

THE PIRASHVILI–BRUN–HÖNING SPECTRAL SEQUENCE

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We discuss THH-calculations using the Brun spectral sequence, illustrating duality in $\mathrm{THH}(B, k)/\mu_m$ and $\mathrm{THH}(B, k)[\mu_m^{-1}]$ for ‘regular local $\mathbb{E}_3$ -rings’. This is a precursor to Hodge and prismatic, or Ausoni–Bökstedt, duality in $V \otimes \mathrm{THH}(B)/\mu_n$ and $V \otimes \mathrm{THH}(B)[\mu_n^{-1}]$ for ‘height $(n - 1)$ local number rings’. The novel parts represent joint work with Sanath Devalapurkar and Jeremy Hahn.

1. CYCLIC INVARIANCE OF THH

Friedhelm Waldhausen [Wal79] credited the following lemma to Keith Dennis.

Lemma 1.1. *Let B and C be $\mathbb{E}_1$ -rings, let M be a B - C -bimodule, and let N be a C - B -bimodule, so that $M \otimes_C N$ is a B -bimodule and $N \otimes_B M$ is a C -bimodule. There is a natural equivalence*

$$\mathrm{THH}(B, M \otimes_C N) \simeq \mathrm{THH}(C, N \otimes_B M).$$

If $B \rightarrow C$ is an $\mathbb{E}_1$ -map, then there is a natural equivalence

$$\mathrm{THH}(B, C) \simeq \mathrm{THH}(C, C \otimes_B C).$$

Proof. The bisimplicial object

$$[p], [q] \longmapsto B^{\otimes p} \otimes M \otimes C^{\otimes q} \otimes N \cong C^{\otimes q} \otimes N \otimes B^{\otimes p} \otimes M$$

partially realizes to

$$[p] \longmapsto B^{\otimes p} \otimes (M \otimes_C N)$$

as well as to

$$[q] \longmapsto C^{\otimes q} \otimes (N \otimes_B M),$$

so its total realization is equivalent to both $\mathrm{THH}(B, M \otimes_C N)$ and $\mathrm{THH}(C, N \otimes_B M)$. The final statement corresponds to $M = N = C$, viewed as a B - C - or C - B -bimodule via the map $B \rightarrow C$ and the multiplication in C . $\square$

This cyclic invariance can be axiomatized to characterize THH as a functor of stable ∞ -categories and categorical bimodules. See the discussions of THH for ring functors in [DM96], shadows in [Pon10], and laced categories in [HNS].

2. THE BRUN SPECTRAL SEQUENCE

Let $B \rightarrow C$ be a map of $\mathbb{E}_1$ -rings. The C -bimodule $C \otimes_B C$ has a Whitehead filtration

$$\cdots \rightarrow \tau_{\geq s+1}(C \otimes_B C) \rightarrow \tau_{\geq s}(C \otimes_B C) \rightarrow \cdots \rightarrow C \otimes_B C.$$

Applying $\mathrm{THH}(C, -)$ we obtain a filtration

$$\cdots \rightarrow \mathrm{THH}(C, \tau_{\geq s+1}(C \otimes_B C)) \xrightarrow{\alpha_s} \mathrm{THH}(C, \tau_{\geq s}(C \otimes_B C)) \rightarrow \cdots$$

of $\mathrm{THH}(C, C \otimes_B C)$, with cofibers

$$\mathrm{cof}(\alpha_s) \simeq \mathrm{THH}(C, \Sigma^s \pi_s(C \otimes_B C)).$$

If C is connective, then the associated half-plane spectral sequence

$$\mathcal{E}_{-s,t}^1 = \pi_{t-2s} \mathrm{THH}(C, \pi_s(C \otimes_B C)) \Longrightarrow \pi_{s+t} \mathrm{THH}(C, C \otimes_B C)$$

is strongly convergent. It can be Serre regraded as

$$\mathcal{E}_{s,t}^2 = \pi_s \mathrm{THH}(C, \pi_t(C \otimes_B C)) \Longrightarrow \pi_{s+t} \mathrm{THH}(C, C \otimes_B C).$$

If $B \rightarrow C$ is an $\mathbb{E}_2$ -map, then these are multiplicative filtrations and spectral sequences.

Suppose now that B is connective, $C = k$ is a commutative ring, and $\pi_0(B) \rightarrow k$ is surjective. Then $\pi_0(k \otimes_{\mathbb{S}} k) \rightarrow \pi_0(k \otimes_B k)$ agrees with the underived ring multiplication $k \otimes k \rightarrow k$, and the k -bimodule structure on $\pi_t(k \otimes_B k)$ is restricted from its left k -module structure. If k is a field, or more generally $\pi_t(k \otimes_B k)$ is flat over k , then we obtain a natural isomorphism

$$\pi_s \mathrm{THH}(k) \otimes_k \pi_t(k \otimes_B k) \xrightarrow{\cong} \pi_s \mathrm{THH}(k, \pi_t(k \otimes_B k)).$$

(Recall here that $\mathrm{THH}(k) = \mathrm{THH}(k, k)$.) Finally, if k is a perfect field of characteristic $p > 0$, we know that $\pi_* \mathrm{THH}(k) = k[\mu_0]$ with μ_0 in degree 2 detected by $\sigma \bar{\tau}_0$ (or $\sigma \bar{\zeta}_1$ for $p = 2$) in the Bökstedt spectral sequence.

Proposition 2.1. *Let B be a connective $\mathbb{E}_1$ -ring, and let $\pi_0(B) \rightarrow k$ be a surjective ring homomorphism to a field k . Then there is a strongly convergent spectral sequence*

$$\mathcal{E}_{*,*}^2 = \pi_* \mathrm{THH}(k) \otimes_k \pi_*(k \otimes_B k) \Longrightarrow \pi_* \mathrm{THH}(k, k \otimes_B k) \cong \pi_* \mathrm{THH}(B, k).$$

If B is an $\mathbb{E}_2$ -ring, then this is a multiplicative spectral sequence.

This was constructed for discrete rings B by Morten Brun [Bru00, Theorem 6.2.10] and for $\mathbb{E}_\infty$ -rings B by Eva Hönig [Hön20, Theorem 1.1]. A dual version, without multiplicative structure, was developed earlier by Teimuraz Pirashvili [Pir95] in the equivalent context of MacLane (co-)homology. We follow Hönig and call this the Brun spectral sequence.

Lemma 2.2. *When $B = \mathbb{S} \rightarrow \mathbb{F}_p = k$, so that $\pi_* \mathrm{THH}(k) = \mathbb{F}_p[\mu_0]$, $\pi_*(k \otimes_B k) = \mathcal{A}_*$ is the dual Steenrod algebra, and $\mathrm{THH}(B, k) = \mathbb{F}_p$, the Brun spectral sequence*

$$\mathcal{E}_{*,*}^2 = \mathbb{F}_p[\mu_0] \otimes \mathcal{A}_* \Longrightarrow \mathbb{F}_p$$

has differentials for $p = 2$ generated by

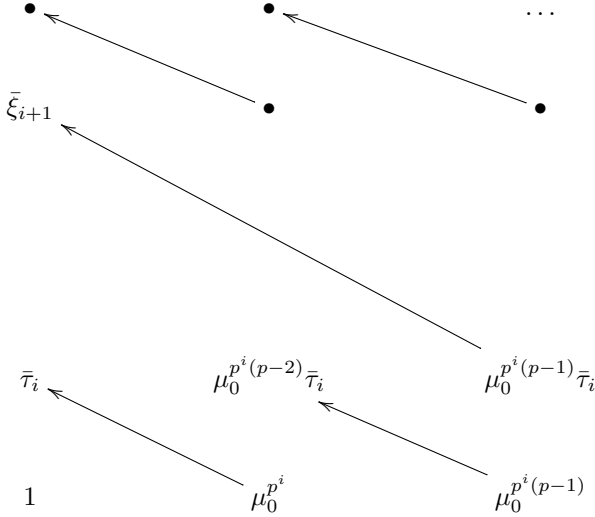
$$d^{2^{i+1}}(\mu_0^{2^i}) = \bar{\zeta}_{i+1}$$

for $i \geq 0$, and differentials for p odd generated by

$$\begin{aligned} d^{2^p i}(\mu_0^{p^i}) &\doteq \bar{\tau}_i \\ d^{2^p i(p-1)}(\mu_0^{p^i(p-1)} \bar{\tau}_i) &\doteq \bar{\xi}_{i+1} \end{aligned}$$

for $i \geq 0$.

Proof. By induction on total degrees, this is the only multiplicative differential pattern leading to $\mathcal{E}^\infty = \mathbb{F}_p$ concentrated in bidegree $(0, 0)$. $\square$



3. HOPF ALGEBROIDS AND HOPF ALGEBRAS

Let $\eta: B \rightarrow C$ be an $\mathbb{E}_2$ -map, so that $C \otimes_B C$ is an $\mathbb{E}_1$ -ring, and suppose that $H_* := \pi_*(C \otimes_B C)$ is flat as a (left or right, it does not matter) $C_* := \pi_*(C)$ -module. Then there is an isomorphism

$$H_* \otimes_{C_*} H_* \xrightarrow{\cong} \pi_*(C \otimes_B C \otimes_B C)$$

and the map

$$C \otimes_B C \cong C \otimes_B B \otimes_B C \xrightarrow{1 \otimes \eta \otimes 1} C \otimes_B C \otimes_B C$$

induces a coproduct

$$\psi: H_* \longrightarrow H_* \otimes_{C_*} H_*.$$

The maps

$$C \cong C \otimes_B B \xrightarrow{1 \otimes \eta} C \otimes_B C$$

$$C \cong B \otimes_B C \xrightarrow{\eta \otimes 1} C \otimes_B C$$

together with the multiplication $C \otimes_B C \rightarrow C$ and the twist isomorphism $C \otimes_B C \cong C \otimes_B C$ induce left and right unit maps $\eta_L, \eta_R: C_* \rightarrow H_*$, an augmentation $\epsilon: H_* \rightarrow C_*$, and a conjugation $\chi: H_* \rightarrow H_*$, respectively.

If H_* is commutative (which is certainly the case if $B \rightarrow C$ is an $\mathbb{E}_3$ -map), then these satisfy relations [Ada69, Lecture 3] exhibiting (H_*, C_*) as a Hopf algebroid. It corepresents groupoids, in the sense that for any graded commutative ring R , $\text{Hom}(C_*, R)$ and $\text{Hom}(H_*, R)$ are the object set and morphism set of a small groupoid, with source, target, identity and inverse induced by η_R, η_L, ϵ and χ , while the composition is induced by ψ .

If $\eta_L = \eta_R$, then H_* is a Hopf C_* -algebra, corepresenting functors from commutative C_* -algebras to groups. This is always the case if B is connective and $C = k$ is a residue field of $\pi_0(B)$, since then the two unit maps $\eta_L, \eta_R: k \rightarrow \pi_0(k \otimes_B k)$ are both sections to the isomorphism $\epsilon: \pi_0(k \otimes_B k) \rightarrow k$.

In the case $B = \mathbb{S} \rightarrow \mathbb{F}_p = k$, this recovers the dual Steenrod (Hopf) algebra $\mathcal{A}_*$. When $B = \mathbb{S} \rightarrow \mathrm{MU} = C$ we get the Novikov Hopf algebroid $(\mathrm{MU}_* \mathrm{MU}, \mathrm{MU}_*)$, and similarly with the Brown–Peterson spectrum BP in place of MU.

4. POINCARÉ DUALITY ALGEBRAS

Let B be a regular local $\mathbb{E}_3$ -ring, meaning that B is connective, $\pi_0(B)$ is a local ring with residue field k , and k is perfect as a B -module, i.e., lies in the thick subcategory Perf_B of Mod_B generated by B . By base change along $B \rightarrow k$ it follows that $k \otimes_B k$ is perfect as a k -module. Hence $H_* = \pi_*(k \otimes_B k)$ is a finite-dimensional, connected, graded commutative Hopf k -algebra. When k is a perfect field, these were classified by Hopf (in characteristic 0) and Borel (in positive characteristic). (Recall that every field of characteristic 0 is perfect, and a field of characteristic $p > 0$ is perfect if and only if the Frobenius endomorphism $x \mapsto x^p$ is an automorphism.)

Theorem 4.1 ([Bor53, §II.6], [MM65, Theorem 7.11]). *Let k be a perfect field of characteristic $p > 0$, and let H_* be a finite-dimensional, connected, graded commutative Hopf k -algebra. Then the underlying k -algebra of H_* is isomorphic to a finite tensor product*

$$\Lambda(x_1, \dots, x_a) \otimes k[y_1, \dots, y_b] / (y_1^{p^{h_1}}, \dots, y_b^{p^{h_b}})$$

of algebras of the forms

- $\Lambda(x)$ with $|x|$ odd;
- $k[y] / (y^{p^h})$ with $|y|$ even or $p = 2$.

We refer to these as finite commutative Hopf algebras. They are in particular Poincaré duality algebras.

Definition 4.2. Let k be a (graded) commutative ring and H_* a graded commutative k -algebra. We say that H_* is a Poincaré duality k -algebra if it comes equipped with a k -linear evaluation homomorphism $\int: H_* \rightarrow \Sigma^d k$ for some d , such that the composite

$$H_* \otimes_k H_* \longrightarrow H_* \xrightarrow{\int} \Sigma^d k$$

is a perfect pairing, in the sense that the adjoint k -linear map

$$e: H_* \xrightarrow{\cong} \mathrm{Hom}_k(H_*, \Sigma^d k) = \Sigma^d H_*^\vee$$

is an isomorphism. If k is a (graded) field, it then follows that H_* is finite-dimensional over k . If H_* is connected, so that $k \cong H_0$, then also $H_d \cong k$, and there is a unique generator $\omega \in H_d$ with $\int \omega = 1$.

For example, the cohomology $H^*(M)$ of a closed, oriented d -manifold, together with evaluation $\langle -, [M] \rangle: H^d(M) \rightarrow \mathbb{Z}$ on the fundamental class, is a Poincaré duality algebra. If M is connected, then there is a unique generator $\omega \in H^d(M) \cong \mathbb{Z}$ with $\langle \omega, [M] \rangle = 1$. We can refer to ω as an orientation form.

The underlying algebra

$$H_* \cong \Lambda(x_1, \dots, x_a) \otimes k[y_1, \dots, y_b] / (y_1^{p^{h_1}}, \dots, y_b^{p^{h_b}})$$

of any finite commutative Hopf algebra has a single generator

$$\omega = x_1 \cdots x_a \cdot y_1^{p^{h_1}-1} \cdots y_b^{p^{h_b}-1}$$

in the top nontrivial degree $d = |\omega|$, and the homomorphism given by $\int \omega = 1$ defines a perfect pairing. Hence, with this choice H_* is a Poincaré duality algebra.

More generally, Browder–Spanier [BS62, Lemma] proved that any finite and connected, but not necessarily commutative, Hopf algebra also admits such a perfect pairing.

5. DUALITY IN $\mathrm{THH}(B, k)$

The following overlaps with results of Greenlees [Gre16, Theorem 1.1].

Theorem 5.1. *Let B be a regular local $\mathbb{E}_3$ -ring with perfect residue field k of characteristic $p > 0$. Let d be the highest degree in which $\pi_*(k \otimes_B k)$ is nontrivial. There is an $m \geq 0$, a class $\mu_m \in \pi_{2p^m} \mathrm{THH}(B, k)$ mapping to $\mu_0^{p^m} \in \pi_{2p^m} \mathrm{THH}(k)$, and a class $\omega \in \pi_{2p^m - 2 + d} \mathrm{THH}(B, k)$, such that*

- (1) $\pi_* \mathrm{THH}(B, k)$ is a free $k[\mu_m]$ -module;
- (2) $\pi_* \mathrm{THH}(B, k)/\mu_m$ is a Poincaré duality k -algebra with integral $\int_1$ evaluating to 1 on the quotient image ω_1 of ω ;
- (3) $\pi_* \mathrm{THH}(B, k)[\mu_m^{\pm 1}]$ is a Poincaré duality $k[\mu_m^{\pm 1}]$ -algebra with integral $\int_2$ evaluating to 1 on the localized image ω_2 of ω .
- (4) The combined map

$$\mathrm{THH}(B, k) \xrightarrow{f} \mathrm{THH}(B, k)/\mu_m \oplus \mathrm{THH}(B, k)[\mu_m^{\pm 1}]$$

is Lagrangian, or presents a Poincaré pair, in the sense that there is a cofiber sequence

$$\begin{aligned} \mathrm{THH}(B, k) &\xrightarrow{f} \mathrm{THH}(B, k)/\mu_m \oplus \mathrm{THH}(B, k)[\mu_m^{\pm 1}] \\ &\quad \simeq \downarrow e_1 \oplus e_2 \\ &\quad I^{|\omega|}(\mathrm{THH}(B, k)/\mu_m \oplus \mathrm{THH}(B, k)[\mu_m^{\pm 1}]) \xrightarrow{I^{|\omega|}(f)} I^{|\omega|} \mathrm{THH}(B, k). \end{aligned}$$

The first three parts of the theorem are immediate consequences of the following more detailed sequence of assertions. We may return to the fourth at a later occasion.

Proposition 5.2. *Let B be a regular local $\mathbb{E}_3$ -ring with perfect residue field k of characteristic $p > 0$.*

- (1) *The connected Hopf k -algebra $\pi_*(k \otimes_B k)$ is finite and commutative, hence a Poincaré duality algebra concentrated in a range of degrees $0 \leq * \leq d$, with evaluation $\int_0: \pi_*(k \otimes_B k) \rightarrow \Sigma^d k$ taking the value 1 on a generator ω_0 of the top nontrivial degree.*
- (2) *The Brun spectral sequence*

$$\mathcal{E}_{*,*}^2 = k[\mu_0] \otimes_k \pi_*(k \otimes_B k) \implies \pi_* \mathrm{THH}(k, k \otimes_B k) \cong \pi_* \mathrm{THH}(B, k)$$

is a commutative algebra spectral sequence. It collapses at the $\mathcal{E}^{d+2} = \mathcal{E}^\infty$ -page. Let $m \geq 0$ be minimal with $2p^m \geq d + 2$. Then $\mu_0^{p^m}$ is an infinite cycle, detecting an element $\mu_m \in \pi_{2p^m} \mathrm{THH}(B, k)$.

- (3) *Each $\mathcal{E}^r$ -page of the Brun spectral sequence is free as a $k[\mu_m]$ -module. It maps onto the quotient spectral sequence*

$$\mathcal{E}_{*,*}^2/\mu_m = k[\mu_0]/\mu_0^{p^m} \otimes_k \pi_*(k \otimes_B k) \implies \pi_* \mathrm{THH}(B, k)/\mu_m,$$

and maps into the localized spectral sequence

$$\mu_m^{-1}\mathcal{E}_{*,*}^2 = k[\mu_0^{\pm 1}] \otimes_k \pi_*(k \otimes_B k) \implies \pi_*\mathrm{THH}(B, k)[\mu_m^{-1}].$$

- (4) The quotient $\mathcal{E}^2$ -page is a bigraded Poincaré duality k -algebra, concentrated in bidegrees between $(0, 0)$ and $(2p^m - 2, d)$, with evaluation $\int': \mathcal{E}_{*,*}^2/\mu_m \rightarrow \Sigma^{2p^m-2+d}k$ taking the value 1 on the generator $\omega' := \mu_0^{p^m-1}\omega_0$ of the top nontrivial bidegree.
- (5) The previous item remains true for each $\mathcal{E}^r/\mu_m$ with $2 \leq r \leq \infty$, and the abutment $\pi_*\mathrm{THH}(B, k)/\mu_m$ is a Poincaré duality k -algebra, concentrated in degrees $0 \leq * \leq 2p^m - 2 + d$, with evaluation $\int_1: \pi_*\mathrm{THH}(B, k)/\mu_m \rightarrow \Sigma^{2p^m-2+d}k$ taking the value 1 on the generator ω_1 of the top nontrivial bidegree, which is detected by $\omega' \in \mathcal{E}_{2p^m-2,d}^\infty/\mu_m$.
- (6) The localized $\mathcal{E}^2$ -page is a bigraded Poincaré duality $k[\mu_m^{\pm 1}]$ -algebra, with evaluation $\int'': \mu_m^{-1}\mathcal{E}_{*,*}^2 \rightarrow \Sigma^{2p^m-2+d}k[\mu_m^{\pm 1}]$ taking the value 1 on the generator $\omega'' := \mu_0^{p^m-1}\omega_0$.
- (7) The previous item remains true for each $\mu_m^{-1}\mathcal{E}^r$ with $2 \leq r \leq \infty$, and the abutment $\pi_*\mathrm{THH}(B, k)[\mu_m^{-1}]$ is a Poincaré duality $k[\mu_m^{\pm 1}]$ -algebra, with evaluation $\int_2: \pi_*\mathrm{THH}(B, k)[\mu_m^{-1}] \rightarrow \Sigma^{2p^m-2+d}k[\mu_m^{\pm 1}]$ taking the value 1 on the generator ω_2 detected by $\omega'' \in \mu_m^{-1}\mathcal{E}_{2p^m-2,d}^\infty$.

Proof. (1) These claims follow from the discussion in Section 4.

(2) The Brun spectral sequence is concentrated in bidegrees (s, t) with $0 \leq t \leq d$, so $d^r = 0$ for $r \geq d + 2$. By Lemma 2.2, we know that $d^{2p^m}(\mu_0^m)$ is the image of $\bar{\tau}_m$ in stem $2p^m - 1 > d$, hence is zero.

(3) All Brun differentials reduce the μ_0 -exponent, and repeat $\mu_0^{p^m}$ -periodically. It follows that no nonzero differentials leave or enter the region with $0 \leq s \leq 2p^m - 2$ and $0 \leq t \leq d$. These agree with the differentials in the quotient spectral sequence. Moreover, they repeat by multiplication by all positive powers of μ_m in the original spectral sequence, as well as for all integer powers of μ_m in the localized spectral sequence.

(4) This follows from the expression for $\mathcal{E}^2/\mu_m$, since $k[\mu_0]/\mu_0^{p^m}$ is a Poincaré duality k -algebra with evaluation taking the value 1 on the top degree class $\mu_0^{p^m-1}$.

(5) Let $V^\vee := \mathrm{Hom}_k(V, k)$ denote the k -linear dual of any k -vector space V . For bidegree reasons, no Brun differential can hit the bidegree $(2p^m - 2, d)$ of ω' . Hence $\int'$ and the algebra product define a pairing

$$\mathcal{E}_{*,*}^r/\mu_m \otimes_k \mathcal{E}_{*,*}^r/\mu_m \xrightarrow{\quad} \mathcal{E}_{*,*}^r/\mu_m \xrightarrow{\int'} \Sigma^{2p^m-2+d}k$$

of spectral sequences, converging to the pairing

$$\pi_*\mathrm{THH}(B, k)/\mu_m \otimes_k \pi_*\mathrm{THH}(B, k)/\mu_m \xrightarrow{\quad} \pi_*\mathrm{THH}(B, k)/\mu_m \xrightarrow{\int_1} \Sigma^{2p^m-2+d}k.$$

The former pairing is adjoint to a map

$$\mathcal{E}_{*,*}^r/\mu_m \xrightarrow{e} \Sigma^{2p^m-2,d}(\mathcal{E}_{*,*}^r/\mu_m)^\vee$$

of spectral sequences, with target a shifted k -linear dual of $\mathcal{E}_{*,*}^r/\mu_m$. The latter pairing is adjoint to a filtered homomorphism

$$\pi_*\mathrm{THH}(B, k)/\mu_m \xrightarrow{e} \Sigma^{2p^m-2+d}(\pi_*\mathrm{THH}(B, k)/\mu_m)^\vee,$$

with associated graded given by the map of $\mathcal{E}^\infty$ -pages. Since the pairing of $\mathcal{E}^2$ -pages is perfect, the map of $\mathcal{E}^2$ -pages is an isomorphism, which implies that the map of $\mathcal{E}^r$ -pages is an isomorphism for each $2 \leq r \leq \infty$. Hence the pairing of $\mathcal{E}^r$ -pages is perfect for each $2 \leq r \leq \infty$. Moreover, by strong convergence it follows that the filtered homomorphism is an isomorphism, so that the pairing of abutments is also perfect.

(6) This follows from the expression for $\mu_m^{-1}\mathcal{E}^2$, since $k[\mu_0^{\pm 1}]$ is a Poincaré duality $k[\mu_m^{\pm 1}]$ -algebra, here with the evaluation taking the value 1 on the class $\mu_0^{p^m-1}$.

(7) This proof is similar to that for (5). $\square$

6. EXAMPLES

Example 6.1 (Bökstedt, ca. 1986). For $B = \mathbb{Z} \rightarrow \mathbb{F}_p = k$ we have $\pi_*(\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p) = \Lambda(\bar{\tau}_0)$. The Brun spectral sequence has $d^2(\mu_0) \doteq \bar{\tau}_0$, leaving

$$\mathcal{E}^3 = \mathcal{E}^\infty = \Lambda(\mu_0^{p-1}\bar{\tau}_0) \otimes \mathbb{F}_p[\mu_0^p].$$

Let $\lambda_1, \mu_1 \in \pi_*\mathrm{THH}(\mathbb{Z}, \mathbb{F}_p)$ in stems $2p-1$ and $2p$ be detected by $\mu_0^{p-1}\bar{\tau}_0$ and μ_0^p , respectively. Then

$$\pi_*\mathrm{THH}(\mathbb{Z}, \mathbb{F}_p) \cong \Lambda(\lambda_1) \otimes \mathbb{F}_p[\mu_1]$$

with $\omega = \lambda_1$, $\pi_*\mathrm{THH}(\mathbb{Z}, \mathbb{F}_p)/\mu_1 \cong \Lambda(\lambda_1)$ and $\pi_*\mathrm{THH}(\mathbb{Z}, \mathbb{F}_p)[\mu_1^{-1}] \cong \Lambda(\lambda_1) \otimes \mathbb{F}_p[\mu_1^{\pm 1}]$. The results are the same for $B = \mathbb{Z}_{(p)}$ or $B = \mathbb{Z}_p$.

Example 6.2 (McClure–Staffeldt [MS93]). For the Adams $\mathbb{E}_\infty$ -ring $B = \ell \rightarrow \mathbb{F}_p = k$, a summand of p -local connective complex K -theory $\mathrm{ku}_{(p)}$, we have $\pi_*(\mathbb{F}_p \otimes_{\ell} \mathbb{F}_p) = \Lambda(\bar{\tau}_0, \bar{\tau}_1)$. The Brun spectral sequence has $d^2(\mu_0) \doteq \bar{\tau}_0$ and $d^{2p}(\mu_0^p) \doteq \bar{\tau}_1$, leaving

$$\mathcal{E}^{2p+1} = \mathcal{E}^\infty = \Lambda(\mu_0^{p-1}\bar{\tau}_0, \mu_0^{p^2-p}\bar{\tau}_1) \otimes \mathbb{F}_p[\mu_0^p].$$

Let $\lambda_1, \lambda_2, \mu_2 \in \pi_*\mathrm{THH}(\ell, \mathbb{F}_p)$ in stems $2p-1$, $2p^2-1$ and $2p^2$ be detected by $\mu_0^{p-1}\bar{\tau}_0$, $\mu_0^{p^2-p}\bar{\tau}_1$ and $\mu_0^{p^2}$, respectively. Then

$$\pi_*\mathrm{THH}(\ell, \mathbb{F}_p) \cong \Lambda(\lambda_1, \lambda_2) \otimes \mathbb{F}_p[\mu_2]$$

with $\omega = \lambda_1\lambda_2$, $\pi_*\mathrm{THH}(\ell, \mathbb{F}_p)/\mu_2 \cong \Lambda(\lambda_1, \lambda_2)$ and $\pi_*\mathrm{THH}(\ell, \mathbb{F}_p)[\mu_2^{-1}] \cong \Lambda(\lambda_1, \lambda_2) \otimes \mathbb{F}_p[\mu_2^{\pm 1}]$. This also applies for $p=2$, writing $\bar{\zeta}_i$ in place of $\bar{\tau}_i$. The results are the same for the summand $B = \ell_p$ of ku_p .

Example 6.3 ([MS93], [BM13]). For the Brown–Peterson $\mathbb{E}_4$ -ring $B = \mathrm{BP} \rightarrow \mathbb{F}_p = k$, a summand of p -local complex bordism $\mathrm{MU}_{(p)}$, we have $\pi_*(\mathbb{F}_p \otimes_{\mathrm{BP}} \mathbb{F}_p) = \Lambda(\bar{\tau}_i \mid i \geq 0)$, so this example is not regular. The Brun spectral sequence has $d^{2p^i}(\mu_0^{p^i}) \doteq \bar{\tau}_i$ for each $i \geq 0$, leaving

$$\mathcal{E}^\infty = \Lambda(\mu_0^{p^i(p-1)}\bar{\tau}_i \mid i \geq 0).$$

Let $\lambda_{i+1} \in \pi_*\mathrm{THH}(\mathrm{BP}, \mathbb{F}_p)$ in stem $2p^{i+1}-1$ be detected by $\mu_0^{p^i(p-1)}\bar{\tau}_i$ for each $i \geq 0$. Then

$$\pi_*\mathrm{THH}(\mathrm{BP}, \mathbb{F}_p) \cong \Lambda(\lambda_i \mid i \geq 1).$$

This also applies for $p=2$, writing $\bar{\zeta}_i$ in place of $\bar{\tau}_i$. The results are the same for the summand $B = \mathrm{BP}_p$ of MU_p . (McClure–Staffeldt also calculate $\pi_*\mathrm{THH}(\mathrm{BP})$ and $\pi_*\mathrm{THH}(\mathrm{MU})$.)

Example 6.4 (Brun [Bru00]). For $B = \mathbb{Z}/p^n \rightarrow \mathbb{F}_p = k$ with $n \geq 2$ we have $\pi_*(\mathbb{F}_p \otimes_{\mathbb{Z}/p^n} \mathbb{F}_p) = \Lambda(\bar{\tau}_0) \otimes \Gamma(\phi_n)$ with $|\phi_n| = 2$, so this example is not regular. The Brun spectral sequence has $d^2(\mu_0) = \bar{\tau}_0$, leaving

$$\mathcal{E}^3 = \mathcal{E}^\infty = \Gamma(\phi_n) \otimes \Lambda(\mu_0^{p-1} \bar{\tau}_0) \otimes k[\mu_0^p].$$

Naturality with respect to $\mathbb{Z} \rightarrow \mathbb{Z}/p^n$ shows that there are no further differentials, and there is no room for multiplicative extensions with the divided power algebra, so

$$\pi_* \mathrm{THH}(\mathbb{Z}/p^n, \mathbb{F}_p) \cong \Gamma(\phi_n) \otimes \Lambda(\lambda_1) \otimes k[\mu_1].$$

(Brun in fact calculates $\pi_* \mathrm{THH}(\mathbb{Z}/p^n)$. See also [KN22].)

Example 6.5 (Lindenstrauss–Madsen [LM00]). For the valuation ring $B = \mathcal{O}_L$ of a p -adic number field, with uniformizer ϖ and finite residue field k , we have $\pi_*(k \otimes_{\mathcal{O}_L} k) \cong \Lambda(\sigma\varpi)$. Here $p \doteq \varpi^e$, with $e \geq 1$ the ramification index. The Brun spectral sequence has $d^2(\mu_0) \doteq \sigma p \doteq e \cdot \varpi^{e-1} \cdot \sigma\varpi$. In the unramified case $e = 1$, when $\mathcal{O}_L \cong W(k)$ is the ring of Witt vectors, this equals $\sigma\varpi$, leaving

$$\mathcal{E}^3 = \mathcal{E}^\infty = \Lambda(\mu_0^{p-1} \cdot \sigma\varpi) \otimes k[\mu_0^p].$$

Let $\lambda_1, \mu_1 \in \pi_* \mathrm{THH}(\mathcal{O}_L, k)$ in stems $2p-1$ and $2p$ be detected by $\mu_0^{p-1} \cdot \sigma\varpi$ and μ_0^p , respectively. Then

$$\pi_* \mathrm{THH}(\mathcal{O}_L)/p \cong \pi_* \mathrm{THH}(\mathcal{O}_L, k) \cong \Lambda(\lambda_1) \otimes k[\mu_1].$$

In the ramified cases $e \geq 2$, we have $e \cdot \varpi^{e-1} \cdot \sigma\varpi = 0$, so that

$$\mathcal{E}^2 = \mathcal{E}^\infty = \Lambda(\sigma\varpi) \otimes k[\mu_0].$$

The ϖ -adic filtration of $\mathcal{O}_L/p$, with associated graded $k[\varpi]/\varpi^e$, then leads to a second spectral sequence

$$\pi_* \mathrm{THH}(\mathcal{O}_L, k) \otimes_k k[\varpi]/\varpi^e \Longrightarrow \pi_* \mathrm{THH}(\mathcal{O}_L)/p$$

with $d^{e-1}(\mu_0) \doteq \sigma p \doteq e \cdot \sigma\varpi \cdot \varpi^{e-1}$. In the wildly ramified case $p \mid e$, these are 0, so

$$\pi_* \mathrm{THH}(\mathcal{O}_L)/p \cong \Lambda(\sigma\varpi) \otimes \mathcal{O}_L/p[\mu_0].$$

In the tamely ramified case $p \nmid e \neq 1$, the differentials leave

$$(\mathcal{O}_L/p\{1, \sigma\varpi \cdot \mu_0^{p-1}\} \oplus \mathcal{O}_L/\varpi^{e-1}\{\sigma\varpi \cdot \mu_0^{i-1}, \varpi \cdot \mu_0^i \mid 0 < i < p\})[\mu_0^p].$$

In particular, $\pi_* \mathrm{THH}(\mathcal{O}_L)/(p, \mu_1)$ is a (non-flat) Poincaré duality $\mathcal{O}_L/p$ -algebra, with integral evaluating to 1 on the orientation form $\sigma\varpi \cdot \mu_0^{p-1}$. It is also a Poincaré duality k -algebra, with orientation form $\lambda_1 \doteq \sigma\varpi \cdot \varpi^{e-1} \cdot \mu_0^{p-1}$. (See also [KN22].)

Example 6.6 (Ausoni [Aus05]). For connective complex K -theory $B = \mathrm{ku} \rightarrow \mathbb{F}_p = k$ at an odd prime p we have $\pi_*(\mathbb{F}_p \otimes_{\mathrm{ku}} \mathbb{F}_p) = \Lambda(\bar{\tau}_0, \sigma u)$. The Brun spectral sequence has $d^2(\mu_0) \doteq \bar{\tau}_0$ but $d^{2p}(\mu_0^p) = \sigma(u^{p-1}) = 0$, leaving

$$\mathcal{E}^3 = \mathcal{E}^\infty = \Lambda(\sigma u, \mu_0^{p-1} \bar{\tau}_0) \otimes \mathbb{F}_p[\mu_0^p].$$

Let $\sigma u, \lambda_1, \mu_1 \in \pi_* \mathrm{THH}(\mathrm{ku}, \mathbb{F}_p)$ in stems $3, 2p-1$ and $2p$ be detected by $\sigma u, \mu_0^{p-1} \bar{\tau}_0$ and μ_0^p , respectively. Then

$$\pi_* \mathrm{THH}(\mathrm{ku}, \mathbb{F}_p) \cong \Lambda(\sigma u, \lambda_1) \otimes \mathbb{F}_p[\mu_1]$$

with $\omega = \sigma u \cdot \lambda_1$, $\pi_* \mathrm{THH}(\mathrm{ku}, \mathbb{F}_p)/\mu_1 \cong \Lambda(\sigma u, \lambda_1)$ and $\pi_* \mathrm{THH}(\mathrm{ku}, \mathbb{F}_p)[\mu_1^{-1}] \cong \Lambda(\sigma u, \lambda_1) \otimes \mathbb{F}_p[\mu_1^{\pm 1}]$. Here one could also take $m = 2$ non-minimal with $\omega = \sigma u \cdot \lambda_1 \cdot \mu_1^{p-1}$, $\pi_* \mathrm{THH}(\mathrm{ku}, \mathbb{F}_p)/\mu_2 \cong \Lambda(\sigma u, \lambda_1) \otimes \mathbb{F}_p[\mu_1]/\mu_1^p$ and $\pi_* \mathrm{THH}(\mathrm{ku}, \mathbb{F}_p)[\mu_2^{-1}] \cong$

$\Lambda(\sigma u, \lambda_1) \otimes \mathbb{F}_p[\mu_1, \mu_2^{\pm 1}]/(\mu_1^p = \mu_2)$. The results are the same for $B = \mathrm{ku}_{(p)}$ or $B = \mathrm{ku}_p$. At $p = 2$, $\mathrm{ku}_{(2)} = \ell$ and $\mathrm{ku}_2 = \ell_2$, which were discussed above.

(Ausoni proceeds to use the u -adic filtration of $\mathrm{ku}/(p, v_1)$, with associated graded $\mathbb{F}_p[u]/u^{p-1}$, to obtain a spectral sequence calculating $\pi_*\mathrm{THH}(\mathrm{ku})/(p, v_1)$, with $\pi_*\mathrm{THH}(\mathrm{ku})/(p, v_1, \mu_2)$ a (non-flat) Poincaré duality $\mathbb{F}_p[u]/u^{p-1}$ -algebra.) It is also a Poincaré duality $\mathbb{F}_p$ -algebra, with orientation form $\lambda_1 \cdot \lambda_2 \doteq \lambda_1 \cdot u^{p-2} \cdot \sigma u$.)

Example 6.7 (Angeltveit–R. [AR05]). For connective real K -theory $B = \mathrm{ko} \rightarrow \mathbb{F}_2 = k$ at $p = 2$ we have $\pi_*(\mathbb{F}_2 \otimes_{\mathrm{ko}} \mathbb{F}_2) = \mathcal{A}(1)_* = \mathbb{F}_2[\bar{\zeta}_1, \bar{\zeta}_2]/(\bar{\zeta}_1^4, \bar{\zeta}_2^2)$. The Brun spectral sequence has $d^2(\mu_0) = \bar{\zeta}_1$ and $d^4(\mu_0^2) = \bar{\zeta}_2$, leaving

$$\mathcal{E}^5 = \mathcal{E}^\infty = \Lambda(\mu_0 \bar{\zeta}_1^3, \mu_0^2 \bar{\zeta}_2) \otimes \mathbb{F}_2[\mu_0^4].$$

Let $\lambda'_1, \lambda_2, \mu_2 \in \pi_*\mathrm{THH}(\mathrm{ko}, \mathbb{F}_2)$ in stems 5, 7 and 8 be detected by $\mu_0 \bar{\zeta}_1^3$, $\mu_0^2 \bar{\zeta}_2$ and μ_0^4 , respectively. Then

$$\pi_*\mathrm{THH}(\mathrm{ko}, \mathbb{F}_2) \cong \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu_2]$$

with $\omega = \lambda'_1 \lambda_2$, $\pi_*\mathrm{THH}(\mathrm{ko}, \mathbb{F}_2)/\mu_2 \cong \Lambda(\lambda'_1, \lambda_2)$ and $\pi_*\mathrm{THH}(\mathrm{ko}, \mathbb{F}_2)[\mu_2^{-1}] \cong \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu_2^{\pm 1}]$. The results are the same for $B = \mathrm{ko}_{(2)}$ or $B = \mathrm{ko}_2$. At $p = 3$, $\mathrm{ko}_{(3)} = \ell$ and $\mathrm{ko}_3 = \ell_3$, which were discussed above.

Example 6.8. For p -complete connective real K -theory $B = \mathrm{ko}_p \rightarrow \mathbb{F}_p = k$ at p odd we have $\pi_*(\mathbb{F}_p \otimes_{\mathrm{ko}} \mathbb{F}_p) = \Lambda(\bar{\tau}_0, \sigma A)$, where $|A| = 4$. The Brun spectral sequence has $d^2(\mu_0) = \bar{\tau}_0$ and no later differentials (except in the case of $\mathrm{ko}_3 = \ell_3$ at $p = 3$, which we discussed above), leaving

$$\mathcal{E}^3 = \mathcal{E}^\infty = \Lambda(\sigma A, \mu_0^{p-1} \bar{\tau}_0) \otimes \mathbb{F}_p[\mu_0^p].$$

Let λ_1 and $\mu_1 \in \pi_*\mathrm{THH}(\mathrm{ko}, \mathbb{F}_2)$ in stems $2p-1$ and $2p$ be detected by $\mu_0^{p-1} \bar{\tau}_0$ and μ_0^p , respectively. Then

$$\pi_*\mathrm{THH}(\mathrm{ko}_p, \mathbb{F}_p) \cong \Lambda(\sigma A, \lambda_1) \otimes \mathbb{F}_2[\mu_1]$$

with $\omega = \sigma A \cdot \lambda_1$, $\pi_*\mathrm{THH}(\mathrm{ko}_p, \mathbb{F}_p)/\mu_1 \cong \Lambda(\sigma A, \lambda_1)$ and $\pi_*\mathrm{THH}(\mathrm{ko}_p, \mathbb{F}_p)[\mu_1^{-1}] \cong \Lambda(\sigma A, \lambda_1) \otimes \mathbb{F}_2[\mu_1^{\pm 1}]$. The results are the same for $B = \mathrm{ko}$ or $B = \mathrm{ko}_{(p)}$.

Example 6.9 ([AR05], [HW22]). For the p -local truncated Brown–Peterson $\mathbb{E}_3$ -ring $B = \mathrm{BP}\langle n-1 \rangle \rightarrow \mathbb{F}_p = k$ we have $\pi_*(\mathbb{F}_p \otimes_{\mathrm{BP}\langle n-1 \rangle} \mathbb{F}_p) = \Lambda(\bar{\tau}_0, \dots, \bar{\tau}_{n-1})$. The Brun spectral sequence has $d^{2^i}(\mu_0^{2^i}) \doteq \bar{\tau}_i$ for $0 \leq i < n$, leaving

$$\mathcal{E}^{2p^{n-1}+1} = \mathcal{E}^\infty = \Lambda(\mu_0^{p^i(p-1)} \bar{\tau}_i \mid 0 \leq i < n) \otimes \mathbb{F}_p[\mu_0^{p^n}].$$

Let $\lambda_1, \dots, \lambda_n$ and $\mu_n \in \pi_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_p)$ in stems $2p-1, \dots, 2p^n-1$ and $2p^n$ be detected by $\mu_0^{p-1} \bar{\tau}_0, \dots, \mu_0^{p^n-p^{n-1}} \bar{\tau}_{n-1}$ and $\mu_0^{p^n}$, respectively. Then

$$\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_p) \cong \Lambda(\lambda_1, \dots, \lambda_n) \otimes \mathbb{F}_p[\mu_n]$$

with orientation form $\omega = \lambda_1 \cdots \lambda_n$, $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_p)/\mu_n \cong \Lambda(\lambda_1, \dots, \lambda_n)$ and $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1 \rangle, \mathbb{F}_p)[\mu_n^{-1}] \cong \Lambda(\lambda_1, \dots, \lambda_n) \otimes \mathbb{F}_p[\mu_n^{\pm 1}]$. This also applies for $p = 2$, writing $\bar{\zeta}_i$ in place of $\bar{\tau}_i$. The results are the same for $B = \mathrm{BP}\langle n-1 \rangle_p$.

Example 6.10. For the connective Lubin–Tate $\mathbb{E}_\infty$ -ring $B = e_n = \tau_{\geq 0} E_n \rightarrow \mathbb{F}_{p^n} = k$, with $\pi_*(e_n) = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]] [u]$ where $|u_i| = 0$ and $|u| = 2$, we have $\pi_*(\mathbb{F}_{p^n} \otimes_{e_n} \mathbb{F}_{p^n}) = \Lambda(\bar{\tau}_0, \sigma u_1, \dots, \sigma u_{n-1}, \sigma u)$. Excluding the case of $e_1 = \mathrm{ku}_p$ at

$p = 2$, the Brun spectral sequence has $d^2(\mu_0) \doteq \bar{\tau}_0$ and no later differentials (except in the case of $e_1 = \ell_2$ at $p = 2$, which we discussed above), leaving

$$\mathcal{E}^3 = \mathcal{E}^\infty = \Lambda(\sigma u_1, \dots, \sigma u_{n-1}, \sigma u, \mu_0^{p-1} \bar{\tau}_0) \otimes \mathbb{F}_p[\mu_0^p].$$

Let λ_1 and $\mu_1 \in \pi_* \mathrm{THH}(e_n, \mathbb{F}_p)$ in stems $2p - 1$ and $2p$ be detected by $\mu_0^{p-1} \bar{\tau}_0$ and μ_0^p , respectively. Then

$$\pi_* \mathrm{THH}(e_n, \mathbb{F}_p) \cong \Lambda(\sigma u_1, \dots, \sigma u_{n-1}, \sigma u, \lambda_1) \otimes \mathbb{F}_p[\mu_1]$$

with $\omega = \sigma u_1 \cdots \sigma u_{n-1} \cdot \sigma u \cdot \lambda_1$, $\pi_* \mathrm{THH}(e_n, \mathbb{F}_p)/\mu_1 \cong \Lambda(\sigma u_1, \dots, \sigma u_{n-1}, \sigma u, \lambda_1)$ and $\pi_* \mathrm{THH}(e_n, \mathbb{F}_p)[\mu_1^{-1}] \cong \Lambda(\sigma u_1, \dots, \sigma u_{n-1}, \sigma u, \lambda_1) \otimes \mathbb{F}_p[\mu_1^{\pm 1}]$. This also applies for $p = 2$, writing $\hat{\zeta}_i$ in place of $\bar{\tau}_i$.

((ETC: $\mathrm{tmf}_1(3)$, tmf at $p = 2$, $\mathrm{tmf}_0(2)$, tmf at $p = 3$, tmf at $p \geq 5$.)

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