

THE BÖKSTEDT SPECTRAL SEQUENCE

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We discuss THH-calculations using the Bökstedt spectral sequence, illustrating duality in THH for regular local $\mathbb{E}_3$ -rings, as emphasized by Greenlees [Gre16]. Later we shall extend this to duality in TC and H_{syn} (syntomic cohomology) for higher local number rings, as in current work with Devalapurkar and Hahn.

Definition 0.1. A connective $\mathbb{E}_3$ -ring B is *regular local* if $\pi_0(B)$ is local with perfect residue field k of prime characteristic p , such that k lies in the thick subcategory Perf_B of Mod_B generated by B . It is *complete* if moreover $B \simeq \text{End}_{\text{End}_B(k)}(k)$.

A *higher local number ring* is a complete regular local $\mathbb{E}_3$ -ring B with finite residue field k , such that there exists a finite spectrum V for which $V \otimes B$ and k generate the same thick subcategory of Mod_B .

In other words, for B to be regular local, it must be possible to finitely build k from B in Mod_B , while to be a higher local number ring, the maps in the construction must be induced up from Sp .

There are MU-synthetic versions of these definitions, replacing spectra with $\text{fil}_{\text{ev}}^* \mathbb{S}$ -modules. To effectively analyze these, we assume that $\pi_* \text{HH}(k/B)$ is concentrated in even degrees. A theorem of Bott [Bot58] about $H^*(\Omega G)$ suggests that this is no additional hypothesis, i.e., that this evenness holds for all regular local $\mathbb{E}_3$ -rings.

For regular local $\mathbb{E}_3$ -rings $B \rightarrow k$ we shall see that $\text{THH}(B) \rightarrow \text{THH}(k)$ is Gorenstein of shift 0, meaning that

$$\text{Hom}_{\text{THH}(B)}(\text{THH}(k), \text{THH}(B)) \simeq \text{THH}(k),$$

and that $\pi_* \text{THH}(B, k)/\mu_n$ is a Poincaré duality algebra. Here $\pi_* \text{THH}(k) = k[\mu_0]$ with $|\mu_0| = 2$, and μ_n maps to $\mu_0^{p^n}$ for some n . Today we will present some examples of higher local number rings, and calculational techniques that illustrate this duality.

1. THE CYCLIC BAR CONSTRUCTION

For a monoid R in a symmetric (or $\mathbb{E}_2$) monoidal category $\mathcal{C}$ with enough colimits, the *cyclic bar construction* [Wal79] is the simplicial object

$$B_{\bullet}^{\text{cy}}(R): [q] \longmapsto R^{\otimes 1+q}$$

with faces and degeneracies

$$\begin{aligned} d_i(r_0, \dots, r_q) &= \begin{cases} (r_0, \dots, r_i r_{i+1}, \dots, r_q) & \text{for } 0 \leq i < q, \\ (r_q r_0, r_1, \dots, r_{q-1}) & \text{for } i = q \end{cases} \\ s_j(r_0, \dots, r_q) &= (r_0, \dots, r_j, 1, r_{j+1}, \dots, r_q). \end{aligned}$$

More precisely, this is a cyclic object [Con83], and its realization

$$B^{\text{cy}}(R) = \operatorname{colim}_{\Delta^{\text{op}}} B_{\bullet}^{\text{cy}}(R)$$

inherits a natural S^1 -action. Another presentation is as the factorization homology

$$B^{\text{cy}}(R) = \int_{S^1} R,$$

using that S^1 is a framed 1-manifold, cf. [Fra13]. When R is commutative, $B^{\text{cy}}(R)$ can also be described in terms of the tensored structure of $\mathcal{C}$ over simplicial sets, as the realization $S^1 \otimes R$ of the simplicial object

$$S_{\bullet}^1 \otimes R: [q] \mapsto R^{\otimes S_q^1}.$$

Here $S_q^1 = \Delta_q^1 / \partial \Delta_q^1$ has $1 + q$ elements. Interpolating between these cases: if R is an $\mathbb{E}_k$ -monoid then $B^{\text{cy}}(R)$ is an $\mathbb{E}_{k-1}$ -monoid, by Dunn additivity.

Letting $B(R, R, R)$ denote the two-sided bar construction, giving an R -bimodule resolution of R , we can write

$$B(R, R, R) \otimes_{R \otimes R^{\text{op}}} R = B^{\text{cy}}(R),$$

with $(r_0, r_1, \dots, r_q, r_{q+1}) \otimes r'$ corresponding to $(r_{q+1} r' r_0, r_1, \dots, r_q)$. This is a model for the relative tensor product

$$R \otimes_{R \otimes R^{\text{op}}} R,$$

where both tensor products $\otimes = \otimes^{\mathbb{L}}$ are derived, but this model does not make the S^1 -action explicit. More generally, for an R -bimodule M , we let

$$B^{\text{cy}}(R, M) = B(R, R, R) \otimes_{R \otimes R^{\text{op}}} M: [q] \mapsto M \otimes R^{\otimes q}$$

be the cyclic bar construction of R with coefficients in M . Then

$$B^{\text{cy}}(R, M) \simeq R \otimes_{R \otimes R^{\text{op}}} M.$$

2. HOCHSCHILD HOMOLOGY

Let k be a commutative ring. A k -algebra R is a monoid in the category of k -modules, with $\otimes = \otimes_k$. Its cyclic bar construction is the Hochschild homology

$$\mathrm{HH}^k(R) = B^{\text{cy}}(R): [q] \mapsto R^{\otimes 1+q}.$$

Its homotopy groups

$$\mathrm{HH}_*^k(R) = \pi_* \mathrm{HH}^k(R) \cong H_*(C_*(R), \partial)$$

agree with the homology groups of the associated Hochschild complex, with $C_q(R) = R^{\otimes 1+q}$ and $\partial_q = \sum_{i=0}^q (-1)^i d_i$.

$$\begin{aligned} \dots &\longrightarrow R \otimes R \otimes R \xrightarrow{\partial_2} R \otimes R \xrightarrow{\partial_1} R \longrightarrow 0 \\ \partial_2: a \otimes b \otimes c &\longmapsto ab \otimes c - a \otimes bc + ca \otimes b \\ \partial_1: a \otimes b &\longmapsto ab - ba \end{aligned}$$

In particular, $\mathrm{HH}_0^k(R) = R/[R, R]$. If R is commutative, then $\mathrm{HH}_1^k(R) \cong \Omega_{R/k}$ agrees with the R -module of Kähler differentials, via $a \otimes b \mapsto a db$. See e.g. [Lod92]. The Hochschild complex is quasi-isomorphic to the normalized quotient complex with $\bar{C}_q(R) = R \otimes (R/k)^{\otimes q}$, so either one can be used to calculate $\mathrm{HH}_*^k(R)$.

The S^1 -action

$$S_+^1 \wedge \mathrm{HH}^k(R) \longrightarrow \mathrm{HH}^k(R)$$

induces a suspension operator

$$\sigma: \mathrm{HH}_*^k(R) \longrightarrow \mathrm{HH}_{*+1}^k(R)$$

taking the homology class of $r \in \mathrm{HH}_0$ to the class of $1 \otimes r \in \mathrm{HH}_1$. When R is commutative, $\sigma(r)$ corresponds to $dr \in \Omega_{R/k}$. In this case, σ is also a differential ($\sigma\sigma = 0$) and a derivation ($\sigma(xy) = \sigma(x)y + x\sigma(y)$, with a sign in the graded case).

From the derived tensor product description, we obtain

$$\mathrm{HH}_*^k(R) \cong \mathrm{Tor}_*^{R \otimes_k R^{\mathrm{op}}}(R, R).$$

For $R = k[x] = k\{1, x, x^2, \dots\}$ a polynomial ring on one generator (in even internal degree), the two-sided bar resolution can be replaced with the short free bimodule resolution

$$0 \rightarrow k[x] \otimes k[x] \xrightarrow{1 \otimes x - x \otimes 1} k[x] \otimes k[x] \xrightarrow{\epsilon} k[x] \rightarrow 0,$$

so that $\mathrm{HH}_*^k(k[x])$ is the homology of the complex

$$0 \rightarrow k[x] \xrightarrow{1-1} k[x] \rightarrow 0,$$

concentrated in degrees 0 and 1.

Lemma 2.1. $\mathrm{HH}_*^k(k[x]) = k[x] \otimes \Lambda(\sigma x)$ with $\sigma x \in \mathrm{HH}_1$.

For $R = \Lambda(x) = k[x]/(x^2)$ an exterior algebra on one generator (in odd internal degree), the normalized Hochschild complex $\bar{C}_*(\Lambda(x))$ is

$$\dots \longrightarrow \Lambda(x) \otimes k\{x\} \otimes k\{x\} \xrightarrow{0} \Lambda(x) \otimes k\{x\} \xrightarrow{0} \Lambda(x) \rightarrow 0.$$

Lemma 2.2. $\mathrm{HH}_*^k(\Lambda(x)) = \Lambda(x) \otimes \Gamma(\sigma x)$ with $\sigma x \in \mathrm{HH}_1$.

Here $\Gamma(\sigma x) = k\{\gamma_j \sigma x \mid j \geq 0\}$ is the *divided power algebra* on σx , with $\gamma_j \sigma x \in \mathrm{HH}_j$ the class of $1 \otimes x^{\otimes j}$, and

$$\gamma_i \sigma x \cdot \gamma_j \sigma x = \frac{(i+j)!}{i!j!} \gamma_{i+j} \sigma x.$$

The pairing in Hochschild homology is given by the Eilenberg–Zilber shuffle product, and the binomial coefficient counts the number of (i, j) -shuffles.

When k has prime characteristic p , there is a factorization

$$\Gamma(\sigma x) \cong \bigotimes_{e \geq 0} \frac{k[\gamma_{p^e} \sigma x]}{(\gamma_{p^e} \sigma x)^p} = k[\sigma x]/(\sigma x)^p \otimes k[\gamma_p \sigma x]/(\gamma_p \sigma x)^p \otimes \dots$$

of the divided power algebra as a tensor product of polynomial rings truncated at height p .

Lemma 2.3. $\mathrm{HH}_*^k(R_1 \otimes R_2) \cong \mathrm{HH}_*^k(R_1) \otimes \mathrm{HH}_*^k(R_2)$.

It follows that

$$\mathrm{HH}_*^k(k[x_1, \dots, x_n]) = k[x_1, \dots, x_n] \otimes \Lambda(\sigma x_1, \dots, \sigma x_n).$$

More generally, the Hochschild–Kostant–Rosenberg theorem says that for any smooth k -algebra R ,

$$\mathrm{HH}_*^k(R) \cong \Omega_{R/k}^* := \Lambda_R^* \Omega_{R/k}$$

agrees with the algebra of Kähler forms. As emphasized by Connes, the suspension operator σ then corresponds to the exterior derivative $\omega \mapsto d\omega$.

We write $\mathrm{HH}^k(R, M) = B^{\mathrm{cy}}(R, M)$ for the Hochschild homology of a k -algebra R with coefficients in an R -bimodule M . Then

$$\mathrm{HH}_*^k(R, M) \cong \mathrm{Tor}_*^{R \otimes_k R^{\mathrm{op}}}(R, M).$$

For example,

$$\mathrm{HH}_*^k(k[x_1, \dots, x_n], k) = \Lambda(\sigma x_1, \dots, \sigma x_n)$$

is a finite Poincaré duality k -algebra.

3. TOPOLOGICAL HOCHSCHILD HOMOLOGY

Let $\mathbb{S}$ be the sphere spectrum. An $\mathbb{S}$ -algebra B is a monoid in the category of spectra, with $\otimes = \wedge$. Its cyclic bar construction is the *topological Hochschild homology*

$$\mathrm{THH}(B) = \mathrm{HH}^{\mathbb{S}}(B) = B^{\mathrm{cy}}(B).$$

This is a spectrum with S^1 -action. Later we shall see that it is a cyclotomic spectrum, say in the sense of Nikolaus–Scholze [NS18], with associated topological cyclic homology spectrum $\mathrm{TC}(B) \simeq \mathrm{CycSp}(\mathrm{THH}(\mathbb{S}), \mathrm{THH}(B))$, originally introduced by Bökstedt–Hsiang–Madsen [BHM93].

Note that $\mathrm{THH}(B)$ realizes the cyclic spectrum

$$\mathrm{THH}_\bullet(B): [q] \longmapsto B^{\otimes^{1+q}}.$$

For $s \geq -1$, let $\Delta_{\leq s} \subset \Delta$ be the full subcategory on the objects $[q]$ with $0 \leq q \leq s$. The *skeleton filtration*

$$0 = \mathrm{sk}_{-1} \mathrm{THH}(B) \longrightarrow B = \mathrm{sk}_0 \mathrm{THH}(B) \longrightarrow \dots \longrightarrow \mathrm{sk}_s \mathrm{THH}(B) \longrightarrow \dots$$

of $\mathrm{THH}(B)$ is the sequence of spectra given by

$$\mathrm{sk}_s \mathrm{THH}(B) = \mathrm{colim}_{\Delta_{\leq s}^{\mathrm{op}}} \mathrm{THH}_\bullet(B).$$

The associated graded spectra are

$$\frac{\mathrm{sk}_s \mathrm{THH}(B)}{\mathrm{sk}_{s-1} \mathrm{THH}(B)} \simeq \Sigma^s B \otimes (B/\mathbb{S})^{\otimes s},$$

where $B/\mathbb{S}$ is the cofiber of the unit map $\eta: \mathbb{S} \rightarrow B$.

(If one instead uses the fat geometric realization, given by $\mathrm{colim}_{M^{\mathrm{op}}}$ where $M \subset \Delta$ is the subcategory of monomorphisms, with skeleton filtration derived from the subcategories $M_{\leq s} = M \cap \Delta_{\leq s}$, then the s -th associated graded spectrum is the unnormalized $\Sigma^s B^{\otimes^{1+s}} = \Sigma^s \bar{B} \otimes B^{\otimes s}$.)

Let p be a prime, and write $H_*(X)$ for $H_*(X; \mathbb{F}_p)$. Then $H_*(X) \otimes H_*(Y) \cong H_*(X \otimes Y)$. Applying H_* to the skeleton filtration of $\mathrm{THH}(B)$ we obtain an unrolled exact couple with

$$\begin{aligned} A_s &= H_*(\mathrm{sk}_s \mathrm{THH}(B)) \\ E_s &= H_*(B \otimes (B/\mathbb{S})^{\otimes s}) \cong H_*(B) \otimes H_*(B/\mathbb{S})^{\otimes s} = \bar{C}_s(H_*(B)). \end{aligned}$$

Here $A_s = 0$ for $s < 0$, so the exact couple is conditionally convergent, with abutment $A_\infty = G = H_* \mathrm{THH}(B)$. The d^1 -differential of the associated spectral sequence agrees with the Hochschild boundary ∂ , so we obtain the Bökstedt spectral sequence

$$\mathcal{E}_{s,t}^2 = \mathrm{HH}_s(H_*(B))_{s+t} \Longrightarrow H_{s+t} \mathrm{THH}(B).$$

It is concentrated in the half-plane $s \geq 0$, with exiting differentials, hence is strongly convergent. If B is an $\mathbb{E}_2$ -ring, so that $B \otimes B \rightarrow B$ is a map of $\mathbb{E}_1$ -rings, then $\phi: \mathrm{THH}(B) \otimes \mathrm{THH}(B) \rightarrow \mathrm{THH}(B)$ is a pairing of filtered spectra, in the sense that it restricts to maps

$$\phi_{s,s'}: \mathrm{sk}_s \mathrm{THH}(B) \otimes \mathrm{sk}_{s'} \mathrm{THH}(B) \longrightarrow \mathrm{sk}_{s+s'} \mathrm{THH}(B).$$

Hence the Bökstedt spectral sequence is then an $\mathbb{F}_p$ -algebra spectral sequence, with the pairing

$$\phi^2: \mathrm{HH}_s(H_*(B)) \otimes \mathrm{HH}_{s'}(H_*(B)) \longrightarrow \mathrm{HH}_{s+s'}(H_*(B))$$

of $\mathcal{E}^2$ -pages being given by the shuffle product, converging to the algebra product

$$\phi_*: H_*\mathrm{THH}(B) \otimes H_*\mathrm{THH}(B) \longrightarrow H_*\mathrm{THH}(B).$$

In this case, $H_*(B)$ is (graded) commutative, so $\mathrm{HH}_0(H_*(B)) = H_*(B)$, while $\mathrm{HH}_1(H_*(B)) \cong \Omega_{H_*(B)/\mathbb{F}_p}$. If B is connective, the $\mathcal{E}^2$ -page begins:

$$\begin{array}{ccccccc}
 & \dots & & \dots & & & \\
 H_1(B) & & \dots & & \dots & & \\
 & \nwarrow d^2 & & & & & \\
 H_0(B) & & \Omega_{H_0(B)/\mathbb{F}_p} & & \mathrm{HH}_2(H_0(B)) & & \dots
 \end{array}$$

Let $\mathcal{A}$ denote the mod p Steenrod algebra, and $\mathcal{A}_*$ its dual. The natural left $\mathcal{A}$ -action on $H^*(X)$ dualizes to a left $\mathcal{A}_*$ -coaction ν on $H_*(X)$. The Bökstedt spectral sequence thus takes values in the category of $\mathcal{A}_*$ -comodules, meaning that each column $\mathcal{E}_{s,*}^r$ is an $\mathcal{A}_*$ -comodule, each differential d^r is $\mathcal{A}_*$ -colinear, the filtration $F_s G$ of $G = H_*\mathrm{THH}(B)$ is by $\mathcal{A}_*$ -subcomodules, and the isomorphisms $F_s G / F_{s-1} G \cong \mathcal{E}_{s,*}^\infty$ are $\mathcal{A}_*$ -colinear. If B is $\mathbb{E}_2$, then it is an $\mathcal{A}_*$ -comodule algebra spectral sequence.

The S^1 -action on $\mathrm{THH}(B)$ induces a suspension operator

$$\sigma: \pi_* \mathrm{THH}(B) \longrightarrow \pi_{*+1} \mathrm{THH}(B),$$

but the specification of a lift in $\pi_1^S(S_+^1)$ of the generator of $H_1(S_+^1)$ leads to the formula

$$\sigma \circ \sigma = \eta \cdot \sigma,$$

where $\eta \in \pi_1(\mathbb{S})$ arises as the Hopf construction on the product $S^1 \times S^1 \rightarrow S^1$. The suspension operator in homology,

$$\sigma: H_* \mathrm{THH}(B) \longrightarrow H_{*+1} \mathrm{THH}(B),$$

is a differential, as in the algebraic cases. If B is $\mathbb{E}_2$, then σ is a derivation. The topological suspension

$$H_*(B) \subset H_*\mathrm{THH}(B) \xrightarrow{\sigma} H_{*+1}\mathrm{THH}(B)$$

is compatible with the algebraic suspension

$$H_*(B) \rightarrow \mathrm{HH}_0(H_*(B)) \xrightarrow{\sigma} \mathrm{HH}_1(H_*(B)),$$

in the sense that the latter yields an infinite cycle that detects the former.

4. THE CASE $B = \mathbb{F}_2$

Let $p = 2$ and consider the Eilenberg–MacLane $\mathbb{E}_\infty$ -ring $B = \mathbb{F}_2$. Then

$$H_*(\mathbb{F}_2) = \mathcal{A}_* = \mathbb{F}_2[\zeta_i \mid i \geq 1] = \mathbb{F}_2[\bar{\zeta}_i \mid i \geq 1]$$

with $|\zeta_i| = 2^i - 1$ for each $i \geq 1$. Here $\bar{\zeta}_i = \chi(\zeta_i)$ is the conjugate generator, in the same degree as ζ_i . The coproduct in $\mathcal{A}_*$ is given by

$$\psi(\zeta_k) = \sum_{i+j=k} \zeta_i^{2^j} \otimes \zeta_j \quad \text{and} \quad \psi(\bar{\zeta}_k) = \sum_{i+j=k} \bar{\zeta}_i \otimes \bar{\zeta}_j^{2^i}$$

where $\zeta_0 = \bar{\zeta}_0 = 1$. Recall also the Dyer–Lashof operation

$$Q_1(\bar{\zeta}_i) = Q^{2^i}(\bar{\zeta}_i) = \bar{\zeta}_{i+1}.$$

The indices correspond to the relations $2|\bar{\zeta}_i| + 1 = |\bar{\zeta}_i| + 2^i = |\bar{\zeta}_{i+1}|$.

Since $\mathrm{THH}(\mathbb{F}_2)$ is an $\mathbb{E}_\infty$ $\mathbb{F}_2$ -algebra, and in particular an $\mathbb{F}_2$ -module, it splits additively as a sum of suspensions of $\mathbb{F}_2$. Hence $H_*\mathrm{THH}(\mathbb{F}_2)$ is a commutative $\mathcal{A}_*$ -algebra in $\mathcal{A}_*$ -comodules, so that

$$\mathcal{A}_* \otimes \pi_*\mathrm{THH}(\mathbb{F}_2) \cong H_*\mathrm{THH}(\mathbb{F}_2)$$

in this category. The Hurewicz homomorphism identifies $\pi_*\mathrm{THH}(\mathbb{F}_2)$ with the subalgebra of $\mathcal{A}_*$ -comodule primitives in $H_*\mathrm{THH}(\mathbb{F}_2)$. Recall that $x \in H_*(X)$ is $\mathcal{A}_*$ -comodule primitive if $\nu(x) = 1 \otimes x$ in $\mathcal{A}_* \otimes H_*(X)$. Hence we can determine $\pi_*\mathrm{THH}(\mathbb{F}_2)$ by calculating $H_*\mathrm{THH}(\mathbb{F}_2)$ with its $\mathcal{A}_*$ -coaction.

For this we use the Bökstedt spectral sequence

$$\mathcal{E}_{*,*}^2 = \mathrm{HH}_*(\mathcal{A}_*) \Longrightarrow H_*\mathrm{THH}(\mathbb{F}_2).$$

It begins at

$$\begin{aligned} \mathcal{E}_{*,*}^2 &= \mathrm{HH}_*(\mathbb{F}_2[\bar{\zeta}_i \mid i \geq 1]) \cong \bigotimes_{i \geq 1} \mathrm{HH}_*(\mathbb{F}_2[\bar{\zeta}_i]) = \bigotimes_{i \geq 1} \mathbb{F}_2[\bar{\zeta}_i] \otimes \Lambda(\sigma\bar{\zeta}_i) \\ &\cong \mathcal{A}_* \otimes \Lambda(\sigma\bar{\zeta}_i \mid i \geq 1). \end{aligned}$$

The factor $\mathcal{A}_* = \mathrm{HH}_0(\mathcal{A}_*)$ is the edge $\mathcal{E}_{0,*}^2$ at $s = 0$. The exterior generator $\sigma\bar{\zeta}_i$ has bidegree $(s, t) = (1, 2^i - 1)$ and total degree 2^i , and sits on the line $\mathcal{E}_{1,*}^2$ at $s = 1$. The s -fold product $\sigma\bar{\zeta}_{i_1} \cdots \sigma\bar{\zeta}_{i_s}$ with $i_1 < \cdots < i_s$ sits in filtration degree s . These form a basis for $\mathcal{E}_{*,*}^2$ as a free $\mathcal{A}_*$ -module.

$$\begin{array}{ccc} & & \sigma\bar{\zeta}_1\sigma\bar{\zeta}_2 \\ & \cdot & \\ & \cdot & \sigma\bar{\zeta}_2 \quad 0 \\ & \cdot & \\ & \cdot & 0 \\ & \cdot & \sigma\bar{\zeta}_1 \quad 0 \\ 1 & 0 & 0 \quad \dots \end{array}$$

For each $r \geq 2$ we must have $d^r(\sigma\bar{\zeta}_i) = 0$, since this differential takes value in filtration $s = 1 - r < 0$, where the $\mathcal{E}^2$ -page is trivial. Likewise $d^r = 0$ on each class in $\mathcal{A}_*$. Since the Bökstedt spectral sequence is multiplicative, it follows that $d^r = 0$ for all r , and so $\mathcal{E}^2 = \mathcal{E}^\infty$. In other words, the spectral sequence collapses at the $\mathcal{E}^2$ -page.

This means that the commutative $\mathcal{A}_*$ -comodule algebra $G = H_*\mathrm{THH}(\mathbb{F}_2)$ has a multiplicative filtration by $\mathcal{A}_*$ -comodule submodules $F_s G$, with associated graded isomorphic to $\mathcal{E}_{*,*}^\infty = \mathcal{E}_{*,*}^2 \cong \mathcal{A}_* \otimes \Lambda(\sigma\bar{\zeta}_i \mid i \geq 1)$.

There will, however, be multiplicative extensions. This can be deduced retroactively from the fact that $H_*\mathrm{THH}(\mathbb{F}_2) \cong \mathcal{A}_* \otimes \pi_*\mathrm{THH}(\mathbb{F}_2)$ (since $\mathrm{THH}(\mathbb{F}_2)$ is an $\mathbb{F}_2$ -algebra) and $\pi_*\mathrm{THH}(\mathbb{F}_2) \cong \mathbb{F}_2[\mu_0]$ with $|\mu_0| = 2$. We instead give a direct argument.

Lemma 4.1 (Bökstedt, [AR05, Prop. 5.9]). *Let B be an $\mathbb{E}_k$ -ring, with $k \geq 2$.*

$$\sigma: H_*(B) \subset H_*\mathrm{THH}(B) \longrightarrow H_{*+1}(\mathrm{THH}(B))$$

commutes with the available Dyer–Lashof operations, so that

$$\sigma Q^s(x) = Q^s(\sigma x)$$

for $s < |x| + k$.

Proof. The adjoint suspension map

$$\tilde{\sigma}: B \longrightarrow \mathrm{Map}(S_+^1, \mathrm{THH}(B)) \simeq D(S_+^1) \otimes \mathrm{THH}(B)$$

is an $\mathbb{E}_k$ -ring map. Apply the Cartan formula, using that the Dyer–Lashof operations in $H_*(D(S_+^1))$ are the Steenrod operations in $H^{-*}(S^1)$, which are trivial. $\square$

Consider $\bar{\zeta}_1 \in H_1(\mathbb{F}_2) \subset H_1\mathrm{THH}(\mathbb{F}_2)$ and form $\sigma\bar{\zeta}_1 \in H_2\mathrm{THH}(\mathbb{F}_2)$, detected by $\sigma\bar{\zeta}_1 \in \mathcal{E}_{1,1}^\infty$. The $\mathcal{A}_*$ -coaction ν is induced by the coproduct ψ , so that

$$\nu(\sigma\bar{\zeta}_1) = (1 \otimes \sigma)(\psi(\bar{\zeta}_1)) = (1 \otimes \sigma)(1 \otimes \bar{\zeta}_1 + \bar{\zeta}_1 \otimes 1) = 1 \otimes \sigma\bar{\zeta}_1$$

since $\sigma(1) = 0$. Hence $\sigma\bar{\zeta}_1$ is $\mathcal{A}_*$ -comodule primitive. Since $\mathrm{THH}(\mathbb{F}_2)$ is an $\mathbb{F}_2$ -module, this means that $\sigma\bar{\zeta}_1$ lifts uniquely over the Hurewicz homomorphism

$$h: \pi_*\mathrm{THH}(\mathbb{F}_2) \longrightarrow H_*\mathrm{THH}(\mathbb{F}_2)$$

to a class $\mu_0 \in \pi_2\mathrm{THH}(\mathbb{F}_2)$.

Lemma 4.2. *For each $i \geq 1$, the Hurewicz image of $\mu_0^{2^{i-1}} \in \pi_*\mathrm{THH}(\mathbb{F}_2)$ is $\sigma\bar{\zeta}_i \in H_*\mathrm{THH}(\mathbb{F}_2)$, which is detected by $\sigma\bar{\zeta}_i \in \mathcal{E}_{1,*}^\infty$.*

Proof. We know this for $i = 1$. Suppose it is known for one $i \geq 1$. Then the Hurewicz image of $\mu_0^{2^i} = (\mu_0^{2^{i-1}})^2$ is

$$(\sigma\bar{\zeta}_i)^2 = Q^{2^i}(\sigma\bar{\zeta}_i) = \sigma Q^{2^i}(\bar{\zeta}_i) = \sigma\bar{\zeta}_{i+1},$$

by the lemma and the known Dyer–Lashof operations in $\mathcal{A}_*$. This completes the inductive step. $\square$

We can write each $m \geq 0$ uniquely as a sum

$$m = 2^{i_1-1} + \dots + 2^{i_s-1}$$

with $s \geq 0$ and $1 \leq i_1 < \dots < i_s$. Then

$$\mu_0^m = \mu_0^{2^{i_1-1}} \dots \mu_0^{2^{i_s-1}} \in \pi_*\mathrm{THH}(\mathbb{F}_2)$$

has Hurewicz image

$$\sigma\bar{\zeta}_{i_1} \cdots \sigma\bar{\zeta}_{i_s} \in H_*\mathrm{THH}(\mathbb{F}_2)$$

and is detected by the $\mathcal{A}_*$ -module generator

$$\sigma\bar{\zeta}_{i_1} \cdots \sigma\bar{\zeta}_{i_s} \in \mathcal{E}_{s,*}^\infty.$$

Hence the free $\mathcal{A}_*$ -module generated by the Hurewicz images of the μ_0^m precisely accounts for all of $\mathcal{E}_{*,*}^\infty$, and therefore also for all of $H_*\mathrm{THH}(\mathbb{F}_2)$.

Theorem 4.3 (Bökstedt). *The Hurewicz homomorphism identifies*

$$\pi_*\mathrm{THH}(\mathbb{F}_2) = \mathbb{F}_2[\mu_0]$$

with the $\mathcal{A}_*$ -comodule primitives in $H_*\mathrm{THH}(\mathbb{F}_2) \cong \mathcal{A}_* \otimes \mathbb{F}_2[\mu_0]$, mapping μ_0 to $\sigma\bar{\zeta}_1$.

5. THE CASE $B = \mathbb{F}_p$

Next, let p be an odd prime and consider the Eilenberg–MacLane $\mathbb{E}_\infty$ -ring $B = \mathbb{F}_p$. Then

$$H_*(\mathbb{F}_p) = \mathcal{A}_* = \mathbb{F}_p[\xi_i \mid i \geq 1] \otimes \Lambda(\tau_i \mid i \geq 0) \cong \mathbb{F}_p[\bar{\xi}_i \mid i \geq 1] \otimes \Lambda(\bar{\tau}_i \mid i \geq 0)$$

with $|\xi_i| = 2p^i - 2$ and $|\tau_i| = 2p^i - 1$ for each i . We write $\bar{\xi}_i = \chi(\xi_i)$ and $\bar{\tau}_i = \chi(\tau_i)$ for the conjugate generators. The coproduct is given by

$$\psi(\xi_k) = \sum_{i+j=k} \xi_i^{p^j} \otimes \xi_j \quad \text{and} \quad \psi(\tau_k) = \tau_k \otimes 1 + \sum_{i+j=k} \xi_i^{p^j} \otimes \tau_j,$$

which in the conjugate form appears as

$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{p^i} \quad \text{and} \quad \psi(\bar{\tau}_k) = 1 \otimes \bar{\tau}_k + \sum_{i+j=k} \bar{\tau}_i \otimes \bar{\xi}_j^{p^i}.$$

Here $\xi_0 = \bar{\xi}_0 = 1$. In particular, $\beta(\bar{\tau}_k) = \bar{\xi}_k$. Recall also the Dyer–Lashof operations

$$Q_{2p-2}(\bar{\xi}_k) = Q^{p^k}(\bar{\xi}_k) = \bar{\xi}_{k+1} \quad \text{and} \quad Q_{p-1}(\bar{\tau}_k) = Q^{p^k}(\bar{\tau}_k) = \bar{\tau}_{k+1}.$$

The indices correspond to the relations $p|\bar{\xi}_k| + 2p - 2 = |\bar{\xi}_k| + p^k(2p - 2) = |\bar{\xi}_{k+1}|$ and $p|\bar{\tau}_k| + p - 1 = |\bar{\tau}_k| + p^k(2p - 2) = |\bar{\tau}_{k+1}|$.

Since $\mathrm{THH}(\mathbb{F}_p)$ is a sum of suspensions of $\mathbb{F}_p$,

$$H_*\mathrm{THH}(\mathbb{F}_p) \cong \mathcal{A}_* \otimes \pi_*\mathrm{THH}(\mathbb{F}_p)$$

as a commutative $\mathcal{A}_*$ -algebra in $\mathcal{A}_*$ -comodules, with $\pi_*\mathrm{THH}(\mathbb{F}_p)$ identified with the $\mathcal{A}_*$ -comodule primitives in $H_*\mathrm{THH}(\mathbb{F}_p)$.

The Bökstedt spectral sequence

$$\mathcal{E}_{*,*}^2 = \mathrm{HH}_*(\mathcal{A}_*) \Longrightarrow H_*\mathrm{THH}(\mathbb{F}_p)$$

begins

$$\begin{aligned} \mathcal{E}_{*,*}^2 &= \mathrm{HH}_*(\mathbb{F}_p[\bar{\xi}_i \mid i \geq 1] \otimes \Lambda(\bar{\tau}_i \mid i \geq 0)) \\ &\cong \bigotimes_{i \geq 1} \mathrm{HH}_*(\mathbb{F}_p[\bar{\xi}_i]) \otimes \bigotimes_{i \geq 0} \mathrm{HH}_*(\Lambda(\bar{\tau}_i)) \\ &= \bigotimes_{i \geq 1} \mathbb{F}_p[\bar{\xi}_i] \otimes \Lambda(\sigma\bar{\xi}_i) \otimes \bigotimes_{i \geq 0} \Lambda(\bar{\tau}_i) \otimes \Gamma(\sigma\bar{\tau}_i) \\ &\cong \mathcal{A}_* \otimes \Lambda(\sigma\bar{\xi}_i \mid i \geq 1) \otimes \Gamma(\sigma\bar{\tau}_i \mid i \geq 0). \end{aligned}$$

The factors $\mathcal{A}_*$ and $\Lambda(\sigma\bar{\xi}_i \mid i \geq 1)$ are generated by classes in filtration $s = 0$ and $s = 1$, respectively. All differentials on these generators land in trivial bidegrees, so all differentials on these factors are zero.

Recall that

$$\Gamma(\sigma\bar{\tau}_i) \cong \bigotimes_{e \geq 0} \mathbb{F}_p[\gamma_{p^e} \sigma\bar{\tau}_i] / (\gamma_{p^e} \sigma\bar{\tau}_i)^p$$

is generated as an algebra by $\gamma_{p^e} \sigma\bar{\tau}_i$ in filtration $s = p^e$, for each $e \geq 0$. Hence the $e = 0$ factor $\mathbb{F}_p[\sigma\bar{\tau}_i] / (\sigma\bar{\tau}_i)^p$ consists of infinite cycles, but there is room for differentials on the generators $\gamma_{p^e} \sigma\bar{\tau}_i$ for $e \geq 1$.

Lemma 5.1. *There are nonzero differentials*

$$d^{p-1}(\gamma_p \sigma\bar{\tau}_i) \doteq \sigma\bar{\xi}_{i+1}$$

for each $i \geq 0$. Hence

$$d^{p-1}(\gamma_j \sigma\bar{\tau}_i) \doteq \sigma\bar{\xi}_{i+1} \cdot \gamma_{j-p} \sigma\bar{\tau}_i$$

for each $j \geq p$, and

$$\mathcal{E}_{*,*}^p \cong \mathcal{A}_* \otimes \bigotimes_{i \geq 0} \mathbb{F}_p[\sigma\bar{\tau}_i] / (\sigma\bar{\tau}_i)^p.$$

Proof. More generally, Hunter [Hun96] shows that $d^{p-1}(\gamma_p \sigma x) \doteq \beta Q_{p-1}(x)$ for $|x|$ odd. A direct argument due to Ausoni goes as follows: Let $\sigma\bar{\tau}_i \in H_* \mathrm{THH}(\mathbb{F}_p)$. Then

$$(\sigma\bar{\tau}_i)^p = Q^{p^i}(\sigma\bar{\tau}_i) = \sigma Q^{p^i}(\bar{\tau}_i) = \sigma\bar{\tau}_{i+1},$$

so its Bockstein image

$$\beta((\sigma\bar{\tau}_i)^p) = \beta(\sigma\bar{\tau}_{i+1}) = \sigma\bar{\xi}_{i+1}$$

must be 0, by the Leibniz rule in characteristic p . Hence $\sigma\bar{\xi}_{i+1} \in \mathcal{E}_{1,2p^{i+1}-1}^2$ must be a boundary, i.e., in the image of some differential. By induction on i , the only possible source of this differential is $\gamma_p \sigma\bar{\tau}_i$. See Figure 1.

There is a Hopf algebra structure on this Bökstedt spectral sequence, induced as in [AR05] by a coproduct

$$\mathrm{THH}(\mathbb{F}_p) \simeq S^1 \otimes \mathbb{F}_p \longrightarrow (S^1 \vee S^1) \otimes \mathbb{F}_p \simeq \mathrm{THH}(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{THH}(\mathbb{F}_p).$$

It follows that d^{p-1} satisfies a co-Leibniz rule with respect to the coproduct

$$\psi(\gamma_k \sigma x) = \sum_{i+j=k} \gamma_i \sigma x \otimes \gamma_j \sigma x$$

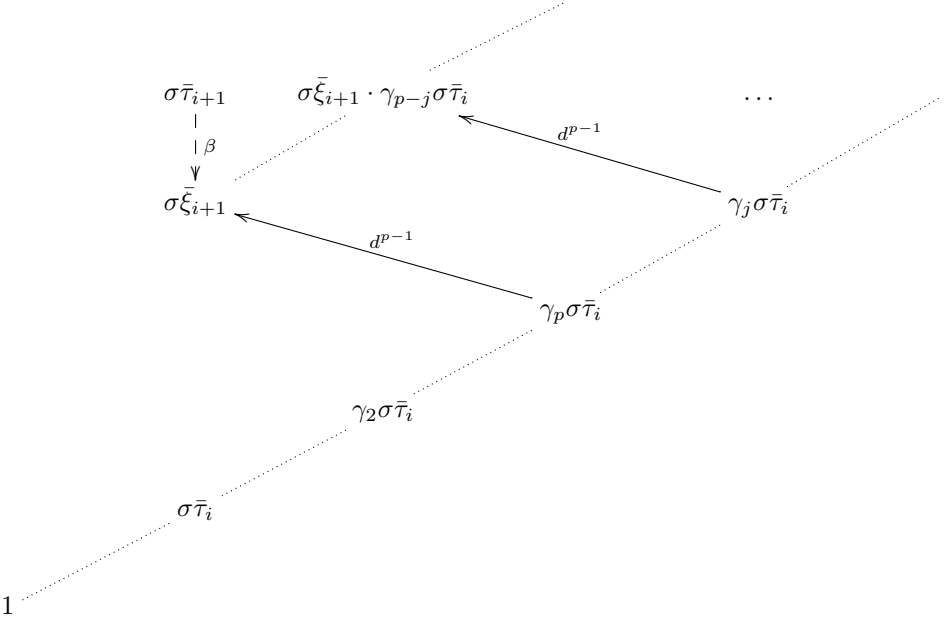
in $\Gamma(\sigma x)$. (This is dual to the product $y^i \cdot y^j = y^{i+j}$ in $\Gamma(\sigma x)^\vee = \mathbb{F}_p[y]$ with y dual to σx . The dual spectral sequence may also be easier to visualize.) This leads to the formula for d^{p-1} on $\gamma_j \sigma\bar{\tau}_i$, and that $\mathcal{E}^2 = \mathcal{E}^{p-1}$.

The formula for $\mathcal{E}^p$ follows from the Künneth isomorphism and the calculation that the homology of $\Lambda(\sigma\bar{\xi}_{i+1}) \otimes \Gamma(\sigma\bar{\tau}_i)$ is $\mathbb{F}_p[\sigma\bar{\tau}_i] / (\sigma\bar{\tau}_i)^p$. $\square$

Since the algebra generators of $\mathcal{E}^p$ are concentrated in filtrations $s \in \{0, 1\}$, it follows that $\mathcal{E}^p = \mathcal{E}^\infty$. Consider $\bar{\tau}_0 \in H_1(\mathbb{F}_p) \subset H_1 \mathrm{THH}(\mathbb{F}_p)$ and form $\sigma\bar{\tau}_0 \in H_2 \mathrm{THH}(\mathbb{F}_p)$. Then

$$\nu(\sigma\bar{\tau}_0) = (1 \otimes \sigma)(1 \otimes \bar{\tau}_0 + \bar{\tau}_0 \otimes 1) = 1 \otimes \sigma\bar{\tau}_0,$$

so $\sigma\bar{\tau}_0$ is $\mathcal{A}_*$ -comodule primitive, hence of the form $h(\mu_0)$ for well-defined class $\mu_0 \in \pi_2 \mathrm{THH}(\mathbb{F}_p)$.

FIGURE 1. d^{p-1} for $\mathrm{THH}(\mathbb{F}_p)$

Lemma 5.2. *For each $i \geq 0$, the Hurewicz image of $\mu_0^{p^i} \in \pi_* \mathrm{THH}(\mathbb{F}_p)$ is $\sigma\bar{\tau}_i \in H_* \mathrm{THH}(\mathbb{F}_p)$, which is detected by $\sigma\bar{\tau}_i \in \mathcal{E}_{1,*}^\infty$.*

Proof. By induction, the Hurewicz image of $\mu_0^{p^{i+1}} = (\mu_0^{p^i})^p$ is

$$(\sigma\bar{\tau}_i)^p = Q^{p^i}(\sigma\bar{\tau}_i) = \sigma Q^{p^i}(\bar{\tau}_i) = \sigma\bar{\tau}_{i+1}.$$

□

We can write each $m \geq 0$ uniquely as a finite sum

$$m = \sum_{i \geq 0} m_i p^i$$

with $m_i \in \{0, \dots, p-1\}$ for each i . Then

$$\mu_0^m = \prod_{i \geq 0} (\mu_0^{p^i})^{m_i} \in \pi_* \mathrm{THH}(\mathbb{F}_p)$$

has Hurewicz image

$$\prod_{i \geq 0} (\sigma\bar{\tau}_i)^{m_i} \in H_* \mathrm{THH}(\mathbb{F}_p)$$

and is detected by the $\mathcal{A}_*$ -module generator

$$\prod_{i \geq 0} (\sigma\bar{\tau}_i)^{m_i} \in \mathcal{E}_{\alpha(m),*}^\infty.$$

Here $\alpha(m) = \sum_{i \geq 0} m_i$ is the sum of the base p ‘digits’ of m . Hence the free $\mathcal{A}_*$ -module generated by the Hurewicz images of the μ_0^m precisely accounts for all of $\mathcal{E}_{*,*}^\infty$, hence also for all of $H_* \mathrm{THH}(\mathbb{F}_p)$.

Theorem 5.3 (Bökstedt). *For p odd, the Hurewicz homomorphism identifies*

$$\pi_* \mathrm{THH}(\mathbb{F}_p) = \mathbb{F}_p[\mu_0]$$

with the $\mathcal{A}_*$ -comodule primitives in $H_* \mathrm{THH}(\mathbb{F}_p) \cong \mathcal{A}_* \otimes \mathbb{F}_p[\mu_0]$, mapping μ_0 to $\sigma \bar{\tau}_0$.

Corollary 5.4. *For any perfect (e.g. finite) field k of prime characteristic $p > 0$,*

$$\pi_* \mathrm{THH}(k) \cong k[\mu_0]$$

with $|\mu_0| = 2$.

Proof. This follows from the case $k = \mathbb{F}_p$, using that $\mathrm{HH}_*^{\mathbb{F}_p}(k) = k$ concentrated in degree 0. $\square$

Remark 5.5. We shall see that for a class of ‘fp-type $n-1$ ’ connective $\mathbb{E}_3$ -rings B with residue field k , there is a degree $2p^n$ class $\mu_n \in \pi_* \mathrm{THH}(B, k)$ mapping to $\mu_0^{p^n}$, such that the quotient ring $\pi_* \mathrm{THH}(B, k)/\mu_n$ is a Poincaré duality k -algebra, i.e., that there is a homomorphism $\int: \pi_* \mathrm{THH}(B, k)/\mu_n \rightarrow \Sigma^s k$ for a suitable stem/shift s , such that the composite

$$\pi_* \mathrm{THH}(B, k)/\mu_n \otimes \pi_* \mathrm{THH}(B, k)/\mu_n \xrightarrow{\cdot} \pi_* \mathrm{THH}(B, k)/\mu_n \xrightarrow{\int} \Sigma^s k$$

is a perfect pairing. Thus $\pi_* \mathrm{THH}(B, k)/\mu_n$ is k -linearly dual to $\pi_{s-*} \mathrm{THH}(B, k)/\mu_n$. In current work with Sanath Devalapurkar and Jeremy Hahn, we establish this for a class of ‘higher local number rings’, which satisfy that there is a finite ‘type n ’ spectrum V such that k and $V \otimes B$ generate the same thick subcategory of B -modules. Moreover, the duality extends to $\mathrm{THH}(B)$ and $\mathrm{TC}(B)$, suitably interpreted.

6. TRUNCATED BROWN–PETERSON SPECTRA AT $p = 2$

The Eilenberg–MacLane $\mathbb{E}_\infty$ -ring $B = \mathbb{Z}$ has

$$H_*(\mathbb{Z}) = \mathbb{F}_2[\bar{\zeta}_1^2, \bar{\zeta}_i \mid i \geq 2],$$

with the inclusion in $\mathcal{A}_*$ being induced by $\mathbb{Z} \rightarrow \mathbb{F}_2$.

More generally, there are p -local truncated Brown–Peterson spectra $B = \mathrm{BP}\langle n-1 \rangle$ with

$$H_*(\mathrm{BP}\langle n-1 \rangle) = \mathbb{F}_2[\bar{\zeta}_1^2, \dots, \bar{\zeta}_n^2, \bar{\zeta}_i \mid i \geq n+1].$$

At any prime p , Hahn–Wilson [HW22] show that these can be realized by $\mathbb{E}_3$ BP-algebras, in such a way that the composite ring homomorphism

$$\mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}] \subset \pi_* \mathrm{BP} \longrightarrow \pi_* \mathrm{BP}\langle n-1 \rangle$$

is an isomorphism. Note that $\mathrm{BP}\langle -1 \rangle = \mathbb{F}_p$ and $\mathrm{BP}\langle 0 \rangle = \mathbb{Z}_{(p)}$, while $\mathrm{BP}\langle \infty \rangle = \mathrm{BP}$ is the original p -local Brown–Peterson spectrum with $\pi_* \mathrm{BP} = \mathbb{Z}_{(p)}[v_n \mid n \geq 1]$, which is a summand of the p -local complex bordism spectrum $\mathrm{MU}_{(p)}$.

For $p = 2$, we have $\mathrm{BP}\langle 1 \rangle = \mathrm{ku}_{(2)}$, the 2-local connective complex K -theory spectrum. The integral spectrum has $\pi_* \mathrm{ku} = \mathbb{Z}[u]$ with $|u| = 2$, and u maps to v_1 upon 2-localization. We also have $\mathrm{BP}\langle 2 \rangle = \mathrm{tmf}_1(3)_{(2)}$, the 2-local topological modular forms spectrum for elliptic curves with $\Gamma_1(3)$ level structure. The integral spectrum has $\pi_* \mathrm{tmf}_1(3) \cong \mathbb{Z}[a_1, a_3]$ with $|a_1| = 2$ and $|a_3| = 6$, and $v_1 = -a_1$ and $v_2 = -7a_3$ after 2-localization. (See [BR21, §9.3] for references.) These particular examples are in fact all $\mathbb{E}_\infty$ -rings.

The following calculations for $\mathrm{BP}\langle 0 \rangle = \mathbb{Z}_{(p)}$ and $\mathrm{BP}\langle 1 \rangle$ are due to Bökstedt, and McClure–Staffeldt [MS93], respectively. For $n \geq 3$, the more commutative structure on $\mathrm{BP}\langle n-1 \rangle$ was not available at that time.

To calculate $H_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$ as an $\mathcal{A}_*$ -comodule algebra, we use the Bökstedt spectral sequence

$$\mathcal{E}_{*,*}^2 = \mathrm{HH}_*(H_*(\mathrm{BP}\langle n-1\rangle)) \implies H_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)),$$

with

$$\begin{aligned} \mathcal{E}_{*,*}^2 &= \mathrm{HH}_*(\mathbb{F}_2[\bar{\zeta}_1^2, \dots, \bar{\zeta}_n^2, \bar{\zeta}_i \mid i > n]) \\ &\cong \bigotimes_{i=1}^n \mathrm{HH}(\mathbb{F}_2[\bar{\zeta}_i^2]) \otimes \bigotimes_{i>n} \mathrm{HH}(\mathbb{F}_2[\bar{\zeta}_i]) \\ &\cong \bigotimes_{i=1}^n \mathbb{F}_2[\bar{\zeta}_i^2] \otimes \Lambda(\sigma\bar{\zeta}_i^2) \otimes \bigotimes_{i>n} \mathbb{F}_2[\bar{\zeta}_i] \otimes \Lambda(\sigma\bar{\zeta}_i) \\ &\cong H_*(\mathrm{BP}\langle n-1\rangle) \otimes \Lambda(\sigma\bar{\zeta}_1^2, \dots, \sigma\bar{\zeta}_n^2, \sigma\bar{\zeta}_i \mid i > n). \end{aligned}$$

All algebra generators are in filtrations $s \in \{0, 1\}$, so all differentials on these generators land in trivial groups, and therefore $\mathcal{E}^2 = \mathcal{E}^\infty$.

For each $1 \leq k \leq n$, consider $\bar{\zeta}_k^2 \in H_*(\mathrm{BP}\langle n-1\rangle)$ and form

$$\sigma\bar{\zeta}_k^2 \in H_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$$

in degree $2^{k+1} - 1$. Then

$$\begin{aligned} \nu(\sigma\bar{\zeta}_k^2) &= (1 \otimes \sigma)(\psi(\bar{\zeta}_k^2)) = (1 \otimes \sigma) \sum_{i+j=k} \bar{\zeta}_i^2 \otimes \bar{\zeta}_j^{2^{i+1}} \\ &= \sum_{i+j=k} \bar{\zeta}_i^2 \otimes \sigma(\bar{\zeta}_j^{2^{i+1}}) = 1 \otimes \sigma\bar{\zeta}_k^2, \end{aligned}$$

since σ is a derivation so that $\sigma(x^2) = 0 \pmod{2}$. Hence $\sigma\bar{\zeta}_k^2$ is an $\mathcal{A}_*$ -comodule primitive detected by the $\mathcal{E}^\infty$ -class with the same name, in bidegree $(1, 2(2^k - 1))$. Moreover,

$$(\sigma\bar{\zeta}_k^2)^2 = Q^{2^{k+1}-1}(\sigma\bar{\zeta}_k^2) = \sigma Q^{2^{k+1}-1}(\bar{\zeta}_k^2) = \sigma(0) = 0,$$

since $Q^s(x^2) = \sum_{i+j=s} Q^i(x)Q^j(x)$ by the Cartan formula, and all terms in this sum cancel pairwise when s is odd. Hence $\sigma\bar{\zeta}_1^2, \dots, \sigma\bar{\zeta}_n^2$ generate an exterior subalgebra of $H_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$.

Next, consider $\bar{\zeta}_{n+1} \in H_*\mathrm{BP}\langle n-1\rangle$ and form

$$\sigma\bar{\zeta}_{n+1} \in H_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$$

in degree 2^{n+1} . Then

$$\begin{aligned} \nu(\sigma\bar{\zeta}_{n+1}) &= (1 \otimes \sigma)(\psi(\bar{\zeta}_{n+1})) = (1 \otimes \sigma) \sum_{i+j=n+1} \bar{\zeta}_i \otimes \bar{\zeta}_j^{2^i} \\ &= \sum_{i+j=n+1} \bar{\zeta}_i \otimes \sigma(\bar{\zeta}_j^{2^i}) = 1 \otimes \sigma\bar{\zeta}_{n+1} + \bar{\zeta}_1 \otimes \sigma\bar{\zeta}_n^2. \end{aligned}$$

This means that $\beta_*(\sigma\bar{\zeta}_{n+1}) = \sigma\bar{\zeta}_n^2$, where β_* denotes the homology Bockstein dual to $\beta = \mathrm{Sq}^1$. In particular, $\sigma\bar{\zeta}_{n+1}$ is not $\mathcal{A}_*$ -comodule primitive, hence is not in the Hurewicz image from $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$.

As in the case of $\mathrm{THH}(\mathbb{F}_2)$, $(\sigma\bar{\zeta}_i)^2 = \sigma\bar{\zeta}_{i+1}$ in $H_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$ for all $i > n$. Hence each non-negative power of $\sigma\bar{\zeta}_{n+1}$ corresponds to a unique product of distinct classes $\sigma\bar{\zeta}_i$ with $i > n$. These multiplicative extensions lead to the following formula.

Proposition 6.1 ([AR05, Cor. 5.13]). *At $p = 2$,*

$$H_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle) \cong H_*(\mathrm{BP}\langle n-1 \rangle) \otimes \Lambda(\sigma\bar{\zeta}_1^2, \dots, \sigma\bar{\zeta}_n^2) \otimes \mathbb{F}_2[\sigma\bar{\zeta}_{n+1}]$$

in commutative $\mathcal{A}_*$ -comodule algebras. The $\mathcal{A}_*$ -coaction on $H_*(\mathrm{BP}\langle n-1 \rangle)$ is the restriction of the coproduct on $\mathcal{A}_*$, the classes $\sigma\bar{\zeta}_1^2, \dots, \sigma\bar{\zeta}_n^2$ are $\mathcal{A}_*$ -comodule primitive, and $\nu(\sigma\bar{\zeta}_{n+1}) = 1 \otimes \sigma\bar{\zeta}_{n+1} + \bar{\zeta}_1 \otimes \sigma\bar{\zeta}_n^2$.

For small n and sufficiently large primes p , there exist Smith–Toda complexes $V(n-1)$, i.e., p -local finite spectra with $\mathrm{BP}_*V(n-1) = \mathrm{BP}_*/(p, \dots, v_{n-1})$. Equivalently, $H_*V(n-1) \cong \Lambda(\zeta_1, \dots, \zeta_n)$, and $H^*V(n-1) \cong \Lambda(Q_0, \dots, Q_{n-1})$, cf. [Tod71], where the Q_i defined by $Q_0 = \mathrm{Sq}^1$ and $Q_i = [\mathrm{Sq}^{2^i}, Q_{i-1}]$ are the Minor primitives in $\mathcal{A}_*$. In particular, $V(-1) = \mathbb{S}$ and $V(0) = \mathbb{S}/p$. When $V(n-1)$ admits a self-map v_n inducing multiplication by v_n in BP-homology, $V(n)$ can be constructed as the cofiber

$$\Sigma^{2p^n-2}V(n-1) \xrightarrow{v_n} V(n-1) \longrightarrow V(n).$$

Informally, we can write $V(n) = \mathbb{S}/(p, \dots, v_n)$. By the Smith–Hopkins periodicity theorem [HS98], $V(n-1)$ always admits a self-map inducing multiplication by some positive power v_n^e in BP-homology, but here we are assuming $e = 1$. Note that $V(n-1) \otimes \mathrm{BP}\langle n-1 \rangle \simeq \mathbb{F}_p$ and $H_*(V(n-1)) \otimes H_*(\mathrm{BP}\langle n-1 \rangle) \cong \mathcal{A}_*$. We write $V(n-1)_*X$ for $\pi_*(V(n-1) \otimes X)$ when thinking of $V(n-1)$ as defining a homology theory, generalizing stable homotopy and stable homotopy mod p .

There is a unique augmentation map $\mathrm{BP}\langle n-1 \rangle \rightarrow \mathbb{F}_p$, which is as highly structured as $\mathrm{BP}\langle n-1 \rangle$. Hence we can view $\mathbb{F}_p$ as a $\mathrm{BP}\langle n-1 \rangle$ -bimodule, and form $\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_p)$. When $V(n-1)$ exists, we have equivalences

$$\begin{aligned} V(n-1) \otimes \mathrm{THH}(\mathrm{BP}\langle n-1 \rangle) &\simeq \mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, V(n-1) \otimes \mathrm{BP}\langle n-1 \rangle) \\ &\simeq \mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_p), \end{aligned}$$

so that $V(n-1)_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle) \cong \pi_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_p)$.

There is a Bökstedt spectral sequence with coefficients

$$\mathcal{E}_{*,*}^2 = \mathrm{HH}_*(H_*(B), H_*(M)) \implies H_*\mathrm{THH}(B, M).$$

With $B = \mathrm{BP}\langle n-1 \rangle$ and $M = \mathbb{F}_2$, we have

$$\mathcal{E}_{*,*}^2 = \mathcal{A}_* \otimes \Lambda(\sigma\bar{\zeta}_1^2, \dots, \sigma\bar{\zeta}_n^2, \sigma\bar{\zeta}_i \mid i > n)$$

converging to

$$H_*(\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_2)) \cong \mathcal{A}_* \otimes \Lambda(\sigma\bar{\zeta}_1^2, \dots, \sigma\bar{\zeta}_n^2) \otimes \mathbb{F}_2[\sigma\bar{\zeta}_{n+1}].$$

Since $\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_2)$ is an $\mathbb{F}_2$ -module, the Hurewicz homomorphism h identifies $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_2)$ with the $\mathcal{A}_*$ -comodule primitives in this abutment. Our calculations then show that there are homotopy classes λ_k with

$$h(\lambda_k) = 1 \otimes \sigma\bar{\zeta}_k^2$$

for $1 \leq k \leq n$, and a homotopy class μ_n with

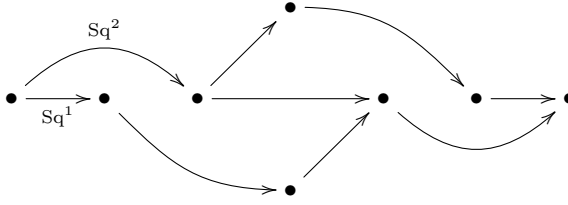
$$h(\mu_n) = 1 \otimes \sigma\bar{\zeta}_{n+1} + \bar{\zeta}_1 \otimes \sigma\bar{\zeta}_n^2,$$

such that

$$\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_2) = \Lambda(\lambda_1, \dots, \lambda_n) \otimes \mathbb{F}_2[\mu_n]$$

as a commutative $\mathbb{F}_2$ -algebra. Here we use that

$$\nu(\zeta_1 \otimes \sigma\bar{\zeta}_n^2) = 1 \otimes (\zeta_1 \otimes \sigma\bar{\zeta}_n^2) + \bar{\zeta}_1 \otimes (1 \otimes \sigma\bar{\zeta}_n^2)$$

FIGURE 2. The subalgebra $\mathcal{A}(1)$ of $\mathcal{A}$

and $\bar{\zeta}_1 = -\zeta_1$.

Proposition 6.2. *The Hurewicz homomorphism identifies*

$$\pi_* \mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_2) = \Lambda(\lambda_1, \dots, \lambda_n) \otimes \mathbb{F}_2[\mu_n]$$

with the $\mathcal{A}_*$ -comodule primitives in

$$H_* \mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_2) = \mathcal{A}_* \otimes \Lambda(\sigma \bar{\zeta}_1^2, \dots, \sigma \bar{\zeta}_n^2) \otimes \mathbb{F}_2[\sigma \bar{\zeta}_{n+1}],$$

mapping λ_k to $1 \otimes \sigma \bar{\zeta}_k^2$ and μ_n to $1 \otimes \sigma \bar{\zeta}_{n+1} + \zeta_1 \otimes \sigma \bar{\zeta}_n^2$. If $V(n-1)$ exists as a spectrum (resp. ring spectrum), then this is also

$$V(n-1)_* \mathrm{THH}(\mathrm{BP}\langle n-1 \rangle)$$

as a graded $\mathbb{F}_2$ -module (resp. $\mathbb{F}_2$ -algebra).

7. REAL K -THEORY AT $p = 2$

The connective real (topological) K -theory $\mathbb{E}_\infty$ -ring ko has

$$H_*(\mathrm{ko}) = \mathbb{F}_2[\bar{\zeta}_1^4, \bar{\zeta}_2^2, \bar{\zeta}_i \mid i \geq 3],$$

with the inclusion in $\mathcal{A}_*$ induced by the unique unital $(\mathbb{E}_\infty)$ -map $\mathrm{ko} \rightarrow \mathbb{F}_2$. The Bökstedt spectral sequence has

$$\mathcal{E}_{*,*}^2 = H_*(\mathrm{ko}) \otimes \Lambda(\sigma \bar{\zeta}_1^4, \sigma \bar{\zeta}_2^2, \sigma \bar{\zeta}_i \mid i \geq 3).$$

The classes with the same names in $H_* \mathrm{THH}(\mathrm{ko})$ satisfy $(\sigma \bar{\zeta}_1^4)^2 = 0$, $(\sigma \bar{\zeta}_2^2)^2 = 0$ and $(\sigma \bar{\zeta}_i)^2 = \sigma \bar{\zeta}_{i+1}$ for $i \geq 3$, by the same Dyer–Lashof arguments as before, and their $\mathcal{A}_*$ -coactions again follow from the formulas for ψ and commutation with σ .

Proposition 7.1 ([AR05, Thm. 6.2(a)]). *At $p = 2$,*

$$H_* \mathrm{THH}(\mathrm{ko}) \cong H_*(\mathrm{ko}) \otimes \Lambda(\sigma \bar{\zeta}_1^4, \sigma \bar{\zeta}_2^2) \otimes \mathbb{F}_2[\sigma \bar{\zeta}_3]$$

in commutative $\mathcal{A}_*$ -comodule algebras. The $\mathcal{A}_*$ -coaction on $H_*(\mathrm{ko})$ is the restriction of the coproduct on $\mathcal{A}_*$, while

$$\begin{aligned} \nu(\sigma \bar{\zeta}_1^4) &= 1 \otimes \sigma \bar{\zeta}_1^4 \\ \nu(\sigma \bar{\zeta}_2^2) &= 1 \otimes \sigma \bar{\zeta}_2^2 + \bar{\zeta}_1^2 \otimes \sigma \bar{\zeta}_1^4 \\ \nu(\sigma \bar{\zeta}_3) &= 1 \otimes \sigma \bar{\zeta}_3 + \bar{\zeta}_1 \otimes \sigma \bar{\zeta}_2^2 + \bar{\zeta}_2 \otimes \sigma \bar{\zeta}_1^4. \end{aligned}$$

As observed by Davis–Mahowald [DM81], there exist four finite spectra $A(1)$ with $\mathrm{ko}_* A(1) = \mathbb{F}_2$ and $H^* A(1) = \mathcal{A}(1)$ the finite subalgebra of $\mathcal{A}$ generated by Sq^1 and Sq^2 . See Figure 2. We have $H^*(\mathrm{ko}) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{F}_2$. Dually, $H_* A(1) = \mathbb{F}_2[\zeta_1, \zeta_2]/(\zeta_1^4, \zeta_2^2)$, but $A(1)$ does not admit a ring spectrum structure.

Proposition 7.2. *The Hurewicz homomorphism identifies*

$$\pi_* \mathrm{THH}(\mathrm{ko}, \mathbb{F}_2) = \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu_2]$$

with the $\mathcal{A}_*$ -comodule primitives in

$$H_* \mathrm{THH}(\mathrm{ko}, \mathbb{F}_2) = \mathcal{A}_* \otimes \Lambda(\sigma \bar{\zeta}_1^4, \sigma \bar{\zeta}_2^2) \otimes \mathbb{F}_2[\sigma \bar{\zeta}_3],$$

mapping λ'_1 in degree 5, λ_2 in degree 7 and μ_2 in degree 8 to

$$\begin{aligned} h(\lambda'_1) &= 1 \otimes \sigma \bar{\zeta}_1^4 \\ h(\lambda_2) &= 1 \otimes \sigma \bar{\zeta}_2^2 + \zeta_1^2 \otimes \sigma \bar{\zeta}_1^4 \\ h(\mu_2) &= 1 \otimes \sigma \bar{\zeta}_3 + \zeta_1 \otimes \sigma \bar{\zeta}_2^2 + \zeta_2 \otimes \sigma \bar{\zeta}_1^4, \end{aligned}$$

respectively. This agrees with $A(1)_* \mathrm{THH}(\mathrm{ko})$ as a graded $\mathbb{F}_2$ -module.

8. TOPOLOGICAL MODULAR FORMS AT $p = 2$

The connective topological modular forms $\mathbb{E}_\infty$ -ring tmf has

$$H_*(\mathrm{tmf}) = \mathbb{F}_2[\bar{\zeta}_1^8, \bar{\zeta}_2^4, \bar{\zeta}_3^2, \bar{\zeta}_i \mid i \geq 4],$$

with the inclusion in $\mathcal{A}_*$ induced by the unique unital $(\mathbb{E}_\infty)$ -map $\mathrm{tmf} \rightarrow \mathbb{F}_2$. The Bökstedt spectral sequence has

$$\mathcal{E}_{*,*}^2 = H_*(\mathrm{ko}) \otimes \Lambda(\sigma \bar{\zeta}_1^8, \sigma \bar{\zeta}_2^4, \sigma \bar{\zeta}_3^2, \sigma \bar{\zeta}_i \mid i \geq 4).$$

The classes with the same names in $H_* \mathrm{THH}(\mathrm{ko})$ satisfy $(\sigma \bar{\zeta}_1^8)^2 = 0$, $(\sigma \bar{\zeta}_2^4)^2 = 0$, $(\sigma \bar{\zeta}_3^2)^2 = 0$ and $(\sigma \bar{\zeta}_i)^2 = \sigma \bar{\zeta}_{i+1}$ for $i \geq 4$, by the same Dyer–Lashof arguments as before, and their $\mathcal{A}_*$ -coactions again follow from the formulas for ψ and commutation with σ .

Proposition 8.1 ([AR05, Thm. 6.2(b)]). *At $p = 2$,*

$$H_* \mathrm{THH}(\mathrm{tmf}) \cong H_*(\mathrm{tmf}) \otimes \Lambda(\sigma \bar{\zeta}_1^8, \sigma \bar{\zeta}_2^4, \sigma \bar{\zeta}_3^2) \otimes \mathbb{F}_2[\sigma \bar{\zeta}_4]$$

in commutative $\mathcal{A}_*$ -comodule algebras. The $\mathcal{A}_*$ -coaction on $H_*(\mathrm{tmf})$ is the restriction of the coproduct on $\mathcal{A}_*$, while

$$\begin{aligned} \nu(\sigma \bar{\zeta}_1^8) &= 1 \otimes \sigma \bar{\zeta}_1^8 \\ \nu(\sigma \bar{\zeta}_2^4) &= 1 \otimes \sigma \bar{\zeta}_2^4 + \bar{\zeta}_1^4 \otimes \sigma \bar{\zeta}_1^8 \\ \nu(\sigma \bar{\zeta}_3^2) &= 1 \otimes \sigma \bar{\zeta}_3^2 + \bar{\zeta}_1^2 \otimes \sigma \bar{\zeta}_2^4 + \bar{\zeta}_2^2 \otimes \sigma \bar{\zeta}_1^8 \\ \nu(\sigma \bar{\zeta}_4) &= 1 \otimes \sigma \bar{\zeta}_4 + \bar{\zeta}_1 \otimes \sigma \bar{\zeta}_3^2 + \bar{\zeta}_2 \otimes \sigma \bar{\zeta}_2^4 + \bar{\zeta}_3 \otimes \sigma \bar{\zeta}_1^8. \end{aligned}$$

Let $\mathcal{A}(2)$ be the 64-dimensional subalgebra of $\mathcal{A}$ generated by Sq^1 , Sq^2 and Sq^4 . As proved by Marilyn J. Roth (now Blockus) [Rot77], there are 1600 $\mathcal{A}$ -module structures on $\mathcal{A}(2)$ extending the given free $\mathcal{A}(2)$ -module structure. The explicit enumeration of these structures given by Bruner [Bru], together with Toda's obstruction theory [Tod71, Lem. 3.1], and a machine calculation with Bruner's **ext**, shows that each $\mathcal{A}$ -module structure on $\mathcal{A}(2)$ is realized in cohomology by a unique spectrum $A(2)$. Hence there are 1600 such spectra, with $H_* A(2) = \mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3]/(\zeta_1^8, \zeta_2^4, \zeta_3^2)$ and $A(2) \otimes \mathrm{tmf} \simeq \mathbb{F}_2$.

Proposition 8.2. *The Hurewicz homomorphism identifies*

$$\pi_* \mathrm{THH}(\mathrm{tmf}, \mathbb{F}_2) = \Lambda(\lambda_1'', \lambda_2', \lambda_3) \otimes \mathbb{F}_2[\mu_3]$$

with the $\mathcal{A}_*$ -comodule primitives in

$$H_* \mathrm{THH}(\mathrm{tmf}, \mathbb{F}_2) = \mathcal{A}_* \otimes \Lambda(\sigma \bar{\zeta}_1^8, \sigma \bar{\zeta}_2^4, \sigma \bar{\zeta}_3^2) \otimes \mathbb{F}_2[\sigma \bar{\zeta}_4],$$

mapping λ_1'' in degree 9, λ_2' in degree 13, λ_3 in degree 15 and μ_3 in degree 16 to

$$h(\lambda_1'') = 1 \otimes \sigma \bar{\zeta}_1^8$$

$$h(\lambda_2') = 1 \otimes \sigma \bar{\zeta}_2^4 + \zeta_1^4 \otimes \sigma \bar{\zeta}_1^8$$

$$h(\lambda_3) = 1 \otimes \sigma \bar{\zeta}_3^2 + \zeta_1^2 \otimes \sigma \bar{\zeta}_2^4 + \zeta_2^2 \otimes \sigma \bar{\zeta}_1^8$$

$$h(\mu_3) = 1 \otimes \sigma \bar{\zeta}_4 + \zeta_1 \otimes \sigma \bar{\zeta}_3^2 + \zeta_2 \otimes \sigma \bar{\zeta}_2^4 + \zeta_3 \otimes \sigma \bar{\zeta}_1^8,$$

respectively. This agrees with $A(2)_* \mathrm{THH}(\mathrm{tmf})$ as a graded $\mathbb{F}_2$ -module.

9. TRUNCATED BROWN–PETERSON SPECTRA AT p ODD

Let p be an odd prime. The Eilenberg–MacLane $\mathbb{E}_\infty$ -ring $B = \mathbb{Z}$ has

$$H_*(\mathbb{Z}) = \mathbb{F}_p[\bar{\xi}_i \mid i \geq 1] \otimes \Lambda(\bar{\tau}_i \mid i \geq 1),$$

with the inclusion in $\mathcal{A}_*$ being induced by $\mathbb{Z} \rightarrow \mathbb{F}_p$. More generally, the p -local truncated Brown–Peterson spectra $B = \mathrm{BP}\langle n-1 \rangle$ have

$$H_*(\mathrm{BP}\langle n-1 \rangle) = \mathbb{F}_p[\bar{\xi}_i \mid i \geq 1] \otimes \Lambda(\bar{\tau}_i \mid i \geq n).$$

There is an $\mathbb{E}_\infty$ -ring map $\mathrm{BP}\langle 1 \rangle \rightarrow \mathrm{ku}_{(p)}$, formally similar to the cyclotomic extension $\mathbb{Z} \rightarrow \mathbb{Z}[\zeta_p]$, and an additive splitting

$$\mathrm{ku}_{(p)} = \bigvee_{i=0}^{p-2} \Sigma^{2i} \mathrm{BP}\langle 1 \rangle$$

of p -local connective complex K -theory as a sum of suspensions of $\mathrm{BP}\langle 1 \rangle$.

To calculate $H_* \mathrm{THH}(\mathrm{BP}\langle n-1 \rangle)$ as an $\mathcal{A}_*$ -comodule algebra, we use the Bökstedt spectral sequence

$$\mathcal{E}_{*,*}^2 = \mathrm{HH}_*(H_*(\mathrm{BP}\langle n-1 \rangle)) \Longrightarrow H_*(\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle)),$$

with

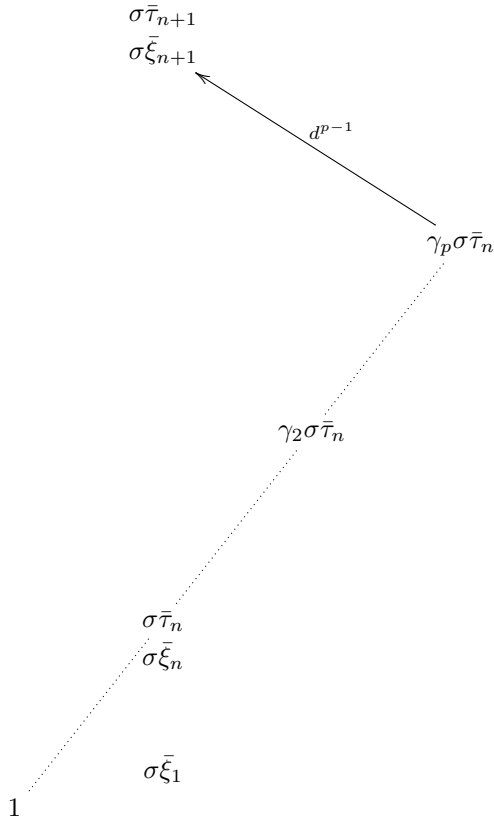
$$\begin{aligned} \mathcal{E}_{*,*}^2 &= \mathrm{HH}_*(\mathbb{F}_p[\bar{\xi}_i \mid i \geq 1] \otimes \Lambda(\bar{\tau}_i \mid i \geq n)) \\ &\cong \bigotimes_{i \geq 1} \mathrm{HH}_*(\mathbb{F}_p[\bar{\xi}_i]) \otimes \bigotimes_{i \geq n} \mathrm{HH}_*(\Lambda(\bar{\tau}_i)) \\ &= \bigotimes_{i \geq 1} \mathbb{F}_p[\bar{\xi}_i] \otimes \Lambda(\sigma \bar{\xi}_i) \otimes \bigotimes_{i \geq n} \Lambda(\bar{\tau}_i) \otimes \Gamma(\sigma \bar{\tau}_i) \\ &\cong H_*(\mathrm{BP}\langle n-1 \rangle) \otimes \Lambda(\sigma \bar{\xi}_i \mid i \geq 1) \otimes \Gamma(\sigma \bar{\tau}_i \mid i \geq n). \end{aligned}$$

Lemma 9.1. *There are nonzero differentials*

$$d^{p-1}(\gamma_p \sigma \bar{\tau}_i) \doteq \sigma \bar{\xi}_{i+1}$$

for each $i \geq n$. Hence

$$d^{p-1}(\gamma_j \sigma \bar{\tau}_i) \doteq \sigma \bar{\xi}_{i+1} \cdot \gamma_{j-p} \sigma \bar{\tau}_i$$

FIGURE 3. d^{p-1} for $\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle)$

for each $j \geq p$, and

$$\mathcal{E}_{*,*}^p \cong H_*(\mathrm{BP}\langle n-1 \rangle) \otimes \Lambda(\sigma\bar{\xi}_i \mid 1 \leq i \leq n) \otimes \bigotimes_{i \geq n} \mathbb{F}_p[\sigma\bar{\tau}_i]/(\sigma\bar{\tau}_i)^p.$$

Proof. This follows by Lemma 5.1, together with naturality of the Bökstedt spectral sequence with respect to the map $\mathrm{BP}\langle n-1 \rangle \rightarrow \mathbb{F}_p$. See Figure 3. $\square$

For each $1 \leq k \leq n$, consider $\bar{\xi}_k \in H_*(\mathrm{BP}\langle n-1 \rangle)$ and form

$$\sigma\bar{\xi}_k \in H_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle)$$

in degree $2p^k - 1$. Then

$$\begin{aligned} \nu(\sigma\bar{\xi}_k) &= (1 \otimes \sigma)(\psi(\bar{\xi}_k)) = (1 \otimes \sigma) \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^p \\ &= \sum_{i+j=k} \bar{\xi}_i \otimes \sigma(\bar{\xi}_j^p) = 1 \otimes \sigma\bar{\xi}_k, \end{aligned}$$

since σ is a derivation so that $\sigma(x^p) = 0 \pmod{p}$. Hence $\sigma\bar{\xi}_k$ is an $\mathcal{A}_*$ -comodule primitive detected by the $\mathcal{E}^\infty$ -class with the same name, in bidegree $(1, 2p^k - 2)$.

Moreover, $(\sigma\bar{\xi}_k)^2 = 0$ by graded commutativity, since the degree $2p^k - 1$ is odd. Hence $\sigma\bar{\xi}_1, \dots, \sigma\bar{\xi}_n$ generate an exterior subalgebra of $H_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle)$.

Next, consider $\bar{\tau}_n \in H_*\mathrm{BP}\langle n-1 \rangle$ and form

$$\sigma\bar{\tau}_n \in H_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle)$$

in degree $2p^n$. Then

$$\begin{aligned} \nu(\sigma\bar{\tau}_n) &= (1 \otimes \sigma)(\psi(\bar{\tau}_n)) = (1 \otimes \sigma)\left(1 \otimes \bar{\tau}_n + \sum_{i+j=n} \bar{\tau}_i \otimes \bar{\xi}_j^{p^i}\right) \\ &= 1 \otimes \sigma\bar{\tau}_n + \sum_{i+j=n} \bar{\tau}_i \otimes \sigma(\bar{\xi}_j^{p^i}) = 1 \otimes \sigma\bar{\tau}_n + \bar{\tau}_0 \otimes \sigma\bar{\xi}_n. \end{aligned}$$

This means that $\beta_*(\sigma\bar{\tau}_n) = \sigma\bar{\xi}_n$, where β_* denotes the homology Bockstein dual to β . In particular, $\sigma\bar{\tau}_n$ is not $\mathcal{A}_*$ -comodule primitive, hence is not in the Hurewicz image from $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle)$.

As in the case of $\mathrm{THH}(\mathbb{F}_p)$, $(\sigma\bar{\tau}_i)^p = \sigma\bar{\tau}_{i+1}$ in $H_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle)$ for all $i \geq n$. Hence

$$(\sigma\bar{\tau}_n)^m = \prod_{i \geq n} (\sigma\bar{\tau}_{n+i})^{m_i} \in H_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle)$$

for each $m = \sum_{i \geq 0} m_i p^i \geq 0$ with $m_i \in \{0, \dots, p-1\}$. These multiplicative extensions lead to the following formula.

Proposition 9.2 ([AR05, Cor. 5.13]). *For p odd,*

$$H_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle) \cong H_*(\mathrm{BP}\langle n-1 \rangle) \otimes \Lambda(\sigma\bar{\xi}_1, \dots, \sigma\bar{\xi}_n) \otimes \mathbb{F}_p[\sigma\bar{\tau}_n]$$

in commutative $\mathcal{A}_*$ -comodule algebras. The $\mathcal{A}_*$ -coaction on $H_*(\mathrm{BP}\langle n-1 \rangle)$ is the restriction of the coproduct on $\mathcal{A}_*$, the classes $\sigma\bar{\xi}_1, \dots, \sigma\bar{\xi}_n$ are $\mathcal{A}_*$ -comodule primitive, and $\nu(\sigma\bar{\tau}_n) = 1 \otimes \sigma\bar{\tau}_n + \bar{\tau}_0 \otimes \sigma\bar{\xi}_n$.

At odd primes p , the Smith–Toda complexes $V(n-1)$ satisfy $H_*V(n-1) \cong \Lambda(\tau_0, \dots, \tau_{n-1})$, and $H^*V(n-1) \cong \Lambda(Q_0, \dots, Q_{n-1})$, where the Q_i defined by $Q_0 = \beta$ and $Q_{i+1} = [P^{p^i}, Q_i]$ are the Minor primitives in $\mathcal{A}_*$. When $V(n-1)$ exists, we have $V(n-1) \otimes \mathrm{BP}\langle n-1 \rangle \simeq \mathbb{F}_p$ and

$$V(n-1) \otimes \mathrm{THH}(\mathrm{BP}\langle n-1 \rangle) \simeq \mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_p).$$

The Bökstedt spectral sequence with coefficients

$$\mathcal{E}_{*,*}^2 = \mathrm{HH}_*(H_*(\mathrm{BP}\langle n-1 \rangle), H_*(\mathbb{F}_p)) \implies H_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_p).$$

has

$$\mathcal{E}_{*,*}^2 \cong \mathcal{A}_* \otimes \Lambda(\sigma\bar{\xi}_i \mid i \geq 1) \otimes \Gamma(\sigma\bar{\tau}_i \mid i \geq n)$$

and

$$\mathcal{E}_{*,*}^p \cong \mathcal{A}_* \otimes \Lambda(\sigma\bar{\xi}_i \mid 1 \leq i \leq n) \otimes \bigotimes_{i \geq n} \mathbb{F}_p[\sigma\bar{\tau}_i] / (\sigma\bar{\tau}_i)^p$$

converging to

$$H_*(\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_p)) \cong \mathcal{A}_* \otimes \Lambda(\sigma\bar{\xi}_1, \dots, \sigma\bar{\xi}_n) \otimes \mathbb{F}_2[\sigma\bar{\tau}_n].$$

Since $\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_p)$ is an $\mathbb{F}_p$ -module, the Hurewicz homomorphism h identifies $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_p)$ with the $\mathcal{A}_*$ -comodule primitives in this abutment. Our calculations then show that there are homotopy classes λ_k with

$$h(\lambda_k) = 1 \otimes \sigma\bar{\xi}_k$$

for $1 \leq k \leq n$, and a homotopy class μ_n with

$$h(\mu_n) = 1 \otimes \sigma \bar{\tau}_n + \tau_0 \otimes \sigma \bar{\xi}_n,$$

such that

$$\pi_* \mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_p) = \Lambda(\lambda_1, \dots, \lambda_n) \otimes \mathbb{F}_p[\mu_n]$$

as a commutative $\mathbb{F}_p$ -algebra. Here we use that $\tau_0 = -\bar{\tau}_0$ is coalgebra primitive.

Proposition 9.3. *The Hurewicz homomorphism identifies*

$$\pi_* \mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_p) = \Lambda(\lambda_1, \dots, \lambda_n) \otimes \mathbb{F}_p[\mu_n]$$

with the $\mathcal{A}_*$ -comodule primitives in

$$H_* \mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_p) = \mathcal{A}_* \otimes \Lambda(\sigma \bar{\xi}_1, \dots, \sigma \bar{\xi}_n) \otimes \mathbb{F}_p[\sigma \bar{\tau}_n],$$

mapping λ_k to $1 \otimes \sigma \bar{\xi}_k$ and μ_n to $1 \otimes \sigma \bar{\tau}_n + \tau_0 \otimes \sigma \bar{\xi}_n$. If $V(n-1)$ exists as a spectrum (resp. ring spectrum), then this is also

$$V(n-1)_* \mathrm{THH}(\mathrm{BP}\langle n-1 \rangle)$$

as a graded $\mathbb{F}_p$ -module (resp. $\mathbb{F}_p$ -algebra).

Remark 9.4. To summarize,

$$\pi_* \mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_p) / \mu_n \cong \Lambda(\lambda_1, \dots, \lambda_n)$$

for $n \geq 0$ and all primes p , while

$$\pi_* \mathrm{THH}(\mathrm{ko}, \mathbb{F}_2) / \mu_2 \cong \Lambda(\lambda'_1, \lambda_2)$$

and

$$\pi_* \mathrm{THH}(\mathrm{tmf}, \mathbb{F}_2) / \mu_3 \cong \Lambda(\lambda''_1, \lambda'_2, \lambda_3)$$

at $p = 2$. In each case, $|\lambda_k| = 2p^k - 1$ and $|\mu_n| = 2p^n$, while $|\lambda'_1| = 5$, $|\lambda''_1| = 9$ and $|\lambda'_2| = 13$.

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