparent restriction of the calculational method to degrees above k is illusory. This improvement played an essential role in [AKAC⁺25].

The cyclotomic structure map can be fairly directly defined in these terms. Recall that $\mathrm{THH}(B)$ realizes the cyclic spectrum

$$[q] \longmapsto \mathrm{THH}_q(B) = B^{\otimes 1+q}.$$

By [BHM93] there is an S^1 -equivariant equivalence

$$\mathrm{THH}(B) \simeq \mathrm{sd}_p \mathrm{THH}(B),$$

where $\mathrm{sd}_p \mathrm{THH}(B)$ realizes the p -fold edgewise subdivision

$$[q] \longmapsto \mathrm{sd}_p \mathrm{THH}_q(B) = \mathrm{THH}_{-1+p(1+q)}(B) = B^{\otimes p(1+q)}.$$

For each C the construction $R_+(C) := (C^{\otimes p})^{tC_p}$ is a topological model [LNR12] for the so-called Singer construction in the setting of modules over the Steenrod algebra. There is a natural structure map

$$\epsilon_C: C \longrightarrow R_+(C) = (C^{\otimes p})^{tC_p}$$

realizing Singer’s homomorphism $\epsilon^*: R_+(M) \rightarrow M$ in (continuous) cohomology, and which [NS18] rename the ‘Tate diagonal’ Δ_p . In the case of a suspension spectrum $C = \Sigma^\infty X$, it is given by the composite

$$\Sigma^\infty X \cong \Sigma^\infty((X^{\wedge p})^{C_p}) \longrightarrow ((\Sigma^\infty X)^{\otimes p})^{hC_p} \longrightarrow ((\Sigma^\infty X)^{\otimes p})^{tC_p},$$

and the general case follows by writing C as a colimit of shifts of suspension spectra.

The Singer structure maps $\epsilon_{B^{\otimes 1+q}}$ for $q \geq 0$ combine to a cyclic map, with S^1 -equivariant realization

$$\begin{aligned} \mathrm{THH}(B) &= \mathrm{colim}_{[q]} B^{\otimes 1+q} \longrightarrow \mathrm{colim}_{[q]} (B^{\otimes p(1+q)})^{tC_p} \\ &\xrightarrow{\kappa} (\mathrm{colim}_{[q]} B^{\otimes p(1+q)})^{tC_p} = \mathrm{sd}_p \mathrm{THH}(B)^{tC_p} \simeq \mathrm{THH}(B)^{tC_p}. \end{aligned}$$

This composite map defines φ_p .

7. CALCULATIONS FOR $B = \mathbb{F}_p$

Let us calculate $\pi_*\mathrm{TC}(\mathbb{F}_p)$, for which the published reference is [HM97]. Recall that $\pi_*\mathrm{THH}(\mathbb{F}_p) = \mathbb{F}_p[\mu]$ where $\mu = \mu_0$ has degree 2 and Hurewicz image $\sigma\bar{\tau}_0$ (or $\sigma\zeta_1$, when $p = 2$).

Lemma 7.1. *The S^1 -homotopy fixed point spectral sequence*

$$\mathcal{E}^2(S^1) = H^{-*}(S^1, \pi_*\mathrm{THH}(\mathbb{F}_p)) = \mathbb{F}_p[t, \mu] \implies \pi_*\mathrm{TC}^-(\mathbb{F}_p)$$

collapses. The image of $p \in \pi_0(\mathbb{S})$ under the unit map $\mathbb{S} \rightarrow \mathrm{TC}^-(\mathbb{F}_p)$ is detected by $t\mu \in \mathcal{E}_{-2,2}^\infty$. Hence

$$\pi_*\mathrm{TC}^-(\mathbb{F}_p) \cong \mathbb{Z}_p[t, \mu]/(t\mu = p)$$

with $|t| = -2$ and $|\mu| = 2$.

Proof. Let X/p denote the cofiber of $p: X \rightarrow X$. In the mod p spectral sequence

$$H^{-*}(S^1, \pi_*\mathrm{THH}(\mathbb{F}_p)/p) \cong \mathbb{F}_p[t, \mu] \otimes \Lambda(\bar{\tau}_0) \implies \pi_*\mathrm{TC}^-(\mathbb{F}_p)/p$$

we have $d^2(\bar{\tau}_0) = t \cdot \sigma\bar{\tau}_0 = t\mu$, so $t\mu$ maps to 0 mod p . Hence $t\mu$ detects a unit times p in the integral spectral sequence $\mathcal{E}^*(S^1)$, and we can normalize t (or μ) to make this unit equal to $+1$. $\square$

Lemma 7.2. *The S^1 -Tate spectral sequence*

$$\hat{\mathcal{E}}^2(S^1) = \hat{H}^{-*}(S^1, \pi_*\mathrm{THH}(\mathbb{F}_p)) = \mathbb{F}_p[t^{\pm 1}, \mu] \implies \pi_*\mathrm{TP}(\mathbb{F}_p)$$

collapses. Hence

$$\pi_*\mathrm{TP}(\mathbb{F}_p) \cong \mathbb{Z}_p[t^{\pm 1}]$$

with $|t| = -2$. The canonical map induces

$$\mathrm{can}_*: \mathbb{Z}_p[t, \mu]/(t\mu = p) \longrightarrow \mathbb{Z}_p[t^{\pm 1}]$$

given by $t \mapsto t$ and $\mu \mapsto pt^{-1}$.

Proof. This follows by naturality along can from the previous lemma. $\square$

Lemma 7.3. *The C_p -Tate spectral sequence*

$$\hat{\mathcal{E}}^2(C_p) = \hat{H}^{-*}(C_p, \pi_*\mathrm{THH}(\mathbb{F}_p)) = \mathbb{F}_p[u_1, t_1^{\pm 1}, \mu] \implies \pi_*\mathrm{THH}(\mathbb{F}_p)^{tC_p}$$

has a differential $d^3(u_1 t_1^{-1}) \doteq t_1 \mu$ and collapses at $\hat{\mathcal{E}}^4(C_p) = \mathbb{F}_p[t_1^{\pm 1}]$. Hence

$$\pi_*\mathrm{THH}(\mathbb{F}_p)^{tC_p} \cong \mathbb{F}_p[t_1^{\pm 1}]$$

with $|t_1| = -2$.

Proof. The composite $\mathbb{S} \rightarrow \mathrm{TP}(\mathbb{F}_p) \xrightarrow{F} \mathrm{THH}(\mathbb{F}_p)^{tC_p}$ factors through $\mathrm{THH}(\mathbb{F}_p)$, where $p = 0$ in $\pi_0\mathrm{THH}(\mathbb{F}_p)$, so $t_1\mu \in \hat{\mathcal{E}}^2(C_p)$ must be a boundary. The only possible source of this differential is $u_1 t_1^{-1} \in \mathcal{E}_{1,0}^2$. $\square$

Lemma 7.4. *The cyclotomic structure map induces*

$$\varphi_{p*}: \mathbb{F}_p[\mu] \longrightarrow \mathbb{F}_p[t_1^{\pm 1}]$$

given by $\mu \mapsto t_1^{-1}$ (up to a unit in $\mathbb{F}_p$), hence factors as the localization inverting μ , followed by an isomorphism

$$\mathbb{F}_p[\mu^{\pm 1}] = \pi_*\mathrm{THH}(\mathbb{F}_p)[\mu^{-1}] \cong \pi_*\mathrm{THH}(\mathbb{F}_p)^{tC_p}.$$

Proof. We know that $G_* \circ \text{can}_*$ and $\varphi_{p*}^{hS^1}$ agree on the image of $p \in \pi_0(\mathbb{S})$. Since

$$\mathbb{Z}_p \cong \pi_0 \text{TC}^-(\mathbb{F}_p) \xrightarrow{\text{can}_*} \pi_0 \text{TP}(\mathbb{F}_p) \xrightarrow{G_*} \pi_0(\text{THH}(\mathbb{F}_p)^{tC_p})^{hS^1}$$

are isomorphisms, the image of p in the target must generate a subgroup of index p . Since the image of p under $\varphi_{p*}^{hS^1}$ is detected by $t\varphi_p(\mu)$ in the S^1 -homotopy fixed point spectral sequence for $\text{THH}(\mathbb{F}_p)^{tC_p}$ (which collapses at $\mathcal{E}^2$, see the next lemma), this class must be nonzero, so $\varphi_p(\mu)$ is a unit times t_1^{-1} . $\square$

Lemma 7.5. *The S^1 -homotopy fixed point spectral sequence*

$$\mu^{-1}\mathcal{E}^2(S^1) = H^{-*}(S^1, \pi_* \text{THH}(\mathbb{F}_p)^{tC_p}) \cong \mathbb{F}_p[t, \mu^{\pm 1}] \implies \pi_*(\text{THH}(\mathbb{F}_p)^{tC_p})^{hS^1}$$

collapses. Hence

$$\pi_*(\text{THH}(\mathbb{F}_p)^{tC_p})^{hS^1} \cong \mathbb{Z}_p[\mu^{\pm 1}]$$

with $|\mu| = 2$. The map $\varphi_p^{hS^1}$ induces

$$\varphi_{p*}^{hS^1}: \mathbb{Z}_p[t, \mu]/(t\mu = p) \longrightarrow \mathbb{Z}_p[\mu^{\pm 1}]$$

given by $t \mapsto p\mu^{-1}$ and $\mu \mapsto \mu$.

Proof. This follows by naturality along $\varphi_p^{hS^1}$ from the previous lemma. $\square$

Lemma 7.6. *The equivalence G induces the isomorphism*

$$G_*: \mathbb{Z}_p[t^{\pm 1}] \xrightarrow{\cong} \mathbb{Z}_p[\mu^{\pm 1}]$$

given by $t \mapsto \mu^{-1}$, up to a unit in $\mathbb{Z}_p$. Hence

$$(\text{can} - \varphi)_*: \mathbb{Z}_p[t, \mu]/(t\mu = p) \longrightarrow \mathbb{Z}_p[t^{\pm 1}]$$

is given by

$$\begin{cases} t^i \mapsto t^i \mod p & \text{in degree } -2i < 0, \\ \mu^j \mapsto -t^{-j} \mod p & \text{in degree } 2j > 0, \text{ up to unit in } \mathbb{Z}_p, \\ 1 \mapsto 1 - 1 = 0 & \text{in degree } 0. \end{cases}$$

Proof. Being a graded ring isomorphism, G_* must take t to a unit times μ^{-1} . $\square$

Theorem 7.7 ([HM97]).

$$\pi_* \text{TC}(\mathbb{F}_p)_p^\wedge \cong \Lambda_{\mathbb{Z}_p}(\partial) = \mathbb{Z}_p\{1, \partial\}$$

with $|\partial| = -1$.

Proof. This follows from the previous lemma, using the long exact sequence associated to (6). $\square$

Since φ_p for $B = \mathbb{F}_p$ is an equivalence above degree $k = -2$, these calculations also apply for the original construction of TC , in degrees $* \geq -1$. On the other hand, each $\text{THH}(\mathbb{F}_p)^{C_{p^n}}$ is connective, and it is easy to show from this that $\text{TC}(\mathbb{F}_p)$ is (-1) -connective, recovering the statement in all degrees.

Since $\mathbb{F}_p$ is an $\mathbb{E}_\infty$ -ring, $\text{TC}(\mathbb{F}_p)$ is also an $\mathbb{E}_\infty$ -ring. While it is equivalent to $\mathbb{Z}_p \oplus \Sigma^{-1}\mathbb{Z}_p$ as a spectrum, the $\mathbb{E}_\infty$ -ring structure is not that of a graded commutative ring. By [BR10], we have $Q_{p-1}(\partial) = Q^0(\partial) = \partial \neq 0$, at least for $p = 2$.

8. A REDUCED DECOMPOSITION

Thinking informally of ring spectra as the rings of functions on suitable geometric objects, the definition of $\mathrm{TC}(B)$ as the equalizer of the two localization maps in (5) can be interpreted geometrically as the formation of the coequalizer $\mathrm{Spec} \mathrm{TC}(B) \sim B^{\mathrm{syn}}$ of two open immersions

$$B^\Delta \sim \mathrm{Spec} \mathrm{TP}(B) \xrightleftharpoons[\mathrm{can}^{\mathrm{op}}]{\varphi^{\mathrm{op}}} \mathrm{Spec} \mathrm{TC}^-(B) \sim B^{\mathrm{Nyg}}.$$

The reduction $\mathrm{TC}(B) \rightarrow \mathrm{TC}(B)/p$ is then dual to the inclusion of a divisor

$$B_{p=0}^{\mathrm{syn}} \sim \mathrm{Spec} \mathrm{TC}(B)/p \subset \mathrm{Spec} \mathrm{TC}(B) \sim B^{\mathrm{syn}}$$

given by the vanishing locus of p , and similarly for $\mathrm{TC}^-(B)/p$ and $\mathrm{TP}(B)/p$. In the context of thinking about prismatic and syntomic cohomology for $B = \mathbb{Z}_p$ as coherent cohomology over suitable stacks, Drinfeld [Dri24] observed that the reduced stack corresponding to $\mathrm{TC}^-(\mathbb{Z}_p)/p, v_1$ decomposes as a union of two divisors. As observed by Bhatt and Lurie [Bha22], this leads to an alternative view of $\mathbb{Z}_{p=v_1=0}^{\mathrm{syn}} \sim \mathrm{Spec} \mathrm{TC}(\mathbb{Z})/p, v_1$ as the sum of two Poincaré pairs. In the simpler case of $B = \mathbb{F}_p$, this corresponds to a decomposition of $\mathrm{TC}(\mathbb{F}_p)/p$, which we now outline. Both cases are generalized in current work with Devalapurkar and Hahn, concerning Poincaré duality in $\mathrm{TC}(\mathrm{BP}\langle n-1 \rangle)/p, \dots, v_n$ and related examples.

The relation $p = t\mu$ in $\pi_* \mathrm{TC}^-(\mathbb{F}_p)$ leads to a pullback square

$$\begin{array}{ccc} \mathrm{TC}^-(\mathbb{F}_p)/p & \xrightarrow{\mathrm{HT}} & \mathrm{TC}^-(\mathbb{F}_p)/t \\ \mathrm{dR} \downarrow & \lrcorner & \downarrow i \\ \mathrm{TC}^-(\mathbb{F}_p)/\mu & \xrightarrow{F} & \mathrm{TC}^-(\mathbb{F}_p)/(t, \mu). \end{array}$$

Here $\mathrm{TC}^-(\mathbb{F}_p)/t \simeq \mathrm{THH}(\mathbb{F}_p)$ and $\mathrm{TC}^-(\mathbb{F}_p)/(t, \mu) \simeq \mathrm{THH}(\mathbb{F}_p)/\mu \simeq \mathbb{F}_p$. Hence we can rewrite this square as in the larger diagram below.

$$\begin{array}{ccccc} & \mathrm{TC}(\mathbb{F}_p)/p & & \mathrm{THH}(\mathbb{F}_p)^{tC_p} & \leftarrow \\ & \searrow \pi/p & & \nearrow \varphi_p^{hS^1}/p & \\ \Sigma \mathrm{TC}(\mathbb{F}_p)/p & & \mathrm{TC}^-(\mathbb{F}_p)/p & \xrightarrow{\mathrm{HT}} & \mathrm{THH}(\mathbb{F}_p) \\ & \swarrow \partial/p & \downarrow \mathrm{dR} \lrcorner & \downarrow i & \\ & \mathrm{TP}(\mathbb{F}_p)/p & \xleftarrow{j} \mathrm{TC}^-(\mathbb{F}_p)/\mu & \xrightarrow{F} & \mathrm{THH}(\mathbb{F}_p)/\mu \\ & & & & \times \end{array} \quad \begin{array}{l} \nearrow \varphi_p \\ \simeq G/p \end{array}$$

In the target of j we have used that $t \in \pi_* \mathrm{TP}(\mathbb{F}_p)$ is a unit, so that $\mathrm{TP}(\mathbb{F}_p)/p \simeq \mathrm{TP}(\mathbb{F}_p)/\mu$. The map j then inverts t . In the target of $\varphi_p^{hS^1}/p$ we have used that $\mu \in \pi_*(\mathrm{THH}(\mathbb{F}_p)^{tC_p})^{hS^1}$ is a unit, so that $(\mathrm{THH}(\mathbb{F}_p)^{tC_p})^{hS^1}/p \simeq (\mathrm{THH}(\mathbb{F}_p)^{tC_p})^{hS^1}/t \simeq \mathrm{THH}(\mathbb{F}_p)^{tC_p}$. The symbol ‘ $\times$ ’ indicates that only the subdiagram without G/p commutes. Note that π/p and ∂/p are the fiber and cofiber of $(\mathrm{can} - \varphi)/p = \mathrm{can}/p - (G/p)^{-1} \circ \varphi_p^{hS^1}/p$, respectively, not e.g. of can/p .

At the level of homotopy groups, this appears as follows.

$$\begin{array}{ccccc}
 & \Lambda(\partial) & & \mathbb{F}_p[\mu^{\pm 1}] & \leftarrow \\
 & \searrow & & \uparrow & \\
 \Sigma\Lambda(\partial) & & \mathbb{F}_p[t, \mu]/(t\mu) & \xrightarrow{\quad} & \mathbb{F}_p[\mu] \\
 & \swarrow & \downarrow \text{J} & & \downarrow \\
 & \mathbb{F}_p[t^{\pm 1}] & \mathbb{F}_p[t] & \xrightarrow{\quad} & \mathbb{F}_p \\
 & \uparrow & & & \uparrow \\
 & & & & \mathbb{R} \\
 & & & & \times
 \end{array}$$

We obtain an alternative description of $\mathrm{TC}(\mathbb{F}_p)/p$ as the limit of

$$\begin{array}{ccc}
 \mathrm{THH}(\mathbb{F}_p) & & \mathrm{TC}^-(\mathbb{F}_p)/\mu \\
 \downarrow i & \swarrow \varphi_p \quad \searrow F & \downarrow G/p \circ j \\
 \mathrm{THH}(\mathbb{F}_p)/\mu & & \mathrm{THH}(\mathbb{F}_p)^{tC_p}
 \end{array}$$

or as the fiber of

$$\mathrm{THH}(\mathbb{F}_p) \oplus \mathrm{TC}^-(\mathbb{F}_p)/\mu \xrightarrow{\begin{bmatrix} -i & F \\ -\varphi_p & G/p \circ j \end{bmatrix}} \mathrm{THH}(\mathbb{F}_p)/\mu \oplus \mathrm{THH}(\mathbb{F}_p)^{tC_p}.$$

Geometrically, we are gluing two lines $\mathrm{Spec} \mathbb{F}_p[\mu]$ and $\mathrm{Spec} \mathbb{F}_p[t]$ along the closed point $\mathrm{Spec} \mathbb{F}_p$ where $\mu = t = 0$, as well as along the open complements $\mathrm{Spec} \mathbb{F}_p[\mu^{\pm 1}] \cong \mathrm{Spec} \mathbb{F}_p[t^{\pm 1}]$. This presentation is convenient for exhibiting duality in $\mathrm{TC}(B)$ or syntomic cohomology for more general B .

9. THE DUNDAS–GOODWILLIE–MCCARTHY THEOREM

((ETC: Relative K -theory vs. relative TC [DGM13], [HM97] and [CMM21].))

10. TR REVISITED

((ETC: Formula for TR. Antieau–Nikolaus t -structure [AN21].))

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