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THE FIBER OF THE LINEARIZATION MAP $A(*) \to K(\mathbb{Z})$

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1. INTRODUCTION

Waldhausen's *algebraic K-theory of spaces* is a homotopy functor $X \mapsto A(X)$ from spaces to infinite loop spaces. It is related to diffeomorphisms of smooth manifolds through stable smooth pseudoisotopies, or concordances. Recall that when Y is a smooth manifold its smooth pseudoisotopy space $P(Y)$ is the space of self-diffeomorphisms of $Y \times I$ relative to $Y \times 0 \cup \partial Y \times I$. The *stable* smooth pseudoisotopy space is defined as $\mathscr{P}(Y) = \operatorname{hocolim}_N P(Y \times I^N)$, where the maps in the colimit are certain suspension maps. The smooth Whitehead space functor $X \mapsto \mathrm{Wh}^{\mathrm{Diff}}(X)$ is a homotopy functor defined in [33], satisfying $\Omega^2 \mathrm{Wh}^{\mathrm{Diff}}(Y) \simeq \mathscr{P}(Y)$ when Y is a smooth manifold.

Let $Q(X_+) = \operatorname{hocolim}_N \Omega^N \Sigma^N (X_+)$ represent the unreduced stable homotopy of X. Then $A(X)$ satisfies the following theorem, proved in [34]:

THEOREM 1.1. (Waldhausen). $A(X) \simeq Q(X_+) \times \mathrm{Wh}^{\mathrm{Diff}}(X)$.

In the particular case $X = *$ this asserts that $A(*) \simeq Q(S^0) \times \mathrm{Wh}^{\mathrm{Diff}}(*)$, with

$$\Omega^2 \mathrm{Wh}^{\mathrm{Diff}}(*) \simeq \mathscr{P}(*) \simeq \operatorname{hocolim}_N \mathrm{Diff}(D^{N+1} \text{ rel } D^N).$$

Here $D^N \subset S^N$ can be viewed as a hemispherical disc on the boundary of D^{N+1}. Furthermore, there is a fibration sequence

$$\mathrm{Diff}(D^{N+1} \text{ rel } \partial D^{N+1}) \to \mathrm{Diff}(D^{N+1} \text{ rel } D^N) \to \mathrm{Diff}(D^N \text{ rel } \partial D^N)$$

(when restricted to the path components in the image of the second map). These spaces clearly have geometric interest. On the level of homotopy groups we find

$$\pi_i A(*) \cong \pi_i Q(S^0) \oplus \pi_{i-2} \mathscr{P}(*)$$

$$\pi_{i-2} \mathscr{P}(*) \cong \operatorname{colim}_N \pi_{i-2} \mathrm{Diff}(D^{N+1} \text{ rel } D^N).$$

By Igusa's stability theorem [20], the colimit is actually achieved for some finite $N \gg i$. Hence homotopy in $\mathscr{P}(*)$ reappears as homotopy in $\mathrm{Diff}(D^N \text{ rel } \partial D^N)$ for suitable (large) N.

Quillen's higher *algebraic K-theory of a ring R* is represented by the K-theory space $K(R) = K_0(R) \times BGL(R)^+$, where $GL(R) = \operatorname{colim}_k GL_k(R)$ and the superscript $+$ on $BGL(R)$ denotes Quillen's plus construction, which abelianizes the fundamental group and

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leaves homology unchanged [4]. The higher algebraic K-groups are $K_i(R) = \pi_i K(R)$, and e.g. in the case when $R = \mathcal{O}_E$ is the ring of integers in a number field E they encode number theoretic information about R. The rational K-groups $K_i(\mathcal{O}_E) \otimes \mathbb{Q}$ were computed by Borel in [7]. In the case $E = \mathbb{Q}$, $R = \mathbb{Z}$, the result is

$$K_i(\mathbb{Z}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q}, & i = 0 \quad \text{or} \quad i \equiv 1 \bmod 4 \text{ and } i \geqslant 5 \\ 0, & \text{otherwise.} \end{cases}$$

There is a linearization map $L\colon A(*) \to K(\mathbb{Z})$ arising from thinking of $A(*)$ as the K-theory of the sphere spectrum viewed as a "ring up to homotopy" [32]. One definition of $A(*)$ which makes it clear how to define this map goes as follows. Let $\mathrm{Hteq}(\bigvee_k S^n)$ be the (topological) monoid of based self-homotopy equivalences of $\bigvee_k S^n = S^n \vee \cdots \vee S^n$ (k summands). There is a map $\mathrm{Hteq}(\bigvee_k S^n) \to GL_k(\mathbb{Z})$ taking a homotopy equivalence $f\colon \bigvee_k S^n \to \bigvee_k S^n$ to its induced isomorphism $\tilde{H}_n(f)\colon \mathbb{Z}^k \to \mathbb{Z}^k$ on nth reduced homology. This map takes a point to its path component. There are stabilization maps increasing k and n, given respectively by wedge sum with additional sphere summands, and by suspension. Hence we may form $\mathrm{hocolim}_{n,k} B\mathrm{Hteq}(\bigvee_k S^n)$, with fundamental group $GL(\mathbb{Z})$, and define

$$A(*) = \mathbb{Z} \times \mathrm{hocolim}_{n,k} B\,\mathrm{Hteq}\Big(\bigvee_k S^n\Big)^+.$$

The *linearization map* $L\colon A(*) \to K(\mathbb{Z})$ is then induced by naturality of the plus construction with respect to the map to path components. It is a rational equivalence, by Proposition 2.2 of [32]. Hence

$$\pi_i \mathscr{P}(*) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{for } i \equiv 3 \bmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

In fact $\pi_i \mathrm{Diff}(D^N, \partial D^N) \otimes \mathbb{Q} \cong \mathbb{Q}$ if $i \equiv 3 \bmod 4$ and $N \gg i$ is odd, and zero otherwise; compare [18].

The issue of torsion remains. It appears to be extremely difficult to explicitly determine the torsion groups in $\pi_i A(*)$ and $K_i(\mathbb{Z})$. However, we may also pose the relative question of by how much these groups differ, and this is the subject of the present paper.

Let $\mathrm{hofib}(L)$ be the *homotopy fiber of* L, fitting into a fiber sequence

$$\mathrm{hofib}(L) \to A(*) \xrightarrow{L} K(\mathbb{Z}).$$

Then $\mathrm{hofib}(L)$ is rationally trivial, and since the homotopy groups of $A(*)$ and $K(\mathbb{Z})$ are finitely generated [28, 16], its homotopy groups are finite in each degree. Thus no information is lost by localizing or completing $\mathrm{hofib}(L)$ at one prime p at a time. The first nonvanishing homotopy group of $\mathrm{hofib}(L)$ completed at p was determined by Waldhausen, e.g. in Corollary 3.7 of [33]. We recall the result below.

Let $Q_0(S^0)$ and $Q_1(S^0)$ be the base point and identity components of $Q(S^0)$. We identify $Q_1(S^0)$ with the monoid SG of stable orientation-preserving self-homotopy equivalences of spheres, cf. Definition 5.1. There is then an inclusion $BSG \to \mathrm{hocolim}_{n,k} B\,\mathrm{Hteq}(\bigvee_k S^n)$ corresponding to the inclusion of special (1×1)-matrices $BSL_1(R) \to BGL(R)$, which induces a map $BSG \to A(*)$. The self maps in SG induce the identity map on homology, so the composite

$$BSG \to A(*) \xrightarrow{L} K(\mathbb{Z})$$

is constant, and induces a map $w\colon BSG \to \mathrm{hofib}(L)$.

THEOREM 1.2 (Waldhausen). *After completion at p the homotopy fiber* hofib(L) *is* $(2p-3)$*-connected, and the natural map* $w: BSG \to \mathrm{hofib}(L)$ *induces an isomorphism*

$$\mathbb{Z}/p \cong \pi_{2p-2} BSG \to \pi_{2p-2}\,\mathrm{hofib}(L).$$

In fact w is $(4p-3)$-connected for odd p, by our Corollary 4.4 and Theorem 6.4 below. For a sketch why w is at least $(4p-6)$-connected see [13].

We extend this result at odd primes p to a complete calculation of $\pi_*\,\mathrm{hofib}(L)$ in the initial range of degrees where the (complex) j-map $U \to Q(S^0)$ induces isomorphisms on homotopy, i.e. for $* < \deg(\beta_1) = 2p(p-1) - 2$. As a corollary we obtain torsion classes in $\pi_{i-2}\mathscr{P}(*)$ coming from $\pi_i A(*)$ and mapping to zero in $K_i(\mathbb{Z})$, which represent new families of diffeomorphisms of discs as described above. These families were not directly detectable by the linear K-theoretic invariants. See Theorem 1.3 and Corollary 1.4 below for precise statements.

Furthermore, in the range of degrees where the S^1-transfer map $Q(\Sigma_+ \mathbb{C}P^\infty) \to Q(S^0)$ is surjective, i.e. for $* < \deg(\beta_{p+1}) = 2p(p+2)(p-1) - 2$, we reduce the analysis of $\pi_*\,\mathrm{hofib}(L)$ essentially to the calculation of the stable homotopy groups of $\mathbb{C}P^\infty$. These results are summarized in Theorem 6.4, and we refer the reader to that theorem for the complete results.

Let p be an odd prime, and suppose all spaces and groups are completed at p.

THEOREM 1.3. (i) *When* $* < \deg(\beta_1) = 2(p-1)p - 2$, *the homotopy fiber of the linearization map* $L: A(*) \to K(\mathbb{Z})$ *has homotopy groups*

$$\pi_*\,\mathrm{hofib}(L) \cong \begin{cases} \mathbb{Z}/p & \text{if } * = 2n \text{ where } m(p-1) \leqslant n < mp \text{ for some } 1 \leqslant m < p \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *The classes in degrees* $2n$ *with* $n \equiv 0 \bmod p-1$ *are in the image of the natural map* $w: BSG \to \mathrm{hofib}(L)$, *and map to zero in* $\pi_{2n}A(*)$.

(iii) *The remaining classes, in degrees* $2n$ *with* $m(p-1) < n < mp$ *for some* $1 < m < p$, *inject onto direct summands in* $\pi_{2n}A(*)$.

Thus $\pi_*\,\mathrm{hofib}(L) \cong \mathbb{Z}/p$ precisely in the even degrees

$$2p-2,\ 4p-4,\ 4p-2,\ 6p-6,\ 6p-4,\ 6p-2,\ \ldots, 2(p-1)^2,\ \ldots,\ 2(p-1)p-4$$

and is trivial otherwise, as long as $* < 2(p-1)p - 2$.

COROLLARY 1.4. (i) *When* $n = m(p-1)$ *and* $1 \leqslant m < p$ *there are classes of infinite order in* $K_{2n+1}(\mathbb{Z})$ *mapping to classes of order* p *in* $\pi_{2n}\mathrm{hofib}(L)$ *under the connecting map* $\Omega K(\mathbb{Z}) \to \mathrm{hofib}(L)$.

(ii) *There are torsion classes of order* p *in* $\pi_{2n-2}\mathscr{P}(*)$ *when* n *satisfies* $m(p-1) < n < mp$ *for some* $1 < m < p$, *which come from* $\pi_{2n}A(*)$ *and map to zero in* $K_{2n}(\mathbb{Z})$. *These classes are detected in* $\pi_* TC(*)$ *(which will be reviewed momentarily) as the torsion classes of* $\pi_* Q(\Sigma\mathbb{C}P^\infty)$.

The theorem and corollary follow by specialization from Theorem 6.4, and are proved in Section 6. Again we emphasize that Theorem 6.4 below gives further information along these lines in the range of degrees from $\deg(\beta_1)$ to $\deg(\beta_{p+1})$.

The method of proof uses the topological cyclic homology functor of Bökstedt *et al.* [12], and the associated cyclotomic trace map from K-theory. These are defined for suitable rings up to homotopy, namely those arising from "functors with smash product", or FSPs, which were introduced in [11]. (See [12] for a published definition.) For such an FSP R one can define its K-theory $K(R)$, its topological cyclic homology $TC(R)$, and the cyclotomic trace map $trc\colon K(R) \to TC(R)$. These constructions are natural in R, and include the cases discussed above as special cases. Namely, the sphere spectrum is the ring up to homotopy associated with the identity FSP, with K-theory equal to $A(*)$ and topological cyclic homology denoted $TC(*)$. The integers are also the ring up to homotopy associated with an FSP, whose K-theory is $K(\mathbb{Z})$ and whose topological cyclic homology is denoted $TC(\mathbb{Z})$. The linearization map is induced by a map of FSPs, representing the path component map $Q(S^0) \to \pi_0 Q(S^0) \cong \mathbb{Z}$ on underlying ring spaces. Hence there is a commutative square of infinite loop spaces and maps

$$\begin{array}{ccc} A(*) & \xrightarrow{trc} & TC(*) \\ \downarrow L & & \downarrow L \\ K(\mathbb{Z}) & \xrightarrow{trc} & TC(\mathbb{Z}). \end{array} \tag{1}$$

We can access $\mathrm{hofib}(L\colon A(*) \to K(\mathbb{Z}))$ by means of the following recent theorem due to Dundas [15]:

THEOREM 1.5 (Dundas). *The square above is homotopy cartesian after completion at any prime p. Hence there is a homotopy equivalence*

$$\mathrm{hofib}(L\colon A(*) \to K(\mathbb{Z})) \to \mathrm{hofib}(L\colon TC(*) \to TC(\mathbb{Z}))$$

after p-adic completion.

This allows us to reduce the problem posed to the study of $TC(*)$, $TC(\mathbb{Z})$ and the linearization map between them. Furthermore, unlike the largely unknown spaces $A(*)$ and $K(\mathbb{Z})$, the homotopy types of these spaces are known, by Theorem 5.17 of [12] in the case of $TC(*)$, and by Theorem 9.17 of Bökstedt and Madsen [13] in the case of $TC(\mathbb{Z})$ at odd primes p. Hence only the identification of the linearization map in topological cyclic homology remains. This is achieved in a range of degrees by our Theorems 1.3 and 6.4.

In general Dundas' theorem holds for the square diagram induced by a map $R_1 \to R_2$ of FSPs inducing a nilpotent extension on π_0, i.e. when $\pi_0 R_1 \to \pi_0 R_2$ is a surjective ring homomorphism with nilpotent kernel. This is a generalization to FSPs of a similar theorem due to McCarthy [25], which holds for maps of simplicial rings. Both of these theorems may be viewed as p-adic versions of a rational theorem due to Goodwillie [19], asserting that in the case of simplicial rings, if topological cyclic homology is replaced by negative cyclic homology, then the corresponding square is rationally homotopy cartesian.

The paper is organized as follows. Section 2 fixes notation and recalls the image of J and cokernel of J spaces. Section 3 reviews the calculations of $TC(*)$ and $TC(\mathbb{Z})$. Proposition 3.6 gives a first product splitting of $\mathrm{hofib}(L)$. The difficult piece is related to the homotopy fiber of the S^1-transfer map $Q(\Sigma_+ \mathbb{C}P^\infty) \to Q(S^0)$. Section 4 reviews Segal's splitting of the stable homotopy of $\mathbb{C}P^\infty$ as BU and a space F with finite homotopy groups in each degree. Corollary 4.4 gives the homotopy groups of F in a beginning range of degrees. Section 5 collects previous geometric results on the j-maps and the S^1-transfer, summarized in diagram (15). This is applied in Corollary 5.13 to give, essentially, a splitting of U off from the homotopy fiber of the S^1-transfer map, and a description of the remainder term. In

Section 6 we assemble the available information to prove our main Theorem 6.4, from which Theorem 1.3 and Corollary 1.4 above follow. Finally Section 7 contains some speculative remarks on the space level structure of hofib(L).

2. STABLE HOMOTOPY OF SPHERES AND THE IMAGE OF J

We begin by presenting our conventions.

Let p be an odd prime. All groups, spaces and spectra will hereafter be implicitly completed at p. After this section we will only work with infinite loop spaces, thought of as connective spectra, and hence we will write $Q(S^0)$ for (the underlying space of) the sphere spectrum, etc. In particular, $TC(R)$ denotes the underlying space of the topological cyclic homology spectrum of an FSP R. By the *infinite loop space cofiber* of a map $f: X \to Y$ of infinite loop spaces we mean the infinite loop space Z fitting into a fiber sequence $X \xrightarrow{f} Y \to Z$, so that $\Omega Z \simeq \text{hofib}(f)$. Similarly, by the *infinite loop space smash product* $X \wedge Y$ we mean the underlying space of the spectrum level smash product of the spectra representing X and Y.

Let k be a topological generator for the p-adic units $\mathbb{Z}_p^\times$, i.e. a generator of $(\mathbb{Z}/p^2)^\times \cong \mathbb{Z}/(p-1)p$. Then the *image of J space* $\operatorname{Im} J$, is defined by the fiber sequence

$$\operatorname{Im} J \to \mathbb{Z} \times BU \xrightarrow{\psi^k - 1} BU. \tag{2}$$

If k is chosen as a prime power then $\operatorname{Im} J \simeq K(\mathbb{F}_k)$ by [27]. The Adams operation ψ^k acts as multiplication by k^i in degree $2i$, so

$$\pi_* \operatorname{Im} J \cong \begin{cases} \mathbb{Z} & \text{for } * = 0 \\ \mathbb{Z}/(k^i - 1) & \text{for } * = 2i - 1 > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Here the p-adic valuations are

$$v_p(k^i - 1) = \begin{cases} 1 + v_p(i) & \text{for } i \equiv 0 \bmod p - 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The K-theory unit $Q(S^0) \to \mathbb{Z} \times BU$ lifts through an infinite loop map $e: Q(S^0) \to \operatorname{Im} J$, which we call the *Adams e-invariant*. For any element $x \in \pi_* Q(S^0)$ the image $e(x) \in \pi_* \operatorname{Im} J$ has the same order as the complex Adams e-invariant $e_{\mathbb{C}}(x) \in \mathbb{Q}/\mathbb{Z}$, by [29]. We can identify $\operatorname{Im} J$ with the connective cover $L_K Q(S^0)[0, \infty)$ of the Bousfield K-localization of the sphere spectrum [9], and then e corresponds to the connective cover of the localization map

$$e: Q(S^0) \to L_K Q(S^0)[0, \infty) \simeq \operatorname{Im} J. \tag{5}$$

Let the *cokernel of J space* $\operatorname{Cok} J$, be defined by the fiber sequence

$$\operatorname{Cok} J \to Q(S^0) \xrightarrow{e} \operatorname{Im} J. \tag{6}$$

By the solution to the Adams conjecture [1], e admits a section $\alpha: \operatorname{Im} J \to Q(S^0)$ up to homotopy, so

$$\pi_* Q(S^0) \cong \pi_* \operatorname{Im} J \oplus \pi_* \operatorname{Cok} J. \tag{7}$$

There is a j-map $U \to Q(S^0)$ taking a unitary isometry $A: \mathbb{C}^n \xrightarrow{\cong} \mathbb{C}^n$ to the induced map of one-point compactifications $j(A): S^{2n} \xrightarrow{\cong} S^{2n}$, inducing the J-homomorphism

$$\pi_*(j): \pi_* U \to \pi_* Q(S^0).$$

For p odd $\mathrm{Im}(\pi_*(j)) = \pi_* \mathrm{Im}\, J \subset \pi_* Q(S^0)$ under the splitting above; hence the terminology.

Let $J_\oplus$ and $J_\otimes$ be the zero- and one-components of $\mathrm{Im}\, J$, with respectively the additive and multiplicative infinite loop space structures. Then there are infinite loop space fiber sequences

$$\begin{aligned} C_\oplus \to Q_0(S^0) \xrightarrow{e_\oplus} J_\oplus \\ C_\otimes \to SG \xrightarrow{e_\otimes} J_\otimes. \end{aligned} \tag{8}$$

Here $C_\oplus$ and $C_\otimes$ are both homotopy equivalent to $\mathrm{Cok}\, J$ as spaces.

The first elements in $\pi_* \mathrm{Cok}\, J$ are generated by the β-family. These are classes $\beta_i \in \pi_* \mathrm{Cok}\, J \subset \pi_* Q(S^0)$ for $i \geqslant 1$, with $\deg(\beta_1) = 2(p-1)p - 2$ and $\deg(\beta_i) = \deg(\beta_1) + 2(i-1)(p^2-1)$. The class β_1 is the first nontrivial class in $\pi_* \mathrm{Cok}\, J$, so $e: Q(S^0) \to \mathrm{Im}\, J$ induces an isomorphism on homotopy groups for $* < \deg(\beta_1)$. We shall also have reason to consider β_{p+1} in degree $\deg(\beta_{p+1}) = 2(p-1)p(p+2) - 2$, since by a theorem of Knapp (see Theorem 5.11 and the remarks following that theorem) the S^1-transfer map $Q(\Sigma_+ \mathbb{C}P^\infty) \to Q_0(S^0)$ induces a surjection on homotopy groups up to this degree.

3. REVIEW OF TOPOLOGICAL CYCLIC HOMOLOGY

The topological cyclic homology of the identity FSP, or conceptually of the sphere spectrum, was determined in [12]. Likewise the topological cyclic homology of the integers, completed at an odd prime p, was determined in [13]. We now review these results.

THEOREM 3.1 (Bökstedt–Hsiang–Madsen). *There is a homotopy cartesian square of infinite loop spaces and maps*

$$\begin{array}{ccc} TC(*) & \longrightarrow & Q(\Sigma_+ \mathbb{C}P^\infty) \\ \downarrow & & \downarrow {\scriptstyle trf_{S^1}} \\ Q(S^0) & \xrightarrow{\simeq *} & Q_0(S^0) \end{array}$$

after p-adic completion.

Here $\Sigma_+ X = \Sigma(X_+)$ is the unreduced suspension of X, and the notation $\simeq *$ denotes a null-homotopic map.

Remark 3.2. The image of the S^1-transfer map trf_{S^1} is connected and thus contained in the zero-component $Q_0(S^0)$ of $Q(S^0)$. In Bökstedt, Hsiang and Madsen's non-connective spectrum-level formulation of the theorem the lower right hand space is $Q(S^0)$, and $\pi_0 Q(S^0) \cong \mathbb{Z}$ maps isomorphically by a connecting map to the single nontrivial negative homotopy group $TC_{-1}(*) \cong \mathbb{Z}$.

Hence $TC(*) \simeq Q(S^0) \times \mathrm{hofib}(trf_{S^1})$. There is a cofiber sequence

$$\Sigma\mathbb{C}P^\infty \to \Sigma_+\mathbb{C}P^\infty \to \Sigma_+ * = S^1$$

split by a choice of a point in $\mathbb{C}P^\infty$, e.g. the point $\mathbb{C}P^0$. Hence we have a product splitting $Q(\Sigma_+\mathbb{C}P^\infty) \simeq Q(\Sigma\mathbb{C}P^\infty) \times Q(S^1)$.

Definition 3.3. Let $t: Q(\Sigma\mathbb{C}P^\infty) \to Q_0(S^0)$ be the *restricted transfer map* given as the composite

$$Q(\Sigma\mathbb{C}P^\infty) \to Q(\Sigma_+\mathbb{C}P^\infty) \xrightarrow{trf_{S^1}} Q_0(S^0).$$

Then trf_{S^1} splits as the sum of t and the Hopf map $\eta: Q(S^1) \to Q_0(S^0)$ in terms of the splitting above. (To see this, use a version of diagram (15) below restricted to $\mathbb{C}P^0 \subset \mathbb{C}P^\infty$ to see that trf_{S^1} restricted over $S^1 \subset Q(S^1) \subset Q(\Sigma_+\mathbb{C}P^\infty)$ is homotopic to the complex j-map $S^1 \cong U(1) \to \Omega^2 S^2 \to Q(S^0)$. The latter is well known to represent $\eta \in \pi_1 Q(S^0)$.)

The Hopf map is null-homotopic when completed at an odd prime, and so we obtain a further splitting $\mathrm{hofib}(trf_{S^1}) \simeq \mathrm{hofib}(t) \times Q(S^1)$. Hence

$$TC(*) \simeq Q(S^0) \times Q(S^1) \times \mathrm{hofib}(t). \tag{9}$$

At odd primes p, $TC(\mathbb{Z})$ is shown in [13] to be the connective cover of its K-localization (modulo a slight correction in degrees zero and one). Thus we have a factorization of the inclusion of the first two terms of the splitting above, through the connective covers of their K-localizations. Hence we have a commutative diagram

$$\begin{array}{ccc} Q(S^0) \times Q(S^1) & \to & TC(*) \\ \downarrow{\scriptstyle e \times Be} & & \downarrow{\scriptstyle L} \\ \mathrm{Im}\, J \times B\, \mathrm{Im}\, J & \to & TC(\mathbb{Z}). \end{array} \tag{10}$$

The mod p calculations in [13] suffice to show that the infinite loop space cofiber of the bottom map is SU, and that the resulting fiber sequence is split, as explained in [30]. This gives the following theorem:

THEOREM 3.4 (Bökstedt–Madsen). *There is a product splitting of infinite loop spaces*

$$TC(\mathbb{Z}) \simeq \mathrm{Im}\, J \times B\, \mathrm{Im}\, J \times SU$$

for each odd prime p.

To complete the picture, we review from [10] that there is a natural map

$$\Phi: K(\mathbb{Z}) \to JK(\mathbb{Z}) \simeq \mathrm{Im}\, J \times BBO$$

at odd primes p. Bökstedt has noted that according to the Lichtenbaum–Quillen conjecture for $K(\mathbb{Z})$ [17, 26] the homotopy fiber of Φ has the homotopy groups of $\Omega\, \mathrm{hofib}(\rho^k)$. Here $\rho^k: B\,Spin \to BSpin_\otimes$ is the Bott cannibalistic class [2, 24], which is an equivalence for primes p which are regular in the sense of number theory. Writing this extension as a twisted product ($\tilde{\times}$) and the conjectured equivalence as $\sim$, we can rewrite the homotopy cartesian square (1) as follows:

$$\begin{array}{ccccccc} Q(S^0) \times \mathrm{Wh}^{\mathrm{Diff}}(*) & \simeq & A(*) & \xrightarrow{trc} & TC(*) & \simeq & Q(S^0) \times Q(S^1) \times \mathrm{hofib}(t) \\ & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle L} & & \\ \mathrm{Im}\, J \times BBO\, \tilde{\times}\, \Omega\, \mathrm{hofib}(\rho^k) & \sim & K(\mathbb{Z}) & \xrightarrow{trc} & TC(\mathbb{Z}) & \simeq & \mathrm{Im}\, J \times B\, \mathrm{Im}\, J \times SU. \end{array} \tag{11}$$

We now turn to analyzing the vertical map on the right-hand side.

Definition 3.5. Let $l: \mathrm{hofib}(t) \to SU$ be the infinite loop map induced on infinite loop space cofibers by $e \times Be$ and L in (10) above.

Then we have a diagram of infinite loop space fiber sequences:

$$\begin{array}{ccccccc} & & \Omega \operatorname{Im} J \times \operatorname{Im} J & & & & \\ & & \downarrow & & & & \\ \Omega\,\mathrm{hofib}(l) & \xrightarrow{a} & \operatorname{Cok} J \times B \operatorname{Cok} J & \to & \mathrm{hofib}(L) & \to & \mathrm{hofib}(l) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega\,\mathrm{hofib}(t) & \xrightarrow{\simeq *} & Q(S^0) \times Q(S^1) & \to & TC(*) & \to & \mathrm{hofib}(t) \\ & & \downarrow{\scriptstyle e \times Be} & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle l} \\ & & \operatorname{Im} J \times B \operatorname{Im} J & \to & TC(\mathbb{Z}) & \to & SU \end{array} \tag{12}$$

PROPOSITION 3.6. *The map a above is null-homotopic as a map of spaces. Hence there is a space level splitting*

$$\Omega\,\mathrm{hofib}(L) \simeq \Omega \operatorname{Cok} J \times \operatorname{Cok} J \times \Omega\,\mathrm{hofib}(l)$$

and

$$\pi_*\,\mathrm{hofib}(L) \cong \pi_* \operatorname{Cok} J \oplus \pi_* B \operatorname{Cok} J \oplus \pi_*\,\mathrm{hofib}(l).$$

Presumably this splitting can be improved to an infinite loop space splitting by means of the multiplicative sequence in eq. (8).

Proof. The map a factors over $\Omega \operatorname{Im} J \times \operatorname{Im} J$ because the composite to $Q(S^0) \times Q(S^1)$ factors through the null-homotopic map $\Omega\,\mathrm{hofib}(t) \to Q(S^0) \times Q(S^1)$. By the solution to the Adams conjecture there is a space level homotopy section $\alpha: \operatorname{Im} J \to Q(S^0)$ to the map e in the fiber sequence below:

$$\operatorname{Cok} J \to Q(S^0) \xrightarrow{e} \operatorname{Im} J \xrightarrow{\simeq *} B \operatorname{Cok} J.$$

Hence the connecting map on the right-hand side is null-homotopic as a map of spaces. Thus a factors through a product of null-homotopic maps, and must be null-homotopic on the space level. The lemma follows. □

It remains to identify $t: Q(\Sigma \mathbb{C}P^\infty) \to Q_0(S^0)$ and $l: \mathrm{hofib}(t) \to SU$ on the homotopy group level. In the next section we commence this project by recalling a splitting of the source of t.

4. THE SEGAL SPLITTING

Consider the natural map

$$\mathbb{C}P^\infty_+ = BU(1)_+ \to \coprod_{n \geq 0} BU(n) \to \mathbb{Z} \times BU \tag{13}$$

induced by viewing line bundles as virtual vector bundles. There is an infinite loop space extension of (13) using the additive infinite loop space structure on $\mathbb{Z} \times BU$, which we denote as

$$\varepsilon: Q(\mathbb{C}P^\infty_+) \to \mathbb{Z} \times BU. \tag{14}$$

Let ε' be the infinite loop map (unique up to homotopy) making the diagram below homotopy commute:

$$\begin{array}{ccc} Q(\mathbb{C}P^{\infty}) & \xrightarrow{\varepsilon'} & BU \\ \downarrow & & \downarrow \\ Q(\mathbb{C}P^{\infty}_{+}) & \xrightarrow{\varepsilon} & \mathbb{Z}\times BU. \end{array}$$

Definition 4.1. Let F by the infinite loop space defined as the homotopy fiber of the map $\varepsilon' : Q(\mathbb{C}P^{\infty}) \to BU$.

THEOREM 4.2 (Segal). *The map $\varepsilon' : Q(\mathbb{C}P^{\infty}) \to BU$ is a rational equivalence and a retraction up to homotopy. Hence there is a space level splitting*

$$Q(\mathbb{C}P^{\infty}) \simeq F \times BU$$

using the H-space structure on $Q(\mathbb{C}P^{\infty})$. In each degree $\pi_* F$ is thus identified with the torsion subgroup of $\pi_* Q(\mathbb{C}P^{\infty})$, which is finite.

Segal gave a proof using representation theory in [31], and Becker gave a proof using the Becker–Gottlieb transfer in [5]. Theorem 5.2 below, due to Crabb and Knapp, deloops this result once, while also giving an explicit right homotopy inverse.

The natural map $e : Q(S^0) \to \operatorname{Im} J$ induces a map $Q(\mathbb{C}P^{\infty}) \to \operatorname{Im} J \wedge \mathbb{C}P^{\infty}$ (infinite loop space smash product) which induces an isomorphism up to degree $\deg(\beta_1) + 2 = 2(p-1)p$ on homotopy groups. The order of the homotopy groups of the target are known, e.g. by Theorem 1 of [21].

THEOREM 4.3 (Knapp). *The group $\pi_*(\operatorname{Im} J \wedge \mathbb{C}P^{\infty}) = (\operatorname{Im} J)_*(\mathbb{C}P^{\infty})$ is infinite cyclic in positive even degrees, and finite in odd degrees. Up to a p-adic unit its order is*

$$\#(\operatorname{Im} J)_{2n-1}(\mathbb{C}P^{\infty}) = \frac{1}{n!} \prod_{i=1}^{n-1} (k^i - 1),$$

which has p-adic valuation

$$v_p(\#(\operatorname{Im} J)_{2n-1}(\mathbb{C}P^{\infty})) = -\sum_{j=1}^{[\log_p(n)]} \left[\frac{n}{p^j}\right] + \sum_{j=1}^{[(n-1)/(p-1)]} (1 + v_p(j)).$$

Here k denotes a topological generator of the p-adic units, as in Section 2. We sketch the argument, to give some indication of where the torsion classes appearing in Theorem 1.3 and Corollary 1.4 come from.

Proof (*sketch*). Let $ku_*(X)$ and $bu_*(X)$ denote the connective and connected reduced K-homology groups of X. These are the homology theories represented by $\mathbb{Z} \times BU$ and BU. The cofiber sequence $\mathbb{C}P^{n-1} \to \mathbb{C}P^n \to S^{2n}$ and the operation $\psi^k - 1 : \mathbb{Z} \times BU \to BU$ give rise to the map of short exact sequences

$$\begin{array}{ccccccccc} 0 \to & ku_{2n}(\mathbb{C}P^{n-1}) & \to & ku_{2n}(\mathbb{C}P^n) & \to & ku_{2n}(S^{2n}) & \to 0 \\ & \downarrow \psi^k - 1 & & \downarrow \psi^k - 1 & & \downarrow \psi^k - 1 & \\ 0 \to & bu_{2n}(\mathbb{C}P^{n-1}) & \to & bu_{2n}(\mathbb{C}P^n) & \to & bu_{2n}(S^{2n}) & \to 0. \end{array}$$

Abstractly the upper row is $\mathbb{Z}^{n-1} \to \mathbb{Z}^n \to \mathbb{Z}$, and the lower row is $\mathbb{Z}^{n-1} \to \mathbb{Z}^{n-1} \to 0$. The vertical kernels and cokernels give the following exact sequence of Im J-homology groups

$$0 \to (\mathrm{Im}\, J)_{2n}(\mathbb{C}P^{n-1}) \to (\mathrm{Im}\, J)_{2n}(\mathbb{C}P^n) \to (\mathrm{Im}\, J)_{2n}(S^{2n})$$

$$\to (\mathrm{Im}\, J)_{2n-1}(\mathbb{C}P^{n-1}) \to (\mathrm{Im}\, J)_{2n-1}(\mathbb{C}P^n) \to (\mathrm{Im}\, J)_{2n-1}(S^{2n}) \to 0.$$

Let $x = H - 1 \in \tilde{K}^0(\mathbb{C}P^\infty)$ where H is the Hopf line bundle, and let $u \in K^{-2}(*)$ be the Bott periodicity element. Then $K^0(\mathbb{C}P^\infty) \cong \mathbb{Z}[[x]]$ with $\psi^k(x) = (x+1)^k - 1$ and $\psi^k(u) = ku$. Let $b_i \in K_0(\mathbb{C}P^\infty)$ be dual to x^i, so $\langle x^i, b_j\rangle = \delta_{ij}$ (Kronecker delta). Then $\tilde{K}^{2n}(\mathbb{C}P^{n-1}) \cong \mathbb{Z}^{n-1}$ with basis $\{u^{-n}x, \dots, u^{-n}x^{n-1}\}$, and dually $ku_{2n}(\mathbb{C}P^{n-1}) \cong bu_{2n}(\mathbb{C}P^{n-1}) \cong \mathbb{Z}^{n-1}$ with basis $\{u^n b_1, \dots, u^n b_{n-1}\}$. In K-cohomology $(\psi^k - 1)(u^{-n}x^i) = (k^{i-n} - 1)u^{-n}x^i$ plus terms involving higher powers of x. Thus $\psi^k - 1$ is represented by a lower triangular matrix, with diagonal entries $k^{i-n} - 1$ for $i = 1, \dots, n-1$. Dually, $\psi^k - 1$ in homology is represented by the transposed matrix, with determinant $k^{-\binom{n}{2}} \prod_{i=1}^{n-1}(k^i - 1)$. Hence $(\mathrm{Im}\, J)_{2n}(\mathbb{C}P^{n-1}) = 0$ and $(\mathrm{Im}\, J)_{2n-1}(\mathbb{C}P^{n-1})$ has order $\prod_{i=1}^{n-1}(k^i - 1)$ up to a p-adic unit.

Clearly $(\mathrm{Im}\, J)_{2n-1}(S^{2n}) = 0$ and $(\mathrm{Im}\, J)_{2n}(S^{2n}) \cong \mathbb{Z}$, so by a diagram chase $(\mathrm{Im}\, J)_{2n}(\mathbb{C}P^n) \cong (\mathrm{Im}\, J)_{2n}(\mathbb{C}P^\infty) \cong \mathbb{Z}$. We claim that the pinch map $\mathbb{C}P^n \to S^{2n}$ induces multiplication by $n!$ on $(\mathrm{Im}\, J)_{2n}$. The fundamental class of $\mathbb{C}P^1 \subset \mathbb{C}P^\infty$ represents the generator $[\mathbb{C}P^1] \in (\mathrm{Im}\, J)_2(\mathbb{C}P^\infty)$, and maps to ub_1 in $ku_2(\mathbb{C}P^\infty)$. Using the Pontrjagin product induced by the H-space structure on $\mathbb{C}P^\infty$, we can form

$$[\mathbb{C}P^1]^n \in (\mathrm{Im}\, J)_{2n}(\mathbb{C}P^\infty) \cong (\mathrm{Im}\, J)_{2n}(\mathbb{C}P^n),$$

mapping to $(ub_1)^n \in ku_{2n}(\mathbb{C}P^\infty)$. By a calculation (involving Stirling numbers) which we omit, $(ub_1)^n = u^n b_1 + \cdots + n! u^n b_n$ in the given basis. Thus $[\mathbb{C}P^1]^n$ is not divisible, since the $u^n b_1$-coefficient is a unit, and must be a generator of $(\mathrm{Im}\, J)_{2n}(\mathbb{C}P^n)$. The pinch map takes $(ub_1)^n$ to $n! u^n b_n$, which is $n!$ times the generator of $(\mathrm{Im}\, J)_{2n}(S^{2n})$. The claim follows.

Hence $(\mathrm{Im}\, J)_{2n-1}(\mathbb{C}P^n) \cong (\mathrm{Im}\, J)_{2n-1}(\mathbb{C}P^\infty)$ has order $1/n! \cdot \prod_{i=1}^{n-1}(k^i - 1)$ up to a p-adic unit, as claimed. □

Corollary 4.4. *In degrees $* < \deg(\beta_1) + 2 = 2(p-1)p$ the torsion group $\pi_* F$ is cyclic of order p when $*$ satisfies $* = 2n - 1$ with $m(p-1) < n < mp$ for some $1 < m < p$, and trivial otherwise.*

Proof. When $n \leqslant (p-1)p$ we have $(n-1)/(p-1) < p$ and $\log_p(n) < 2$, so $\pi_{2n-1} Q(\mathbb{C}P^\infty) \cong \pi_{2n-1} F$ has finite order with p-adic valuation $[(n-1)/(p-1)] - [n/p]$. This difference equals 1 if $n/p < m \leqslant (n-1)/(p-1)$ for some integer m, and 0 otherwise, in this range. The corollary is a simple reformulation of this fact. □

Remark 3.7. Using mod p methods, a closed formula for the number of cyclic summands in $(\mathrm{Im}\, J)_*(\mathbb{C}P^\infty)$ can be determined. This is, however, not sufficient to determine the structure of these groups in higher degrees. Furthermore, beginning in degree $\deg(\beta_1) + 2$ the cokernel of J enters into the torsion in $\pi_* Q(\mathbb{C}P^\infty)$ and thus in $\pi_* F$. Thus it is rather complicated to describe $\pi_* F$ in any higher degrees. We will not proceed any further in this direction.

5. THE S^1-TRANSFER AND j-MAPS

We now use classical results on the S^1-transfer and the J-homomorphism to split off a copy of SU from hofib(t), at least on the level of homotopy groups, in a range. This amounts to fibering a delooped Segal splitting $Q(\Sigma\mathbb{C}P^\infty) \simeq BF \times SU$ over the restricted transfer map t.

Definitions 5.1. Let SG be the monoid of stable unbased orientation-preserving self-homotopy equivalences of spheres, under composition. The natural inclusion $Q_1(S^0) \to SG$ is a homotopy equivalence.

Let $SG(S^1)$ be the monoid of stable orientation preserving S^1-equivariant self-homotopy equivalences of free S^1-spheres, under composition. $SG(S^1)$ is called the *Becker–Schultz space*, and it is proved in [6] that there is a homotopy equivalence $SG(S^1) \simeq Q(\Sigma_+\mathbb{C}P^\infty)$.

An explicit such homotopy equivalence is constructed in chapters 2, 3 and 6 of Crabb's book [14], as a *difference class* of relative Euler classes. This is a homotopy equivalence $\zeta_{S^1}: SG(S^1) \to Q(\Sigma_+\mathbb{C}P^\infty)$. There is a similar homotopy equivalence in the non-equivariant case $\zeta: SG \to Q_0(S^0)$, which is homotopic to the component shift map

$$SG \simeq Q_1(S^0) \xrightarrow{[-1]} Q_0(S^0)$$

using the additive H-space structure on $Q(S^0)$, by Lemma 3.11 of *loc. cit.*

There is a forgetful monoid map $i^*: SG(S^1) \to SG$ induced by restricting the S^1-actions to the trivial group. It is compatible under ζ_{S^1} and ζ with the S^1-transfer trf_{S^1}, almost by definition. For these claims, see p. 18 and p. 70 of *loc. cit.*

A unitary map $A: \mathbb{C}^n \to \mathbb{C}^n$ restricts to an S^1-homotopy equivalence of the S^1-free unit spheres $S^{2n-1} \subset \mathbb{C}^n$, which defines the *$S^1$-equivariant j-map* $j_{S^1}: U \to SG(S^1)$. This is an H-map from the additive H-structure on U to the multiplicative H-structure on $SG(S^1)$. Composing with the forgeful map we obtain the *complex j-map* $j: U \to SG$ with $i^* \circ j_{S^1} = j$.

The *local obstruction* $\theta: U \to Q(\Sigma_+\mathbb{C}P^\infty)$ is defined in a similar manner as ζ_{S^1}, and satisfies $\theta = \zeta_{S^1} \circ j_{S^1}$ (see p. 70 of *loc. cit.*). Next there is a *rotation map* $R: \Sigma_+\mathbb{C}P^\infty \to U$ taking a pair (L, z) with $L \in \mathbb{C}P^\infty$ and $z \in S^1$ to the unitary transformation rotating by z along L and fixing the orthogonal complement $L^\perp$. The composite $\theta \circ R$ is homotopic to the stabilization map $\iota: \Sigma_+\mathbb{C}P^\infty \to Q(\Sigma_+\mathbb{C}P^\infty)$ by a theorem of James (see Proposition 2.7 of *loc. cit.*).

Let $\Delta: U \to J_\oplus$ be the connecting map in the fiber sequence (8) defining $J_\oplus$. A solution to the complex Adams conjecture gives a map $\alpha: J_\oplus \to SG$ such that $\alpha \circ \Delta \simeq j$. Let the restrictions of the Adams e-invariant $e: Q(S^0) \to \operatorname{Im} J$ to the zero- and one-components be $e_\oplus: Q_0(S^0) \to J_\oplus$ and $e_\otimes: SG \simeq Q_1(S^0) \to J_\otimes$. Then $e_\oplus \circ [-1] \simeq [-1] \circ e_\otimes$ since e is a ring map. The *cannibalistic class* $\rho^k: J_\oplus \to J_\otimes$ is the composite $\rho^k = e_\otimes \circ \alpha$, and is an infinite loop map homotopy equivalence (see e.g. Ch. V.3 of [24]).

Finally recall the extension $\varepsilon: Q(\Sigma_+\mathbb{C}P^\infty) \to U$ of R over ι, using the additive infinite loop space structure on U. Thus $\varepsilon \circ \iota = R$. This ε deloops that of (14), as is seen by choosing an explicit Bott periodicity equivalence $\Omega U \simeq \mathbb{Z} \times BU$ as in [8].

The following result appears as Proposition 6.10 of [14].

THEOREM 5.2 (Crabb–Knapp). *The composite $\varepsilon \circ \theta : U \to U$ is homotopic to the identity map.*

The proof is an application of the splitting principle in K-theory. First $\varepsilon \circ \theta$ is shown to be an H-map. Thus it suffices to check the result on line bundles, which amounts to noting that the restrictions over R agree.

We have now produced the diagram below. The maps $\Delta, j_{S^1}, j, \rho^k, i^*, e_\otimes, trf_{S^1}, \varepsilon$ and $e_\oplus$ are defined on the infinite loop space level, while the remaining maps are only defined on the space level. By the discussion above the diagram homotopy commutes, with one exception: *The bottom right hand square does not commute.*†

The horizontal composite $\varepsilon \circ \theta : U \to U$ is the identity, while the composite $e_\oplus \circ \zeta \circ \alpha : J_\oplus \to J_\oplus$ is a homotopy equivalence $[-1] \circ \rho^k$ different from the identity. Thus the composite maps $\Delta \circ \varepsilon$ and $e_\oplus \circ trf_{S^1}$ are not homotopic, because the restrictions over ι are, respectively, $\Delta \circ R$ and $[-1] \circ \rho^k \circ \Delta \circ R$, which are different.

Remark 5.3. One might wish to use this diagram to split $\Delta : U \to J_\oplus$ off from $trf_{S^1} : Q(\Sigma_+ \mathbb{C}P^\infty) \to Q_0(S^0)$. The fact that the right hand square only commutes

$$
\begin{tikzcd}
U \arrow[dr, "js^1"] \arrow[drr, "\theta"] \arrow[ddr, "j"'] \arrow[dd, "\Delta"'] & & \Sigma_+\mathbb{C}P^\infty \arrow[ll, "R"'] \arrow[d, "\iota"] \arrow[dr, "R"] & \\
& SG(S^1) \arrow[r, "\zeta_{s^1}", "\simeq"'] \arrow[d, "i^*"] & Q(\Sigma_+\mathbb{C}P^\infty) \arrow[r, "\varepsilon"] \arrow[d, "trf_{s^1}"] & U \arrow[d, "\Delta"] \\
J_\oplus \arrow[r, "\alpha"] \arrow[dr, "\rho^k"', "\simeq"] & SG \arrow[r, "\zeta", "\simeq"'] \arrow[d, "e_\otimes"] & Q_0(S^0) \arrow[r, "e_\oplus"] & J_\oplus \\
& J_\otimes \arrow[urr, "\simeq", "{[-1]}"'] & &
\end{tikzcd}
\tag{15}
$$

up to an automorphism shows that this might not be possible. Instead we will settle for a splitting on the level of homotopy groups.

We will need the following maps related to θ, ε and Δ.

Definition 5.4. Let $\theta' : SU \to Q(\Sigma\mathbb{C}P^\infty)$ be the composite

$$SU \to U \xrightarrow{\theta} Q(\Sigma_+\mathbb{C}P^\infty) \to Q(\Sigma\mathbb{C}P^\infty),$$

let $\varepsilon' : Q(\Sigma\mathbb{C}P^\infty) \to SU$ be the unique lift of $Q(\Sigma\mathbb{C}P^\infty) \to Q(\Sigma_+\mathbb{C}P^\infty) \xrightarrow{\varepsilon} U$ over the universal covering $SU \to U$, and let $\Delta' : SU \to J_\oplus$ be the composite $SU \to U \xrightarrow{\Delta} J_\oplus$.

† The authors thank Karl-Heinz Knapp for pointing out to us that this square fails to commute.

Then we have a diagram as below:

$$
\begin{tikzcd}
SU \arrow[r, "\theta'"] \arrow[d] & Q(\Sigma \mathbb{C}P^{\infty}) \arrow[r, "\varepsilon'"] \arrow[d, shift left] & SU \arrow[d] \\
U \arrow[r, "\theta"] & Q(\Sigma_{+} \mathbb{C}P^{\infty}) \arrow[r, "\varepsilon"] \arrow[u, shift left] \arrow[d, shift left] & U \\
& Q(S^1) \arrow[u, shift left] &
\end{tikzcd}
\tag{16}
$$

The following lemmas are clear.

LEMMA 5.5. *The difference of $\varepsilon' \circ \theta'$ and id_{SU} factors as $SU \to Q(S^1) \to SU$, hence is zero on homotopy groups. Thus $\pi_*(\varepsilon' \circ \theta')$ is the identity homomorphism, and $\varepsilon' \circ \theta'$ is a homotopy equivalence.*

In fact $\varepsilon' \circ \theta'$ is homotopic to the identity, as self maps of SU are detected on rational homotopy (see, e.g. V.2.9 of [24]).

LEMMA 5.6. *The difference of $SU \to U$ composed with $trf_{S^1} \circ \theta : U \to Q_0(S^0)$, and $t \circ \theta' : SU \to Q_0(S^0)$ factors through the Hopf map*

$$\eta : Q(S^1) \to Q(\Sigma_{+} \mathbb{C}P^{\infty}) \xrightarrow{trf_{S^1}} Q_0(S^0),$$

hence is null-homotopic when completed at an odd prime p.

LEMMA 5.7. *There is a fiber sequence*

$$SU \xrightarrow{\Omega(\psi^k - 1)} SU \xrightarrow{\Delta'} J_{\oplus} \to BSU \xrightarrow{\psi^k - 1} BSU$$

at odd primes p.

PROPOSITION 5.8. *There is a homotopy commutative square*

$$
\begin{array}{ccc}
SU & \xrightarrow{\theta'} & Q(\Sigma \mathbb{C}P^{\infty}) \\
\downarrow{\scriptstyle \Delta'} & & \downarrow{\scriptstyle t} \\
J_{\oplus} & \xrightarrow{\zeta\alpha} & Q_0(S^0)
\end{array}
$$

where Δ' and t are infinite loop maps, while θ' and $\zeta\alpha$ are not H-maps. Both maps θ' and $\zeta\alpha$ are split injective up to homotopy.

Proof. We simply combine diagrams (15) and (16). The map θ' does not lift θ, but the composites to $Q_0(S^0)$ are homotopic nonetheless, by Lemma 5.6. Furthermore, θ' is split injective up to homotopy by Lemma 5.5. The left homotopy inverse to $\zeta\alpha$ is visible in diagram (15). □

The diagram in Proposition 5.8 gives rise to the following lattice of fiber sequences:

$$\begin{array}{ccccccc} \Omega\,\mathrm{hofib}(t) & \xrightarrow{b} & F' & \to & SU & \to & \mathrm{hofib}(t) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \Omega(\psi^k - 1)} & & \downarrow \\ Q(\mathbb{C}P^\infty) & \xrightarrow{c} & F & \xrightarrow{\simeq *} & SU & \xrightarrow{\theta'} & Q(\Sigma\mathbb{C}P^\infty) \\ \downarrow{\scriptstyle \Omega t} & & \downarrow & & \downarrow{\scriptstyle \Delta'} & & \downarrow{\scriptstyle t} \\ \Omega Q_0(S^0) & \xrightarrow{d} & \Omega\,\mathrm{Cok}\,J & \xrightarrow{\simeq *} & J_\oplus & \xrightarrow{\zeta\alpha} & Q_0(S^0). \end{array} \tag{17}$$

For $\varepsilon' \circ \theta'$ is a homotopy equivalence, so the homotopy fiber of θ' is homotopy equivalent to the loop space of the homotopy fiber of $\varepsilon' : Q(\Sigma\mathbb{C}P^\infty) \to SU$, which is BF by Definition 4.1. Similarly the homotopy fiber of $\zeta\alpha$ is homotopy equivalent to the loop space of the homotopy fiber of $e_\oplus : Q_0(S^0) \to J_\oplus$, which is $\mathrm{Cok}\, J$ by eq. (6). In particular the maps c and d admit right homotopy inverses. However, we do not know that these homotopy sections can be chosen compatibly. Hence we cannot directly conclude that b admits a right homotopy inverse.

Definition 5.9. A given choice of commuting homotopy in Proposition 5.8 induces a map of horizontal homotopy fibers $F \to \Omega\,\mathrm{Cok}\,J$ in the diagram above. Let F' be defined as the homotopy fiber of this map.

Clearly F' has finite homotopy groups in each degree, like F. It is not clear whether F' can be defined so as to admit a delooping.

PROPOSITION 5.10. *The map labeled b in* (17) *induces a split surjection on the level of homotopy groups, at least in degrees* $* < \deg(\beta_{p+1}) - 2$.

For the proof, we use the following theorem.

THEOREM 5.11 (Knapp). *At odd primes p the* (*restricted*) *S^1-transfer map*

$$t : Q(\Sigma\mathbb{C}P^\infty) \to Q_0(S^0)$$

induces a surjection on homotopy in all degrees up to $\deg(\beta_{p+1})$.

A complete proof of this result has yet to be published. Diagram (15) shows that the image of $\pi_*(t)$ contains the image of the J-homomorphism, and Theorem 6 of [21] asserts that the image also contains the β-elements $\beta_1, \ldots, \beta_p$, but not β_{p+1}. Hence all products of α- and β-elements in the range $* < \deg(\beta_{p+1})$ are hit by $\pi_*(t)$. In particular the first γ-element is of this form, and is thus hit. The remaining elements in $\pi_* Q(S^0)$ up to degree $\deg(\beta_{p+1})$ are expressible as Toda brackets in α's and β's, and unpublished calculations of Knapp show that also all these classes are hit by $\pi_*(t)$. The theorem follows from this assertion. We are told by Knapp that the remaining calculations will appear shortly as [22].

COROLLARY 5.12. *The long exact sequence in homotopy for the fibration $F' \to F \to \Omega\,\mathrm{Cok}\,J$ splits into short exact sequences*

$$0 \to \pi_{*-1} F' \to \pi_* BF \to \pi_*\,\mathrm{Cok}\,J \to 0$$

for $* < \deg(\beta_{p+1})$.

Proof of Proposition 5.10. By a diagram chase the composite map

$$f: F' \to F \to Q(\mathbb{C}P^\infty) \xrightarrow{\Omega t} \Omega Q_0(S^0)$$

has a lift over $\Omega(\zeta\alpha)$. Here $F \to Q(\mathbb{C}P^\infty)$ is a homotopy section to c. Thus f is null-homotopic precisely if the further composite with $\Omega e_\oplus : \Omega Q_0(S^0) \to \Omega J_\oplus$ is null. We do not know if this is true, but at least we claim that the map $\Omega e_\oplus \circ f$ induces the zero homomorphism on homotopy.

To see this, use the fiber sequence

$$\mathrm{Cok}\, J \wedge \mathbb{C}P^\infty \to Q(\mathbb{C}P^\infty) \xrightarrow{e \wedge \mathrm{id}} \mathrm{Im}\, J \wedge \mathbb{C}P^\infty.$$

Here $\wedge$ denotes the infinite loop space smash product, given by forming a smash product of spectra and taking the resulting underlying space. The source $\mathrm{Cok}\, J \wedge \mathbb{C}P^\infty$ is connective and K-acyclic, and $\Omega J_\oplus$ is the connective cover of its K-localization, so there are no essential maps $\mathrm{Cok}\, J \wedge \mathbb{C}P^\infty \to \Omega J_\oplus$. Hence $\Omega e_\oplus \circ f$ factors through a map $\mathrm{Im}\, J \wedge \mathbb{C}P^\infty \to \Omega J_\oplus$.

Now $\pi_* \Omega J_\oplus$ is concentrated in even degrees. As we saw in the proof of Theorem 4.3, the even homotopy of $\mathrm{Im}\, J \wedge \mathbb{C}P^\infty$ is torsion free. Thus $F' \to F \to \mathrm{Im}\, J \wedge \mathbb{C}P^\infty$ induces the zero map on homotopy in even degrees, since $\pi_* F'$ is torsion, and so our claim that $\pi_*(f)$ is zero in all degrees follows.

Finally $\pi_* \Omega\, \mathrm{hofib}(t) \to \pi_* Q(\mathbb{C}P^\infty)$ is injective in our range of degrees, by Knapp's theorem. Thus $\pi_* F' \to \pi_* F \to \pi_* Q(\mathbb{C}P^\infty)$ factors uniquely through a homomorphism $\pi_* F' \to \pi_* \Omega\, \mathrm{hofib}(t)$, which is a homotopy section to b, compatible with the homotopy section to c. □

COROLLARY 5.13. *For p odd there is a short exact sequence*

$$0 \to \pi_* SU \to \pi_* \mathrm{hofib}(t) \to \pi_{*-1} F' \to 0$$

which is split at least for $* < \deg(\beta_{p+1}) - 1$.

6. CONCLUDING ARGUMENTS

Definition 6.1. Let $l': SU \to SU$ be the space level map given as the composite in the diagram below:

$$\begin{tikzcd} F' \arrow[r] & SU \arrow[r] \arrow[dr, "l'"'] & \mathrm{hofib}(t) \arrow[d, "l"] \\ & & SU \end{tikzcd}$$

We think of $\Omega l': BU \to BU$ as an additive reduced K-theory operation. Hence, by [23], it can be written as a power series in Adams operations ψ^i for various i.

LEMMA 6.2. *In degrees* $* < \deg(\beta_{p+1}) - 1$ *there is a short exact sequence*

$$0 \to \pi_* \mathrm{hofib}(l') \to \pi_* \mathrm{hofib}(l) \to \pi_{*-1} F' \to 0.$$

The subgroup $\pi_* \mathrm{hofib}(l')$ *vanishes in odd degrees.*

Proof. Using Corollary 5.13 the long exact sequence in homotopy associated to the fiber sequence defining hofib(l) splits into exact sequences

$$0 \to \pi_{2n+1}\,\mathrm{hofib}(l) \to \pi_{2n+1}SU \oplus \pi_{2n}F' \xrightarrow{l_*} \pi_{2n+1}SU \to \pi_{2n}\mathrm{hofib}(l) \to \pi_{2n-1}F' \to 0.$$

Here the second map l_* is a rational equivalence, and is zero on the torsion group $\pi_{2n}F'$. We can clearly identify its cokernel with $\pi_{2n}\mathrm{hofib}(l')$. □

PROPOSITION 6.3. (i) *$l': SU \to SU$ induces a p-adic equivalence on homotopy in degrees $* = 2n + 1$ with $n \not\equiv 0 \bmod p - 1$.*

(ii) *In degrees $* < \deg(\beta_1) = 2(p-1)p - 2$ its homotopy fiber has homotopy groups*

$$\pi_*\,\mathrm{hofib}(l') \cong \begin{cases} \mathbb{Z}/p & \text{if } * \equiv 0 \bmod 2p-2, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *In the same range of degrees $\pi_{2n+1}\,\mathrm{hofib}(l) = 0$, and*

$$\pi_{2n}\mathrm{hofib}(l) \cong \pi_{2n-1}F' \oplus \pi_{2n}\mathrm{hofib}(l').$$

Proof. Recall the map $w: BSG \to \mathrm{hofib}(L)$ discussed in Theorem 1.2, which induces an isomorphism on homotopy groups through degree $2p - 2$. By Proposition 3.6 there are isomorphisms $\pi_*\,\mathrm{hofib}(L) \cong \pi_*\,\mathrm{hofib}(l)$ for $* < \deg(\beta_1)$.

Composition with the Adams map $A: \Sigma^{2p-2}S^0/p \to S^0/p$ induces the v_1-action on the mod p homotopy of infinite loop spaces. Thus such groups are graded $\mathbb{Z}/p[v_1]$-modules. Now $l: \mathrm{hofib}(t) \to SU$ is an infinite loop map, and thus induces a $\mathbb{Z}/p[v_1]$-module homomorphism on mod p homotopy. The mod p homotopy of the target is a free $\mathbb{Z}/p[v_1]$-module on $p - 1$ generators in degrees $3, 5, \dots, 2p - 1$, corresponding to the Adams splitting of SU [3]. Since the fiber of l is $(2p - 3)$-connected and has bottom homotopy group $\mathbb{Z}/p$ in degree $2p - 2$, we find that $\pi_*(l; \mathbb{Z}/p)$ maps onto the generators in degrees $3, 5, \dots, 2p - 3$ and is trivial in degree $2p - 1$. By Corollaries 4.4 and 5.12 the space F' is $(4p - 4)$-connected, so $\pi_*SU \to \pi_*\mathrm{hofib}(t)$ is an isomorphism well beyond the range $* \leqslant 2p - 1$. Hence

$$\pi_*(l'; \mathbb{Z}/p): \pi_*(SU; \mathbb{Z}/p) \to \pi_*(SU; \mathbb{Z}/p)$$

maps the mod p homotopy of the first $p - 2$ Adams summands surjectively (and thus isomorphically), and maps the last summand trivially. Thus $\pi_*(l')$ is a p-adic equivalence in all degrees $* = 2n + 1$ with $n \not\equiv 0 \bmod p - 1$, and is trivial mod p in the remaining degrees. The first claim follows.

Now restrict to degrees $* < \deg(\beta_1)$. By Definition 5.9 and Corollary 4.4 the homotopy groups of F and F' agree in this range, and are concentrated in odd degrees $* \not\equiv 1 \bmod 2p - 2$. Hence for $n \equiv 0 \bmod p - 1$ we have $\mathbb{Z} \cong \pi_{2n\pm}SU \cong \pi_{2n\pm1}\,\mathrm{hofib}(t)$ and $\pi_{2n}\mathrm{hofib}(t) = 0$, so

$$\pi_{2n+1}(\mathrm{hofib}(t); \mathbb{Z}/p) \cong \pi_{2n-1}(SU; \mathbb{Z}/p) \cong \mathbb{Z}/p$$

and

$$\pi_{2n}(\mathrm{hofib}(t); \mathbb{Z}/p) = 0.$$

We now restrict attention to these degrees with $n \equiv 0 \bmod p - 1$. By the $\mathbb{Z}/p[v_1]$-module structure the map

$$l': \pi_{2n+1}(SU; \mathbb{Z}/p) \xrightarrow{\cong} \pi_{2n+1}(\mathrm{hofib}(t); \mathbb{Z}/p) \xrightarrow{l} \pi_{2n+1}(SU; \mathbb{Z}/p)$$

is null. Thus the boundary map $\pi_{2n+1}(SU; \mathbb{Z}/p) \to \pi_{2n}(\mathrm{hofib}(l); \mathbb{Z}/p)$ is an isomorphism, from a free $\mathbb{Z}/p[v_1]$-module on one generator. Since $\pi_{2n}(BSG; \mathbb{Z}/p) \to \pi_{2n}(\mathrm{hofib}(l); \mathbb{Z}/p)$ maps onto the generator in degree $2n = 2p - 2$, it is also surjective throughout the range, by the $\mathbb{Z}/p[v_1]$-module structure. Thus integrally $\mathbb{Z}/p \cong \pi_{2n} BSG \to \pi_{2n}\,\mathrm{hofib}(l)$ is a homomorphism inducing an isomorphism mod p, and hence it is a p-adic equivalence. This settles claim (ii).

Regarding claim (iii), there are no extension problems, because at least one of the groups $\pi_{2n-1}F'$ and $\pi_{2n}\,\mathrm{hofib}(l')$ is always zero in the initial range $* < \deg(\beta_1)$. □

We can now assemble the pieces of the calculation. Let p be an odd prime, and let all spaces and groups be completed at p.

THEOREM 6.4. (i) *The homotopy groups of the homotopy fiber of the linearization map* $L: A(*) \to K(\mathbb{Z})$ *satisfy*

$$\pi_*\,\mathrm{hofib}(L) \cong \pi_*\,\mathrm{Cok}\,J \oplus \pi_* B\,\mathrm{Cok}\,J \oplus \pi_*\,\mathrm{hofib}(l).$$

(ii) *When* $* < \deg(\beta_{p+1}) - 1$ *the homotopy groups of* $\mathrm{hofib}(l)$ *fit into a short exact sequence*

$$0 \to \pi_*\,\mathrm{hofib}(l') \to \pi_*\,\mathrm{hofib}(l) \to \pi_{*-1}F' \to 0.$$

This extension is always trivial when $* < \deg(\beta_1)$.

(iii) *When* $* < \deg(\beta_1)$ *the first term in* (ii) *satisfies*

$$\pi_*\,\mathrm{hofib}(l') \cong \pi_* BJ_\otimes \cong \begin{cases} \mathbb{Z}/p & \text{if } * \equiv 0 \bmod 2p-2, \\ 0 & \text{otherwise.} \end{cases}$$

(iv) *When* $* < \deg(\beta_{p+1})$ *the third term in* (ii) *fits into a short exact sequence*

$$0 \to \pi_{*-1}F' \to \pi_* BF \to \pi_*\,\mathrm{Cok}\,J \to 0.$$

When $* < \deg(\beta_1)$ *the right hand term is trivial and the left hand map is an isomorphism. In every degree* $*$ *the homotopy group* $\pi_* BF$ *is isomorphic to the torsion subgroup of* $\pi_* Q(\Sigma\mathbb{C}P^\infty)$.

Proof. Claim (i) follows by Dundas' Theorem 1.5 and Proposition 3.6.

The short exact sequence of claim (ii) comes from Lemma 6.2, and when $* < \deg(\beta_1)$ either $\pi_*\,\mathrm{hofib}(l')$ or $\pi_{*-1}F'$ is zero by Proposition 6.3(i) and Corollary 4.4.

The calculation in (iii) is the result of Proposition 6.3(ii).

The short exact sequence in (iv) comes from Corollary 5.12. Since β_1 is the first nontrivial class in $\pi_*\,\mathrm{Cok}\,J$, it is clear that $\pi_*\,\mathrm{Cok}\,J = 0$ for $* < \deg(\beta_1)$. By Segal's Theorem 3.4 the torsion subgroup of $\pi_* Q(\mathbb{C}P^\infty)$ can be identified with $\pi_* F$. □

Remark 6.5. This result reduces the determination of $\pi_*\,\mathrm{hofib}(L)$ in the range $* < \deg(\beta_{p+1}) - 1$ to the identification of $l': SU \to SU$, the calculation of the torsion subgroup $\pi_* BF$ in $\pi_* Q(\Sigma\mathbb{C}P^\infty)$, the calculation of the surjection $\pi_* BF \to \pi_*\,\mathrm{Cok}\,J$, and the resolution of the extension problem in claim (ii).

Proof of Theorem 1.3. When $* < \deg(\beta_1) = 2(p-1)p - 2$ the theorem above identifies $\pi_*\,\mathrm{hofib}(L)$ with the sum of $\pi_* BF$ and $\pi_*\,\mathrm{hofib}(l')$. The former groups are given in Corollary 4.4, the latter in Proposition 6.3 (ii). Combining these results gives claim (i) of the theorem.

Regarding claim (ii), the classes in degrees $2n$ with $n \equiv 0 \bmod p-1$ constitute the summand $\pi_* \operatorname{hofib}(l')$, which is hit isomorphically from $\pi_{2n} BSG$ by the proof of Proposition 6.3. These classes are in the delooped image of J, so map to zero through $\pi_{2n}A(*) \to \pi_{2n} \mathrm{Wh}^{\mathrm{Diff}}(*)$ by Corollary 3.3 of [33]. Hence they land in $\pi_{2n}Q(S^0) \subseteq \pi_{2n}A(*)$, but this group has no p-torsion in this range.

For claim (iii), the remaining classes map nontrivially to $\pi_{*-1}F'$, which is a split summand in $\pi_* \operatorname{hofib}(t)$ by Corollary 5.13, and thus in $\pi_* TC(*)$ by (9). Thus the composite

$$\pi_* \operatorname{hofib}(L) \to \pi_* A(*) \xrightarrow{trc} \pi_* TC(*)$$

maps these classes onto a direct summand in $\pi_* TC(*)$ and thus also in $\pi_* A(*)$, in this range. □

Proof of Corollary 1.4. When $n \equiv 0 \bmod p-1$ the classes of order p in $\pi_{2n} \operatorname{hofib}(L)$ map to zero in $\pi_{2n}A(*)$, and thus come from classes in $K_{2n+1}(\mathbb{Z})$ through the connecting map. We claim that these classes are of infinite order. The connecting map factors as

$$\Omega K(\mathbb{Z}) \xrightarrow{trc} \Omega TC(\mathbb{Z}) \to \operatorname{hofib}(L).$$

The torsion classes in $TC_{2n+1}(\mathbb{Z})$ come from the $\operatorname{Im} J$ and $B \operatorname{Im} J$ terms, which are hit by L and map to zero under the connecting map. Hence the classes detected in $K_{2n+1}(\mathbb{Z})$ map to nonzero classes in $\pi_{2n+1}SU$, and are of infinite order.

The remaining classes are detected as the even-degree homotopy of $\operatorname{hofib}(t)$, which injects onto the torsion in $\pi_* Q(\Sigma \mathbb{C}P^\infty)$, which in turn is a summand in $\pi_* Q(\Sigma_+ \mathbb{C}P^\infty)$. This detecting map factors as

$$\operatorname{hofib}(L) \to A(*) \xrightarrow{trc} TC(*) \to Q(\Sigma_+ \mathbb{C}P^\infty).$$

The claim follows. □

7. A CONJECTURE

We conclude with some speculations suggesting a simpler description of the homotopy fiber of the linearization map.

The map $w: BSG \to \operatorname{hofib}(L)$ may be projected away from the terms $\operatorname{Cok} J$ and $B \operatorname{Cok} J$ to give a map $BSG \to \operatorname{hofib}(l)$. The target sits in the fiber sequence $F' \to \operatorname{hofib}(l') \to \operatorname{hofib}(l)$, where $l': SU \to SU$ can be viewed as a K-theory operation. The map $BSG \to \operatorname{hofib}(l)$ is at least $(2p-1)$-connected, and F' is $(4p-4)$-connected, so on the level of homotopy groups there is a $(2p-1)$-connected homomorphism $\pi_* BSG \to \pi_* \operatorname{hofib}(l')$. If we assume that this also holds on the space level, then the essential properties of the map l' can be determined.

Proposition 7.1. *Suppose there is a* $(2p-1)$*-connected map* $BSG \to \operatorname{hofib}(l')$. *Then* $\Omega l'$ *is homotopic to* $\psi^k - 1: BU \to BU$ *up to pre- and post-composition with homotopy equivalences of* BU. *Hence* $\operatorname{hofib}(l') \simeq BJ_\otimes$.

Proof. The target is K-local to well below the connectivity of BSG, so this hypothetical map extends over $Be_{\otimes}: BSG \to BJ_{\otimes}$. Thus there are horizontal fiber sequences

$$\begin{array}{ccccccc} BU & \xrightarrow{\psi^k - 1} & BU & \longrightarrow & BJ_{\otimes} & \longrightarrow & SU \\ \downarrow c & & \downarrow b & & \downarrow a & & \\ BU & \xrightarrow{\Omega l'} & BU & \longrightarrow & \mathrm{hofib}(l') & \longrightarrow & SU \end{array}$$

with a vertical map a as indicated. Now $K^{-1}(BU) = 0$, so the composite $BU \to BJ_{\otimes} \to \mathrm{hofib}(l') \to SU$ is null-homotopic. Hence there is a lift $b: BU \to BU$ compatible with a. We may alter the choice of b by any map factoring through $\Omega l'$, which is an equivalence on the first $p-2$ of the $p-1$ Adams summands in BU. Hence we may assume b is an equivalence on these summands. Let c be the induced map of homotopy fibers. Then we have a map of fiber ssequences as above. The maps $\psi^k - 1$ and $\Omega l'$ are equivalences on the first $p-2$ Adams summands, so c inherits the same property from b. Finally consider the last Adams summand. By hypothesis the map a induces a mod p equivalence in degree $2p-2$. Hence so does b, and by v_1-periodicity the map b is a p-adic equivalence, also on the last Adams summand. Compatibility of c and b implies that c is also an equivalence on this final summand. For $\psi^k - 1$ does not admit any nontrivial factorizations through BU. Hence b and c are p-adic equivalences, and $\Omega l'$ agrees with $\psi^k - 1$ up to such automorphisms. Thus $\mathrm{hofib}(l') \simeq BJ_{\otimes}$. □

We can now map Cok J and BSG into hofib(L). First Cok J has been identified as a term in a product splitting of hofib(L). Next the subspace BCok J in BSG might imaginably map through w by an equivalence to the subspace BCok J in hofib(L). Then the remaining $BJ_{\otimes}$ might map to hofib(l) by the same map as hofib(l') does. Then only BF', if it exists, remains as the infinite loop space cofiber. We express these observations as a conjectural fiber sequence

$$BSG \times \mathrm{Cok}\, J \to \mathrm{hofib}(L) \to BF'. \tag{18}$$

If BF' does not exist we can instead extend the sequence to the left by F'. By definition F' and Cok J are related by the fiber sequence $F' \to F \to \Omega\,\mathrm{Cok}\, J$. Hence hofib($L$) seems to consist of BSG and the two components BF' and Cok J of BF, reassembled in an untwisted fashion.

This ends our speculations.

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