

COMPOSITIO MATHEMATICA

Algebraic K -theory of real topological K -theory

Gabriel Angelini-Knoll , Christian Ausoni  and John Rognes 

Compositio Math. **162** (2026), 1760–1817.

doi: [10.1017/S0010437X26103418](https://doi.org/10.1017/S0010437X26103418)



FOUNDATION
COMPOSITIO
MATHEMATICA



LONDON
MATHEMATICAL
SOCIETY
EST. 1865





Algebraic K -theory of real topological K -theory

Gabriel Angelini-Knoll^{id}, Christian Ausoni^{id} and John Rognes^{id}

ABSTRACT

We determine the $A(1)$ -homotopy of the topological cyclic homology of the connective real K -theory spectrum ko . The answer has an associated graded that is a free $\mathbb{F}_2[v_2^4]$ -module of rank 52, on explicit generators in stems $-1 \leq * \leq 30$. The calculation is achieved by using prismatic and syntomic cohomology of ko as introduced by Hahn, Raksit and Wilson, extending work of Bhatt, Morrow and Scholze from the case of classical commutative rings to $\mathbb{E}_\infty$ -rings. A new feature in our case is that there are nonzero differentials in the motivic spectral sequence from syntomic cohomology to topological cyclic homology.

1. Introduction

Work of Hahn, Raksit and Wilson [HRW22] extends the notions of prismatic cohomology and syntomic cohomology to the setting of $\mathbb{E}_\infty$ -rings. This produces a new tool for computing topological cyclic homology and, consequently, algebraic K -theory. In the present paper, we use this tool to compute the $A(1)$ -homotopy (cf. Notation 2.20) of the algebraic K -theory of the $\mathbb{E}_\infty$ -ring ko , known as connective real K -theory. Throughout, we work at the prime $p = 2$.

This paper continues the program from [AR02], examining the arithmetic of ring spectra through the lens of telescopically localized algebraic K -theory. In particular, the second and third authors have put forward, in a bundle of predictions known as the redshift conjectures [AR08], the assertion that algebraic K -theory increases chromatic complexity by one. This has now been proven in a qualitative form, for all $\mathbb{E}_\infty$ -rings, in a *tour de force* by Burklund, Schlank and Yuan [BSY22], building on [Yua26, CMN⁺24] and [LMM⁺24].

To better understand the arithmetic of $\mathbb{E}_\infty$ -rings, it is still desirable to prove more quantitative forms of the redshift conjectures, such as the one originally appearing as the ‘chromatic redshift problem’ in [Rog00]. In the present paper, we solve this problem in the case of ko . Explicitly, we prove the following theorem.

Received 20 March 2024, accepted in final form 13 May 2026.

2020 Mathematics Subject Classification 14F30, 19D50, 19D55, 55Q51, 55P43 (primary), 13D03, 19E20, 55N15, 55Q10, 55T25 (secondary).

Keywords: topological cyclic homology; algebraic K -theory; real topological K -theory; chromatic redshift; telescope conjecture; prismatic cohomology; syntomic cohomology; even filtration; motivic filtration; evenly free descent; homotopy fixed points; Tate construction; Adams spectral sequence; Novikov spectral sequence; v_2 -periodicity.

© The Author(s), 2026. This is an Open Access article, distributed under the terms of the Creative Commons Attribution-NonCommercial licence (<https://creativecommons.org/licenses/by-nc/4.0/>), which permits non-commercial re-use, distribution, and reproduction in any medium, provided the original article is properly cited. Written permission must be obtained prior to any commercial use. *Compositio Mathematica* is © Foundation Compositio Mathematica.

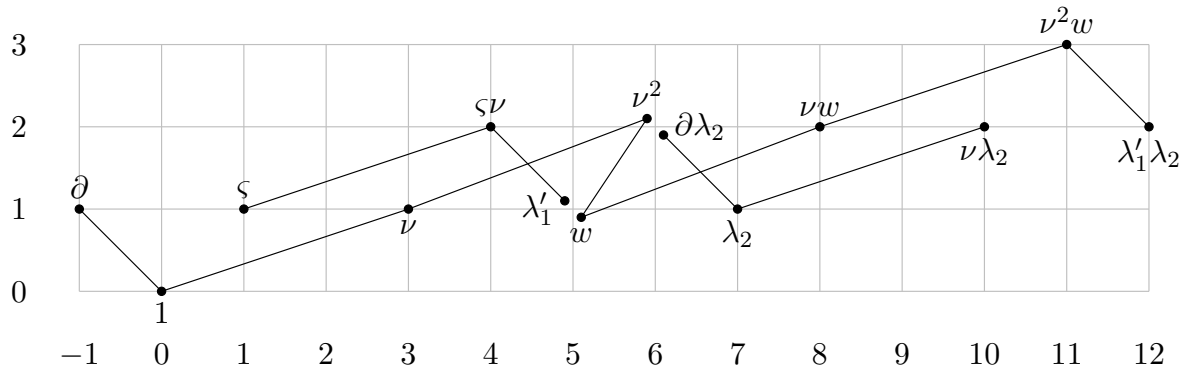


FIGURE 1. The $\mathbb{F}_2[v_2]$ -basis for the syntomic cohomology modulo $(2, \eta, v_1)$ of ko , with lines of slope -1 , 1 and $1/3$ indicating multiplication by ∂ , η and ν , respectively.

THEOREM A (Theorem 7.2). *The $A(1)$ -homotopy $A(1)_* \mathrm{TC}(ko)$ of the topological cyclic homology of ko is a $\mathbb{Z}/4[v_2^{32}]$ -module. The associated graded*

$$\begin{aligned} \mathrm{Gr}_{\mathrm{mot}}^* A(1)_* \mathrm{TC}(ko) &\cong \mathbb{F}_2\{v_2^i \mid 0 \leq i \equiv 0, 1 \pmod{4}\} \\ &\quad \oplus \mathbb{F}_2[v_2]\{\partial, \varsigma, \nu, \lambda_1', w, \lambda_2\} \\ &\quad \oplus \mathbb{F}_2[v_2]\{\varsigma\nu, \nu^2, \partial\lambda_2, \nu w, \nu\lambda_2, \lambda_1'\lambda_2\} \\ &\quad \oplus \mathbb{F}_2\{v_2^j \nu^2 w \mid 0 \leq j \equiv 2, 3 \pmod{4}\} \end{aligned}$$

of its descending motivic filtration $\mathrm{Fil}_{\mathrm{mot}}^* A(1)_* \mathrm{TC}(ko)$ is a finitely generated and free $\mathbb{F}_2[v_2^4]$ -module of rank 52. Here, $|\partial| = -1$, $|\varsigma| = 1$, $|\nu| = 3$, $|\lambda_1'| = |w| = 5$, $|\nu\lambda_2| = 6$ and $|\lambda_2| = 7$.

This calculation is carried out by first computing the syntomic cohomology with $A(1)$ -coefficients, *alias* modulo $(2, \eta, v_1)$, of connective real K -theory.

THEOREM B (Theorem 6.12). *The syntomic cohomology modulo $(2, \eta, v_1)$ of ko is*

$$\begin{aligned} \overline{A}(1)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(ko) &:= \pi_*(\overline{A}(1) \otimes \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(ko)) \\ &\cong \mathbb{F}_2[v_2]\{1, \partial, \varsigma, \nu, \lambda_1', w, \lambda_2, \varsigma\nu, \nu^2, \partial\lambda_2, \nu w, \nu\lambda_2, \lambda_1'\lambda_2, \nu^2 w\}, \end{aligned}$$

where the (stem, motivic filtration) bidegrees of the $\mathbb{F}_2[v_2]$ -module generators are as displayed in Figure 1 and Table 1.

See Notation 7.1 for the algebraic filtration $\mathrm{Fil}_{\mathrm{mot}}^*(-)$ on $A(1)_* \mathrm{TC}(ko)$ and its associated graded $\mathrm{Gr}_{\mathrm{mot}}^*(-)$, see Definition 2.18 for the spectrum level filtration $\mathrm{fil}_{\mathrm{mot}}^*(-)$ on $\mathrm{TC}(ko)$ and its associated graded $\mathrm{gr}_{\mathrm{mot}}^*(-)$, and see Construction 2.29 for the meaning of $\overline{A}(1) \otimes (-) := \mathrm{gr}_{\mathrm{ev}/\mathbb{S}}^* A(1) \otimes_{\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}} (-)$.

As a consequence, we determine the $A(1)$ -homotopy of the algebraic K -theory of ko and of its 2-completion $ko_2^\wedge$.

THEOREM C (Theorems 7.4 and 7.5). *There are exact sequences of $\mathbb{Z}/4[v_2^{32}]$ -modules*

$$0 \rightarrow \Sigma^3 \mathbb{F}_2 \rightarrow A(1)_* K(ko) \xrightarrow{\mathrm{trc}} A(1)_* \mathrm{TC}(ko) \rightarrow \mathbb{F}_2\{\partial, \varsigma\} \rightarrow 0$$

and

$$0 \rightarrow \Sigma^1 \mathbb{F}_2 \oplus \Sigma^3 \mathbb{F}_2 \rightarrow A(1)_* K(ko_2^\wedge) \xrightarrow{\mathrm{trc}} A(1)_* \mathrm{TC}(ko) \rightarrow \mathbb{F}_2\{\partial, \varsigma\} \rightarrow 0,$$

with $|\partial| = -1$ and $|\varsigma| = 1$.

TABLE 1. Bidegrees and detecting classes for the $\mathbb{F}_2[v_2]$ -module generators of $\overline{A}(1)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko})$.

Generator	Bidegree	Detecting class
1	(0, 0)	1
∂	(-1, 1)	∂
ς	(1, 1)	$t^2 \lambda'_1$
ν	(3, 1)	$t \lambda'_1$
w	(5, 1)	$t \lambda_2$
λ'_1	(5, 1)	λ'_1
λ_2	(7, 1)	λ_2
$\varsigma \nu$	(4, 2)	$\partial \lambda'_1$
$\partial \lambda_2$	(6, 2)	$\partial \lambda_2$
ν^2	(6, 2)	$t^3 \lambda'_1 \lambda_2 \bmod \partial \lambda_2$
νw	(8, 2)	$t^2 \lambda'_1 \lambda_2$
$\nu \lambda_2$	(10, 2)	$t \lambda'_1 \lambda_2$
$\lambda'_1 \lambda_2$	(12, 2)	$\lambda'_1 \lambda_2$
$\nu^2 w$	(11, 3)	$\partial \lambda'_1 \lambda_2$

Using this, we show in Corollary 7.6 that there is no $\mathbb{E}_0$ -ring map $K(\mathrm{ko}) \rightarrow \mathrm{tmf}$, i.e., no spectrum map inducing an isomorphism on π_0 . In contrast, an $\mathbb{E}_\infty$ -ring map $K(\mathrm{ko}) \rightarrow E_2$ is guaranteed to exist by [BSY22]. Here, tmf denotes the topological modular forms spectrum, and E_2 is a suitable height 2 Lubin–Tate theory.

As usual, we let $K(n)$ denote the (2-primary) height n Morava K -theory, with coefficient ring $K(n)_* = \mathbb{F}_2[v_n^{\pm 1}]$, and we let L_n and L_n^f denote Bousfield localization at $K(0) \oplus \cdots \oplus K(n)$ and at $T(0) \oplus \cdots \oplus T(n)$, respectively, where $T(m) = v_m^{-1} F(m)$ is the telescope of any v_m self-map of a type m finite 2-local spectrum $F(m)$.

We say that a spectrum X satisfies the (strong, 2-primary) *height n telescope conjecture* if the canonical map $L_n^f X \rightarrow L_n X$ is an equivalence. By recent groundbreaking work of Burklund, Hahn, Levy and Schlank [BHL⁺23], we know for each $n \geq 2$ that not all spectra have this property. However, it is still interesting to consider the question of which spectra do satisfy this conjecture (see, e.g., [MR99, Conjecture 7.3]). In Theorem 7.7, we deduce from [HRW22] and [CMN⁺20] that the spectra $K(\mathrm{ko})$, $K(\mathrm{ko}_2^\wedge)$ and $\mathrm{TC}(\mathrm{ko})$ all satisfy the height 2 telescope conjecture.

We say that a spectrum X has *telescopic complexity n* if the map $X \rightarrow L_n^f X$ is an equivalence in all sufficiently large degrees. In [AR08], the redshift conjecture was phrased in terms of this property. In Theorem 7.8, we apply our Theorems A and C to confirm that the spectra $K(\mathrm{ko})_{(2)}$, $K(\mathrm{ko}_2^\wedge)_{(2)}$ and $\mathrm{TC}(\mathrm{ko})_{(2)}$ (as well as their 2-completions) have telescopic complexity 2, as predicted.

1.1 Organization

In §2, we give a presentation for the associated graded $\mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})$ of the motivic filtration on $\mathrm{THH}(\mathrm{ko})$, and calculate its bigraded homotopy modulo $(2, \eta, v_1)$ and modulo $(2, v_1)$. In §3, we compare two filtrations leading to spectral sequences calculating the associated graded objects $\mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}^-(\mathrm{ko})$ and $\mathrm{gr}_{\mathrm{mot}}^* \mathrm{TP}(\mathrm{ko})$ of the motivic filtrations on $\mathrm{TC}^-(\mathrm{ko})$ and $\mathrm{TP}(\mathrm{ko})$, with appropriate finite coefficients. In §4, we show how some homotopy classes from $A(1)$ are detected in homotopy modulo $(2, \eta, v_1)$ of $\mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}^-(\mathrm{ko})$. In §5, we study the $\mathbb{T}$ -Tate spectral sequence calculating prismatic cohomology, i.e., the homotopy modulo $(2, v_1, v_2)$ and $(2, \eta, v_1, v_2)$.

of $\mathrm{gr}_{\mathrm{mot}}^* \mathrm{TP}(\mathrm{ko})$. In § 6, we study the $\mathbb{T}$ -homotopy fixed point spectral sequence calculating the homotopy of $\mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}^-(\mathrm{ko})$ modulo $(2, v_1, v_2)$ and $(2, \eta, v_1, v_2)$, and use this to calculate syntomic cohomology, i.e., the homotopy of $\mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko})$ modulo $(2, v_1)$ and $(2, \eta, v_1)$. Finally, in § 7, we determine the differential pattern in the motivic spectral sequence calculating $A(1)_* \mathrm{TC}(\mathrm{ko})$, and use known facts about the cyclotomic trace map to deduce corresponding results for $A(1)_* \mathrm{K}(\mathrm{ko}_2^\wedge)$ and $A(1)_* \mathrm{K}(\mathrm{ko})$.

1.2 Conventions

We let $\mathcal{A}$ and $\mathcal{A}^\vee$ denote the mod 2 Steenrod algebra and its dual, respectively. We set $H_*(X) := H_*(X; \mathbb{F}_2)$ and write

$$\nu: H_*(X) \longrightarrow \mathcal{A}^\vee \otimes H_*(X)$$

for the $\mathcal{A}^\vee$ -coaction. We write $\mathcal{A}(1)$ for the subalgebra of $\mathcal{A}$ generated by Sq^1 and Sq^2 , and note that $\mathcal{A}(1)^\vee = \mathbb{F}_2[\xi_1, \xi_2]/(\xi_1^4, \xi_2^2)$.

We say that a spectrum X is *even* if its homotopy groups $\pi_*(X) = X_*$ are concentrated in even degrees.

Let $\mathbb{Z}^{\mathrm{op}}$ be the category whose objects are integers, such that $\mathrm{Mor}_{\mathbb{Z}^{\mathrm{op}}}(n, m) = *$ if $n \geq m$ and $= \emptyset$ otherwise. Let $\mathbb{Z}^\delta$ be the integers as a discrete category. Given a presentably symmetric monoidal stable ∞ -category $\mathcal{C}$, we write

$$\mathcal{C}^{\mathrm{fil}} := \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C}), \quad \mathcal{C}^{\mathrm{gr}} := \mathrm{Fun}(\mathbb{Z}^\delta, \mathcal{C}).$$

Given $I^* \in \mathcal{C}^{\mathrm{fil}}$ and $J^* \in \mathcal{C}^{\mathrm{gr}}$, we write $I^w := I(w)$ and $J^w := J(w)$. We recall that there is a symmetric monoidal functor

$$\mathrm{gr}^*: \mathcal{C}^{\mathrm{fil}} \longrightarrow \mathcal{C}^{\mathrm{gr}}$$

defined on an object I^* by $\mathrm{gr}^w I = I^w/I^{w+1}$. As in the above notation, for consistency we use the superscript $\star$ as in $I^\star$ for a filtered object, the superscript $*$ as in J^* for a graded object, and a superscript $\bullet$ as in $K^\bullet$ for a cosimplicial object. In § 3, we also use increasing filtrations and homological grading, in which case $\mathrm{fil}_w := \mathrm{fil}^{-w}$ and $\mathrm{gr}_w := \mathrm{gr}^{-w}$.

We use the terminology from [Isa19, Definition 4.1.2] for (hidden) extensions in spectral sequences. We use the term ‘algebra spectral sequence’ in the sense of [HR24, Definition 4.9], and we will also consider module spectral sequences over such algebra spectral sequences.

In §§ 2–6, we use the following conventions. We write $\mathrm{THH}(A/R) = |[q] \mapsto A \otimes_R \cdots \otimes_R A|$ for the relative topological Hochschild homology of an R -algebra A , with $q+1$ copies of A in simplicial degree q , and we simply write $\mathrm{THH}(A)$ when $R = \mathbb{S}$. We implicitly 2-complete each of the following invariants: $\mathrm{THH} := \mathrm{THH}(-)_2^\wedge$, $\mathrm{TC}^- := (\mathrm{THH}(-)^{h\mathbb{T}})_2^\wedge$, $\mathrm{TP} := (\mathrm{THH}(-)^{t\mathbb{T}})_2^\wedge$, $\mathrm{TC} := \mathrm{TC}(-)_2^\wedge$ and $\mathrm{K} := \mathrm{K}(-)_2^\wedge$. We write ko , ku , KU , MU , MUP , $\mathbb{S}$ and BP for the 2-completions of connective real K -theory, connective complex K -theory, periodic complex K -theory, complex cobordism, periodic complex cobordism, the sphere spectrum and the Brown–Peterson spectrum, respectively. We will simply write $\mathbb{Z}$ for the 2-adic integers. We write Sp_2 for the ∞ -category of 2-complete spectra with symmetric monoidal product $\otimes$. Note that our smash product is implicitly 2-completed in §§ 2–6, so that $\otimes_R := (- \otimes_R -)_2^\wedge$ for any 2-complete $\mathbb{E}_\infty$ -ring R , and we omit R from the notation when R is the 2-complete sphere spectrum. We also write $\otimes$ for the (underived) tensor product over the 2-adic integers and expect the intended meaning to be clear from the context. We write $\mathbb{T}$ for the circle regarded as the group of complex numbers of modulus 1, and we set $\mathcal{C}^{B\mathbb{T}} := \mathrm{Fun}(B\mathbb{T}, \mathcal{C})$. We write $C_2 = \{\pm 1\} \subset \mathbb{T}$ for the subgroup of order 2.

In § 7, as in the present § 1, we explicitly include notation for 2-completion, especially in the argument of $\mathrm{K}(-)$. Note that the canonical map $\mathrm{TC}(\mathrm{ko})_2^\wedge \rightarrow \mathrm{TC}(\mathrm{ko}_2^\wedge)_2^\wedge$ is an equivalence by

[Mad94, pp. 274–275] (cf. [NS18, pp. 351–352]), so we can omit 2-completion in the argument of $\mathrm{TC}(-)_2^\wedge$.

2. Hochschild homology and motivic filtrations

We first introduce filtrations on THH , TC^- , TP and TC . The reader is encouraged to read [BHM93, BM94, BM95, HN20] for background on these invariants and [HRW22] for a thorough account of the filtrations that we use in this paper.

DEFINITION 2.1. Given a map $f: A \rightarrow B$ of 2-complete $\mathbb{E}_\infty$ -rings, we write $C^\bullet(B/A)$ for the associated cosimplicial *Amitur complex* with $C^q(B/A) = B^{\otimes_{Aq+1}}$.

Recall that there is a map of $\mathbb{E}_\infty$ -rings $c: \mathrm{ko} \rightarrow \mathrm{ku}$ called the *complexification map*, where $\mathrm{ko}_* = \mathbb{Z}[\eta, A, B]/(2\eta, \eta^3, \eta A, A^2 - 4B)$, $\mathrm{ku}_* = \mathbb{Z}[u]$, $A \mapsto 2u^2$ and $B \mapsto u^4$, with $|\eta| = 1$, $|A| = 4$, $|B| = 8$ and $|u| = 2$.

LEMMA 2.2. *There exists an $\mathbb{E}_\infty$ -ring map $\mathrm{MU} \rightarrow \mathrm{ku}$. We can choose the generators of $\mathrm{MU}_* = \pi_*(\mathrm{MU}) = \mathbb{Z}[x_i \mid i \geq 1]$ so that $x_1 \mapsto u$ and $x_i \mapsto 0$ for each $i \geq 2$. Here, $|x_i| = 2i$.*

Proof. The proof of [HY20, Theorem 4.3] shows the existence of $\mathbb{E}_\infty$ -ring maps $\mathrm{MU} \rightarrow \mathrm{MUP} \rightarrow \mathrm{KU}$. Passing to connective covers gives the factorization through ku . Recall our running convention that $\mathbb{Z}$ denotes the 2-adic integers. Both MU and ku have first Postnikov k -invariant the generator of $H^3(H\mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}/2$, so $\mathbb{Z}\{x_1\} = \pi_2(\mathrm{MU}) \rightarrow \pi_2(\mathrm{ku}) = \mathbb{Z}\{u\}$ must be an isomorphism. Dividing the first choice of x_1 by a unit, we can assume that $x_1 \mapsto u$. For each $i \geq 2$, we can then subtract a multiple of x_1^i from the first choice of x_i to ensure that $x_i \mapsto 0$. $\square$

We hereafter fix $\mathrm{MU} \rightarrow \mathrm{ku}$ and the x_i as in the above lemma. This provides $\mathbb{E}_\infty$ -ring maps $C^q(\mathrm{MU}/\mathbb{S}) \rightarrow C^q(\mathrm{ku}/\mathrm{ko})$, compatibly for all $q \geq 0$. We write

$$\tau_{\geq \star}: \mathrm{Sp}_2 \longrightarrow \mathrm{Sp}_2^{\mathrm{fil}}$$

for the monoidal Whitehead filtration [Lur17, Proposition 1.4.3.6, Example 2.2.1.10]. Note that the circle group $\mathbb{T}$ acts freely on $S^1 \subset S^3 \subset \cdots \subset S^\infty = E\mathbb{T}$. We write $F(S_+^{2\ell+1}, \mathrm{THH})^\mathbb{T}$ for the ℓ th-order approximate fixed points functor, taking A/R to $F(S_+^{2\ell+1}, \mathrm{THH}(A/R))^\mathbb{T} \simeq \lim_{\mathrm{CP}^\ell} \mathrm{THH}(A/R)$, as in [BR05, §2] and [AAC⁺25, §7]. Here, $\mathrm{CP}^\ell \subset \mathrm{CP}^\infty = B\mathbb{T}$. We also write THH^{hC_2} and THH^{tC_2} for the functors obtained by applying C_2 -homotopy fixed points and the C_2 -Tate construction to THH , respectively.

DEFINITION 2.3. For $F \in \{\mathrm{THH}, \mathrm{TC}^-, \mathrm{TP}, F(S_+^{2\ell+1}, \mathrm{THH})^\mathbb{T}, \mathrm{THH}^{hC_2}, \mathrm{THH}^{tC_2}\}$ we define $\mathbb{E}_\infty$ algebras

$$\begin{aligned} \mathrm{fil}_{\mathrm{mot}}^* F(\mathrm{ko}) &:= \mathrm{Tot}(\tau_{\geq 2\star} F(C^\bullet(\mathrm{ku}/\mathrm{ko})/C^\bullet(\mathrm{MU}/\mathbb{S}))), \\ \mathrm{gr}_{\mathrm{mot}}^* F(\mathrm{ko}) &:= \mathrm{gr}^*(\mathrm{fil}_{\mathrm{mot}}^* F(\mathrm{ko})) \end{aligned}$$

in $\mathrm{Sp}_2^{\mathrm{fil}}$ and $\mathrm{Sp}_2^{\mathrm{gr}}$, respectively. Here, Tot denotes the totalization of a cosimplicial object. We refer to $\mathrm{gr}_{\mathrm{mot}}^w \mathrm{TP}(\mathrm{ko})$ as the weight w *prismatic cohomology* spectrum of ko and to $\pi_* \mathrm{gr}_{\mathrm{mot}}^w \mathrm{TP}(\mathrm{ko})$ as the weight w *prismatic cohomology* groups.

Remark 2.4. We will show in Proposition 2.15 that

$$\mathrm{THH}(C^q(\mathrm{ku}/\mathrm{ko})/C^q(\mathrm{MU}/\mathbb{S})) = C^q(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\mathrm{THH}(\mathrm{ko}))$$

is even and 2-torsion free. It follows that $F(C^q(\mathrm{ku}/\mathrm{ko})/C^q(\mathrm{MU}/\mathbb{S}))$ is even for each functor $F \in \{\mathrm{THH}, \mathrm{TC}^-, \mathrm{TP}, F(S_+^{2\ell+1}, \mathrm{THH})^\mathbb{T}, \mathrm{THH}^{hC_2}, \mathrm{THH}^{tC_2}\}$ and $q \geq 0$. Hence the double-speed

Whitehead filtration in the definition of $\mathrm{fil}_{\mathrm{mot}}^* F(\mathrm{ko})$ will be well behaved for each F in the above list.

Remark 2.5. We will verify in Proposition 2.17 that $\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TP}(\mathrm{ko})$ agrees with prismatic cohomology in the sense of [HRW22, Definition 1.3.4], building on [BMS19].

Remark 2.6. Note that when $\mathrm{THH}(A/R)$ is even, we are considering the filtered spectrum defined by $\tau_{\geq 2*} \mathrm{TP}(A/R) = \tau_{\geq 2*} (\mathrm{THH}(A/R)^{t\mathbb{T}})$, which is not the same as the filtered spectrum $(\tau_{\geq 2*} \mathrm{THH}(A/R))^{t\mathbb{T}}$. In particular, the class t in the associated graded of the $\mathbb{T}$ -Tate filtration, lifting the generator of $\pi_{-2}(H\mathbb{F}_2)^{t\mathbb{T}}$, has weight -1 (and motivic filtration 0; see below). The same applies for t lifting the generator of $\pi_{-2}(H\mathbb{F}_2)^{h\mathbb{T}}$, in the associated graded of the $\mathbb{T}$ -homotopy fixed point filtration for $\mathrm{TC}^-(A/R) = \mathrm{THH}(A/R)^{h\mathbb{T}}$.

We also fix terminology for gradings.

DEFINITION 2.7. Given $M^* \in \mathrm{Sp}_2^{\mathrm{gr}}$ and $x \in \pi_n M^w$, we say that x has *stem* n , *weight* w and *motivic filtration* $2w - n$, and write $\|x\| = (n, 2w - n)$. In [HRW22, Definition 1.4.2], stem and motivic filtration are called degree and Adams weight, respectively. We refer to the (Adams indexed, Bousfield–Kan) spectral sequence

$$E_2^{n, 2w-n} = \pi_n \mathrm{gr}_{\mathrm{mot}}^w F(\mathrm{ko}) \implies \pi_n F(\mathrm{ko})$$

for each $F \in \{\mathrm{THH}, \mathrm{TC}^-, \mathrm{TP}, F(S_+^{2\ell+1}, \mathrm{THH})^{\mathbb{T}}, \mathrm{THH}^{hC_2}, \mathrm{THH}^{tC_2}\}$ as the *motivic spectral sequence*. We will show in Lemma 2.19 that the motivic spectral sequence converges in each case.

We plot each motivic spectral sequence with stem on the horizontal axis and motivic filtration on the vertical axis. Given this convention, if $\|x\| = (n, 2w - n)$ then $\|d_r(x)\| = (n - 1, 2w - n + r)$. Note that we write $E_r^{n, 2w-n}$ where it is also standard to write $E_r^{2w-n, 2w}$ in the literature. With these conventions, the motivic spectral sequence $E_r^{n,s}$ -term is concentrated in internal degrees $n + s = 2w$ and, consequently, $d_r = 0$ for all even integers $r \geq 2$.

Our first aim is to show that the filtrations from Definition 2.3 agree with the motivic filtrations considered in [HRW22, Definition 4.2.1].

DEFINITION 2.8 [HRW22, Definition 2.2.15(4)]. A map $A \rightarrow B$ of $\mathbb{E}_\infty$ -rings is *2-completely evenly free* if for each nonzero even $\mathbb{E}_\infty$ A -algebra C of bounded 2-power torsion, the pushout $B \otimes_A C$ is equivalent as a C -module to a nonzero wedge sum of even suspensions of C . In what follows, we will omit ‘2-completely’ and simply refer to such maps of $\mathbb{E}_\infty$ -rings as evenly free maps, to be consistent with our conventions.

Remark 2.9. This condition on $B \otimes_A C$ is equivalent to $\pi_*(B \otimes_A C)$ being a nonzero free C_* -module on generators in even stems. An evenly free map $A \rightarrow B$ of $\mathbb{E}_\infty$ -rings is 2-completely eff (= evenly faithfully flat), in the sense of [HRW22, Definition 2.2.15(2)], as per [HRW22, Remark 2.2.16].

LEMMA 2.10. *The complexification map $\mathrm{ko} \rightarrow \mathrm{ku}$ is evenly free. Moreover, for each even $\mathbb{E}_\infty$ ko -algebra C , there is an isomorphism of C_* -algebras*

$$\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C) \cong C_*[b_1]/(\hat{b}_1^2),$$

where $\hat{b}_1^2 = b_1^2 + c_2 b_1 + c_4$ for some $c_2, c_4 \in C_*$ with $|c_2| = 2$ and $|c_4| = 4$.

Proof. Recall the Wood (homotopy) cofiber sequence

$$\Sigma \mathrm{ko} \xrightarrow{\eta} \mathrm{ko} \xrightarrow{c} \mathrm{ku} \xrightarrow{R} \Sigma^2 \mathrm{ko} \quad (2.1)$$

of ko -modules, where we write R for the map satisfying $R \circ (u \cdot -) = \Sigma^2 r$, with $r: \mathrm{ku} \rightarrow \mathrm{ko}$ the realification map. The resulting cofiber sequence

$$\Sigma C \xrightarrow{\eta} C \longrightarrow \mathrm{ku} \otimes_{\mathrm{ko}} C \xrightarrow{R} \Sigma^2 C$$

induces a short exact sequence of C_* -modules

$$0 \rightarrow C_* \longrightarrow \pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C) \xrightarrow{R} \Sigma^2 C_* \rightarrow 0, \quad (2.2)$$

since C is even, so that $\eta: \Sigma C_* \rightarrow C_*$ is zero. We conclude that $\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C)$ is concentrated in even degrees.

A choice of a class $b_1 \in \pi_2(\mathrm{ku} \otimes_{\mathrm{ko}} C)$ with $R(b_1) = \Sigma^2 1$ is equivalent to a choice of a splitting of (2.2), so that $\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C) \cong C_*\{1, b_1\}$ is free over C_* and nonzero if $C \neq 0$. Writing $\hat{b}_1^2 = b_1^2 + c_2 b_1 + c_4$ for the monic polynomial that vanishes in $\pi_4(\mathrm{ku} \otimes_{\mathrm{ko}} C)$, there is an isomorphism

$$\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C) \cong C_*[b_1]/(\hat{b}_1^2)$$

of C_* -algebras. □

Remark 2.11. Note that c_2 and c_4 in the statement of Lemma 2.10 need not be zero. For example, when $C = \mathrm{ku}$, then b_1 can be chosen as in [DLR22, Lemma 5.1] so that $\hat{b}_1^2 = b_1^2 - ub_1$.

LEMMA 2.12. *Let $A \rightarrow B$ be a map of connective $\mathbb{E}_\infty$ -rings with bounded 2-power torsion. If $\pi_*(B \otimes_A H\pi_0 A)$ is a nonzero, free and even $\pi_0 A$ -module, then $A \rightarrow B$ is evenly free.*

Proof. Let C be a nonzero even $\mathbb{E}_\infty$ A -algebra of bounded 2-power torsion. Associated to the double-speed Whitehead filtration $\tau_{\geq 2*} C$ we have an algebra spectral sequence $E_1(C) \implies \pi_*(C)$ with $E_1^{*,w}(C) = \pi_{*-2w}(H\pi_{2w} C)$. Similarly, the induced filtration $B \otimes_A \tau_{\geq 2*} C$ yields a spectral sequence $E_1(B \otimes_A C) \implies \pi_*(B \otimes_A C)$ with $E_1^{*,w}(B \otimes_A C) = \pi_{*-2w}(B \otimes_A H\pi_{2w} C)$, which we view as a module spectral sequence over the first one. Both spectral sequences are conditionally convergent, since A and B are assumed to be connective. A choice of $\pi_0 A$ -module basis for $\pi_*(B \otimes_A H\pi_0 A)$, and the equivalence

$$B \otimes_A H\pi_{2w} C \simeq (B \otimes_A H\pi_0 A) \otimes_{H\pi_0 A} H\pi_{2w} C,$$

show that $E_1(B \otimes_A C) \cong \pi_*(B \otimes_A H\pi_0 A) \otimes_{\pi_0 A} E_1(C)$ is a nonzero free $E_1(C)$ -module on generators in (filtration 0 and) even stems. Hence both spectral sequences are concentrated in even stems, collapse at $E_1 = E_\infty$, and are strongly convergent by [Boa99, Theorem 8.10]. It follows that the $E_1(C)$ -module generators lift to the abutment, giving a $\pi_*(C)$ -module basis for $\pi_*(B \otimes_A C)$. □

Remark 2.13. Without our implicit 2-completion assumptions and the bounded 2-power torsion assumption, the same argument as in the proof of the above lemma provides a criterion for a map of connective $\mathbb{E}_\infty$ -rings to be evenly free in the sense of [HRW22, Definition 2.2.15(3)].

PROPOSITION 2.14. *The map $\mathrm{THH}(\mathrm{ko}) \rightarrow \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$, induced by the complexification map $c: \mathrm{ko} \rightarrow \mathrm{ku}$ and the unit map $\mathbb{S} \rightarrow \mathrm{MU}$, is evenly free.*

Proof. Since ko , ku and MU are connective and have finite type, it follows that $\mathrm{THH}(\mathrm{ko})$ and $\mathrm{THH}(\mathrm{ku}/\mathrm{MU})$ have finite type, and therefore have bounded 2-power torsion. Hence, by Lemma 2.12, it suffices to show that

$$\pi_*(\mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \otimes_{\mathrm{THH}(\mathrm{ko})} H\mathbb{Z})$$

is a nonzero, free and even $\mathbb{Z}$ -module. The $\mathbb{E}_\infty$ -ring map $\mathrm{THH}(\mathrm{ko}) \rightarrow H\mathbb{Z}$ can be rewritten as $\mathrm{THH}(\mathrm{ko}/\mathbb{S}) \rightarrow \mathrm{THH}(H\mathbb{Z}/H\mathbb{Z})$, which is induced by $\mathrm{ko} \rightarrow H\mathbb{Z}$ over $\mathbb{S} \rightarrow H\mathbb{Z}$. Then

$$\begin{aligned} \mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \otimes_{\mathrm{THH}(\mathrm{ko})} H\mathbb{Z} &\cong \mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \otimes_{\mathrm{THH}(\mathrm{ko})} \mathrm{THH}(H\mathbb{Z}/H\mathbb{Z}) \\ &\simeq \mathrm{THH}(\mathrm{ku} \otimes_{\mathrm{ko}} H\mathbb{Z}/\mathrm{MU} \otimes H\mathbb{Z}), \end{aligned}$$

where the last equivalence holds because THH commutes with pushouts of $\mathbb{E}_\infty$ -rings.

Recall that $\pi_*(\mathrm{MU} \otimes H\mathbb{Z}) \cong H_*(\mathrm{MU}; \mathbb{Z}) = \mathbb{Z}[b_k \mid k \geq 1]$ as $\mathbb{Z}$ -algebras, with $|b_k| = 2k$. Since $\mathrm{MU} \rightarrow \mathrm{ku}$ is 4-connected by Lemma 2.2, and $\mathbb{S} \rightarrow \mathrm{ko}$ is 3-connected, it follows that $\mathrm{MU} \otimes H\mathbb{Z} \rightarrow \mathrm{ku} \otimes_{\mathrm{ko}} H\mathbb{Z}$ is 4-connected. Hence the generators b_k may be chosen so that $\pi_*(\mathrm{MU} \otimes H\mathbb{Z}) \rightarrow \pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} H\mathbb{Z}) = \mathbb{Z}[b_1]/(b_1^2)$ satisfies $b_1 \mapsto b_1$ and $b_k \mapsto 0$ for $k \geq 2$, where b_1 in the target is fixed as in Lemma 2.10.

To determine THH of $\mathrm{ku} \otimes_{\mathrm{ko}} H\mathbb{Z}$ relative to $\mathrm{MU} \otimes H\mathbb{Z}$, we first study the Künneth (or Tor) spectral sequence

$$\begin{aligned} E^2 &= \mathrm{Tor}_*^{\mathbb{Z}[b_k \mid k \geq 1]}(\mathbb{Z}[b_1]/(b_1^2), \mathbb{Z}[b_1]/(b_1^2)) \\ &\implies \pi_*((\mathrm{ku} \otimes_{\mathrm{ko}} H\mathbb{Z}) \otimes_{(\mathrm{MU} \otimes H\mathbb{Z})} (\mathrm{ku} \otimes_{\mathrm{ko}} H\mathbb{Z})). \end{aligned}$$

Since b_1^2 and b_k for $k \geq 2$ act trivially on $\mathbb{Z}[b_1]/(b_1^2)$, we compute that

$$E^2 \cong \mathbb{Z}[b_1]/(b_1^2) \otimes \Lambda(\sigma b_1^2, \sigma b_k \mid k \geq 2)$$

with $\sigma b_1^2, \sigma b_k \in \mathrm{Tor}_1$, where the triviality of the square $(\sigma b_1^2)^2$ follows from the fact that the product in Tor is given by the shuffle product. Since the algebra generators are all in filtration ≤ 1 , the Künneth spectral sequence collapses at the E^2 -term. Note that this is a homological spectral sequence associated to an increasing filtration. Since the E^∞ -term is a nonzero free $\mathbb{Z}$ -module and the abutment is also a $\mathbb{Z}$ -module, we conclude that the abutment is a free and nonzero $\mathbb{Z}$ -module.

We can rule out all potential hidden multiplicative extensions, because Tor_0 splits off from the abutment and all the classes in Künneth filtration 1 are in odd total degree. Therefore, we have an algebra isomorphism

$$D_* := \pi_*((\mathrm{ku} \otimes_{\mathrm{ko}} H\mathbb{Z}) \otimes_{(\mathrm{MU} \otimes H\mathbb{Z})} (\mathrm{ku} \otimes_{\mathrm{ko}} H\mathbb{Z})) \cong \mathbb{Z}[b_1]/(b_1^2) \otimes \Lambda(\sigma b_1^2, \sigma b_k \mid k \geq 2).$$

Second, we apply the Künneth spectral sequence

$$E^2 = \mathrm{Tor}_*^{D_*}(\mathbb{Z}[b_1]/(b_1^2), \mathbb{Z}[b_1]/(b_1^2)) \implies \pi_* \mathrm{THH}(\mathrm{ku} \otimes_{\mathrm{ko}} H\mathbb{Z}/\mathrm{MU} \otimes H\mathbb{Z}).$$

The E^2 -term of this spectral sequence, namely,

$$E^2 = \mathbb{Z}[b_1]/(b_1^2) \otimes \Gamma(\sigma^2 b_1^2, \sigma^2 b_k \mid k \geq 2),$$

is concentrated in even degrees, so the spectral sequence collapses at the E^2 -term and the abutment is also nonzero, free and concentrated in even degrees. $\square$

PROPOSITION 2.15. *For each $q \geq 0$, the spectrum $C^q(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\mathrm{THH}(\mathrm{ko}))$ is an even $\mathbb{E}_\infty$ ku -algebra whose homotopy groups are free as a ku_* -module.*

Proof. By Proposition 2.14 and induction, it suffices to show that $\pi_* \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$ is free as a ku_* -module and concentrated in even degrees. This follows from [HW22, Theorem E], but we recall the argument in our context. The Künneth spectral sequence

$$\begin{aligned} E^2 &= \mathrm{Tor}_*^{\mathrm{MU}_*}(\mathrm{ku}_*, \mathrm{ku}_*) = \mathrm{Tor}_*^{\mathbb{Z}[x_i \mid i \geq 1]}(\mathbb{Z}[u], \mathbb{Z}[u]) \\ &= \mathbb{Z}[u] \otimes \Lambda(\sigma x_i \mid i \geq 2) \implies \pi_*(\mathrm{ku} \otimes_{\mathrm{MU}} \mathrm{ku}) \end{aligned}$$

collapses at the E^2 -term. Again, we can rule out all hidden multiplicative extensions, because Tor_0 splits off from the abutment and all classes in Künneth filtration 1 are in odd total degrees.

Next, we apply the Künneth spectral sequence

$$\begin{aligned} E^2 &= \mathrm{Tor}_*^{\pi_*(\mathrm{ku} \otimes_{\mathrm{MU}} \mathrm{ku})}(\mathrm{ku}_*, \mathrm{ku}_*) = \mathrm{Tor}_*^{\mathbb{Z}[u] \otimes \Lambda(\sigma x_i | i \geq 2)}(\mathbb{Z}[u], \mathbb{Z}[u]) \\ &= \mathbb{Z}[u] \otimes \Gamma(\sigma^2 x_i | i \geq 2) \implies \pi_* \mathrm{THH}(\mathrm{ku}/\mathrm{MU}). \end{aligned}$$

This E^2 -term is also concentrated in even degrees, and is free over $\mathbb{Z}[u] = \mathrm{ku}_*$. Hence $E^2 = E^\infty$, and the abutment is free over ku_* on even degree generators. $\square$

The following fact is stated without proof in [HRW22, Example 1.3.3]. We follow this reference and write $L_{B/A}^{\mathrm{alg}}$ for the algebraic cotangent complex, with homological grading, of a homomorphism $A \rightarrow B$ of (ungraded) commutative rings.

PROPOSITION 2.16. *$\mathrm{ko}_* \mathrm{MU} \cong \mathbb{Z}[u, b_1^2, \bar{b}_k | k \geq 2]$ is a polynomial ring on generators in dimensions 2, 4 and $2k$ for $k \geq 2$. Hence $L_{\mathrm{ko}_* \mathrm{MU}/\mathrm{MU}_*}^{\mathrm{alg}}$ has 2-complete Tor-amplitude contained in $[0, 1]$, so that $\mathrm{MU} \rightarrow \mathrm{ko} \otimes \mathrm{MU}$ is 2-quasi-lci.*

Proof. Since MU_* is free over BP_* , we have a ring isomorphism $\mathrm{ko}_* \mathrm{BP} \otimes_{\mathrm{BP}_*} \mathrm{MU}_* \cong \mathrm{ko}_* \mathrm{MU}$. Here, $H_*(\mathrm{ko}) = \mathcal{A}^\vee \square_{\mathcal{A}(1)^\vee} \mathbb{F}_2$, so $H_*(\mathrm{ko} \otimes \mathrm{BP}) \cong \mathcal{A}^\vee \square_{\mathcal{A}(1)^\vee} H_*(\mathrm{BP})$ and the E_2 -term of the Adams spectral sequence

$${}^{\mathrm{Ad}} E_2 = \mathrm{Ext}_{\mathcal{A}^\vee}(\mathbb{F}_2, H_*(\mathrm{ko} \otimes \mathrm{BP})) \implies \pi_*(\mathrm{ko} \otimes \mathrm{BP})$$

can be rewritten as $\mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2, H_*(\mathrm{BP}))$. Since $H_*(\mathrm{BP}) = \mathbb{F}_2[\xi_k^2 | k \geq 1]$ is concentrated in even degrees, its $\mathcal{A}(1)^\vee$ -coaction factors as

$$\begin{aligned} H_*(\mathrm{BP}) &\xrightarrow{\nu} \Lambda(\xi_1^2) \otimes H_*(\mathrm{BP}) \subset \mathcal{A}(1)^\vee \otimes H_*(\mathrm{BP}), \\ \xi_k^2 &\mapsto \begin{cases} 1 \otimes \xi_1^2 + \xi_1^2 \otimes 1 & \text{for } k = 1, \\ 1 \otimes \xi_k^2 & \text{for } k \geq 2. \end{cases} \end{aligned}$$

To calculate the Adams E_2 -term, we use the Cartan–Eilenberg spectral sequence

$${}^{\mathrm{CE}} E_2 = \mathrm{Ext}_{\Lambda(\xi_1^2)}(\mathbb{F}_2, \mathrm{Ext}_{\Lambda(\xi_1, \xi_2)}(\mathbb{F}_2, H_*(\mathrm{BP}))) \implies \mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2, H_*(\mathrm{BP}))$$

(cf. [CE56, Theorem XVI.6.1]) for the Hopf algebra extension $\Lambda(\xi_1^2) \rightarrow \mathcal{A}(1)^\vee \rightarrow \Lambda(\xi_1, \xi_2)$. Its E_2 -term

$${}^{\mathrm{CE}} E_2 \cong \mathrm{Ext}_{\Lambda(\xi_1^2)}(\mathbb{F}_2, \mathbb{F}_2[v_0, v_1] \otimes H_*(\mathrm{BP})) = \mathbb{F}_2[v_0, v_1] \otimes \mathbb{F}_2[\xi_1^4, \xi_k^2 | k \geq 2]$$

is concentrated in filtration degree 0, so ${}^{\mathrm{CE}} E_2 = {}^{\mathrm{CE}} E_\infty$. Hence the Adams E_2 -term is concentrated in even stems, so we also have ${}^{\mathrm{Ad}} E_2 = {}^{\mathrm{Ad}} E_\infty$. Thus $\pi_*(\mathrm{ko} \otimes \mathrm{BP}) \cong \mathbb{Z}[u, t_1^2, \bar{t}_k | k \geq 2]$, with 2 detected by v_0 , u detected by v_1 , t_1^2 detected by ξ_1^4 , and $\bar{t}_k$ detected by ξ_k^2 . Base change along $\mathrm{BP} \rightarrow \mathrm{MU}$ replaces the degree $2(2^k - 1)$ generators $\bar{t}_k$ for $k \geq 2$ with degree $2k$ polynomial generators $\bar{b}_k$ for $k \geq 2$, while t_1^2 is replaced by b_1^2 .

Since MU_* and $\mathrm{ko}_* \mathrm{MU}$ are polynomial rings, the algebraic cotangent complexes $L_{\mathrm{MU}_*/\mathbb{Z}}^{\mathrm{alg}}$ and $L_{\mathrm{ko}_* \mathrm{MU}/\mathbb{Z}}^{\mathrm{alg}}$ are free modules over MU_* and $\mathrm{ko}_* \mathrm{MU}$, respectively. By the transitivity cofiber sequence

$$\mathrm{ko}_* \mathrm{MU} \otimes_{\mathrm{MU}_*} L_{\mathrm{MU}_*/\mathbb{Z}}^{\mathrm{alg}} \longrightarrow L_{\mathrm{ko}_* \mathrm{MU}/\mathbb{Z}}^{\mathrm{alg}} \longrightarrow L_{\mathrm{ko}_* \mathrm{MU}/\mathrm{MU}_*}^{\mathrm{alg}}$$

associated to the ring homomorphisms $\mathbb{Z} \rightarrow \mathrm{MU}_* \rightarrow \mathrm{ko}_* \mathrm{MU}$ (cf. [Qui70, 5.1]), it follows that $L_{\mathrm{ko}_* \mathrm{MU}/\mathrm{MU}_*}^{\mathrm{alg}}$ has 2-complete Tor-amplitude concentrated in homological degrees $[0, 1]$. By definition [HRW22, Definitions 4.1.1(2) and 4.1.6], this means that $\mathrm{MU} \rightarrow \mathrm{ko} \otimes \mathrm{MU}$ is 2-quasi-lci. $\square$

PROPOSITION 2.17. *The filtrations from Definition 2.3 for $F \in \{\mathrm{THH}, \mathrm{TC}^-, \mathrm{TP}\}$ agree with the motivic filtrations considered in [HRW22, Definition 4.2.1]. Moreover, there exist maps*

$$\mathrm{can}, \varphi: \mathrm{fil}_{\mathrm{mot}}^* \mathrm{TC}^-(\mathrm{ko}) \longrightarrow \mathrm{fil}_{\mathrm{mot}}^* \mathrm{TP}(\mathrm{ko})$$

of $\mathbb{E}_\infty$ algebras in $\mathrm{Sp}_2^{\mathrm{fil}}$, which converge to the canonical map and Frobenius map from [NS18].

Proof. By Proposition 2.14 we can apply [HRW22, Corollary 2.2.17] to $A = \mathrm{THH}(\mathrm{ko})$ and $B = \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$. In this case, $B^{\otimes_A q+1} = C^q(B/A)$ is even by Proposition 2.15, so the totalization of the double-speed Whitehead filtration agrees with the even filtration.

The map of connective $\mathbb{E}_\infty$ -rings $\mathbb{S} \rightarrow \mathrm{ko}$ is chromatically 2-quasi-lci in the sense of [HRW22, Definition 4.1.11], because by Proposition 2.16 we know that both MU and $\mathrm{ko} \otimes \mathrm{MU}$ are even, and the map $\mathrm{MU} \rightarrow \mathrm{ko} \otimes \mathrm{MU}$ is 2-quasi-lci. Hence the even filtrations also define the motivic filtrations, as per [HRW22, Definition 4.2.1], and this confirms the first statement.

The second statement then follows directly from [HRW22, Theorem 1.3.6]. $\square$

DEFINITION 2.18. Motivated by Proposition 2.17, we define $\mathbb{E}_\infty$ algebras

$$\begin{aligned} \mathrm{fil}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko}) &:= \mathrm{eq}(\mathrm{can}, \varphi: \mathrm{fil}_{\mathrm{mot}}^* \mathrm{TC}^-(\mathrm{ko}) \longrightarrow \mathrm{fil}_{\mathrm{mot}}^* \mathrm{TP}(\mathrm{ko})), \\ \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko}) &:= \mathrm{gr}^*(\mathrm{fil}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko})) \end{aligned}$$

in $\mathrm{Sp}_2^{\mathrm{fil}}$ and $\mathrm{Sp}_2^{\mathrm{gr}}$, respectively. By taking the (homotopy) equalizer of can and φ , we retain the multiplicative structure, but additively $\mathrm{fil}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko})$ is also the (homotopy) fiber of $\mathrm{can} - \varphi$. In light of [HRW22, §5] and [BMS19], we refer to $\mathrm{gr}_{\mathrm{mot}}^w \mathrm{TC}(\mathrm{ko})$ and $\pi_* \mathrm{gr}_{\mathrm{mot}}^w \mathrm{TC}(\mathrm{ko})$ as the weight w *syntomic cohomology* spectrum and syntomic cohomology groups of ko , refer to the spectral sequence

$$E_2^{n, 2w-n} = \pi_n \mathrm{gr}_{\mathrm{mot}}^w \mathrm{TC}(\mathrm{ko}) \implies \pi_n \mathrm{TC}(\mathrm{ko})$$

as the motivic spectral sequence, and follow the same grading conventions as in Definition 2.7. This spectral sequence converges, as we now review.

LEMMA 2.19. *For $F \in \{\mathrm{THH}, \mathrm{TC}^-, \mathrm{TP}, \mathrm{TC}, F(S_+^{2\ell+1}, \mathrm{THH})^\mathbb{T}, \mathrm{THH}^{hC_2}, \mathrm{THH}^{tC_2}\}$, the filtration $\mathrm{fil}_{\mathrm{mot}}^s F(\mathrm{ko})$ is (derived) complete and exhaustive. In other words,*

$$\lim \mathrm{fil}_{\mathrm{mot}}^* F(\mathrm{ko}) = 0, \quad \mathrm{colim} \mathrm{fil}_{\mathrm{mot}}^* F(\mathrm{ko}) = F(\mathrm{ko}).$$

Proof. Since limits commute with limits, $\lim \mathrm{fil}_{\mathrm{mot}}^* F(\mathrm{ko}) = 0$ for each

$$F \in \{\mathrm{THH}, \mathrm{TC}^-, \mathrm{TP}, \mathrm{TC}, F(S_+^{2\ell+1}, \mathrm{THH})^\mathbb{T}, \mathrm{THH}^{hC_2}, \mathrm{THH}^{tC_2}\}.$$

It follows from [HRW22, Theorems 1.3.5 and 1.3.6, Corollary 2.2.17] and Propositions 2.14 and 2.17 that $\mathrm{colim} \mathrm{fil}_{\mathrm{mot}}^* F(\mathrm{ko}) = F(\mathrm{ko})$ for $F \in \{\mathrm{THH}, \mathrm{TC}^-, \mathrm{TP}, \mathrm{TC}\}$. It therefore suffices to show that $\mathrm{colim} \mathrm{fil}_{\mathrm{mot}}^* F(\mathrm{ko}) = F(\mathrm{ko})$ for each $F \in \{F(S_+^{2\ell+1}, \mathrm{THH})^\mathbb{T}, \mathrm{THH}^{hC_2}, \mathrm{THH}^{tC_2}\}$. Here, we adapt the proof of [HRW22, Proposition 2.3.5]. Let

$$Y^\bullet := C^\bullet(\mathrm{THH}(\mathrm{ku}/\mathrm{MU}) / \mathrm{THH}(\mathrm{ko}))$$

and note that the canonical map

$$\mathrm{colim}_w \mathrm{fil}_{\mathrm{mot}}^w G(\mathrm{THH}(\mathrm{ko})) = \mathrm{colim}_w \mathrm{Tot} \tau_{\geq 2w} G(Y^\bullet) \xrightarrow{\sim} \mathrm{Tot} G(Y^\bullet)$$

is an equivalence for each $G \in \{F(S_+^{2\ell+1}, -)^\mathbb{T}, (-)^{hC_2}, (-)^{tC_2}\}$ since Tot preserves coconnectivity. Note also that the canonical maps

$$F(S_+^{2\ell+1}, \mathrm{THH}(\mathrm{ko}))^\mathbb{T} \xrightarrow{\sim} \mathrm{Tot}(F(S_+^{2\ell+1}, Y^\bullet)^\mathbb{T})$$

and $\mathrm{THH}(\mathrm{ko})^{hC_2} \xrightarrow{\simeq} \mathrm{Tot}((Y^\bullet)^{hC_2})$ are equivalences, since limits commute with limits. This handles the cases $F = F(S_+^{2\ell+1}, \mathrm{THH})^\mathbb{T}$ and $F = \mathrm{THH}^{hC_2}$.

To see that the canonical map $\mathrm{THH}(\mathrm{ko})^{tC_2} \xrightarrow{\simeq} \mathrm{Tot}((Y^\bullet)^{tC_2})$ is an equivalence, it now suffices to show that the canonical map

$$\mathrm{THH}(\mathrm{ko})_{hC_2} \xrightarrow{\simeq} \mathrm{Tot}((Y^\bullet)_{hC_2})$$

is an equivalence. This will follow from the fact that the fiber I of the map

$$A := \mathrm{THH}(\mathrm{ko}) \rightarrow \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$$

is 1-connective and that $(-)_{hC_2}$ preserves connectivity. In particular, by [MNN17, Proposition 2.14], the fiber of the map

$$\mathrm{THH}(\mathrm{ko})_{hC_2} \longrightarrow \mathrm{Tot}_n((Y^\bullet)_{hC_2})$$

can be identified with $(I^{\otimes_A(n+1)})_{hC_2}$, which is $(n+1)$ -connective, implying the claim after passing to the limit as n grows. Here, we use the fact that Tot_n is a finite limit and therefore commutes with $(-)_{hC_2}$. $\square$

We now introduce the relevant type 2 finite coefficient spectra. Let $V(0) = C2$ denote the cofiber of the map $2: \mathbb{S} \rightarrow \mathbb{S}$ and let $C\eta$ denote the cofiber of the Hopf map $\eta: \Sigma\mathbb{S} \rightarrow \mathbb{S}$. By [DM81, Proposition 2.1], there are precisely four equivalence classes of finite spectra $X = (V(0) \otimes C\eta)/v_1$ with the property that $H^*(X) \cong \mathcal{A}(1)$ as an $\mathcal{A}(1)$ -module, each characterized by the Sq^4 -action in its mod 2 cohomology.

NOTATION 2.20. Following [BEM17, §1], we write $A(1)[ij]$ with $i, j \in \{0, 1\}$ for the spectrum $(V(0) \otimes C\eta)/v_1$, where Sq^4 on the cohomology generator in degree 0 is i times the generator in degree 4, and Sq^4 on the cohomology generator in degree 2 is j times the generator in degree 6. We write $A(1)$ in place of $A(1)[ij]$ when making statements that hold for any choice of $i, j \in \{0, 1\}$. The identity map $A(1) \rightarrow A(1)$ has additive order 4 (see Lemma 4.6) and [BEM17, Main Theorem] proves that $A(1)$ admits a v_2^{32} self-map. Hence the $A(1)$ -homotopy $A(1)_*Y = \pi_*(A(1) \otimes Y)$ of any spectrum Y is naturally a $\mathbb{Z}/4[v_2^{32}]$ -module.

In view of Proposition 2.15 and the fact that $A(1) \otimes \mathrm{ku}$ is even, we know that $A(1) \otimes C^\bullet(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\mathrm{THH}(\mathrm{ko}))$ is even. This motivates the following definition.

DEFINITION 2.21. Let

$$\mathrm{fil}_{\mathrm{mot}}^*(A(1) \otimes \mathrm{THH}(\mathrm{ko})) := \mathrm{Tot}(\tau_{\geq 2*}(A(1) \otimes C^\bullet(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\mathrm{THH}(\mathrm{ko})))),$$

and write $\mathrm{gr}_{\mathrm{mot}}^*(A(1) \otimes \mathrm{THH}(\mathrm{ko})) := \mathrm{gr}^*(\mathrm{fil}_{\mathrm{mot}}^*(A(1) \otimes \mathrm{THH}(\mathrm{ko})))$ for the associated graded spectrum.

Remark 2.22. The spectrum $A(1) \otimes \mathrm{ko} \simeq H\mathbb{F}_2$ admits a unique $\mathbb{E}_\infty$ ko -algebra structure. Hence, for each $\mathbb{E}_\infty$ ko -algebra R , the identification

$$A(1) \otimes R = A(1) \otimes \mathrm{ko} \otimes_{\mathrm{ko}} R \simeq H\mathbb{F}_2 \otimes_{\mathrm{ko}} R$$

lets us regard $A(1) \otimes R$ as an $\mathbb{E}_\infty$ $H\mathbb{F}_2$ -algebra. For example, we will identify $\mathrm{THH}(\mathrm{ko}, H\mathbb{F}_2) := H\mathbb{F}_2 \otimes_{\mathrm{ko}} \mathrm{THH}(\mathrm{ko})$ with $A(1) \otimes \mathrm{THH}(\mathrm{ko})$, and we may therefore consider the latter as an $\mathbb{E}_\infty$ $H\mathbb{F}_2$ -algebra.

LEMMA 2.23. Let $A := A(1) \otimes \mathrm{THH}(\mathrm{ko})$ and let $B := A(1) \otimes \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$. Then there is an isomorphism

$$A_* \cong \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu]$$

of graded $\mathbb{F}_2$ -algebras, where $|\lambda'_1| = 5$, $|\lambda_2| = 7$ and $|\mu| = 8$. There is also an isomorphism

$$B_* \cong \Lambda(\hat{\xi}_1^2) \otimes \mathbb{F}_2[\mu] \otimes P$$

of A_* -algebras, where $|\hat{\xi}_1^2| = 2$, $|\mu| = 8$ and P is a polynomial algebra with generators in even degrees. The A_* -algebra structure is determined by the map $A_* \rightarrow B_*$ sending λ'_1 and λ_2 to zero and mapping μ to μ .

Proof. The complexification map $c: \mathrm{ko} \rightarrow \mathrm{ku}$ and the unique $\mathbb{E}_\infty$ -ring map $\mathrm{ku} \rightarrow H\mathbb{F}_2$ induce monomorphisms

$$H_*(\mathrm{ko}) = \mathbb{F}_2[\xi_1^4, \bar{\xi}_2^2, \bar{\xi}_k \mid k \geq 3] \longrightarrow H_*(\mathrm{ku}) = \mathbb{F}_2[\xi_1^2, \bar{\xi}_2^2, \bar{\xi}_k \mid k \geq 3] \longrightarrow \mathcal{A}^\vee$$

of $\mathcal{A}^\vee$ -comodule algebras. By Milnor's construction of the ξ_i , the composite map $\mathrm{MU} \rightarrow \mathrm{ku} \rightarrow H\mathbb{F}_2$ induces the homomorphism $H_*(\mathrm{MU}) = \mathbb{F}_2[b_i \mid i \geq 1] \rightarrow H_*(H\mathbb{F}_2) = \mathcal{A}^\vee$ given by $b_{2^i-1} \mapsto \xi_i^2$ and $b_j \mapsto 0$ for $j \neq 2^i - 1$. Hence these formulas also hold in $H_*(\mathrm{ku})$. Let $\bar{b}_i = \chi b_i$ denote the conjugate classes in $H_*(\mathrm{MU})$, so that $\bar{b}_{2^i-1} \mapsto \bar{\xi}_i^2$ and $\bar{b}_j \mapsto 0$ for $j \neq 2^i - 1$. Standard Hochschild homology computations (cf. [MS93, Proposition 2]) yield

$$\mathrm{HH}_*(H_*(\mathrm{ko})) = H_*(\mathrm{ko}) \otimes \Lambda(\sigma\xi_1^4, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_k \mid k \geq 3),$$

$$\mathrm{HH}_*(H_*(\mathrm{ku})) = H_*(\mathrm{ku}) \otimes \Lambda(\sigma\xi_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_k \mid k \geq 3).$$

The usual argument for hidden extensions in the Bökstedt spectral sequence (see, e.g., [AR05, Theorem 6.2]) implies that

$$(\sigma\bar{\xi}_3)^{2^{k-3}} = \sigma\bar{\xi}_k,$$

and produces identifications

$$H_*(\mathrm{THH}(\mathrm{ko})) \cong H_*(\mathrm{ko}) \otimes \Lambda(\sigma\xi_1^4, \sigma\bar{\xi}_2^2) \otimes \mathbb{F}_2[\sigma\bar{\xi}_3],$$

$$H_*(\mathrm{THH}(\mathrm{ku})) \cong H_*(\mathrm{ku}) \otimes \Lambda(\sigma\xi_1^2, \sigma\bar{\xi}_2^2) \otimes \mathbb{F}_2[\sigma\bar{\xi}_3].$$

As noted above, $H_*(A(1) \otimes \mathrm{ko}) \cong \mathcal{A}^\vee$. By Remark 2.22 and the evident collapsing Künneth spectral sequence, we have

$$H_*(A(1) \otimes \mathrm{ku}) \cong H_*(H\mathbb{F}_2 \otimes_{\mathrm{ko}} \mathrm{ku}) \cong \mathcal{A}^\vee \otimes_{H_*(\mathrm{ko})} H_*(\mathrm{ku}) \cong \mathcal{A}^\vee \otimes \Lambda(\hat{\xi}_1^2),$$

where $\hat{\xi}_1^2 := 1 \otimes \xi_1^2 + \xi_1^2 \otimes 1$ denotes the $\mathcal{A}^\vee$ -comodule primitive class, so that $A(1)_*(\mathrm{ku}) = \pi_*(A(1) \otimes \mathrm{ku}) = \Lambda(\hat{\xi}_1^2)$. We conclude that

$$A_* = A(1)_* \mathrm{THH}(\mathrm{ko}) = \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu],$$

$$A(1)_* \mathrm{THH}(\mathrm{ku}) = \Lambda(\hat{\xi}_1^2, \lambda_1, \lambda_2) \otimes \mathbb{F}_2[\mu],$$

with λ'_1 , λ_2 and μ in $A(1)_* \mathrm{THH}(\mathrm{ko})$ having Hurewicz images $\sigma\xi_1^4$, $\sigma\bar{\xi}_2^2 + \xi_1^2 \cdot \sigma\xi_1^4$ and $\sigma\bar{\xi}_3 + \xi_1 \cdot \sigma\bar{\xi}_2^2 + \xi_2 \cdot \sigma\xi_1^4$, respectively, while λ_1 , λ_2 and μ in $A(1)_* \mathrm{THH}(\mathrm{ku})$ have Hurewicz images $\sigma\xi_1^2$, $\sigma\bar{\xi}_2^2$ and $\sigma\bar{\xi}_3 + \xi_1 \cdot \sigma\bar{\xi}_2^2$ (cf. [AR05, Proposition 8.7]). Note that $\sigma\xi_1^4 = \sigma\xi_1^2 \cdot \xi_1^2 + \xi_1^2 \cdot \sigma\xi_1^2 = 0$ in the ku -case. Hence the induced homomorphism $A(1)_* \mathrm{THH}(\mathrm{ko}) \rightarrow A(1)_* \mathrm{THH}(\mathrm{ku})$ is given by $\lambda'_1 \mapsto 0$, $\lambda_2 \mapsto \lambda_2$ and $\mu \mapsto \mu$.

Next, we compute $A(1)_* \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$, using the fact that

$$A(1) \otimes \mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \simeq (A(1) \otimes \mathrm{THH}(\mathrm{ku})) \otimes_{\mathrm{THH}(\mathrm{MU})} \mathrm{MU}. \quad (2.3)$$

We know that

$$\pi_* \mathrm{THH}(\mathrm{MU}) \cong \mathrm{MU}_* \otimes \Lambda(\sigma\bar{b}_i \mid i \geq 1)$$

by [MS93, Remark 4.3] (cf. [Rog20, Proposition 4.5]). We expand

$$\mathrm{THH}(\mathrm{MU}) \longrightarrow \mathrm{THH}(\mathrm{ku}) \longrightarrow A(1) \otimes \mathrm{THH}(\mathrm{ku})$$

as the composite

$$\begin{aligned} \mathrm{MU} \otimes_{\mathrm{MU} \otimes \mathrm{MU}} \mathrm{MU} &\longrightarrow \mathrm{ku} \otimes_{\mathrm{ku} \otimes \mathrm{ku}} \mathrm{ku} \\ &\longrightarrow A(1) \otimes \mathrm{ku} \otimes_{A(1) \otimes \mathrm{ku} \otimes \mathrm{ku}} A(1) \otimes \mathrm{ku}, \end{aligned}$$

where

$$\begin{aligned} \mathrm{MU}_*[\bar{b}_i \mid i \geq 1] &= \pi_*(\mathrm{MU} \otimes \mathrm{MU}) \longrightarrow \pi_*(\mathrm{ku} \otimes \mathrm{ku}) \\ &\longrightarrow \pi_*(A(1) \otimes \mathrm{ku} \otimes \mathrm{ku}) = \Lambda(\hat{\xi}_1^2) \otimes H_*(\mathrm{ku}) \end{aligned}$$

takes $\bar{b}_{2^i-1}$ to $\bar{\xi}_i^2$ for $i \geq 1$ and $\bar{b}_j$ to 0 for $j \neq 2^i - 1$. Hence $\pi_* \mathrm{THH}(\mathrm{MU}) \rightarrow A(1)_* \mathrm{THH}(\mathrm{ku})$ is given by $\sigma \bar{b}_1 \mapsto \sigma \bar{\xi}_1^2 = \lambda_1$, $\sigma \bar{b}_3 \mapsto \sigma \bar{\xi}_2^2 = \lambda_2$ and $\sigma \bar{b}_i \mapsto 0$ for $i \notin \{1, 3\}$. (This uses that $\sigma \bar{b}_{2^i-1} \mapsto \sigma \bar{\xi}_i^2 = \sigma \bar{\xi}_i \cdot \bar{\xi}_i + \bar{\xi}_i \cdot \sigma \bar{\xi}_i = 0$ for $i \geq 3$, while $\bar{\xi}_1$ and $\bar{\xi}_2$ do not exist in $H_*(\mathrm{ku})$.) The $\pi_* \mathrm{THH}(\mathrm{MU})$ -algebra structure on MU_* is given by mapping x_i to x_i and mapping $\sigma \bar{b}_i$ trivially for all $i \geq 1$. The Künneth spectral sequence associated to (2.3) therefore has E^2 -term

$$\Lambda(\hat{\xi}_1^2) \otimes \mathbb{F}_2[\mu] \otimes \Gamma(\sigma^2 \bar{b}_{2^i-1} \mid i \geq 3) \otimes \Gamma(\sigma^2 \bar{b}_j \mid j \neq 2^i - 1).$$

This spectral sequence is concentrated in even total degrees and therefore collapses at the E^2 -term. We resolve the hidden multiplicative extensions using Steinberger's computation [BMM⁺86, III.2] of the Dyer–Lashof operations on $H_*(H\mathbb{F}_2) = \mathcal{A}^\vee$ and Kochman's computation [Koc73, Theorem 6] of the Dyer–Lashof operations on $H_*(\mathrm{BU}) \cong H_*(\mathrm{MU})$, as in the proof of [HW22, Lemma 2.4.1]. (Note that Steinberger's result is used for the $\sigma^2 \bar{b}_{2^i-1}$, while Kochman's theorem is used for the remaining $\sigma^2 \bar{b}_j$.) This produces the identification

$$B_* = A(1)_* \mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \cong \Lambda(\hat{\xi}_1^2) \otimes \mathbb{F}_2[\mu] \otimes P,$$

where

$$P := \mathbb{F}_2[w_i \mid i \geq 0] \otimes \mathbb{F}_2[y_{j,i} \mid j \geq 2 \text{ even}, i \geq 0]$$

is a polynomial algebra with algebra generators in even degrees. Here, w_i is any choice of lift of $\gamma_{2^i}(\sigma^2 \bar{b}_7)$ and $y_{j,i}$ is any choice of lift of $\gamma_{2^i}(\sigma^2 \bar{b}_j)$. We conclude that B_* is an even $\mathbb{E}_\infty$ -ring, and the A_* -algebra structure is determined by λ'_1 and λ_2 mapping trivially for degree reasons and μ mapping to μ by the first half of this proof. $\square$

COROLLARY 2.24. *The map $A(1) \otimes \mathrm{THH}(\mathrm{ko}) \rightarrow A(1) \otimes \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$, induced by the complexification map $c: \mathrm{ko} \rightarrow \mathrm{ku}$ and the unit map $\mathbb{S} \rightarrow \mathrm{MU}$, is evenly free.*

Proof. This can be proven directly using Lemma 2.23, but instead we simply point out that it follows from Proposition 2.14 and Remark 2.22 by base change along $\mathrm{ko} \rightarrow H\mathbb{F}_2$, using the following three pushout squares of $\mathbb{E}_\infty$ -rings.

$$\begin{array}{ccccc} \mathrm{ko} & \longrightarrow & \mathrm{THH}(\mathrm{ko}) & \longrightarrow & \mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \\ \downarrow & & \downarrow & & \downarrow \\ H\mathbb{F}_2 & \longrightarrow & A(1) \otimes \mathrm{THH}(\mathrm{ko}) & \longrightarrow & A(1) \otimes \mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \\ & & \downarrow & & \downarrow \\ & & C & \longrightarrow & \mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \otimes_{\mathrm{THH}(\mathrm{ko})} C \end{array} \quad \square$$

CONVENTION 2.25. To be consistent with our implicit 2-completion, we write $\mathrm{fil}_{\mathrm{ev}}^*$ for the functor denoted $\mathrm{fil}_{\mathrm{ev},2}^*$ in [HRW22, Construction 2.1.11].

Remark 2.26. By [HRW22, Corollary 2.2.17], Remark 2.22 and Corollary 2.24, we can identify

$$\mathrm{fil}_{\mathrm{mot}}^*(A(1) \otimes \mathrm{THH}(\mathrm{ko})) \simeq \mathrm{fil}_{\mathrm{ev}}^*(A(1) \otimes \mathrm{THH}(\mathrm{ko})),$$

in the sense of Definition 2.21 and [HRW22, Construction 2.1.11].

THEOREM 2.27. *There is an isomorphism*

$$\pi_* \mathrm{gr}_{\mathrm{mot}}^*(A(1) \otimes \mathrm{THH}(\mathrm{ko})) \cong \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu]$$

of bigraded $\mathbb{F}_2$ -algebras, with $\|\lambda'_1\| = (5, 1)$, $\|\lambda_2\| = (7, 1)$ and $\|\mu\| = (8, 0)$.

Proof. We closely follow [HW22] and [HRW22]. Starting with the proof of [HW22, Proposition 6.1.6], let $A := A(1) \otimes \mathrm{THH}(\mathrm{ko})$ and $B := A(1) \otimes \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$, so that $A_* \cong \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu]$ and $B_* \cong \Lambda(\hat{\xi}_1^2) \otimes \mathbb{F}_2[\mu] \otimes P$ by Lemma 2.23. The descent spectral sequence associated to the cosimplicial Amitsur resolution $C^\bullet(B/A) = B^{\otimes_A \bullet + 1}$ for $A \rightarrow B$ has E_1 -term

$$E_1^q(B/A) = \pi_*(B^{\otimes_A q+1})$$

for $q \geq 0$, and converges to A_* . Since B_* is concentrated in even stems, Corollary 2.24 implies that $\Sigma := \pi_*(B \otimes_A B)$ is even and free over B_* , so that (B_*, Σ) is a flat Hopf algebroid. Let $C_\Sigma^*(B_*, B_*)$ denote the associated cobar complex. It follows by induction on q that the natural homomorphism

$$\begin{aligned} C_\Sigma^q(B_*, B_*) &= \Sigma \otimes_{B_*} \cdots \otimes_{B_*} \Sigma \xrightarrow{\cong} \pi_*((B \otimes_A B) \otimes_B \cdots \otimes_B (B \otimes_A B)) \\ &\cong \pi_*(B \otimes_A \cdots \otimes_A B) = E_1^q(B/A) \end{aligned}$$

is an isomorphism for each $q \geq 0$, since the relevant Künneth spectral sequences collapse. Passing to cohomology, we obtain an isomorphism

$$\mathrm{Ext}_\Sigma^*(B_*, B_*) \cong E_2^*(B/A),$$

identifying the descent spectral sequence E_2 -term with the Hopf algebroid cohomology of (B_*, Σ) . We claim that in each stem this E_2 -term has the same finite order as A_* , so that the descent spectral sequence for $A \rightarrow B$ must collapse at $E_2 = E_\infty$.

By convergence, the descent E_2 -term is an upper bound for A_* . To show that the bound is exact, we consider the multiplicative Whitehead filtrations $\tau_{\geq *A}$ and $\tau_{\geq *B}$ of A and B , respectively. For each $q \geq 0$, we equip $B^{\otimes_A q+1}$ with the relative convolution filtration

$$\mathrm{fil}^* B^{\otimes_A q+1} = (\tau_{\geq *B})^{\otimes_{(\tau_{\geq *A})} q+1},$$

having associated graded $\mathbb{E}_\infty$ -ring

$$\mathrm{gr}^* B^{\otimes_A q+1} = (H\pi_* B)^{\otimes_{(H\pi_* A)} q+1}.$$

Here, $H\pi_* A$ and $H\pi_* B$ are to be interpreted as the graded $\mathbb{E}_\infty$ -rings $\mathrm{gr}^*(\tau_{\geq *A})$ and $\mathrm{gr}^*(\tau_{\geq *B})$, respectively. We have proved in Lemma 2.23 that $A_* \rightarrow B_*$ is given by $\lambda'_1 \mapsto 0$, $\lambda_2 \mapsto 0$ and $\mu \mapsto \mu$, so that

$$\bar{\Sigma} := \pi_*(H\pi_* B \otimes_{H\pi_* A} H\pi_* B) \cong \Gamma(\sigma\lambda'_1, \sigma\lambda_2) \otimes \Lambda(\hat{\xi}_1^2) \otimes P \otimes \Lambda(\hat{\xi}_1^2) \otimes P \otimes \mathbb{F}_2[\mu]$$

is even and free over B_* . Hence $(B_*, \bar{\Sigma})$ is a flat Hopf algebroid, and as above we have compatible isomorphisms

$$C_\Sigma^q(B_*, B_*) = \bar{\Sigma} \otimes_{B_*} \cdots \otimes_{B_*} \bar{\Sigma} \xrightarrow{\cong} (H\pi_* B)^{\otimes_{(H\pi_* A)} q+1}$$

for all $q \geq 0$. Since these bigraded groups are concentrated in even stems, and differentials reduce the stem by one, the convolution filtration spectral sequence

$$\pi_*((H\pi_* B)^{\otimes_{(H\pi_* A)} q+1}) \implies \pi_*(B^{\otimes_A q+1})$$

collapses at this term. This proves that $\pi_*(B^{\otimes_A q+1}) = E_1^q(B/A)$ has a descending filtration with associated graded given by $C_{\overline{\Sigma}}^q(B_*, B_*)$. These filtrations are compatible for varying $q \geq 0$, so the descent E_1 -term is a filtered differential graded algebra with associated graded $E_1 = C_{\overline{\Sigma}}^*(B_*, B_*)$. Passing to cohomology, we obtain the May–Ravenel spectral sequence

$$E_2 = \text{Ext}_{\overline{\Sigma}}(B_*, B_*) \implies \text{Ext}_{\Sigma}(B_*, B_*)$$

converging to the descent E_2 -term (cf. [Rav86, Theorem A1.3.9]).

We now view the Hopf algebroid $(B_*, \overline{\Sigma})$ as the tensor product of the three Hopf algebroids

$$(\mathbb{F}_2, \Gamma(\sigma\lambda'_1, \sigma\lambda_2)), \quad (\Lambda(\hat{\xi}_1^2) \otimes P, \Lambda(\hat{\xi}_1^2) \otimes P \otimes \Lambda(\hat{\xi}_1^2) \otimes P), \quad (\mathbb{F}_2[\mu], \mathbb{F}_2[\mu]).$$

These have cohomology algebras $\Lambda(\lambda'_1, \lambda_2)$, $\mathbb{F}_2$ and $\mathbb{F}_2[\mu]$, respectively, with $\lambda'_1, \lambda_2 \in \text{Ext}^1$ and $\mu \in \text{Ext}^0$. This confirms that the May–Ravenel E_2 -term

$$\text{Ext}_{\overline{\Sigma}}(B_*, B_*) \cong \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu]$$

has the same finite order in each stem as A_* , which implies that the May–Ravenel spectral sequence and the descent spectral sequence both collapse at their E_2 -terms. Moreover, there is no room for hidden multiplicative extensions, since λ'_1 and λ_2 both square to zero in A_* .

We have now established that the descent spectral sequence

$$E_1^q(B/A) = \pi_*(B^{\otimes_A q+1}) \implies A_*$$

is concentrated in even internal degrees $n + q = 2w$, having E_2 -term

$$E_2(B/A) \cong \text{Ext}_{\Sigma}(B_*, B_*) \cong \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu]$$

with (n, q) -bidegrees $\|\lambda'_1\| = (5, 1)$, $\|\lambda_2\| = (7, 1)$ and $\|\mu\| = (8, 0)$. Following [HRW22, Example 4.2.3], we apply [HRW22, Corollary 2.2.17(1)] to the evenly free map $A \rightarrow B$ to see that

$$\text{fil}_{\text{ev}}^* A \xrightarrow{\simeq} \text{Tot}(\text{fil}_{\text{ev}}^* B^{\otimes_A \bullet+1}) = \text{Tot}(\tau_{\geq 2*}(B^{\otimes_A \bullet+1}))$$

is an equivalence. For each integer weight w , there is a spectral sequence converging to

$$\pi_* \text{Tot}(\tau_{\geq 2w}(B^{\otimes_A \bullet+1})),$$

with the E_1 -term given by the part of the descent spectral sequence $E_1(B/A)$ that is located in internal degrees $n + q \geq 2w$. The d_1 -differential preserves this part, so the E_2 -term for weight w is given by the part of $E_2(B/A)$ in the same range of internal degrees. By naturality, this spectral sequence must collapse at the E_2 -term, since the full descent spectral sequence does so. It follows that

$$\pi_* \text{fil}_{\text{ev}}^w A \longrightarrow A_*$$

maps the source isomorphically to the subgroup of classes in internal degree $\geq 2w$, and $\pi_* \text{gr}_{\text{ev}}^w A$ is isomorphic to the summand in A_* consisting of classes in internal degree $= 2w$. Hence

$$\pi_* \text{gr}_{\text{ev}}^* A \cong \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu]$$

as bigraded algebras, with $(n, 2w - n)$ -bidegrees $\|\lambda'_1\| = (5, 1)$, $\|\lambda_2\| = (7, 1)$ and $\|\mu\| = (8, 0)$. $\square$

By [HRW22, Corollary 2.2.21(1)], there are equivalences

$$\text{fil}_{\text{ev}}^* \mathbb{S} \simeq \text{Tot}(\tau_{\geq 2*} C^{\bullet}(\text{MU}/\mathbb{S})), \quad \text{gr}_{\text{ev}}^* \mathbb{S} \simeq \text{Tot} \Sigma^{2*} H\pi_{2*} C^{\bullet}(\text{MU}/\mathbb{S}).$$

The even filtration is extended to module spectra in [HRW22, Appendix A]. As such the even filtration is lax symmetric monoidal [HRW22, Remark A.1.6] and the associated graded functor is symmetric monoidal, so $\text{gr}_{\text{ev}}^* \mathbb{S}$ is an $\mathbb{E}_{\infty}$ algebra in graded 2-complete spectra, and $\text{gr}_{\text{ev}/\mathbb{S}}^*$ is a lax symmetric monoidal functor from 2-complete spectra to $\text{gr}_{\text{ev}}^* \mathbb{S}$ -modules. Note that when

R is an $\mathbb{E}_\infty$ -ring and MU_*R is concentrated in even degrees, then $\mathrm{fil}_{\mathrm{ev}}^* R$ agrees with $\mathrm{fil}_{\mathrm{ev}/\mathbb{S}}^* R$ by [HRW22, Corollary 1.1.6 and Remark 1.1.7].

CONVENTION 2.28. We will simply write $\otimes$ for $\otimes_{\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}}$ when it is clear from the context that we are in the category of modules over $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$.

CONSTRUCTION 2.29. For finite spectra V with $\mathrm{MU}_*(V)$ concentrated in even degrees, we shall write $\bar{V}$ for the $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -module $\mathrm{gr}_{\mathrm{ev}/\mathbb{S}}^* V \simeq \mathrm{Tot} \Sigma^{2*} H\pi_{2*}(C^\bullet(\mathrm{MU}/\mathbb{S}) \otimes V)$. In particular, this defines the modules $\bar{V}(0)$, $\bar{C}\eta$ and $\bar{A}(1)$.

By [GIK⁺22], we can identify $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ with $C\tau$ in the $\mathbb{C}$ -motivic homotopy category $\mathrm{SH}(\mathbb{C})$ (cf. [HRW22, Recollection 6.0.1]). Consequently, the $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -module $\bar{V}$ corresponds to the even $\mathrm{MU}_*\mathrm{MU}$ -comodule $\mathrm{MU}_*(V)$ under the equivalence of [GWX21, Theorem 1.13(2)]. Moreover, by [GWX21, Remark 4.15], this equivalence is symmetric monoidal.

Hence, for each $k \geq 0$, there is an essentially unique $\mathbb{E}_\infty$ $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -algebra $\bar{V}(k)$ that corresponds to the commutative $\mathrm{MU}_*\mathrm{MU}$ -comodule algebra $\mathrm{MU}_*/(2, v_1, \dots, v_k)$. For $k=0$ this gives the previously defined $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -module $\bar{V}(0)$ an $\mathbb{E}_\infty$ algebra structure, even though $V(0) = C2$ is not a ring spectrum, and for $k \geq 1$ it defines the $\mathbb{E}_\infty$ algebras $\bar{V}(k)$, in spite of $V(k)$ not existing as a spectrum. Furthermore, the isomorphism

$$\mathrm{MU}_*(A(1)) \cong \mathrm{MU}_*/(2, v_1) \otimes_{\mathrm{MU}_*} \mathrm{MU}_*(C\eta) \quad (2.4)$$

of $\mathrm{MU}_*\mathrm{MU}$ -comodules exhibits $\mathrm{MU}_*(A(1))$ as an $\mathrm{MU}_*/(2, v_1)$ -module in that category. It follows that we have an equivalence $\bar{A}(1) \simeq \bar{V}(1) \otimes \bar{C}\eta$, exhibiting $\bar{A}(1)$ as a $\bar{V}(1)$ -module in the category of $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -modules. Here, we use the fact that $\eta \in \pi_1 \mathbb{S}$ is detected in Novikov filtration 1, so the cofiber of $\eta: \Sigma^{1,1} \mathrm{gr}_{\mathrm{ev}}^* \mathbb{S} \rightarrow \mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ is equivalent to $\bar{C}\eta$ as defined above. Hence we have a cofiber sequence

$$\Sigma^{1,1} \bar{V}(1) \xrightarrow{\eta} \bar{V}(1) \xrightarrow{i} \bar{A}(1) \xrightarrow{j} \Sigma^{2,0} \bar{V}(1) \quad (2.5)$$

of $\bar{V}(1)$ -modules, mapping to the cofiber sequence

$$\Sigma^{1,1} \bar{V}(2) \xrightarrow{\eta} \bar{V}(2) \xrightarrow{i} \bar{V}(2) \otimes \bar{C}\eta \xrightarrow{j} \Sigma^{2,0} \bar{V}(2) \quad (2.6)$$

of $\bar{V}(2)$ -modules.

When $\bar{V}$ and M^* are $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -modules, we shall write

$$\bar{V}_* M^* := \pi_*(\bar{V} \otimes M^*)$$

for the homotopy groups of the graded spectrum $\bar{V} \otimes M^*$, keeping in mind that this is a bigraded abelian group. Note that we are applying Convention 2.28 throughout this construction.

LEMMA 2.30. *The $\mathrm{MU}_*\mathrm{MU}$ -comodules $\mathrm{MU}_*(C\eta)$ and $\mathrm{MU}_*(A(1))$ do not admit the structure of $\mathrm{MU}_*\mathrm{MU}$ -comodule algebras.*

Proof. In view of (2.4), it suffices to prove this for $\mathrm{MU}_*(A(1))$. Writing $\mathrm{MU}_*(C\eta) = \mathrm{MU}_*\{1, b_1\}$, so that $\mathrm{MU}_*(A(1)) \cong \mathrm{MU}_*/(2, v_1)\{1, b_1\}$, the coaction

$$\nu: \mathrm{MU}_*(A(1)) \longrightarrow \mathrm{MU}_*\mathrm{MU} \otimes_{\mathrm{MU}_*} \mathrm{MU}_*(A(1))$$

satisfies $\nu(b_1) = b_1 \otimes 1 + 1 \otimes b_1$ and $\nu(x_2) = b_2 \otimes 1 + 1 \otimes x_2$ (see, e.g., [Rog20, (2.1)]). If $\mathrm{MU}_*(A(1))$ were an $\mathrm{MU}_*\mathrm{MU}$ -comodule algebra, we would have $\nu(b_1^2) = (b_1 \otimes 1 + 1 \otimes b_1)^2 = b_1^2 \otimes 1 + 1 \otimes b_1^2$. This contradicts each of the possibilities $b_1^2 = 0$ or $b_1^2 = x_2$, and there are no other alternatives. $\square$

LEMMA 2.31. *There is an equivalence*

$$\overline{A}(1) \otimes \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko}) \simeq \mathrm{gr}_{\mathrm{mot}}^*(A(1) \otimes \mathrm{THH}(\mathrm{ko}))$$

of $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -modules.

Proof. Let $v_1: \Sigma^2 V(0) \otimes C\eta \rightarrow V(0) \otimes C\eta$ be one of the (eight) v_1 -maps with cofiber one of the four spectra $A(1)$. Multiplication by 2 acts injectively on $\mathrm{MU}_*(\mathbb{S})$ and on $\mathrm{MU}_*(\mathrm{ku}) \cong \mathrm{ku}_*[b_k \mid k \geq 1]$, so the natural map $\overline{V}(0) \otimes \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku} \rightarrow \mathrm{gr}_{\mathrm{ev}/\mathbb{S}}^*(V(0) \otimes \mathrm{ku})$ is an equivalence by [GIK⁺22, Proposition 3.18]. Since $\mathrm{MU}_*(C\eta)$ is a free MU_* -module on two generators, the maps $\overline{V}(0) \otimes \overline{C}\eta \rightarrow \mathrm{gr}_{\mathrm{ev}/\mathbb{S}}^*(V(0) \otimes C\eta)$ and $\overline{C}\eta \otimes \mathrm{gr}_{\mathrm{ev}/\mathbb{S}}^*(V(0) \otimes \mathrm{ku}) \rightarrow \mathrm{gr}_{\mathrm{ev}/\mathbb{S}}^*(V(0) \otimes C\eta \otimes \mathrm{ku})$ are equivalences by [Pst23, Lemma 4.24]. Thus the lower horizontal arrow in the following commutative square is also an equivalence.

$$\begin{array}{ccc} \overline{V}(0) \otimes \overline{C}\eta \otimes \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku} & \xrightarrow{\simeq} & \overline{C}\eta \otimes \mathrm{gr}_{\mathrm{ev}/\mathbb{S}}^*(V(0) \otimes \mathrm{ku}) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{gr}_{\mathrm{ev}/\mathbb{S}}^*(V(0) \otimes C\eta) \otimes \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku} & \xrightarrow{\simeq} & \mathrm{gr}_{\mathrm{ev}/\mathbb{S}}^*(V(0) \otimes C\eta \otimes \mathrm{ku}) \end{array}$$

Since v_1 acts injectively on $\mathrm{MU}_*(V(0) \otimes C\eta)$ and on $\mathrm{MU}_*(V(0) \otimes C\eta \otimes \mathrm{ku})$ it follows, by [GIK⁺22, Proposition 3.18] again, that the columns in the following diagram are cofiber sequences.

$$\begin{array}{ccc} \mathrm{gr}_{\mathrm{ev}/\mathbb{S}}^*(\Sigma^2 V(0) \otimes C\eta) \otimes \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku} & \xrightarrow{\simeq} & \mathrm{gr}_{\mathrm{ev}/\mathbb{S}}^*(\Sigma^2 V(0) \otimes C\eta \otimes \mathrm{ku}) \\ \downarrow v_1 & & \downarrow v_1 \\ \mathrm{gr}_{\mathrm{ev}/\mathbb{S}}^*(V(0) \otimes C\eta) \otimes \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku} & \xrightarrow{\simeq} & \mathrm{gr}_{\mathrm{ev}/\mathbb{S}}^*(V(0) \otimes C\eta \otimes \mathrm{ku}) \\ \downarrow & & \downarrow \\ \overline{A}(1) \otimes \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku} & \xrightarrow{\simeq} & \mathrm{gr}_{\mathrm{ev}}^*(A(1) \otimes \mathrm{ku}) \end{array}$$

Hence the lower horizontal arrow is also an equivalence.

Using Proposition 2.15, it follows that there are equivalences

$$\overline{A}(1) \otimes \mathrm{gr}_{\mathrm{ev}}^* C^q(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\mathrm{THH}(\mathrm{ko})) \xrightarrow{\simeq} \mathrm{gr}_{\mathrm{ev}}^*(A(1) \otimes C^q(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\mathrm{THH}(\mathrm{ko})))$$

for all $q \geq 0$, compatible with the cosimplicial structure maps. Passing to totalizations, and using that $\overline{A}(1)$ is a finite $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -module, we obtain

$$\begin{aligned} & \overline{A}(1) \otimes \mathrm{Tot}(\mathrm{gr}_{\mathrm{ev}}^* C^\bullet(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\mathrm{THH}(\mathrm{ko}))) \\ & \simeq \mathrm{Tot}(\overline{A}(1) \otimes \mathrm{gr}_{\mathrm{ev}}^* C^\bullet(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\mathrm{THH}(\mathrm{ko}))) \\ & \simeq \mathrm{Tot} \mathrm{gr}_{\mathrm{ev}}^*(A(1) \otimes C^\bullet(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\mathrm{THH}(\mathrm{ko}))). \end{aligned}$$

In view of Definitions 2.3 and 2.21, this establishes the asserted equivalence. $\square$

Remark 2.32. A consequence of Lemma 2.30 is that $\overline{C}\eta$ and $\overline{A}(1)$ are not $\mathbb{E}_1$ algebras in $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -modules. However, by Lemma 2.31, there is an identification of $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -modules

$$\overline{A}(1) \otimes \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko}) \simeq \mathrm{gr}_{\mathrm{mot}}^*(A(1) \otimes \mathrm{THH}(\mathrm{ko})),$$

where the right-hand side is an $\mathbb{E}_\infty \mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -algebra. We therefore use this to equip the left-hand side with an $\mathbb{E}_\infty \mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -algebra structure. Note that the left-hand side also has a canonical action of the circle $\mathbb{T}$, but this $\mathbb{T}$ -action is not an action through $\mathbb{E}_\infty$ -ring maps, because the right-hand

side is not equipped with a compatible $\mathbb{T}$ -action. See Remark 5.7 for an algebraic incarnation of this.

LEMMA 2.33. *Let M^* be a $\mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}$ -module. Then $\overline{C}\eta \otimes M^* \simeq \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku} \otimes_{\mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}} M^*$ and there is a natural, trigraded, η -Bockstein spectral sequence*

$$E_1 = (\overline{C}\eta_* M^*)[\eta] \implies \pi_*(M^*).$$

If M^* is uniformly bounded below, then this spectral sequence is conditionally convergent. If M^* is a $\mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}$ -algebra, then this is an algebra spectral sequence.

Proof. The Wood cofiber sequence (2.1) induces a cofiber sequence

$$\Sigma^{1,1} \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko} \xrightarrow{\eta} \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko} \xrightarrow{c} \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku} \xrightarrow{R} \Sigma^{2,0} \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}$$

in the category of $\mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}$ -modules, where c is a map of $\mathbb{E}_\infty$ algebras. This follows as in [GIK⁺22, Proposition 3.18], since $\mathrm{MU}_*(\mathrm{ko})$ is concentrated in even degrees by Proposition 2.16. Hence $\overline{C}\eta \otimes M^*$ and $\mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku} \otimes_{\mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}} M^*$ are both the cofiber of $\eta: \Sigma^{1,1} M^* \rightarrow M^*$.

The Bousfield–Kan homotopy (= descent) spectral sequence for the cosimplicial $\mathbb{E}_\infty \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}$ -algebra

$$C^\bullet(\mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku} / \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko})$$

is well known to be multiplicative, and converges conditionally (and strongly) to $\pi_* \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}$. The normalized Tot-filtration is the same as the η -adic tower

$$\dots \xrightarrow{\eta} \Sigma^{2,2} \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko} \xrightarrow{\eta} \Sigma^{1,1} \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko} \xrightarrow{\eta} \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko},$$

so we can equally well call this the η -Bockstein spectral sequence. In particular, its E_1 -term is

$$E_1^q = \Sigma^{q,q} \pi_* \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku} \cong (\pi_* \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku})\{\eta^q\}$$

for each $q \geq 0$, and it is concentrated in even internal degrees (= integer weights).

Tensoring over $\mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}$ with M^* , the Bousfield–Kan spectral sequence for the cosimplicial $\mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}$ -module

$$C^\bullet(\mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku} / \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}) \otimes_{\mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}} M^*$$

has abutment $\pi_* M^*$, and is multiplicative if M^* is a $\mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}$ -algebra. The normalized Tot-filtration is the same as the η -adic tower

$$\dots \xrightarrow{\eta} \Sigma^{2,2} M^* \xrightarrow{\eta} \Sigma^{1,1} M^* \xrightarrow{\eta} M^*,$$

and the E_1 -term is

$$E_1^* \cong \pi_*(\overline{C}\eta \otimes M^*)[\eta].$$

If M^* is uniformly bounded below, then its η -adic tower has trivial (homotopy) limit, which ensures conditional convergence. $\square$

Example 2.34. The η -Bockstein spectral sequence

$$E_1 = (\pi_* \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku})[\eta] \implies \pi_* \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}$$

has $E_1 = \mathbb{Z}[\eta, u]$, $d_1(u) = 2\eta$ and $E_2 = E_\infty = \mathbb{Z}[\eta, u^2]/(2\eta) \cong \pi_* \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}$. The motivic (= Novikov) spectral sequence

$$E_2 = \pi_* \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko} \implies \pi_* \mathrm{ko}$$

has $E_2 = \mathbb{Z}[\eta, u^2]/(2\eta)$ with $\|\eta\| = (1, 1)$, $\|u^2\| = (4, 0)$, $d_3(u^2) = \eta^3$ and $E_4 = E_\infty = (\mathbb{Z}\{1, 2u^2\} \oplus \mathbb{Z}/2\{\eta, \eta^2\}) \otimes \mathbb{Z}[u^4]$. Here, $A \in \pi_4(\mathrm{ko})$ and $B \in \pi_8(\mathrm{ko})$ are detected by $2u^2$ and u^4 , respectively.

The bigraded homotopy rings of the $\mathbb{E}_\infty$ $\mathrm{gr}_{\mathrm{ev}}^*$ ko -algebras

$$\begin{array}{ccc} \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko} & \xrightarrow{i} & \overline{C}\eta \otimes \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko} \simeq \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku} \\ \downarrow i_0 & & \downarrow i_0 \\ \overline{V}(0) \otimes \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko} & \xrightarrow{i} & \overline{V}(0) \otimes \overline{C}\eta \otimes \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko} \simeq \overline{V}(0) \otimes \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku} \\ \downarrow i_1 & & \downarrow i_1 \\ \overline{V}(1) \otimes \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko} & \xrightarrow{i} & \overline{A}(1) \otimes \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko} \simeq \overline{V}(1) \otimes \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ku} \end{array}$$

are thus as follows.

$$\begin{array}{ccc} \mathbb{Z}[\eta, u^2]/(2\eta) & \longrightarrow & \mathbb{Z}[u] \\ \downarrow & & \downarrow \\ \mathbb{Z}/2[\eta, u] & \longrightarrow & \mathbb{Z}/2[u] \\ \downarrow & & \downarrow \\ \mathbb{Z}/2[\eta] & \longrightarrow & \mathbb{Z}/2 \end{array}$$

DEFINITION 2.35. Let $v_2: \Sigma^{6,0}\overline{V}(1) \rightarrow \overline{V}(1)$ be the $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -module map corresponding to the $\mathrm{MU}_* \mathrm{MU}$ -comodule homomorphism $v_2: \Sigma^6 \mathrm{MU}_*/(2, v_1) \rightarrow \mathrm{MU}_*/(2, v_1)$, so that there is a cofiber sequence

$$\Sigma^{6,0}\overline{V}(1) \xrightarrow{v_2} \overline{V}(1) \xrightarrow{i_2} \overline{V}(2) \xrightarrow{j_2} \Sigma^{7,-1}\overline{V}(1)$$

of $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -modules. The induced map

$$v_2: \Sigma^{6,0}\overline{V}(1) \otimes \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko} \longrightarrow \overline{V}(1) \otimes \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}$$

of $\mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}$ -modules is null-homotopic, and there is a unique class

$$\varepsilon_2 \in \overline{V}(2)_* \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}$$

in bidegree $\|\varepsilon_2\| = (7, -1)$ with $j_2(\varepsilon_2) = \Sigma^{7,-1}1$. We have $\varepsilon_2^2 = 0$, since the group in bidegree $(14, -2)$ is trivial. Then $\overline{V}(2)_* \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko} \cong \Lambda(\varepsilon_2) \otimes \overline{V}(1)_* \mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}$, and in general we have a natural algebra isomorphism

$$\Lambda(\varepsilon_2) \otimes \overline{V}(1)_* M^* \cong \overline{V}(2)_* M^*$$

for any $\mathrm{gr}_{\mathrm{ev}}^* \mathrm{ko}$ -algebra M^* .

COROLLARY 2.36. *There are preferred isomorphisms of bigraded $\mathbb{F}_2$ -algebras*

$$\overline{A}(1)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko}) \cong \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu]$$

and

$$(\overline{V}(2) \otimes \overline{C}\eta)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko}) \cong \Lambda(\varepsilon_2, \lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu].$$

Proof. The first isomorphism is a direct consequence of Theorem 2.27, Lemma 2.31 and Remark 2.32. The second isomorphism arises as in Definition 2.35 with $M^* = \overline{C}\eta \otimes \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})$. $\square$

In the following proposition, we evaluate the spectral sequence from Lemma 2.33 in the case when $M^* = \overline{V}(1) \otimes \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})$ and $\overline{C}\eta \otimes M^* \cong \overline{A}(1) \otimes \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})$: for an illustration, see

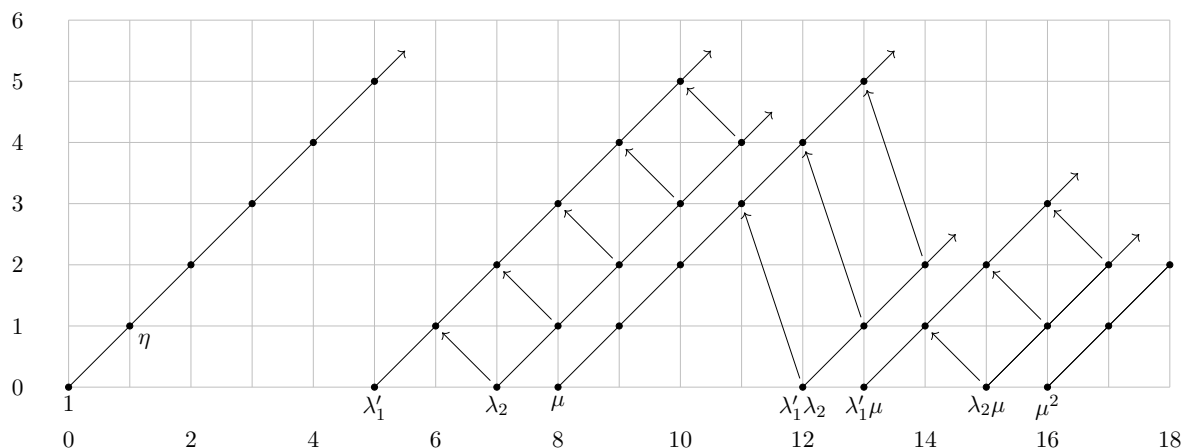

 FIGURE 2. The η -Bockstein $E_1 \Rightarrow \bar{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\mathbf{ko})$.

Figure 2. The authors thank an anonymous referee for pointing out how to remove an indeterminacy in the following statement, which we had previously only resolved later in the paper, in a more indirect manner.

PROPOSITION 2.37. *The η -Bockstein spectral sequence*

$$E_1 = \bar{A}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\mathbf{ko}) [\eta] \Longrightarrow \bar{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\mathbf{ko})$$

has differentials

$$\begin{aligned} d_1(\lambda_2) &= \eta \lambda'_1, \\ d_3(\lambda'_1 \lambda_2) &= \eta^3 \mu \end{aligned}$$

and no further differentials besides those generated by the Leibniz rule. Moreover, there is no room for η -extensions. Consequently, we can identify

$$\bar{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\mathbf{ko}) \cong \frac{\mathbb{F}_2[\eta, \lambda'_1, \mu]}{(\eta \lambda'_1, (\lambda'_1)^2 = \eta^2 \mu)}$$

as a bigraded $\mathbb{F}_2$ -algebra, where $\|\eta\| = (1, 1)$, $\|\mu\| = (8, 0)$ and $\|\lambda'_1\| = (5, 1)$.

Proof. We deduce these differentials using a small part of the known (implicitly 2-complete) computation of $\pi_* \operatorname{THH}(\mathbf{ko})$ from [AHL10, § 7]. The unit $\mathbf{ko} \rightarrow \operatorname{THH}(\mathbf{ko})$ and augmentation $\epsilon: \operatorname{THH}(\mathbf{ko}) \rightarrow \mathbf{ko}$ exhibit $\mathbf{ko}$ as a retract of $\operatorname{THH}(\mathbf{ko})$ in the category of $\mathbb{E}_\infty$ -rings. We write $\operatorname{THH}(\mathbf{ko})/\mathbf{ko}$ for the complementary summand in $\mathbf{ko}$ -modules. In degrees $* < 12$ we have

$$H_*(\mathbf{ko})\{\sigma \bar{\xi}_1^4, \sigma \bar{\xi}_2^2, \sigma \bar{\xi}_3\} \cong H_*(\operatorname{THH}(\mathbf{ko})/\mathbf{ko}),$$

so there is an 11-connected map $\Sigma^5 \mathbf{ksp} \simeq \mathbf{ko} \otimes (S^5 \cup_\eta e^7 \cup_2 e^8) \rightarrow \operatorname{THH}(\mathbf{ko})/\mathbf{ko}$. By [AHL10, Corollary 7.3, Figure 5], the η^2 -multiple in $\pi_6 \mathbf{ksp} \cong \pi_{11} \Sigma^5 \mathbf{ksp}$ maps to zero in THH , so

$$\pi_*(\operatorname{THH}(\mathbf{ko})/\mathbf{ko}) \cong (\mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}/2, 0)$$

for $5 \leq * \leq 11$.

We consider the η -Bockstein spectral sequence

$$E_1 = \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\eta, \mu] \Longrightarrow \bar{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\mathbf{ko})$$

with $\|\lambda'_1\| = (5, 1)$, $\|\lambda_2\| = (7, 1)$, $\|\mu\| = (8, 0)$ and $\|\eta\| = (1, 1)$, the v_1 -Bockstein spectral sequence

$$E_1 = \bar{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\mathbf{ko}) [v_1] \rightarrow \bar{V}(0)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\mathbf{ko})$$

with $\|v_1\| = (2, 0)$, the v_0 -Bockstein spectral sequence

$$E_1 = \overline{V}(0)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})[v_0] \implies \pi_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})$$

with $\|v_0\| = (0, 0)$, and the motivic spectral sequence

$$E_2 = \pi_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko}) \implies \pi_* \operatorname{THH}(\operatorname{ko}).$$

In each case the spectral sequence for ko splits off as a direct summand. Taking this into account, there is no possible source or target for a differential affecting λ'_1 in any of these spectral sequences. Hence λ'_1 survives in bidegree $(5, 1)$ to detect the generator of $\pi_5(\operatorname{THH}(\operatorname{ko})/\operatorname{ko}) \cong \mathbb{Z}$. Since $\pi_6(\operatorname{THH}(\operatorname{ko})/\operatorname{ko}) = 0$, it follows that $\eta\lambda'_1$ in bidegree $(6, 2)$ is an infinite cycle that detects zero, i.e., a boundary in one of these spectral sequences. Since $\eta\lambda'_1$ is not a v_1 - or v_0 -multiple, it cannot be a v_1 -Bockstein or v_0 -Bockstein boundary. Since the motivic E_2 -term is readily seen to be zero in bidegree $(7, 0)$, it can also not be a motivic boundary. Hence $d_1(\lambda_2) = \eta\lambda'_1$ in the η -Bockstein spectral sequence is the only remaining possibility.

By construction, this d_1 -differential can be rewritten as $i(j(\lambda_2)) = \Sigma^{2,0}\lambda'_1$, with i and j induced by the maps in (2.5). Here,

$$i: \overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko}) \longrightarrow \overline{A}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})$$

must be injective (in fact, bijective) in bidegree $(5, 1)$, since there are no nonzero η -multiples there, so $\Sigma^{-2,0}j(\lambda_2)$ in its source is the unique lift over i of λ'_1 , which we also denote by λ'_1 . We shall return to this insight at the end of the proof.

There is no room for other η -Bockstein d_1 -differentials, so the next differential to be determined is $d_3(\lambda'_1\lambda_2) \in \mathbb{F}_2\{\eta^3\mu\}$. On one hand, if $d_3(\lambda'_1\lambda_2) = \eta^3\mu$, then the η -Bockstein E_∞ -term (modulo the summand for ko) will be

$$\mathbb{F}_2\{\lambda'_1, \mu, \eta\mu, \eta^2\mu\}$$

in stems ≤ 12 . On the other hand, if $d_3(\lambda'_1\lambda_2) = 0$ then it will be

$$\mathbb{F}_2\{\lambda'_1, \mu, \eta\mu, \eta^2\mu, \eta^3\mu\}$$

in stems ≤ 11 , with the 12-stem concentrated in motivic filtrations ≥ 2 . In either case this determines $\overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})$ in these stems.

The first nonzero v_1 -Bockstein differential is $d_1(\mu) = v_1\lambda'_1$. If it were not there, then $v_1\lambda'_1$ would survive to $\overline{V}(0)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})$ and $\pi_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})$ to detect a nonzero class in the trivial group $\pi_7(\operatorname{THH}(\operatorname{ko})/\operatorname{ko})$, which is impossible. There is no room for other v_1 -Bockstein differentials affecting stems ≤ 11 , so if $d_3(\lambda'_1\lambda_2) = \eta^3\mu$, then the v_1 -Bockstein E_∞ -term (modulo the summand for ko) will be

$$\mathbb{F}_2\{\lambda'_1, \eta\mu, \eta^2\mu, v_1\eta\mu\}$$

in stems ≤ 11 , while if $d_3(\lambda'_1\lambda_2) = 0$, then it will be

$$\mathbb{F}_2\{\lambda'_1, \eta\mu, \eta^2\mu, \eta^3\mu, v_1\eta\mu\}$$

in these stems. In either case, the 12-stem is concentrated in motivic filtrations ≥ 2 , and these expressions determine $\overline{V}(0)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})$ in this range of stems.

In the v_0 -Bockstein spectral sequence, there is no room for differentials on $(\lambda'_1$ and) $\eta\mu$. Multiplying by η^2 , it follows that $\eta^3\mu$ is an infinite cycle (but possibly zero). Since it is not a v_0 -multiple, it cannot be a v_0 -Bockstein boundary, and since it is in motivic filtration 3, and the motivic E_2 -term is now known to be zero in bidegrees $(12, 0)$ and $(12, 1)$, it cannot be a motivic d_r -boundary for $r \geq 2$. Hence if $d_3(\lambda'_1\lambda_2)$ were zero, then $\eta^3\mu$ would survive to $\overline{V}(0)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})$ and $\pi_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})$ to detect a nonzero class in $\pi_{11}(\operatorname{THH}(\operatorname{ko})/\operatorname{ko}) = 0$, which is impossible.

This contradiction shows that $d_3(\lambda'_1 \lambda_2) = \eta^3 \mu$, as claimed. This leaves the η -Bockstein E_4 -term

$$\frac{\Lambda(\lambda'_1) \otimes \mathbb{F}_2[\eta, \mu]}{(\eta \lambda'_1, \eta^3 \mu)}.$$

There is no room for further differentials, so this is also the E_∞ -term. The only multiplicative ambiguity in the abutment $\overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})$ is whether $(\lambda'_1)^2$ is 0 or $\eta^2 \mu$. To resolve this indeterminacy, observe that

$$j: \overline{A}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko}) \longrightarrow \Sigma^{2,0} \overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})$$

is a map of $\overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})$ -modules, so $j(\lambda_2) = \Sigma^{2,0} \lambda'_1$ implies that

$$j(\lambda'_1 \lambda_2) = \Sigma^{2,0} (\lambda'_1)^2.$$

If $(\lambda'_1)^2$ were 0, then $\lambda'_1 \lambda_2$ would lift over i to a class in $\overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})$, but this cannot occur because it would contradict the nonzero differential $d_3(\lambda'_1 \lambda_2) = \eta^3 \mu$. Hence $(\lambda'_1)^2 = \eta^2 \mu$. $\square$

COROLLARY 2.38. *We can identify*

$$\overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko}) \cong \Lambda(\varepsilon_2) \otimes \frac{\mathbb{F}_2[\eta, \lambda'_1, \mu]}{(\eta \lambda'_1, (\lambda'_1)^2 = \eta^2 \mu)}$$

as a bigraded $\mathbb{F}_2$ -algebra.

Proof. Here, ε_2 is chosen as in Definition 2.35 with $M^* = \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})$. $\square$

3. Filtering the motivic associated graded

In subsequent sections, we will use various filtrations on the associated graded of the motivic filtration on topological negative cyclic homology, topological periodic cyclic homology, and the C_2 -homotopy fixed points and C_2 -Tate construction on topological Hochschild homology. Here, we discuss two filtrations that one might consider and show in what sense they agree.

We first discuss a collection of increasing filtrations $\operatorname{fil}_\star^{\operatorname{GM}}(-)$. Here, we use (subscripted) homological indexing. Recall that the circle group $\mathbb{T}$ acts freely on $S^1 \subset S^3 \subset \cdots \subset S^\infty = E\mathbb{T}$ and C_2 acts freely on $S^0 \subset S^1 \subset \cdots \subset S^\infty = EC_2$.

DEFINITION 3.1. Define a filtered $\mathbb{T}$ -spectrum $\widetilde{E\mathbb{T}}_\star$ by $\widetilde{E\mathbb{T}}_{2k} = \widetilde{E\mathbb{T}}_{2k+1} = S^{k\mathbb{C}}$, and a filtered C_2 -spectrum $(\widetilde{EC_2})_\star$ by $(\widetilde{EC_2})_k = S^{k\mathbb{R}}$, for all integers k . Following Greenlees and May [GM95], for each spectrum X with $\mathbb{T}$ -action, we define a filtered spectrum

$$\operatorname{GM}_\star^\mathbb{T}(X) := [\widetilde{E\mathbb{T}}_\star \otimes F(E\mathbb{T}_+, X)]^\mathbb{T}$$

and for each spectrum X with C_2 -action, we define a filtered spectrum

$$\operatorname{GM}_\star^{C_2}(X) := [(\widetilde{EC_2})_\star \otimes F(EC_{2+}, X)]^{C_2}.$$

In filtration $\star=0$, these specialize to the homotopy fixed point spectra $\operatorname{GM}_0^\mathbb{T}(X) = X^{h\mathbb{T}}$ and $\operatorname{GM}_0^{C_2}(X) = X^{hC_2}$. More generally, we have

$$\operatorname{GM}_{-2k}^\mathbb{T}(X) = \operatorname{GM}_{-2k+1}^\mathbb{T}(X) \simeq F(E\mathbb{T}/S^{2k-1}, X)^\mathbb{T}$$

and $\operatorname{GM}_{-k}^{C_2}(X) \simeq F(EC_2/S^{k-1}, X)^{C_2}$ for all $k \geq 0$ (cf. [Gre87, p. 437] or [HR24, Lemma 6.32]).

For brevity, let

$$Y^\bullet = C^\bullet(\operatorname{THH}(\operatorname{ku}/\operatorname{MU})/\operatorname{THH}(\operatorname{ko})).$$

This is naturally an $\mathbb{E}_\infty$ algebra in $C^\bullet(\mathrm{MU}/\mathbb{S})$ -modules with $\mathbb{T}$ -action. Define a filtered graded spectrum

$$\mathrm{fil}_\star^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TP}(\mathrm{ko}) := \mathrm{Tot} \Sigma^{2*} H\pi_{2*} \mathrm{GM}_\star^\mathbb{T}(Y^\bullet),$$

and define $\mathrm{fil}_\star^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}^-(\mathrm{ko})$ and $\mathrm{fil}_\star^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* F(S_+^{2\ell+1}, \mathrm{THH}(\mathrm{ko}))^\mathbb{T}$ for $\ell \geq 0$ by truncating this filtration to live in the ranges $\star \leq 0$ and $-2\ell \leq \star \leq 0$, respectively. Similarly, define a filtered graded spectrum

$$\mathrm{fil}_\star^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2} := \mathrm{Tot} \Sigma^{2*} H\pi_{2*} \mathrm{GM}_\star^{C_2}(Y^\bullet),$$

and define $\mathrm{fil}_\star^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{hC_2}$ by truncating this filtration to the range $\star \leq 0$.

We know from [BBL⁺14, Proposition 3.8] (cf. [NS18, Lemma II.4.2]) that there is a natural equivalence $G: (Y^q)^{t\mathbb{T}} \simeq ((Y^q)^{tC_2})^{h\mathbb{T}}$ for each $q \geq 0$. Since each Y^q is even and 2-torsion free by Proposition 2.15, we know that each $(Y^q)^{tC_2}$ is even, and we obtain a second filtration of $\mathrm{gr}_{\mathrm{mot}}^* \mathrm{TP}(\mathrm{ko}) \simeq \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathrm{ko})^{tC_2})^{h\mathbb{T}}$ by truncating the filtration $\mathrm{Tot} \Sigma^{2*} H\pi_{2*} \mathrm{GM}_\star^\mathbb{T}((Y^\bullet)^{tC_2})$ to the range $\star \leq 0$. We write $\mathrm{fil}_\star^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathrm{ko})^{tC_2})^{h\mathbb{T}}$ for this filtration. Letting the weight w vary, these are modules over $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S} \simeq \mathrm{Tot} \Sigma^{2*} H\pi_{2*} C^\bullet(\mathrm{MU}/\mathbb{S})$, regarded as a filtered graded spectrum with $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ in filtrations $\star \geq 0$ and 0 in filtrations $\star < 0$, where the structure maps are identities for $\star \geq 0$ and the zero map for $\star < 0$.

We refer to these as Greenlees–May filtrations. Each one is complete and exhaustive, and hence induces a conditionally convergent spectral sequence. We write

$$\mathrm{gr}_s^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^w F(\mathrm{ko}) = \mathrm{fil}_s^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^w F(\mathrm{ko}) / \mathrm{fil}_{s-1}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^w F(\mathrm{ko})$$

for each $F \in \{\mathrm{TC}^-, \mathrm{TP}, F(S_+^{2\ell+1}, \mathrm{THH})^\mathbb{T}, \mathrm{THH}^{hC_2}, \mathrm{THH}^{tC_2}, (\mathrm{THH}^{tC_2})^{h\mathbb{T}}\}$. Letting s and w vary, these are modules over $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$, with action mapping $\mathrm{gr}_s^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^w F(\mathrm{ko})$ to $\mathrm{gr}_s^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^{w+*} F(\mathrm{ko})$.

Alternatively, we can take the approach of [HRW22, Construction 2.1.14] and define the following filtrations of the associated graded of the motivic filtration. Here, we return to our standard conventions and use (superscripted) cohomological indexing.

DEFINITION 3.2. Let $Y^\bullet = C^\bullet(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\mathrm{THH}(\mathrm{ko}))$. We define filtered graded objects

$$\begin{aligned} \mathrm{fil}_+^* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}^-(\mathrm{ko}) &:= \mathrm{Tot} \Sigma^{2*} H\pi_{2*} (\tau_{\geq \star} Y^\bullet)^{h\mathbb{T}}, \\ \mathrm{fil}_+^* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TP}(\mathrm{ko}) &:= \mathrm{Tot} \Sigma^{2*} H\pi_{2*} (\tau_{\geq \star} Y^\bullet)^{t\mathbb{T}}, \\ \mathrm{fil}_+^* \mathrm{gr}_{\mathrm{mot}}^* F(S_+^{2\ell+1}, \mathrm{THH}(\mathrm{ko}))^\mathbb{T} &:= \mathrm{Tot} \Sigma^{2*} H\pi_{2*} F(S_+^{2\ell+1}, \tau_{\geq \star} Y^\bullet)^\mathbb{T}, \\ \mathrm{fil}_+^* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{hC_2} &:= \mathrm{Tot} \Sigma^{2*} H\pi_{2*} (\tau_{\geq \star} Y^\bullet)^{hC_2}, \\ \mathrm{fil}_+^* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2} &:= \mathrm{Tot} \Sigma^{2*} H\pi_{2*} (\tau_{\geq \star} Y^\bullet)^{tC_2}, \\ \mathrm{fil}_+^* \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathrm{ko})^{tC_2})^{h\mathbb{T}} &:= \mathrm{Tot} \Sigma^{2*} H\pi_{2*} (\tau_{\geq \star} ((Y^\bullet)^{tC_2}))^{h\mathbb{T}}, \end{aligned}$$

where $\ell \geq 0$. Each one is complete and exhaustive, by proofs similar to that of Lemma 2.19, and hence induces a conditionally convergent spectral sequence. We refer to these as Whitehead filtrations. They come equipped with an action of the $\mathbb{E}_\infty$ algebra in filtered graded spectra

$$\mathrm{Tot} \Sigma^{2*} H\pi_{2*} (\tau_{\geq \star} C^\bullet(\mathrm{MU}/\mathbb{S})) \simeq \begin{cases} \mathrm{gr}_{\mathrm{ev}}^* \mathbb{S} & \text{for } \star \leq 2*, \\ 0 & \text{for } \star > 2*. \end{cases}$$

We write

$$\mathrm{gr}_+^s \mathrm{gr}_{\mathrm{mot}}^w F(\mathrm{ko}) = \mathrm{fil}_+^s \mathrm{gr}_{\mathrm{mot}}^w F(\mathrm{ko}) / \mathrm{fil}_+^{s+1} \mathrm{gr}_{\mathrm{mot}}^w F(\mathrm{ko})$$

for $F \in \{\mathrm{TC}^-, \mathrm{TP}, F(S_+^{2\ell+1}, \mathrm{THH})^\mathbb{T}, \mathrm{THH}^{hC_2}, \mathrm{THH}^{tC_2}, (\mathrm{THH}^{tC_2})^{h\mathbb{T}}\}$. Letting s and w vary, these are modules over $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$, with action mapping $\mathrm{gr}_+^s \mathrm{gr}_{\mathrm{mot}}^w F(\mathrm{ko})$ to $\mathrm{gr}_+^{s+2*} \mathrm{gr}_{\mathrm{mot}}^{w+*} F(\mathrm{ko})$.

The filtrations from Definition 3.2 are clearly multiplicative, whereas our identification of the spectral sequence starting pages is more clear for the filtrations of Definition 3.1. Luckily, we can prove that they agree, up to an indexing shift that depends upon the weight grading.

PROPOSITION 3.3. *There is an equivalence of filtered $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -modules*

$$\mathrm{fil}_{-2n}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* F(\mathrm{ko}) \longrightarrow \mathrm{fil}_+^{2n+2*} \mathrm{gr}_{\mathrm{mot}}^* F(\mathrm{ko})$$

for $F \in \{\mathrm{TC}^-, \mathrm{TP}, F(S_+^{2\ell+1}, \mathrm{THH})^{\mathbb{T}}, (\mathrm{THH}^{tC_2})^{h\mathbb{T}}\}$.

Proof. Again, let $Y^\bullet = C^\bullet(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\mathrm{THH}(\mathrm{ko}))$. We first discuss the case $F = \mathrm{TP}$. We have

$$\mathrm{gr}_{\mathrm{mot}}^w \mathrm{THH}(\mathrm{ko}) = \mathrm{Tot} \Sigma^{2w} H\pi_{2w} Y^\bullet$$

and

$$\mathrm{gr}_{\mathrm{mot}}^w \mathrm{TP}(\mathrm{ko}) = \mathrm{Tot} \Sigma^{2w} H\pi_{2w} ((Y^\bullet)^{t\mathbb{T}}).$$

The latter is filtered by

$$\mathrm{fil}_m^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{TP}(\mathrm{ko}) = \mathrm{Tot} \Sigma^{2w} H\pi_{2w} \mathrm{GM}_m^{\mathbb{T}}(Y^\bullet)$$

for $m \in \mathbb{Z}$, with associated graded

$$\begin{aligned} \mathrm{gr}_{-2n}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{TP}(\mathrm{ko}) &= \mathrm{Tot} \Sigma^{2w} H\pi_{2w} (\Omega^{2n} Y^\bullet) \\ &\cong (\mathrm{Tot} \Sigma^{2w+2n} H\pi_{2w+2n} Y^\bullet) \{t^n\} \end{aligned}$$

in even gradings, where t^n reduces stem by $2n$. In odd gradings, $\mathrm{gr}_{-2n+1}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{TP}(\mathrm{ko}) = 0$.

The Whitehead filtration

$$\mathrm{fil}_+^{2w+2n} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{TP}(\mathrm{ko}) = \mathrm{Tot} \Sigma^{2w} H\pi_{2w} ((\tau_{\geq 2w+2n} Y^\bullet)^{t\mathbb{T}})$$

has associated graded

$$\begin{aligned} \mathrm{gr}_+^{2w+2n} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{TP}(\mathrm{ko}) &= \mathrm{Tot} \Sigma^{2w} H\pi_{2w} ((\Sigma^{2w+2n} H\pi_{2w+2n} Y^\bullet)^{t\mathbb{T}}) \\ &\cong (\mathrm{Tot} \Sigma^{2w+2n} H\pi_{2w+2n} Y^\bullet) \{t^n\}. \end{aligned}$$

Again, $\mathrm{gr}_+^{2w+2n-1} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{TP}(\mathrm{ko}) = 0$. These constructions are lax symmetric monoidal, by [NS18, Corollary I.4.3].

The composite map

$$\Sigma^{2w} H\pi_{2w} ([\widetilde{E\mathbb{T}}_{-2n} \otimes F(E\mathbb{T}_+, Y^\bullet)]^{\mathbb{T}}) \longrightarrow \Sigma^{2w} H\pi_{2w} ((Y^\bullet)^{t\mathbb{T}}) \longrightarrow \Sigma^{2w} H\pi_{2w} ((\tau_{< 2w+2n} Y^\bullet)^{t\mathbb{T}})$$

factors through

$$\Sigma^{2w} H\pi_{2w} ([\widetilde{E\mathbb{T}}_{-2n} \otimes F(E\mathbb{T}_+, \tau_{< 2w+2n} Y^\bullet)]^{\mathbb{T}}) = 0,$$

and hence the first map in the composite lifts to induce a map of filtrations

$$\mathrm{fil}_{-2n}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{TP}(\mathrm{ko}) \longrightarrow \mathrm{fil}_+^{2w+2n} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{TP}(\mathrm{ko}),$$

both with limit 0 and colimit $\mathrm{gr}_{\mathrm{mot}}^w \mathrm{TP}(\mathrm{ko})$. From the definition of the filtered $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -module structure on either side, it is clear that this is a map of filtered $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -modules. The associated graded spectra are equivalent as $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -modules, and hence so are the filtrations, up to the reindexing from $\mathrm{fil}_{-2n}^{\mathrm{GM}}$ to fil_+^{2w+2n} . Hence the filtrations agree, up to reindexing. By restricting to filtrations $m \leq 0$, or $-2\ell \leq m \leq 0$, the same proof applies when replacing TP with TC^- or $F(S_+^{2\ell+1}, \mathrm{THH})^{\mathbb{T}}$. Since Y^q is even and 2-torsion free, we know $(Y^q)^{tC_2}$ is even for each $q \geq 0$, and again the same proof applies to deduce the result for $F = (\mathrm{THH}^{tC_2})^{h\mathbb{T}}$. $\square$

COROLLARY 3.4. *Let $\bar{V}$ be a $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -module. There are trigraded spectral sequences*

$$\begin{aligned}\bar{V}_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko}) [t] &\Longrightarrow \bar{V}_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}^-(\mathrm{ko}), \\ \bar{V}_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko}) [t^{\pm 1}] &\Longrightarrow \bar{V}_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TP}(\mathrm{ko}), \\ \bar{V}_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko}) [t]/(t^{\ell+1}) &\Longrightarrow \bar{V}_* \mathrm{gr}_{\mathrm{mot}}^* F(S_+^{2\ell+1}, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}, \\ \bar{V}_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2} [t] &\Longrightarrow \bar{V}_* \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathrm{ko})^{tC_2})^{h\mathbb{T}}.\end{aligned}$$

The first is the (algebraic) homotopy fixed point spectral sequence, the second is the (algebraic) Tate spectral sequence, the third is the approximate fixed point spectral sequence, and the last is the μ -inverted homotopy fixed point spectral sequence.

When $\bar{V}$ is a $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -algebra, these are algebra spectral sequences. When $\bar{V}$ is a module over a given $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -algebra, they are module spectral sequences over the corresponding algebra spectral sequences. When $\bar{V}$ is a finite $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -module, they are conditionally convergent.

Proof. We apply $\bar{V} \otimes (-)$ to the filtered $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -modules $\mathrm{fil}_{\star}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* F(\mathrm{ko})$ for

$$F \in \{\mathrm{TC}^-, \mathrm{TP}, F(S_+^{2\ell+1}, \mathrm{THH})^{\mathbb{T}}, (\mathrm{THH}^{tC_2})^{h\mathbb{T}}\}.$$

In the case of $F = \mathrm{TP}$, we have observed in the proof of Proposition 3.3 that there is an equivalence $\mathrm{gr}_{-2n}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TP}(\mathrm{ko}) \simeq \mathrm{gr}_{\mathrm{mot}}^{*+n} \mathrm{THH}(\mathrm{ko})\{t^n\}$, so that we have an equivalence

$$\bar{V} \otimes \mathrm{gr}_{-2n}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TP}(\mathrm{ko}) \simeq \bar{V} \otimes \mathrm{gr}_{\mathrm{mot}}^{*+n} \mathrm{THH}(\mathrm{ko})\{t^n\}$$

for each integer n . The cases $F = \mathrm{TC}^-$ and $F = F(S_+^{2\ell+1}, \mathrm{THH})^{\mathbb{T}}$ are obtained by restricting the filtrations. The case of $F = (\mathrm{THH}^{tC_2})^{h\mathbb{T}}$ is similar to that of TC^- , since each $(Y^q)^{tC_2}$ is even by Proposition 2.15. The fact that these are algebra spectral sequences when $\bar{V}$ is a $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -algebra follows by Proposition 3.3 and the fact that the filtrations in Definition 3.2 are multiplicative. $\square$

PROPOSITION 3.5. *Suppose that $\bar{V}$ is a $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -module such that 2 acts trivially on $\bar{V}$. Then we can identify the C_2 -Tate spectral sequence associated to the filtered $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -module*

$$\bar{V} \otimes \mathrm{fil}_{+}^* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2}$$

with the spectral sequence associated to the filtered $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -module

$$\bar{V} \otimes \mathrm{fil}_{\star}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2}.$$

The spectral sequence is of the form

$$\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko}) [t^{\pm 1}] \otimes \Lambda(u_1) \Longrightarrow \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2},$$

where t in Tate filtration -2 has bidegree $\|t\| = (-2, 0)$ and u_1 in Tate filtration -1 has bidegree $\|u_1\| = (-1, -1)$.

Under the same hypotheses, we can similarly identify the C_2 -homotopy fixed point spectral sequence associated to the filtered $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -module

$$\bar{V} \otimes \mathrm{fil}_{+}^* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{hC_2}$$

with the spectral sequence associated to the filtered $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -module

$$\bar{V} \otimes \mathrm{fil}_{\star}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{hC_2}.$$

In this case, the spectral sequence is of the form

$$\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko}) [t] \otimes \Lambda(u_1) \Longrightarrow \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{hC_2},$$

where t in homotopy fixed point filtration -2 has bidegree $\|t\| = (-2, 0)$ and u_1 in homotopy fixed point filtration -1 has bidegree $\|u_1\| = (-1, -1)$.

In each case, the spectral sequence is an algebra spectral sequence when $\bar{V}$ is a $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -algebra, and is conditionally convergent when $\bar{V}$ is a finite $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -module.

Proof. Let $Y^\bullet = C^\bullet(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\mathrm{THH}(\mathrm{ko}))$. We write $\tau_{[a,b]} = \mathrm{cof}(\tau_{\geq b+1} \rightarrow \tau_{\geq a})$ for integers $a < b$ and similarly we write $\mathrm{GM}_{[m,m+1]}^{C_2}(X) = \mathrm{cof}(\mathrm{GM}_{m-1}^{C_2}(X) \rightarrow \mathrm{GM}_{m+1}^{C_2}(X))$. When m is even, there are equivalences

$$\tau_{[2w,2w+1]}(\mathrm{GM}_{[m,m+1]}^{C_2}(Y^q)) \simeq \tau_{[2w,2w+1]}((\Sigma^{2w-m} H\pi_{2w-m} Y^q)^{tC_2})$$

since Y^q is even for each q . These are compatible with the cosimplicial structure maps, and totalize to an equivalence

$$\mathrm{fil}_{[m,m+1]}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{THH}(\mathrm{ko})^{tC_2} \simeq \mathrm{fil}_+^{2w-m} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{THH}(\mathrm{ko})^{tC_2}.$$

Here, we write $\mathrm{fil}_{[m,m+1]}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{THH}(\mathrm{ko})^{tC_2}$ for the cofiber of

$$\mathrm{fil}_{m-1}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{THH}(\mathrm{ko})^{tC_2} \longrightarrow \mathrm{fil}_{m+1}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{THH}(\mathrm{ko})^{tC_2}.$$

The free C_2 -equivariant $(m+1)$ -cell in $\widetilde{EC_2} = S^\infty$ is attached to the free C_2 -equivariant m -cell by a pinch map $S^m \rightarrow S^m/S^{m-1} \cong C_{2+} \wedge S^m$. Hence there are cofiber sequences

$$\Sigma^{2w} H\pi_{2w}(\Sigma^m Y^q) \longrightarrow \tau_{[2w,2w+1]} \mathrm{GM}_{[m,m+1]}(Y^q) \longrightarrow \Sigma^{2w+1} H\pi_{2w+1}(\Sigma^{m+1} Y^q) \xrightarrow{N} \Sigma(-),$$

with $N = (+1)_* + (-1)_*$ the norm (or trace) map given by the sum of the actions by the elements in C_2 , having totalization a cofiber sequence

$$\mathrm{gr}_m^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{THH}(\mathrm{ko})^{tC_2} \longrightarrow \mathrm{fil}_{[m,m+1]}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{THH}(\mathrm{ko})^{tC_2} \longrightarrow \mathrm{gr}_{m+1}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{THH}(\mathrm{ko})^{tC_2} \xrightarrow{N} \Sigma(-).$$

Since the C_2 -action on each Y^q is the restriction of a $\mathbb{T}$ -action, the map N is homotopic to multiplication by 2.

Using the earlier equivalence, and letting w vary, we get a cofiber sequence

$$\mathrm{gr}_m^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2} \longrightarrow \mathrm{fil}_+^{2*-m} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2} \longrightarrow \mathrm{gr}_{m+1}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2} \xrightarrow{N} \Sigma(-)$$

of $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -modules, where the $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -actions are as in Definitions 3.1 and 3.2. For each $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -module $\bar{V}$, we get another cofiber sequence

$$\begin{aligned} \bar{V} \otimes \mathrm{gr}_m^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2} &\longrightarrow \bar{V} \otimes \mathrm{fil}_+^{2*-m} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2} \\ &\longrightarrow \bar{V} \otimes \mathrm{gr}_{m+1}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2} \xrightarrow{\bar{V} \otimes N} \dots, \end{aligned}$$

where we again consider the $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -actions as in Definitions 3.1 and 3.2. If multiplication by 2 acts trivially on $\bar{V}$, e.g., if this is a $\bar{V}(0)$ -module, then the connecting map $\bar{V} \otimes N$ is null-homotopic, and the cofiber sequence splits. Then the spectral sequence associated to the filtration $\bar{V} \otimes \mathrm{fil}_+^* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2}$ of $\bar{V} \otimes \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2}$ has starting term the sum over $n \in \mathbb{Z}$ of

$$\begin{aligned} &\bar{V}_* \mathrm{gr}_+^{2*+2n} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2} \\ &\cong \bar{V}_* \mathrm{gr}_{-2n}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2} \oplus \bar{V}_* \mathrm{gr}_{-2n+1}^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2} \\ &\cong \bar{V}_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2} \{t^n, u_1 t^{n-1}\}, \end{aligned}$$

with t^n and $u_1 t^{n-1}$ in (stem, motivic filtration) bigrading equal to $(-2n, 0)$ and $(-2n+1, -1)$, respectively. Truncating at $\star \leq 0$ produces the result for $(-)^{hC_2}$ in place of $(-)^{tC_2}$.

Suppose that $\bar{V}$ is a $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -algebra. To confirm the formula

$$\bar{V}_* \mathrm{gr}_*^{\mathrm{GM}} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2} \cong \bar{V}_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko}) [t^{\pm 1}] \otimes \Lambda(u_1)$$

for the E^2 -page of the C_2 -Tate spectral sequence as a trigraded $\mathbb{F}_2$ -algebra, it suffices to check that multiplication by $u_1 t^{-1}$ takes the copy of $\bar{V}_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})$ in Tate filtration 0 to the copy

in Tate filtration 1, since we know that t , in the case of the $\mathbb{T}$ -Tate spectral sequence, acts invertibly on this spectral sequence. However, this is already built into the notation since, in the $n=0$ case, the two module generators there are $1=t^0$ and $u_1 t^{-1}$. Note that u_1^2 lands in a bidegree that is trivial, so the relation $u_1^2=0$ follows. In the case of the C_2 -homotopy fixed point spectral sequence, the algebra structure on the E^2 -page can be determined from the fact that it is a subalgebra of the E^2 -page of the C_2 -Tate spectral sequence. $\square$

4. Detection

The classical mod 2 Adams spectral sequence

$${}^{\text{Ad}}E_2(X) = \text{Ext}_{\mathcal{A}^\vee}(\mathbb{F}_2, H_*(X)) \Longrightarrow \pi_*(X_2^\wedge)$$

is strongly convergent for bounded-below spectra X with $H_*(X)$ of finite type. Its E_2 -term can be calculated as the cohomology of the normalized cobar complex

$$0 \longrightarrow H_*(X) \xrightarrow{d_1^0} \bar{\mathcal{A}}^\vee \otimes H_*(X) \xrightarrow{d_1^1} \bar{\mathcal{A}}^\vee \otimes \bar{\mathcal{A}}^\vee \otimes H_*(X) \longrightarrow \cdots$$

Here, $\bar{\mathcal{A}}^\vee = \text{cok}(\mathbb{F}_2 \rightarrow \mathcal{A}^\vee)$, and we will use the notation $[a]m = a \otimes m \in \bar{\mathcal{A}}^\vee \otimes H_*(X)$. Recall that d_1^0 is given by the normalized $\mathcal{A}^\vee$ -coaction on $H_*(X)$, while d_1^1 also involves the coproduct $\psi: \mathcal{A}^\vee \rightarrow \mathcal{A}^\vee \otimes \mathcal{A}^\vee$.

When $X = A(1)[ij]$ as in Notation 2.20, the Adams E_2 -terms

$${}^{\text{Ad}}E_2(A(1)[ij]) = \text{Ext}_{\mathcal{A}}(H^*(A(1)[ij]), \mathbb{F}_2) \Longrightarrow \pi_* A(1)[ij]$$

are readily calculated in a finite range using Bruner's `ext` software [Bru93, BR21]. The results in stems $* \leq 28$ are shown in Figure 3, with the usual (stem, Adams filtration) bigrading. Lines of bidegree $(0, 1)$, $(1, 1)$ and $(3, 1)$ (dashed) indicate multiplications by h_0 , h_1 and h_2 , respectively.

In each case, the three 1-cochains

$$[\xi_1^4]1, \quad [\xi_2^2]1 + [\xi_1^4]\xi_1^2, \quad [\xi_3]1 + [\xi_2^2]\xi_1 + [\xi_1^4]\xi_2 \quad (4.1)$$

in $\bar{\mathcal{A}}^\vee \otimes \mathcal{A}(1)^\vee$ are cocycles, but not coboundaries; hence they represent nonzero classes in ${}^{\text{Ad}}E_2(A(1)[ij])$ in bidegrees $(3, 1)$, $(5, 1)$ and $(6, 1)$, respectively. For sparsity reasons, these classes all survive to ${}^{\text{Ad}}E_\infty(A(1)[ij])$, and detect nonzero homotopy classes in stems 3, 5 and 6, denoted

$$\nu, \quad w, \quad v_2$$

in $\pi_* A(1)[ij]$, for each $i, j \in \{0, 1\}$. Observe that v_2 is only defined modulo ν^2 , and $2v_2 = 0$ if $i = 0$, while $2v_2 = \nu^2$ if $i = 1$.

LEMMA 4.1. *In the Adams spectral sequences for the $A(1)[ij]$, the differentials originating in stems $* \leq 24$ are all zero. The class $\nu\bar{\kappa} \in \pi_{23}(\mathbb{S})$ maps to zero in $\pi_{23}A(1)[ij]$.*

Proof. This mostly follows from sparsity and the module structure over the Adams spectral sequence for $\mathbb{S}$, using that $d_2(h_4) = h_0 h_3^2$ maps to zero under $\mathbb{S} \rightarrow A(1)$. Only the Adams d_2 -differential from bidegree $(t-s, s) = (19, 2)$ requires special attention, but the Novikov E_2 -term (see Figure 6) shows that $\pi_{19}A(1)$ has order $2^2 = 4$, so there is no room for such an Adams differential.

In each case, the map ${}^{\text{Ad}}E_2(\mathbb{S}) \rightarrow {}^{\text{Ad}}E_2(A(1))$ of Adams E_2 -terms takes the bidegree $(23, 5)$ class $h_2 g$ detecting $\nu\bar{\kappa}$ to zero, as can be checked with `ext`, and the target has no classes in stem 23 and Adams filtration ≥ 6 . It follows that $\nu\bar{\kappa} \mapsto 0$. $\square$

ALGEBRAIC K -THEORY OF REAL TOPOLOGICAL K -THEORY

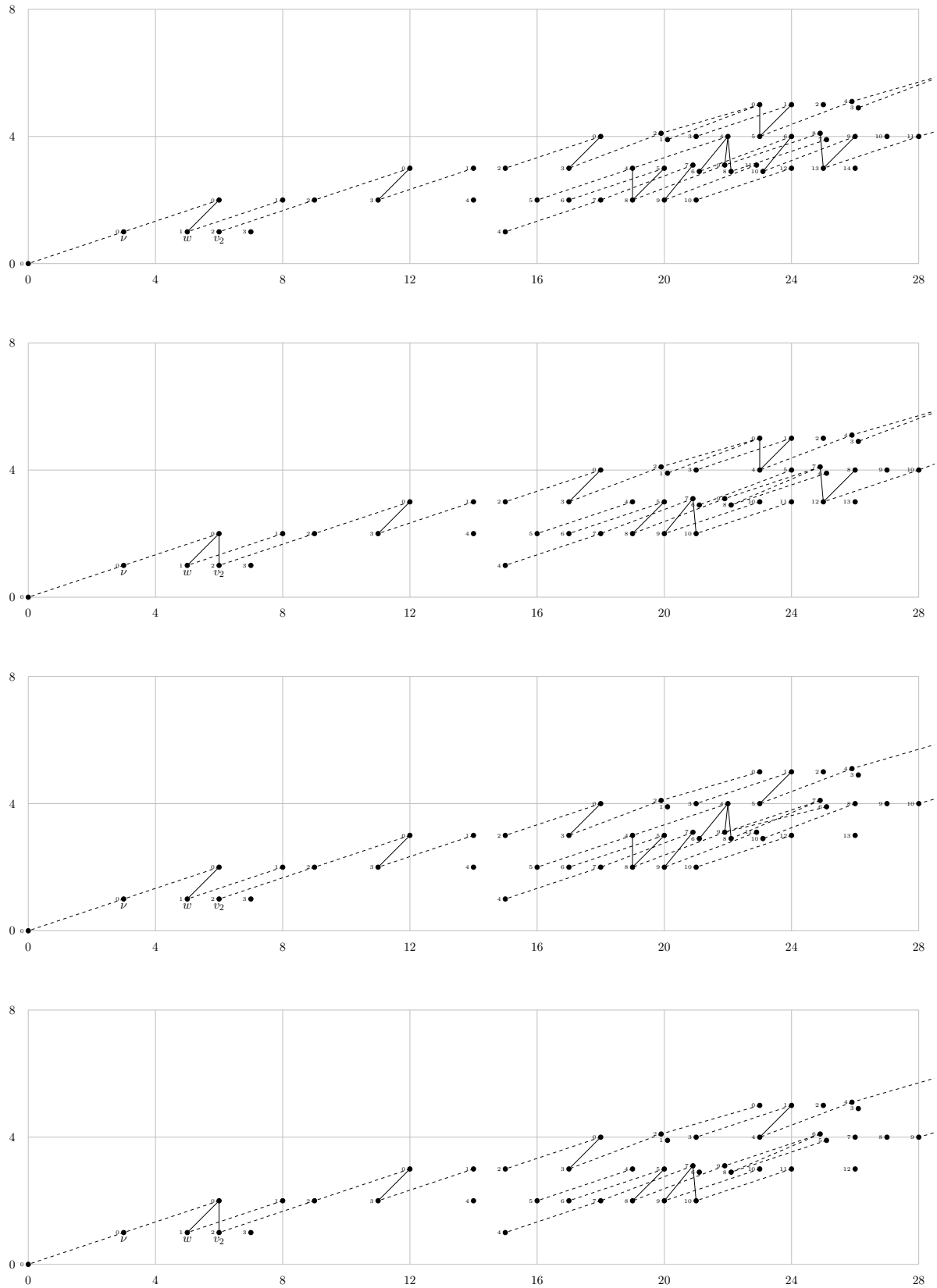


FIGURE 3. Adams E_2 -terms for $A(1)[00]$, $A(1)[10]$, $A(1)[01]$ and $A(1)[11]$ (from top to bottom).

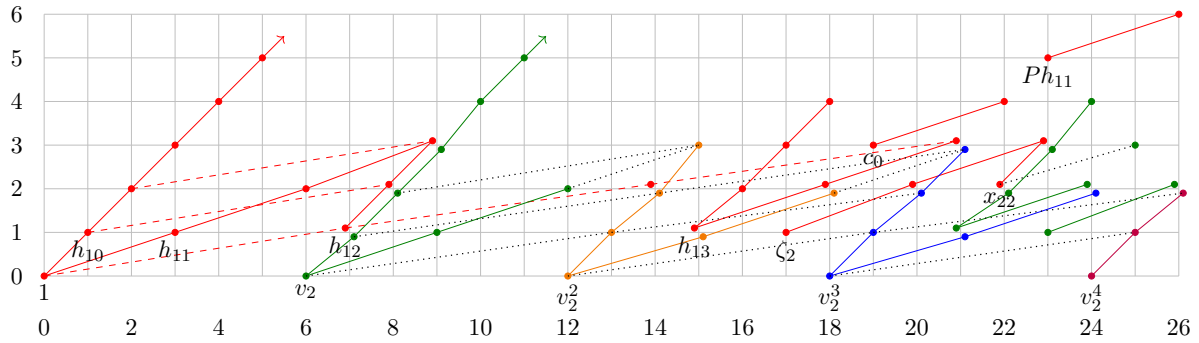


FIGURE 4 (colour online). The v_2 -Bockstein $E_\infty \Rightarrow \text{Ext}_{\text{BP}_*\text{BP}}(\text{BP}_*, \text{BP}_*/I_2)$.

To calculate the Novikov E_2 -term

$$^{\text{Nov}}E_2(A(1)) = \text{Ext}_{\text{MU}_*\text{MU}}(\text{MU}_*, \text{MU}_*A(1)) \cong \text{Ext}_{\text{BP}_*\text{BP}}(\text{BP}_*, \text{BP}_*A(1))$$

for these spectra, we can note that $\text{BP}_*A(1) = \text{BP}_*/I_2\{1, t_1\}$ and use the long exact sequence obtained by applying $\text{Ext}_{\text{BP}_*\text{BP}}(\text{BP}_*, -)$ to the BP_*BP -comodule extension

$$0 \longrightarrow \text{BP}_*/I_2 \longrightarrow \text{BP}_*A(1) \longrightarrow \Sigma^2\text{BP}_*/I_2 \rightarrow 0$$

classified by

$$h_{10} = [t_1] \in \text{Ext}_{\text{BP}_*\text{BP}}(\text{BP}_*, \text{BP}_*)$$

in (stem, Novikov filtration) bidegree $(1, 1)$. The groups

$$\text{Ext}_{\text{BP}_*\text{BP}}(\text{BP}_*, \text{BP}_*/I_2)$$

are calculated in a range as in [Rav86, § 4.4, p. 162], starting with the isomorphism

$$\text{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \text{Ext}_{\text{BP}_*\text{BP}}(\text{BP}_*, \text{BP}_*/I_\infty)$$

that doubles internal degrees, followed by the v_n -Bockstein spectral sequences

$$E_1 = \text{Ext}_{\text{BP}_*\text{BP}}(\text{BP}_*, \text{BP}_*/I_{n+1})[v_n] \Rightarrow \text{Ext}_{\text{BP}_*\text{BP}}(\text{BP}_*, \text{BP}_*/I_n)$$

for descending $n \geq 2$. The v_2 -Bockstein spectral sequence E_∞ -term for BP_*/I_2 in stems $* \leq 26$ is shown in Figure 4, corresponding to [Rav86, Figure 4.4.23(c)]. Lines of bidegree $(1, 1)$, $(3, 1)$ and $(7, 1)$ (dashed) indicate multiplications by $h_{10} = [t_1]$, $h_{11} = [t_1^2]$ and $h_{12} = [t_1^4]$, respectively. (Some) hidden extensions are shown as dotted lines. We emphasize the relation

$$v_2h_{11}^3 = v_2^2h_{10}^3, \quad (4.2)$$

which follows from $v_2h_{12} = v_2^2h_{10}$, which in turn follows from the formula $\eta_R(v_3) \equiv v_3 + v_2t_1^4 + v_2^2t_1 \pmod{I_2}$ for the right unit in BP_*BP (see [Rav86, 4.3.1]).

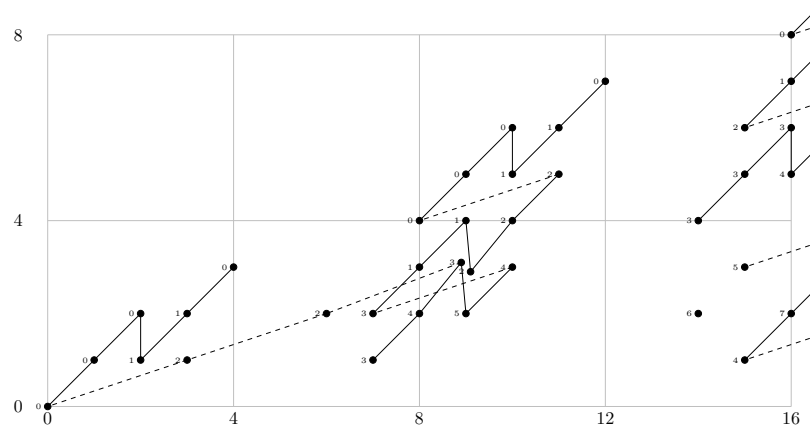
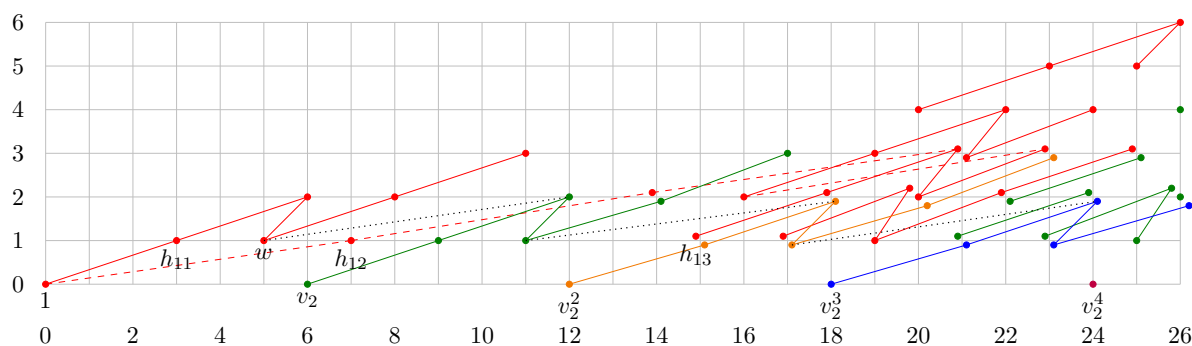
Alternatively, one can start with the internal-degree-doubling isomorphism

$$\text{Ext}_{\mathcal{A}}(H^*(C2), \mathbb{F}_2) \cong \text{Ext}_{\text{BP}_*\text{BP}}(\text{BP}_*, \text{BP}_*/I_\infty\{1, t_1\})$$

and calculate the v_n -Bockstein spectral sequences

$$E_1 = \text{Ext}_{\text{BP}_*\text{BP}}(\text{BP}_*, \text{BP}_*/I_{n+1}\{1, t_1\})[v_n] \Rightarrow \text{Ext}_{\text{BP}_*\text{BP}}(\text{BP}_*, \text{BP}_*/I_n\{1, t_1\})$$

for descending $n \geq 2$. The Adams E_2 -term for $C2$ in stems $* \leq 16$ is shown in Figure 5, and the resulting v_2 -Bockstein E_∞ -term for $\text{BP}_*/I_2\{1, t_1\} = \text{BP}_*A(1)$ in stems $* \leq 26$ is shown in Figure 6. Again, (some) hidden extensions are shown as dotted lines.

FIGURE 5. The Adams E_2 -term for $C2$.FIGURE 6 (colour online). The v_2 -Bockstein $E_\infty \implies \text{Ext}_{\text{BP}_*\text{BP}}(\text{BP}_*, \text{BP}_*A(1))$.

LEMMA 4.2. In the Novikov spectral sequences for the $A(1)[ij]$, the nonzero differentials originating in stems $* \leq 22$ are

$$d_3(v_2^2) = h_{11}^2 w, \quad d_3(v_2^3) = v_2 h_{11}^2 w.$$

In the cases $A(1)[10]$ and $A(1)[11]$, there is a nonzero d_3 from bidegree $(t-s, s) = (23, 1)$.

In every case, $d_3(v_2^4) = 0$ and $d_5(v_2^4) \neq 0$.

Proof. This follows by comparison of the order in each stem of the Adams E_∞ -term, which equals that of the abutment $\pi_* A(1)[ij]$, with the order in each stem of the Novikov E_2 -term. In particular, $\pi_{12} A(1) = \mathbb{Z}/2$ implies that v_2^2 must support a nonzero differential. Similarly, the group $\pi_{18} A(1)$ has order 2^2 , so v_2^3 must support a nonzero differential. The groups $\pi_{22} A(1)[ij]$ have order $2^3 = 8$ for $i = 0$ and order $2^2 = 4$ for $i = 1$, while the groups $\pi_{23} A(1)[ij]$ have order 2^4 for $i = 0$ and 2^3 for $i = 1$. To account for this, the Novikov differential d_3 from bidegree $(t-s, s) = (23, 1)$ to $(22, 4)$ must be nonzero when $i = 1$. Moreover, there must be a rank 1 Novikov differential from the 24-stem to the 23-stem. By h_{11} -linearity, it cannot originate in bidegree $(24, 2)$; hence it is either a d_3 or a d_5 starting on v_2^4 .

Inspection of the Novikov E_2 -term for $\mathbb{S}$ in [Rav86, Figure 4.4.45] shows that $\nu\bar{\kappa} \in \pi_{23}(\mathbb{S})$ is detected by a generator x of the $\mathbb{Z}/8$ in (stem, Novikov filtration) bidegree $(23, 5)$ of ${}^{\text{Nov}}E_2(\mathbb{S})$. The unit map $\mathbb{S} \rightarrow A(1)$ takes this generator x to the generator y of the $\mathbb{Z}/2$ in the same bidegree of ${}^{\text{Nov}}E_2(A(1))$ (see Figure 6). Since $\nu\bar{\kappa}$ maps to zero in $\pi_{23} A(1)$ (by Lemma 4.1) it follows that this nonzero class y is a boundary, and so $d_5(v_2^4) = y \neq 0$ is the only possibility. In particular, we must have $d_3(v_2^4) = 0$. $\square$

There is a cofiber sequence

$$\Sigma^{-2} \mathrm{THH}(\mathrm{ko}) \xrightarrow{i} F(S_+^3, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \xrightarrow{p} \mathrm{THH}(\mathrm{ko}) \xrightarrow{\sigma} \Sigma^{-1} \mathrm{THH}(\mathrm{ko}), \quad (4.3)$$

where σ is induced by the $\mathbb{T}$ -action on $\mathrm{THH}(\mathrm{ko})$, and the following commutative diagram.

$$\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathrm{TC}^-(\mathrm{ko}) & \xrightarrow{q} & F(S_+^3, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \\ \downarrow & & & \searrow & \downarrow p \\ \mathrm{ko} & \longrightarrow & & & \mathrm{THH}(\mathrm{ko}) \end{array}$$

By truncating the ordinary homotopy fixed point spectral sequence

$$E^2 = A(1)_* \mathrm{THH}(\mathrm{ko}) [t] \Longrightarrow A(1)_* \mathrm{TC}^-(\mathrm{ko}),$$

where t is in stem -2 , we obtain a two-column approximate fixed point spectral sequence

$$E^2 = A(1)_* \mathrm{THH}(\mathrm{ko}) \{1, t\} \Longrightarrow A(1)_* F(S_+^3, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}, \quad (4.4)$$

which is really just the long exact sequence in $A(1)$ -homotopy associated to the cofiber sequence (4.3). We have the following analogue of [AR02, Proposition 4.8].

PROPOSITION 4.3. *The unit images in $A(1)_* \mathrm{TC}^-(\mathrm{ko})$ and $A(1)_* F(S_+^3, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$ of the classes ν , w and $v_2 \in \pi_* A(1)$ are detected by*

$$t\lambda'_1, \quad t\lambda_2, \quad t\mu,$$

respectively, in the homotopy fixed point and approximate fixed point spectral sequences.

Proof. By naturality, it suffices to prove this in the approximate fixed point case. The unit map takes the infinite cycles in (4.1), detecting ν , w and v_2 in $\pi_* A(1)$, to the 1-cocycles

$$\begin{aligned} & [\xi_1^4](1 \otimes 1), \\ & [\xi_2^2](1 \otimes 1) + [\xi_1^4](\xi_1^2 \otimes 1), \\ & [\xi_3](1 \otimes 1) + [\xi_2^2](\xi_1 \otimes 1) + [\xi_1^4](\xi_2 \otimes 1) \end{aligned} \quad (4.5)$$

in $\bar{\mathcal{A}}^\vee \otimes \mathcal{A}(1)^\vee \otimes H_*(F(S_+^3, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}})$. We claim that these are not in the image of the coboundary d_1^0 from the 0-cochains

$$\mathcal{A}(1)^\vee \otimes H_*(F(S_+^3, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}),$$

and hence represent nonzero classes in ${}^{\mathrm{Ad}}E_\infty(A(1) \otimes F(S_+^3, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}})$, detecting the (nonzero) images of ν , w and v_2 in $A(1)_* F(S_+^3, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$.

Recall that $H_*(\mathrm{ko}) = \mathbb{F}_2[\bar{\xi}_1^4, \bar{\xi}_2^2, \bar{\xi}_3, \dots]$, $H_* \mathrm{THH}(\mathrm{ko}) = H_*(\mathrm{ko}) \otimes \Lambda(\sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2) \otimes \mathbb{F}_2[\sigma\bar{\xi}_3]$ and $\mathcal{A}(1)^\vee = \mathbb{F}_2[\xi_1, \xi_2]/(\xi_1^4, \xi_2^2)$. In the long exact sequence associated to (4.3), the map σ has kernel $\mathbb{F}_2\{1, \sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2\}$ in degrees ≤ 7 , and the image of i consists of t -multiples. In the extension

$$0 \longrightarrow \mathcal{A}(1)^\vee \otimes \mathrm{im}(i) \longrightarrow \mathcal{A}(1)^\vee \otimes H_* F(S_+^3, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \longrightarrow \mathcal{A}(1)^\vee \otimes \ker(\sigma) \longrightarrow 0$$

the coboundaries on classes in $\mathcal{A}(1)^\vee \otimes \mathrm{im}(i)$ will lie in $\bar{\mathcal{A}}^\vee \otimes \mathcal{A}(1)^\vee \otimes \mathrm{im}(i)$, and hence do not contribute any terms of the form $[a](m \otimes 1)$. The classes 1 , $\sigma\bar{\xi}_1^4$ and $\sigma\bar{\xi}_2^2$ are $\mathcal{A}^\vee$ -comodule primitive in $\ker(\sigma)$, and hence lift to classes in $H_* F(S_+^3, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$ that are $\mathcal{A}^\vee$ -comodule primitive modulo $\mathrm{im}(i)$, so the coboundaries on (the lifts of) $\mathcal{A}(1)^\vee \otimes \mathbb{F}_2\{1, \sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2\}$ also do not contain any terms of the form in (4.5). This proves our claim.

It remains to be determined where in (4.4) the (nonzero) unit images of ν , w and v_2 are detected. Recall that $A(1)_* \mathrm{THH}(\mathrm{ko}) = \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu]$ is equal to $\mathbb{F}_2\{1, \lambda'_1, \lambda_2, \mu\}$ in stems

≤ 8 . The composite map

$$A(1) \longrightarrow A(1) \otimes F(S_+^3, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \xrightarrow{p} A(1) \otimes \mathrm{THH}(\mathrm{ko})$$

factors through $A(1) \otimes \mathrm{ko} \simeq \mathbb{F}_2$, so the images of ν , w and v_2 in $A(1)_* \mathrm{THH}(\mathrm{ko})$ are all zero. (This was obvious for ν and v_2 .) Hence the nonzero images of ν , w and v_2 must all be detected by t -multiples in the approximate fixed point spectral sequence, and for degree reasons the only possible detecting classes are $t\lambda'_1$, $t\lambda_2$ and $t\mu$, respectively. $\square$

Remark 4.4. In light of [HW22, Lemma A.4.1], we may view Proposition 4.3 as confirming that there are unique lifts of ν , w and v_2 so that $\lambda'_1 = \sigma^2\nu$, $\lambda_2 = \sigma^2w$ and $\mu = \sigma^2v_2$, where

$$\sigma^2 : A(1)_* \Sigma^2 \mathrm{fib}(\mathbb{S} \rightarrow \mathrm{ko}) \longrightarrow A(1)_* \mathrm{THH}(\mathrm{ko})$$

is the reduced suspension map of [HW22, Example A.2.4]. Here, uniqueness follows because $A(1)_* \mathrm{ko} = \mathbb{F}_2$.

Since $\mathrm{MU}_* A(1)$ is even, the motivic spectral sequence

$$E_2 = \pi_* \overline{A}(1) \Longrightarrow \pi_* A(1)$$

can be identified with the Novikov spectral sequence

$${}^{\mathrm{Nov}} E_2(A(1)) = \mathrm{Ext}_{\mathrm{BP}_* \mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_* A(1)) \Longrightarrow \pi_* A(1),$$

as in [HRW22, Corollary A.2.6]. The spectral sequence must collapse in stems ≤ 10 , for sparsity reasons, so the three classes denoted h_{11} , w and v_2 in Figure 6 must detect ν , w and v_2 in $\pi_* A(1)$, respectively. We can also identify $\pi_* \overline{V}(1)$ with $\mathrm{Ext}_{\mathrm{BP}_* \mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_2)$, with classes h_{11} and v_2 as shown in Figure 4, but in this case there is no Novikov spectral sequence to $\pi_* V(1)$, since the spectrum $V(1)$ does not exist.

COROLLARY 4.5. *The classes h_{11} , w and v_2 in $\pi_* \overline{A}(1)$ map by the unit to classes in*

$$\overline{A}(1)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}^-(\mathrm{ko})$$

detected by $t\lambda'_1$, $t\lambda_2$ and $t\mu$, respectively. Likewise, the images of h_{11} and w in $\pi_*(\overline{V}(2) \otimes \overline{C}\eta)$ are detected by $t\lambda'_1$ and $t\lambda_2$ in $(\overline{V}(2) \otimes \overline{C}\eta)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}^-(\mathrm{ko})$.

The classes h_{11} and v_2 in $\pi_* \overline{V}(1)$ map by the unit to classes in

$$\overline{V}(1)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}^-(\mathrm{ko})$$

detected by $t\lambda'_1$ and $t\mu$, respectively. Likewise, the image of h_{11} in $\pi_* \overline{V}(2)$ in $\overline{V}(2)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}^-(\mathrm{ko})$ is detected by $t\lambda'_1$.

In each of these cases, for $V \in \{V(1), A(1), V(2), V(2) \otimes C\eta\}$, a unit image detected by a class in $\overline{V}_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}^-(\mathrm{ko})$ is also detected in $\overline{V}_* \mathrm{gr}_{\mathrm{mot}}^* F(S_+^3, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$, by the class with the same name.

Proof. Naturality with respect to

$$q: \mathrm{TC}^-(\mathrm{ko}) = F(E\mathbb{T}_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \longrightarrow F(S_+^3, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$$

shows that it suffices to prove these assertions in the case of the approximate homotopy fixed points. Using Corollary 2.36, the approximate fixed point spectral sequence

$$E^2 = \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu] \{1, t\} \Longrightarrow \overline{A}(1)_* \mathrm{gr}_{\mathrm{mot}}^* F(S_+^3, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$$

and the motivic spectral sequence

$$E_2 = \overline{A}(1)_* \mathrm{gr}_{\mathrm{mot}}^* F(S_+^3, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \Longrightarrow A(1)_* F(S_+^3, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$$

are concentrated in motivic filtrations $0 \leq * \leq 2$ and integer weights; hence the motivic spectral sequence collapses, and the result follows from Proposition 4.3. The claim with coefficients in $\overline{V}(2) \otimes \overline{C}\eta$ follows by passing to cofibers for multiplication by v_2 .

The E^2 -term of the approximate fixed point spectral sequence

$$E^2 = (\overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})) \{1, t\} \Longrightarrow \overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* F(S_+^3, \operatorname{THH}(\operatorname{ko}))^{\mathbb{T}},$$

which computes the E_2 -term of the motivic spectral sequence

$$E_2 = \overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* F(S_+^3, \operatorname{THH}(\operatorname{ko}))^{\mathbb{T}} \Longrightarrow V(1)_* F(S_+^3, \operatorname{THH}(\operatorname{ko}))^{\mathbb{T}},$$

has been determined in Proposition 2.37. Since h_{11} and v_2 in $\pi_* \overline{V}(1)$ map to h_{11} and v_2 in $\pi_* \overline{A}(1)$, the claim with $\overline{V}(1)$ -coefficients follows from the commuting square

$$\begin{array}{ccc} \pi_* \overline{V}(1) & \xrightarrow{i} & \pi_* \overline{A}(1) \\ \downarrow & & \downarrow \\ \overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* F(S_+^3, \operatorname{THH}(\operatorname{ko})) & \xrightarrow{i \otimes 1} & \overline{A}(1)_* \operatorname{gr}_{\operatorname{mot}}^* F(S_+^3, \operatorname{THH}(\operatorname{ko})) \end{array}$$

and the injectivity of the bottom map in the relevant bidegrees, which can be observed from the map of approximate fixed point spectral sequences. Again, the claim with $\overline{V}(2)$ -coefficients follows by passing to cofibers for multiplication by v_2 . $\square$

In § 6, we shall calculate the syntomic cohomology $\overline{A}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TC}(\operatorname{ko})$, and in § 7 we shall use the associated motivic spectral sequence to calculate $A(1)_* \operatorname{TC}(\operatorname{ko})$. The following lemma and corollary will be used to show that the d_3 -differentials in this spectral sequence propagate in a v_2^4 -periodic pattern. Recall from [Lan73, Proposition 2.11] that

$$\mathbb{F}_2[v_2] = \operatorname{Ext}_{\operatorname{BP}_* \operatorname{BP}}^0(\operatorname{BP}_*, \operatorname{BP}_*/(2, v_1)) \subset \operatorname{BP}_*/(2, v_1).$$

LEMMA 4.6. *Let $\operatorname{End} A(1) = F(A(1), A(1))$ be the endomorphism $\mathbb{S}$ -algebra of any one of the spectra $A(1)[ij]$. The induced $\operatorname{BP}_* \operatorname{BP}$ -comodule BP_* -algebra structure on $\operatorname{BP}_* \operatorname{End} A(1)$ descends to a $\operatorname{BP}_* \operatorname{BP}$ -comodule $\operatorname{BP}_*/(2, v_1)$ -algebra structure. Hence the E_2 -term of the Novikov spectral sequence*

$${}^{\operatorname{Nov}} E_2(\operatorname{End} A(1)) = \operatorname{Ext}_{\operatorname{BP}_* \operatorname{BP}}(\operatorname{BP}_*, \operatorname{BP}_* \operatorname{End} A(1)) \Longrightarrow \pi_* \operatorname{End} A(1)$$

is an algebra over $\operatorname{Ext}_{\operatorname{BP}_* \operatorname{BP}}(\operatorname{BP}_*, \operatorname{BP}_*/(2, v_1)) \supset \mathbb{F}_2[v_2]$. In particular,

$$d_2(v_2^2) = 0, \quad d_3(v_2^4) = 0$$

in this Novikov spectral sequence. Moreover, $4 \cdot \operatorname{id} = 0$ in $\pi_0 \operatorname{End} A(1)$.

Proof. To see that the BP_* -algebra structure in $\operatorname{BP}_* \operatorname{BP}$ -comodules descends to a $\operatorname{BP}_*/(2, v_1)$ -algebra structure in that category, it suffices to check that the homomorphism

$$\operatorname{BP}_* \rightarrow \operatorname{BP}_* \operatorname{End} A(1),$$

induced by the $\mathbb{S}$ -algebra unit map, sends 2 and v_1 to zero. This can be verified using the map of Adams spectral sequences

$${}^{\operatorname{Ad}} E_2(\operatorname{BP}) = \operatorname{Ext}_{\mathcal{A}}(H^* \operatorname{BP}, \mathbb{F}_2) \longrightarrow \operatorname{Ext}_{\mathcal{A}}(H^*(\operatorname{BP} \otimes \operatorname{End} A(1)), \mathbb{F}_2) = {}^{\operatorname{Ad}} E_2(\operatorname{BP} \otimes \operatorname{End} A(1)).$$

Standard minimal resolution calculations, which can be obtained from Bruner's `ext` program (see [Bru93, BR21]), show that multiplication by h_0 and $\langle h_0, h_1, - \rangle$ both act trivially from bidegree $(0, 0)$ on the right-hand side, and that there are no classes in stems 0 or 2 that have Adams filtration ≥ 2 . The last fact also implies that $4 \cdot \operatorname{id} = 0$ in $\pi_0 \operatorname{End} A(1)$, and a closer inspection shows that $2 \cdot \operatorname{id} \neq 0$ in each case.

The functor $\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, -)$ is lax symmetric monoidal, so it follows that ${}^{\mathrm{Nov}}E_2(\mathrm{End} A(1))$ is an $\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/(2, v_1))$ -algebra. In particular, it is an $\mathbb{F}_2[v_2]$ -algebra, with v_2 acting centrally and $1 + 1 = 0$. Hence we have equalities $d_2(v_2^2) = d_2(v_2) \cdot v_2 + v_2 \cdot d_2(v_2) = 0$ and $d_3(v_2^4) = d_3(v_2^2) \cdot v_2^2 + v_2^2 \cdot d_3(v_2^2) = 0$. $\square$

The tautological left action of $\mathrm{End} A(1)$ on $A(1)$ induces a left action of the Novikov filtration of $\mathrm{End} A(1)$ on the Novikov filtration of $A(1)$. The latter is equivalent to the even filtration $\mathrm{fil}_{\mathrm{ev}}^* A(1)$, since $\mathrm{MU}_* A(1)$ is concentrated in even degrees. Hence $\mathrm{fil}_{\mathrm{Nov}}^* \mathrm{End} A(1)$ also acts on the convolution product filtration

$$\mathrm{fil}_{\mathrm{ev}}^* A(1) \otimes_{\mathrm{fil}_{\mathrm{ev}}^*} \mathrm{fil}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko}),$$

and induces a left action of the Novikov spectral sequence for $\mathrm{End} A(1)$ on the motivic spectral sequence converging to $A(1)_* \mathrm{TC}(\mathrm{ko})$.

COROLLARY 4.7. *The differentials in the motivic spectral sequence*

$$E_2 = \overline{A}(1)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko}) \implies A(1)_* \mathrm{TC}(\mathrm{ko})$$

satisfy $d_3(v_2^4 \cdot y) = v_2^4 \cdot d_3(y)$ for all $y \in E_3$.

Proof. This follows from the Leibniz rule $d_3(v_2^4 \cdot y) = d_3(v_2^4) \cdot y + v_2^4 \cdot d_3(y)$ for the pairing of spectral sequences

$${}^{\mathrm{Nov}}E_2(\mathrm{End} A(1)) \otimes \overline{A}(1)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko}) \longrightarrow \overline{A}(1)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko}),$$

since $d_3(v_2^4) = 0$ by Lemma 4.6. $\square$

5. Prismatic cohomology

Recall that by Corollary 3.4, there is a $\overline{V}(2)$ -homotopy $\mathbb{T}$ -Tate spectral sequence

$$\begin{aligned} \hat{E}^2(\mathbb{T}) &= \overline{V}(2)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko}) [t^{\pm 1}] \\ &\implies \overline{V}(2)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TP}(\mathrm{ko}) \end{aligned} \quad (5.1)$$

and a $\overline{V}(2) \otimes \overline{C}\eta$ -homotopy $\mathbb{T}$ -Tate spectral sequence

$$\begin{aligned} \hat{E}^2(\mathbb{T}) &= (\overline{V}(2) \otimes \overline{C}\eta)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko}) [t^{\pm 1}] \\ &\implies (\overline{V}(2) \otimes \overline{C}\eta)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TP}(\mathrm{ko}). \end{aligned} \quad (5.2)$$

They can be reindexed as cohomologically graded periodic t -Bockstein spectral sequences, in which case $\hat{E}^{2r}(\mathbb{T})$ and d^{2r} correspond to E_r and d_r . However, we shall need to make a comparison with similar C_2 -Tate spectral sequences, for which our indexing is convenient.

THEOREM 5.1 (Prismatic cohomology modulo $(2, v_1, v_2)$ of ko). *The $\overline{V}(2)$ -homotopy $\mathbb{T}$ -Tate spectral sequence (5.1) is an algebra spectral sequence with E^2 -term*

$$\hat{E}^2(\mathbb{T}) = \Lambda(\varepsilon_2) \otimes \frac{\mathbb{F}_2[\eta, \lambda'_1, \mu]}{(\eta\lambda'_1, (\lambda'_1)^2 = \eta^2\mu)} \otimes \mathbb{F}_2[t^{\pm 1}]$$

and differentials

$$\begin{aligned} d^2(t^{-1}) &= \eta, & d^2(\varepsilon_2) &= t\mu, \\ d^6(t^{-2}) &= t\lambda'_1, & d^6(t^{-1}\lambda'_1) &= t^2(\lambda'_1)^2 = t^2\eta^2\mu, \end{aligned}$$

leading to

$$\hat{E}^\infty(\mathbb{T}) = \mathbb{F}_2\{1, t^2\lambda'_1, (t^2\lambda'_1)^2, \lambda'_1\} \otimes \mathbb{F}_2[t^{\pm 4}].$$

Hence there is a preferred isomorphism

$$\overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TP}(\operatorname{ko}) \cong \mathbb{F}_2\{1, \eta, \eta^2, \lambda'_1\} \otimes \mathbb{F}_2[t^{\pm 4}],$$

where $1, \eta, \eta^2, \lambda'_1$ and $t^{\pm 4}$ in bidegrees $(0, 0), (1, 1), (2, 2), (5, 1)$ and $(\mp 8, 0)$ are detected by $1, t^2\lambda'_1, (t^2\lambda'_1)^2 = t^4\eta^2\mu, \lambda'_1$ and $t^{\pm 4}$, respectively.

THEOREM 5.2 (Prismatic cohomology modulo $(2, \eta, v_1, v_2)$ of ko). *The $\overline{V}(2) \otimes \overline{C}\eta$ -homotopy $\mathbb{T}$ -Tate spectral sequence (5.2) is a module spectral sequence over (5.1), with E^2 -term*

$$\hat{E}^2(\mathbb{T}) = \Lambda(\varepsilon_2) \otimes \Lambda(\lambda'_1)\{1, \lambda_2\} \otimes \mathbb{F}_2[\mu] \otimes \mathbb{F}_2[t^{\pm 1}]$$

and differentials

$$\begin{aligned} d^2(\varepsilon_2) &= t\mu, \\ d^6(t^{-1}) &= t^2\lambda'_1, & d^6(t^{-2}) &= t\lambda'_1, \\ d^6(t^{-1}\lambda_2) &= t^2\lambda'_1\lambda_2, & d^6(t^{-2}\lambda_2) &= t\lambda'_1\lambda_2, \\ d^8(t^{-3}) &= t\lambda_2, & d^8(t^{-1}\lambda'_1) &= t^3\lambda'_1\lambda_2, \end{aligned}$$

leading to

$$\hat{E}^\infty(\mathbb{T}) = \Lambda(\lambda'_1)\{1, \lambda_2\} \otimes \mathbb{F}_2[t^{\pm 4}].$$

Hence there is a preferred isomorphism

$$(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TP}(\operatorname{ko}) \cong \Lambda(\lambda'_1)\{1, \lambda_2\} \otimes \mathbb{F}_2[t^{\pm 4}],$$

where $1, \lambda'_1, \lambda_2, \lambda'_1\lambda_2$ and $t^{\pm 4}$ in bidegrees $(0, 0), (5, 1), (7, 1), (12, 2)$ and $(\mp 8, 0)$ are detected by the classes in the E^∞ -term with the same names.

The proofs of these theorems will occupy the remainder of this section. By Corollary 2.38, we can identify the E^2 -term in (5.1) as stated in Theorem 5.1.

PROPOSITION 5.3. *The spectral sequence (5.1) is multiplicative and has differentials*

$$\begin{aligned} d^2(t^{-1}) &= \eta, & d^2(\eta) &= 0, & d^2(\lambda'_1) &= 0, \\ d^2(\varepsilon_2) &= t\mu, & d^2(\mu) &= t\eta\mu. \end{aligned}$$

Consequently, we can identify

$$\hat{E}^4(\mathbb{T}) = \mathbb{F}_2\{1, t\lambda'_1, \lambda'_1, \eta^2\mu\} \otimes \mathbb{F}_2[t^{\pm 2}]$$

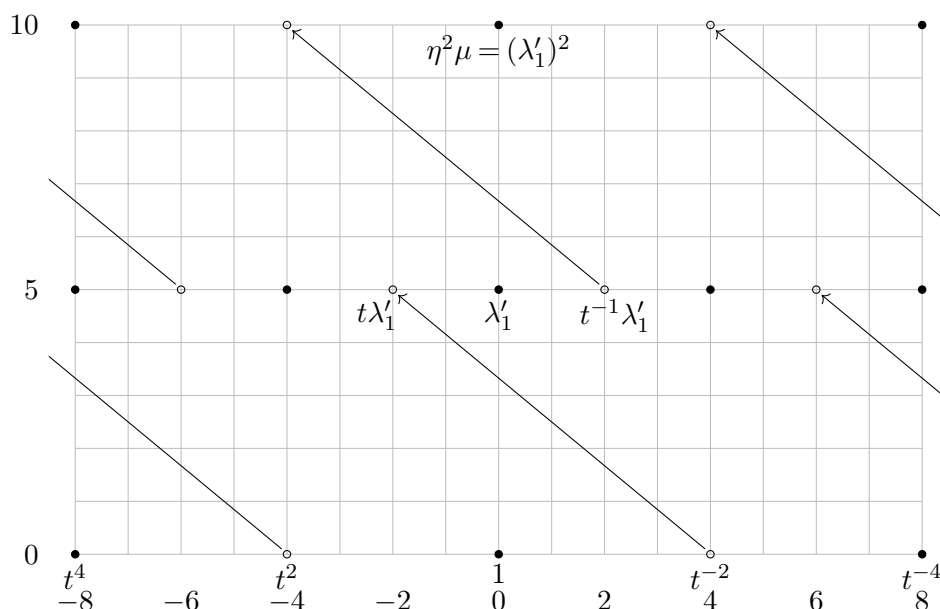
with $\eta^2\mu = \eta^3\varepsilon_2$.

Proof. The first claim follows from Corollary 3.4. Using the $\mathbb{T}$ -equivariant attaching maps of the standard $\mathbb{T}$ -CW complex structure on $S^\infty = E\mathbb{T}$, we compute differentials

$$d^2(t^{-1}) = \eta, \quad d^2(\eta) = 0$$

as in [Hes96, Lemma 1.4.2].

We know that $d^2(t\lambda'_1) = 0$, because $t\lambda'_1$ detects $\nu \in \{h_{11}\}$ by Corollary 4.5. Consequently, $d^2(\lambda'_1) = 0$ by the Leibniz rule and the fact that $\eta \cdot \lambda'_1 = 0$.


 FIGURE 7. The $\mathbb{T}$ -Tate $\hat{E}^4(\mathbb{T}) \Rightarrow \bar{V}(2)_* \text{gr}_{\text{mot}}^* \text{TP}(\text{ko})$.

To show that $d_2(\varepsilon_2) = t\mu$ we apply the graded analogue of [BR22, Proposition 2.3] (a variant of [BG95, Lemma 2.2]) to the following smash product of cofiber sequences.

$$\begin{array}{ccccc}
 \Sigma^{4,-1}\bar{V}(1) \otimes \text{THH}(\text{ko}) & \longrightarrow & \Sigma^{6,0}\bar{V}(1) \otimes F(S_+^3, \text{THH}(\text{ko}))^{\mathbb{T}} & \xrightarrow{p} & \Sigma^{6,0}\bar{V}(1) \otimes \text{THH}(\text{ko}) \\
 \downarrow v_2 & & \downarrow v_2 & & \downarrow v_2 \\
 \Sigma^{-2,-1}\bar{V}(1) \otimes \text{THH}(\text{ko}) & \longrightarrow & \bar{V}(1) \otimes F(S_+^3, \text{THH}(\text{ko}))^{\mathbb{T}} & \xrightarrow{p} & \bar{V}(1) \otimes \text{THH}(\text{ko}) \\
 \downarrow i_2 & & \downarrow i_2 & & \downarrow i_2 \\
 \Sigma^{-2,-1}\bar{V}(2) \otimes \text{THH}(\text{ko}) & \longrightarrow & \bar{V}(2) \otimes F(S_+^3, \text{THH}(\text{ko}))^{\mathbb{T}} & \xrightarrow{p} & \bar{V}(2) \otimes \text{THH}(\text{ko})
 \end{array}$$

Starting in the upper right-hand corner, the unit $\Sigma^{6,0}1$ lifts over the connecting map j_2 to ε_2 in the lower right-hand corner and maps under the connecting map σ to $d^2(\varepsilon_2)$ in the lower left-hand corner. This is the same as the result of lifting to $\Sigma^{6,0}1$ at the top, mapping to v_2 in the center, lifting to $t\mu$ at the left, and pushing to $t\mu$ in the lower left-hand corner.

In particular, $t\mu$ is a d^2 -cycle, and the Leibniz rule implies that $d^2(\mu) = t\eta\mu$. This leads to the E^4 -term shown in Figure 7. $\square$

Remark 5.4. In Figure 7 (respectively, Figure 8), the horizontal axis is the $\mathbb{T}$ -Tate filtration and the vertical axis is the stem of $\bar{V}(2)_* \text{gr}_{\text{mot}}^* \text{THH}(\text{ko})$ (respectively, $(\bar{V}(2) \otimes \bar{C}\eta)_* \text{gr}_{\text{mot}}^* \text{THH}(\text{ko})$). The spectral sequence converges to the sum of the two gradings and the differentials follow the convention of the homological Serre spectral sequence; in other words, the d^r -differential goes from the coordinate (a, b) to $(a - r, b + r - 1)$. The d^r -differentials also increase motivic filtration by one. Similar conventions apply for the $\mathbb{T}$ -homotopy fixed point spectral sequences in Figures 9 and 10.

PROPOSITION 5.5. *The classes $t^{\pm 4}$ and λ'_1 are permanent cycles in the spectral sequence (5.1). Moreover, there are differentials*

$$d^6(t^{-2}) = t\lambda'_1, \quad d^6(t^{-1}\lambda'_1) = t^2\eta^2\mu.$$

The spectral sequence (5.1) collapses at $\hat{E}^8(\mathbb{T}) = \hat{E}^\infty(\mathbb{T})$.

Proof. We know that $t\lambda'_1$ is an infinite cycle in (5.1), because by Corollary 4.5 it detects $\nu \in \{h_{11}\}$ in $\bar{V}(2)_* \text{gr}_{\text{mot}}^* \text{TC}^-(\text{ko})$. By Corollary 3.4 and Proposition 3.5 there is a commutative square of spectral sequences converging to

$$\begin{array}{ccc} \bar{V}(2)_* \text{gr}_{\text{mot}}^* \text{TC}^-(\text{ko}) & \xrightarrow{F^h} & \bar{V}(2)_* \text{gr}_{\text{mot}}^* \text{THH}(\text{ko})^{hC_2} \\ \text{can} \downarrow & & \downarrow \text{can} \\ \bar{V}(2)_* \text{gr}_{\text{mot}}^* \text{TP}(\text{ko}) & \xrightarrow{F^t} & \bar{V}(2)_* \text{gr}_{\text{mot}}^* \text{THH}(\text{ko})^{tC_2} \end{array}$$

with E^2 -terms as follows.

$$\begin{array}{ccc} \bar{V}(2)_* \text{gr}_{\text{mot}}^* \text{THH}(\text{ko}) \otimes \mathbb{F}_2[t] & \longrightarrow & \bar{V}(2)_* \text{gr}_{\text{mot}}^* \text{THH}(\text{ko}) \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t] \\ \downarrow & & \downarrow \\ \bar{V}(2)_* \text{gr}_{\text{mot}}^* \text{THH}(\text{ko}) \otimes \mathbb{F}_2[t^{\pm 1}] & \longrightarrow & \bar{V}(2)_* \text{gr}_{\text{mot}}^* \text{THH}(\text{ko}) \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 1}] \end{array}$$

Recall that in the two right-hand cases, u_1 has (stem, motivic filtration) bidegree $\|u_1\| = (-1, -1)$.

We know that $\nu \in \{h_{11}\}$ maps to zero in $\bar{V}(2)_* \text{gr}_{\text{mot}}^* \text{THH}(\text{ko})$, because the target is zero in the relevant bidegree by Corollary 2.38. A chase in the diagram

$$\begin{array}{ccccc} & & \mathbb{S} & & \\ & & \downarrow & & \\ \text{TF}(\text{ko}) & \xrightarrow{F} & \text{THH}(\text{ko})^{C_2} & \xrightarrow{R} & \text{THH}(\text{ko}) \\ \downarrow \Gamma & & \downarrow \Gamma_1 & & \downarrow \hat{\Gamma}_1 = \varphi_2 \\ \text{TC}^-(\text{ko}) & \xrightarrow{F^h} & \text{THH}(\text{ko})^{hC_2} & \xrightarrow{R^h = \text{can}} & \text{THH}(\text{ko})^{tC_2} \\ \downarrow q & & \downarrow F^h & & \\ F(S^3_+, \text{THH}(\text{ko}))^{\mathbb{T}} & \xrightarrow{p} & \text{THH}(\text{ko}) & & \end{array}$$

similar to the proof of [AR02, Theorem 5.5], shows that $t\lambda'_1$ must be a boundary in the $\bar{V}(2)$ -homotopy C_2 -Tate spectral sequence. There is no earlier C_2 -Tate d^r -differential hitting $t\lambda'_1$, for $2 \leq r \leq 5$, since η is an infinite cycle. Consequently,

$$d^6(t^{-2}) = t\lambda'_1$$

in the $\bar{V}(2)$ -homotopy C_2 -Tate spectral sequence, and therefore also in the $\bar{V}(2)$ -homotopy $\mathbb{T}$ -Tate spectral sequence (cf. Figure 7). Here, we use the fact that the map from the $\bar{V}(2)$ -homotopy $\mathbb{T}$ -Tate spectral sequence to the $\bar{V}(2)$ -homotopy C_2 -Tate spectral sequence is injective in the relevant bidegrees. (In fact, as we will observe in Proposition 6.3, the map of spectral sequences is injective at every page because the class u_1 is a permanent cycle in the target spectral sequence.)

To complete the proof, we use the Leibniz rule to deduce that $d^6(t^{-1}\lambda'_1) = d^6(t\lambda'_1 \cdot t^{-2}) = (t\lambda'_1)^2 = t^2\eta^2\mu$, that $d^6(t^{-4}) = t\lambda'_1 \cdot t^{-2} + t^{-2} \cdot t\lambda'_1 = 0$, and that $d^6(t^4) = 0$. All later differentials are zero, because the target groups are trivial. $\square$

The following corollary immediately follows from Proposition 5.5.

COROLLARY 5.6. *There are isomorphisms*

$$\overline{V}(2)_n \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TP}(\operatorname{ko}) \cong \begin{cases} \mathbb{F}_2\{1\} & \text{for } n = 0, \\ \mathbb{F}_2\{t^2\lambda'_1\} & \text{for } n = 1, \\ \mathbb{F}_2\{t^4\eta^2\mu\} & \text{for } n = 2, \\ 0 & \text{for } n = 3, 4, 6, 7, \\ \mathbb{F}_2\{\lambda'_1\} & \text{for } n = 5. \end{cases}$$

These repeat 8-periodically, via multiplication by $t^{\pm 4}$. $\square$

We now move towards computing the spectral sequence (5.2). By Corollary 2.36, its E^2 -term has the form asserted in Theorem 5.2.

Remark 5.7. In general, the differentials in (5.2) do not satisfy the Leibniz rule with respect to the (non-equivariant) algebra structure from Corollary 2.36. This is commensurable with Remark 2.32.

PROPOSITION 5.8. *The spectral sequence (5.2) is a module over the spectral sequence (5.1). There are differentials*

$$d^2(\varepsilon_2) = t\mu, \quad d^6(t^{-2}) = t\lambda'_1, \quad d^8(t^{-3}) = t\lambda_2$$

in (5.2), and multiplication by $t^{\pm 4}$ and λ'_1 commutes with all differentials in this spectral sequence.

Proof. The unit map $\overline{V}(2) \rightarrow \overline{V}(2) \otimes \overline{C}\eta$ is a map of $\overline{V}(2)$ -modules, so (5.2) is a module spectral sequence over (5.1), and the map from (5.1) to (5.2) respects this module structure. This implies that multiplication by the infinite cycles $t^{\pm 4}$ and λ'_1 will commute with each differential in (5.2). The module structure also implies that $d^2(\varepsilon_2 \cdot 1) = t\mu \cdot 1 - \varepsilon_2 \cdot d^2(1)$, so that $d^2(\varepsilon_2) = t\mu$ in (5.2). It follows that

$$\hat{E}^4(\mathbb{T}) = \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[t^{\pm 1}]$$

(see Figure 8).

We have shown in Corollary 4.5 that the images in $(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TC}^-(\operatorname{ko})$ of $\nu \in \{h_{11}\}$ and w are detected by $t\lambda'_1$ and $t\lambda_2$, respectively, so the same holds in $(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})^{hC_2}$. We also know that the images of ν and w are trivial in $(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})$, so this means that classes detected by $t\lambda'_1$ and $t\lambda_2$ map trivially to $(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})^{tC_2}$. It follows that $t\lambda'_1$ and $t\lambda_2$ must be hit by differentials in the $\overline{V}(2) \otimes \overline{C}\eta$ -homotopy C_2 -Tate spectral sequence. By examination of bidegrees, the only possibility is that $d^6(t^{-2}) = t\lambda'_1$ and $d^8(t^{-3}) = t\lambda_2$. Since the map of spectral sequences converging to

$$F^t: (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TP}(\operatorname{ko}) \longrightarrow (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})^{tC_2}$$

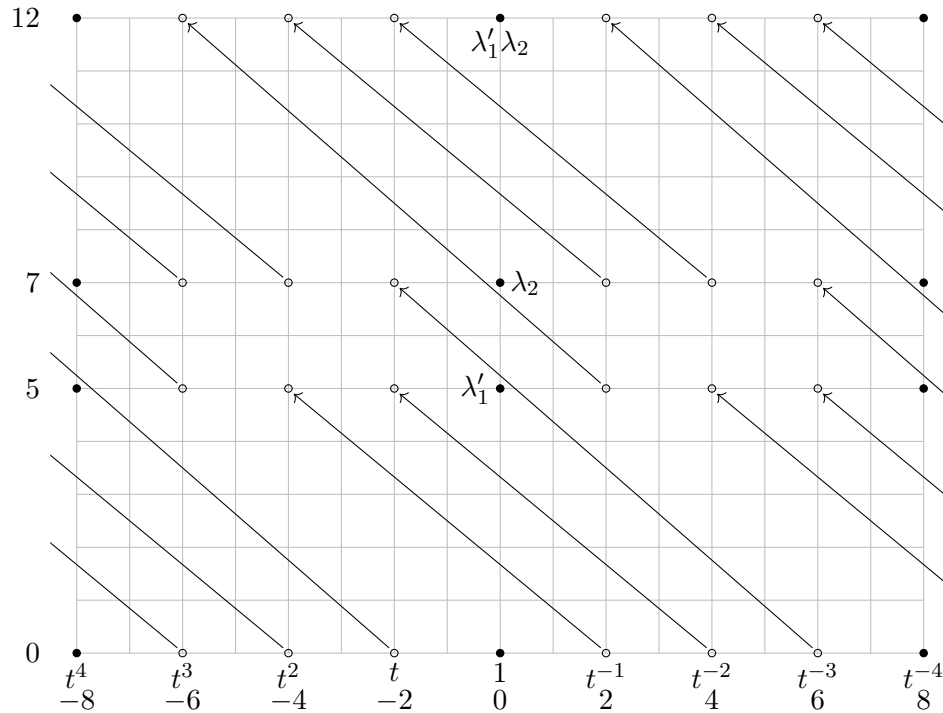
is injective in the relevant bidegrees, we also have the stated differentials in the spectral sequence (5.2). $\square$

PROPOSITION 5.9. *There are differentials*

$$\begin{aligned} d^6(t^{-1}) &= t^2\lambda'_1, & d^6(t^{-1}\lambda_2) &= t^2\lambda'_1\lambda_2, \\ d^6(t^{-2}\lambda_2) &= t\lambda'_1\lambda_2, & d^6(\lambda_2) &= 0 \end{aligned}$$

in the spectral sequence (5.2).

Proof. We must have $d^6(t^{-3}) = 0$ in the spectral sequence (5.2), since t^{-3} survives to its E^8 -term by Proposition 5.8. By Proposition 5.5, we have $d^6(t^{-2}) = t\lambda'_1$ in the spectral sequence (5.1). Using


 FIGURE 8. The T-Tate $\hat{E}^4(\mathbb{T}) \Rightarrow (\bar{V}(2) \otimes \bar{C}\eta)_* \text{gr}_{\text{mot}}^* \text{TP}(\text{ko})$.

the module structure of (5.2) over (5.1), we deduce that $d^6(t^{-5}) = d^6(t^{-2} \cdot t^{-3}) = t\lambda'_1 \cdot t^{-3} + t^{-2} \cdot 0 = t^{-2}\lambda'_1$ and $d^6(t^{-1}) = t^4 \cdot d^6(t^{-5}) = t^2\lambda'_1$.

Since $t\lambda_2$ is a d^8 -boundary by Proposition 5.8, it must be a d^6 -cycle, which implies that $d^6(t^{-1}\lambda_2) = d^6(t^{-2} \cdot t\lambda_2) = t\lambda'_1 \cdot t\lambda_2 + t^{-2} \cdot 0 = t^2\lambda'_1\lambda_2$.

The fact that $t\lambda_2$ is a d^8 -boundary also implies that $\lambda'_1 \cdot t\lambda_2 = t\lambda'_1\lambda_2$ must be a d^r -boundary for some $r \leq 8$. Since $t^{-3}\lambda'_1 = t^{-4} \cdot t\lambda'_1$ is a d^6 -boundary, it cannot be the source of this d^r -differential, so the only remaining possibility is that $d^6(t^{-2}\lambda_2) = t\lambda'_1\lambda_2$.

Using the module structure over the spectral sequence (5.1), we can also conclude that $d^6(\lambda_2) = d^6(t^2 \cdot t^{-2}\lambda_2) = t^5\lambda'_1 \cdot t^{-2}\lambda_2 + t^2 \cdot t\lambda'_1\lambda_2 = 0$. $\square$

COROLLARY 5.10. *There are isomorphisms*

$$(\bar{V}(2) \otimes \bar{C}\eta)_n \text{gr}_{\text{mot}}^* \text{TP}(\text{ko}) \cong \begin{cases} \mathbb{F}_2\{1\} & \text{for } n = 0, \\ 0 & \text{for } n = 1, 2, 3, \\ \mathbb{F}_2\{t^4\lambda'_1\lambda_2\} & \text{for } n = 4, \\ \mathbb{F}_2\{\lambda'_1\} & \text{for } n = 5, \end{cases}$$

and these repeat 8-periodically, via multiplication by $t^{\pm 4}$.

Proof. This follows directly from Propositions 5.8 and 5.9. $\square$

PROPOSITION 5.11.

- The unit images of η and η^2 are detected by $t^2\lambda'_1$ and $t^4\eta^2\mu$, respectively, in the spectral sequence (5.1).
- There is a nonzero differential

$$d^8(t^{-1}\lambda'_1) = t^3\lambda'_1\lambda_2$$

in the $\overline{V}(2) \otimes \overline{C}\eta$ -homotopy $\mathbb{T}$ -Tate spectral sequence (5.2). Hence

$$(\overline{V}(2) \otimes \overline{C}\eta)_n \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TP}(\operatorname{ko}) \cong \begin{cases} 0 & \text{for } n = 6, \\ \mathbb{F}_2\{\lambda_2\} & \text{for } n = 7, \end{cases}$$

repeating 8-periodically via multiplication by $t^{\pm 4}$.

Proof.

(a) The $\overline{V}(2)$ -module cofiber sequence (2.6) induces a long exact sequence

$$\begin{aligned} \cdots \longrightarrow (\overline{V}(2) \otimes \overline{C}\eta)_{n+2} \operatorname{gr}_{\operatorname{mot}}^{*+1} \operatorname{TP}(\operatorname{ko}) &\xrightarrow{j} \overline{V}(2)_n \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TP}(\operatorname{ko}) \\ &\xrightarrow{\eta} \overline{V}(2)_{n+1} \operatorname{gr}_{\operatorname{mot}}^{*+1} \operatorname{TP}(\operatorname{ko}) \xrightarrow{i} (\overline{V}(2) \otimes \overline{C}\eta)_{n+1} \operatorname{gr}_{\operatorname{mot}}^{*+1} \operatorname{TP}(\operatorname{ko}) \longrightarrow \cdots \end{aligned}$$

By the cases $n \in \{1, 2\}$ of Corollary 5.10, and exactness, it follows that η and η^2 generate $\overline{V}(2)_n \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TP}(\operatorname{ko}) \cong \mathbb{F}_2$ for $n = 1$ and 2 , and hence are detected by the only classes in stems 1 and 2 , namely, $t^2\lambda'_1$ and $t^4\eta^2\mu$, in the E^∞ -term of (5.1).

(b) By Corollary 5.6 and exactness, it follows that $(\overline{V}(2) \otimes \overline{C}\eta)_n \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TP}(\operatorname{ko})$ is 0 for $n = 6$ and $\mathbb{F}_2$ for $n = 7$. Hence $t^3\lambda'_1\lambda_2$ in stem 6 cannot survive to the E^∞ -term of (5.2), and since $d^6(\lambda_2) = 0$ by Proposition 5.9 the only possible source of a differential killing it is $t^{-1}\lambda'_1$. Hence $d^8(t^{-1}\lambda'_1) = t^3\lambda'_1\lambda_2$, and the lone surviving class in stem 7 of the E^∞ -term of (5.2) is λ_2 .

In summary, the homomorphisms in the long exact sequence above are given, at the level of detecting classes, by $\eta: 1 \mapsto t^2\lambda'_1$, $\eta: t^2\lambda'_1 \mapsto t^4\eta^2\mu$, $i: 1 \mapsto 1$, $i: \lambda'_1 \mapsto \lambda'_1$, $j: t^4\lambda'_1\lambda_2 \mapsto t^4\eta^2\mu$ and $j: \lambda_2 \mapsto \lambda'_1$. $\square$

Proof of Theorem 5.1. This theorem follows by combining the differentials from Propositions 5.3–5.5 with the detection result from Proposition 5.11(a). $\square$

Proof of Theorem 5.2. This theorem follows by combining the differentials from Propositions 5.8, 5.9 and 5.11(b). $\square$

6. Syntomic cohomology

We shall now calculate the syntomic cohomology modulo $(2, v_1)$ and $(2, \eta, v_1)$ of ko (cf. Definition 2.18). We first carry out these computations in $\overline{V}(2)$ - and $\overline{V}(2) \otimes \overline{C}\eta$ -homotopy, and then use v_2 -Bockstein spectral sequences to lift the results to $\overline{V}(1)$ - and $\overline{A}(1)$ -homotopy.

By Corollary 3.4, there is a $\overline{V}(2)$ -homotopy $\mathbb{T}$ -homotopy fixed point spectral sequence

$$\begin{aligned} E^2(\mathbb{T}) &= \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko}) [t] \\ &= \Lambda(\varepsilon_2) \otimes \frac{\mathbb{F}_2[\eta, \lambda'_1, \mu]}{(\eta\lambda'_1, (\lambda'_1)^2 = \eta^2\mu)} \otimes \mathbb{F}_2[t] \\ &\implies \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TC}^-(\operatorname{ko}) \end{aligned} \tag{6.1}$$

and a $\overline{V}(2) \otimes \overline{C}\eta$ -homotopy $\mathbb{T}$ -homotopy fixed point spectral sequence

$$\begin{aligned} E^2(\mathbb{T}) &= (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko}) [t] \\ &= \Lambda(\varepsilon_2) \otimes \Lambda(\lambda'_1)\{1, \lambda_2\} \otimes \mathbb{F}_2[\mu] \otimes \mathbb{F}_2[t] \\ &\implies (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TC}^-(\operatorname{ko}). \end{aligned} \tag{6.2}$$

The former is an algebra spectral sequence, and the latter is a module spectral sequence over it by Corollary 3.4. They can be reindexed as cohomologically graded t -Bockstein spectral sequences,

but the current indexing is the one inherited from the homologically graded C_2 - and $\mathbb{T}$ -Tate spectral sequences (see Figures 9 and 10).

PROPOSITION 6.1. *There is an isomorphism*

$$\begin{aligned}\overline{V}(2)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}^-(\mathrm{ko}) &\cong \mathbb{F}_2[t^4]\{1, t^2\lambda'_1, \lambda'_1, (\lambda'_1)^2\} \\ &\oplus \mathbb{F}_2\{t\lambda'_1, (t\lambda'_1)^2\} \\ &\oplus \mathbb{F}_2[\eta]\{\eta, \eta^4\varepsilon_2\} \\ &\oplus \mathbb{F}_2[\bar{\mu}]\{\bar{\mu}, \eta\bar{\mu}, \eta^2\bar{\mu}, \lambda'_1\mu\}\end{aligned}$$

with $\bar{\mu} = \mu + \eta\varepsilon_2$, where $(\lambda'_1)^2 = \eta^2\mu \neq \eta^2\bar{\mu}$, $\eta \cdot \eta^2\bar{\mu} = \eta^4\varepsilon_2$ and $\bar{\mu}^2 = \mu^2$.

Proof. The map of spectral sequences induced by $\mathrm{can}: \mathrm{TC}^-(\mathrm{ko}) \rightarrow \mathrm{TP}(\mathrm{ko})$ is given at the E^2 -terms by inverting t , so the differentials in (5.1) from Theorem 5.1 lift to differentials

$$\begin{aligned}d^2(t) &= t^2\eta, & d^2(\varepsilon_2) &= t\mu, \\ d^6(t^2) &= t^5\lambda'_1, & d^6(t^3\lambda'_1) &= t^6(\lambda'_1)^2\end{aligned}$$

in (6.1). Moreover, η , λ'_1 and $t\mu$ are infinite cycles. Some bookkeeping shows that

$$\begin{aligned}E^4(\mathbb{T}) &= \mathbb{F}_2[t^2]\{1, t\lambda'_1, \lambda'_1, (\lambda'_1)^2\} \\ &\oplus \mathbb{F}_2[\eta]\{\eta, \eta^4\varepsilon_2\} \\ &\oplus \mathbb{F}_2[\bar{\mu}]\{\bar{\mu}, \eta\bar{\mu}, \eta^2\bar{\mu}, \lambda'_1\mu\}\end{aligned}$$

with $\bar{\mu} = \mu + \eta\varepsilon_2$ and $\eta \cdot \eta^2\bar{\mu} = \eta^4\varepsilon_2$, and

$$\begin{aligned}E^8(\mathbb{T}) &= E^\infty(\mathbb{T}) = \mathbb{F}_2[t^4]\{1, t^2\lambda'_1, \lambda'_1, (\lambda'_1)^2\} \\ &\oplus \mathbb{F}_2\{t\lambda'_1, (t\lambda'_1)^2\} \\ &\oplus \mathbb{F}_2[\eta]\{\eta, \eta^4\varepsilon_2\} \\ &\oplus \mathbb{F}_2[\bar{\mu}]\{\bar{\mu}, \eta\bar{\mu}, \eta^2\bar{\mu}, \lambda'_1\mu\}.\end{aligned}$$

□

PROPOSITION 6.2. *There is an isomorphism*

$$\begin{aligned}(\overline{V}(2) \otimes \overline{C}\eta)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}^-(\mathrm{ko}) &\cong \frac{\mathbb{F}_2[t^4, \mu]}{(t^4\mu)} \otimes \Lambda(\lambda'_1)\{1, \lambda_2\} \\ &\oplus \mathbb{F}_2\{t^2\lambda'_1, t\lambda'_1, t\lambda_2, t^3\lambda'_1\lambda_2, t^2\lambda'_1\lambda_2, t\lambda'_1\lambda_2\}.\end{aligned}$$

Proof. The differentials in (5.2) from Theorem 5.2 lift over the canonical map to differentials $d^2(\varepsilon_2) = t\mu$ (repeating t -periodically) and

$$\begin{aligned}d^6(t^2) &= t^5\lambda'_1, & d^6(t^3) &= t^6\lambda'_1, \\ d^6(t^2\lambda_2) &= t^5\lambda'_1\lambda_2, & d^6(t^3\lambda_2) &= t^6\lambda'_1\lambda_2, \\ d^8(t) &= t^5\lambda_2, & d^8(t^3\lambda'_1) &= t^7\lambda'_1\lambda_2\end{aligned}$$

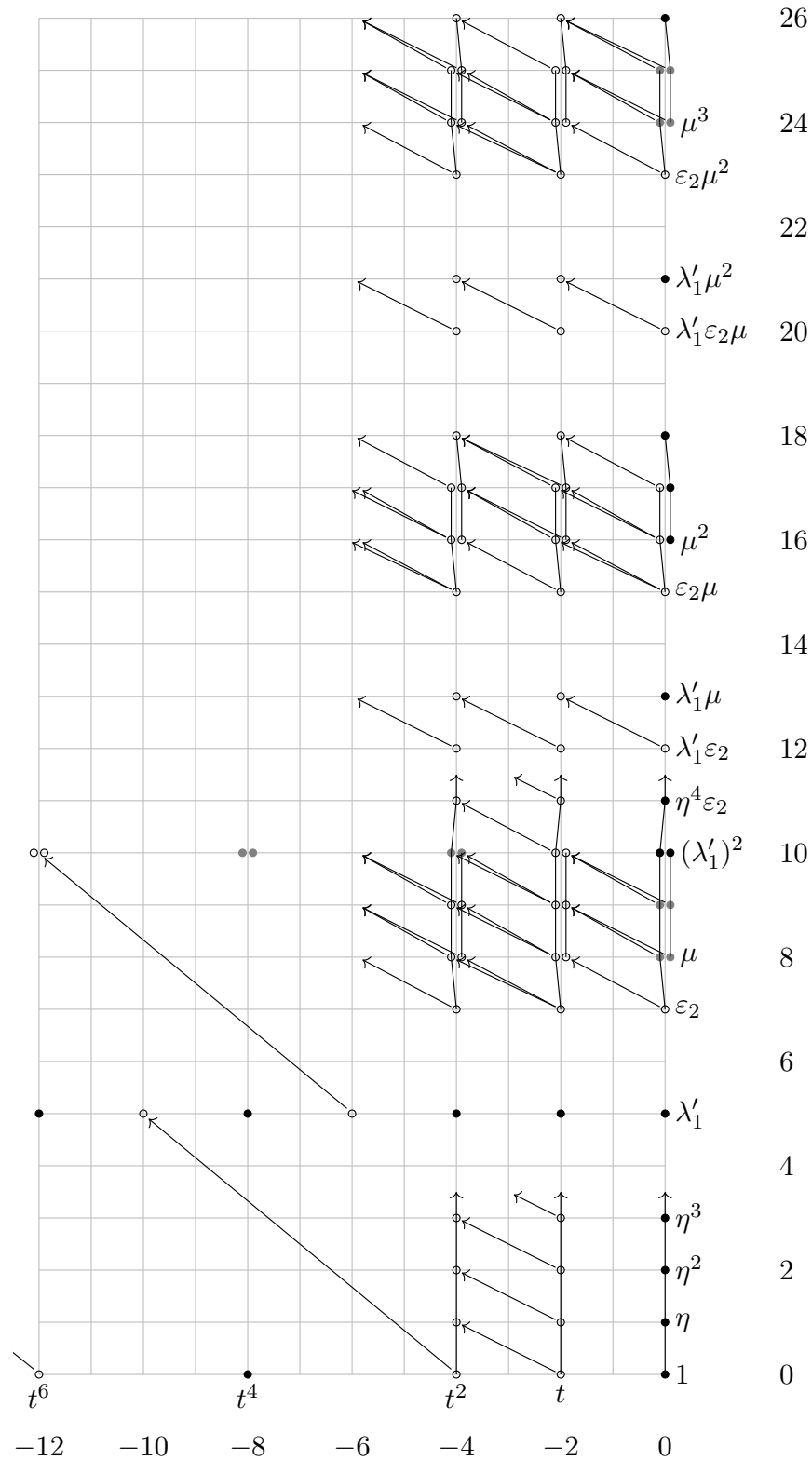
(repeating t^4 -periodically) in (6.2). It follows that

$$E^4(\mathbb{T}) = \mathbb{F}_2[t, \mu]/(t\mu) \otimes \Lambda(\lambda'_1)\{1, \lambda_2\},$$

and $E^{10}(\mathbb{T}) = E^\infty(\mathbb{T})$ is equal to

$$\frac{\mathbb{F}_2[t^4, \mu]}{(t^4\mu)} \otimes \Lambda(\lambda'_1)\{1, \lambda_2\} \oplus \mathbb{F}_2\{t^2\lambda'_1, t\lambda'_1, t\lambda_2, t^3\lambda'_1\lambda_2, t^2\lambda'_1\lambda_2, t\lambda'_1\lambda_2\}.$$

□


 FIGURE 9. The $\mathbb{T}$ -homotopy fixed point spectral sequence converging to $\overline{V}(2)_* \text{gr}_{\text{mot}}^* \text{TC}^-(\text{ko})$.

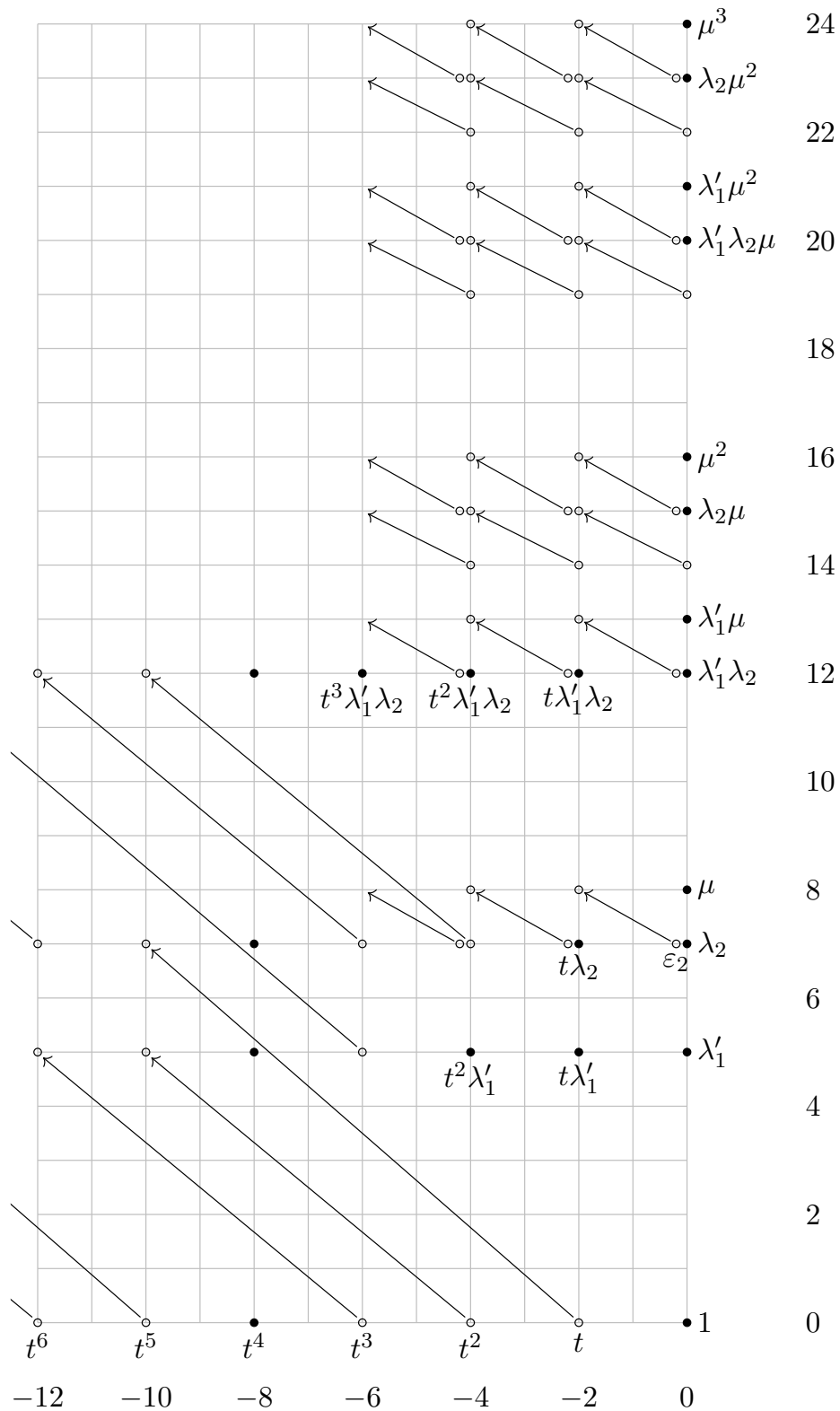


FIGURE 10. The $\mathbb{T}$ -homotopy fixed point spectral sequence converging to the bigraded $\mathbb{F}_2$ -module $(\overline{V}(2) \otimes \overline{C}\eta)_* \mathfrak{g}_{\text{mot}}^* \text{TC}^-(\text{ko})$.

As discussed in the proof of Proposition 5.5, there is a $\overline{V}(2)$ -homotopy C_2 -Tate spectral sequence

$$\begin{aligned}\hat{E}^2(C_2) &= \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko}) \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 1}] \\ &= \Lambda(\varepsilon_2) \otimes \frac{\mathbb{F}_2[\eta, \lambda'_1, \mu]}{(\eta\lambda'_1, (\lambda'_1)^2 = \eta^2\mu)} \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 1}] \\ &\implies \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})^{tC_2}.\end{aligned}\tag{6.3}$$

Similarly, we have a $\overline{V}(2) \otimes \overline{C}\eta$ -homotopy C_2 -Tate spectral sequence

$$\begin{aligned}\hat{E}^2(C_2) &= (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko}) \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 1}] \\ &= \Lambda(\varepsilon_2) \otimes \Lambda(\lambda'_1)\{1, \lambda_2\} \otimes \mathbb{F}_2[\mu] \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 1}] \\ &\implies (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})^{tC_2}.\end{aligned}\tag{6.4}$$

There is a map F^t of algebra spectral sequences from (5.1) to (6.3), and (6.4) is a module spectral sequence over (6.3).

PROPOSITION 6.3. *There is an isomorphism*

$$\overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})^{tC_2} \cong \mathbb{F}_2\{1, \eta, \eta^2, \lambda'_1\} \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 4}],$$

where η, η^2 and λ'_1 are detected by $t^2\lambda'_1$, $(t^2\lambda'_1)^2$ and λ'_1 , respectively. Under this correspondence, the cyclotomic structure map

$$\varphi_2: \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko}) \longrightarrow \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})^{tC_2}$$

is given by $\varepsilon_2 \mapsto u_1 t^{-4}$, $\eta \mapsto \eta$, $\lambda'_1 \mapsto \lambda'_1$ and $\mu \mapsto t^{-4}$, and hence can be identified with the localization homomorphism

$$\overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko}) \longrightarrow \mu^{-1} \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})$$

that inverts μ .

Proof. We shall use naturality with respect to the complexification map $c: \operatorname{ko} \rightarrow \operatorname{ku}$ to access the cyclotomic structure map φ_2 for ko and the differentials in (6.3), so we first review the results of Hahn and Wilson about the complex case.

By [HW22, Proposition 6.1.6], we have $\overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ku}) \cong \Lambda(\lambda_1, \lambda_2) \otimes \mathbb{F}_2[\mu]$, with $\|\lambda_1\| = (3, 1)$, $\|\lambda_2\| = (7, 1)$ and $\|\mu\| = (8, 0)$. Hence $\overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ku}) \cong \Lambda(\varepsilon_2, \lambda_1, \lambda_2) \otimes \mathbb{F}_2[\mu]$, with $\|\varepsilon_2\| = (7, -1)$. The $\overline{V}(1)$ -homotopy C_2 -Tate spectral sequence

$$\begin{aligned}\hat{E}^2(C_2, \operatorname{ku}) &= \overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ku}) \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 1}] \\ &\implies \overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ku})^{tC_2}\end{aligned}$$

is an algebra spectral sequence with differentials $d^4(t^{-1}) = t\lambda_1$, $d^8(t^{-2}) = t^2\lambda_2$ and $d^9(u_1 t^{-4}) = t\mu$, leaving $\hat{E}^\infty(C_2, \operatorname{ku}) = \Lambda(\lambda_1, \lambda_2) \otimes \mathbb{F}_2[t^{\pm 4}]$, and the cyclotomic structure map is given in $\overline{V}(1)$ -homotopy by $\lambda_1 \mapsto \lambda_1$, $\lambda_2 \mapsto \lambda_2$ and $\mu \mapsto t^{-4}$, and hence is identified with the ring homomorphism that inverts μ , as in [HRW22, Theorem 6.1.4].

It follows that the $\overline{V}(2)$ -homotopy C_2 -Tate spectral sequence

$$\begin{aligned}\hat{E}^2(C_2, \operatorname{ku}) &= \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ku}) \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 1}] \\ &\implies \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ku})^{tC_2}\end{aligned}$$

has differentials $d^2(\varepsilon_2) = t\mu$, $d^4(t^{-1}) = t\lambda_1$ and $d^8(t^{-2}) = t^2\lambda_2$, leaving behind the E^∞ -term $\hat{E}^\infty(C_2, \operatorname{ku}) = \Lambda(\lambda_1, \lambda_2) \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 4}]$. By exactness of localization, the cyclotomic structure map

$$\varphi_2: \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ku}) \longrightarrow \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ku})^{tC_2}$$

must also agree with the ring homomorphism that inverts μ . Hence it is given on λ_1 , λ_2 and μ as in the $\overline{V}(1)$ -case, while $\varphi_2(\varepsilon_2)$ can only be detected by $u_1 t^{-4}$.

Next, we appeal to naturality. We have seen in the proof of Lemma 2.23 that

$$c: \overline{A}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko}) \longrightarrow \overline{A}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ku})$$

is given by $\lambda'_1 \mapsto 0$, $\lambda_2 \mapsto \lambda_2$ and $\mu \mapsto \mu$. It follows by naturality with respect to $i: \overline{V}(1) \rightarrow \overline{A}(1)$ that $c: \overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko}) \rightarrow \overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ku})$ is given by $\eta \mapsto 0$, $\lambda'_1 \mapsto 0$ and $\mu \mapsto \mu$, and similarly with $\overline{V}(2)$ -coefficients, where also $\varepsilon_2 \mapsto \varepsilon_2$. Chasing ε_2 and μ around the commutative diagram

$$\begin{array}{ccc} \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko}) & \xrightarrow{\varphi_2} & \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})^{tC_2} \\ \downarrow c & & \downarrow c \\ \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ku}) & \xrightarrow{\varphi_2} & \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ku})^{tC_2} \end{array}$$

we see that $\varphi_2(\varepsilon_2)$ and $\varphi_2(\mu)$ in the real case must be detected in the same, or higher, Tate filtration as the detecting classes in the complex case, namely, $u_1 t^{-4}$ and t^{-4} . There are no classes of higher Tate filtration in the same total degrees, so the only possibility is that $\varphi_2(\varepsilon_2)$ is detected by $u_1 t^{-4}$ and $\varphi_2(\mu)$ is detected by t^{-4} , also in the real case.

In particular, this shows that $u_1 = t^4 \cdot u_1 t^{-4}$ is a permanent cycle in the spectral sequence (6.3). By naturality with respect to the map F^t , we deduce that we have the same differentials in the C_2 -Tate spectral sequence as in the $\mathbb{T}$ -Tate spectral sequence (5.1), listed in Theorem 5.1. This leaves

$$\hat{E}^\infty(C_2) = \mathbb{F}_2\{1, t^2 \lambda'_1, (t^2 \lambda'_1)^2, \lambda'_1\} \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 4}].$$

It is clear that the $\mathbb{S}$ -algebra map φ_2 takes η to η , which we have seen is detected by $t^2 \lambda'_1$. The relation $(\lambda'_1)^2 = \eta^2 \mu$ now shows that $\varphi_2(\lambda'_1)^2$ must be detected by $(t^2 \lambda'_1)^2 \cdot t^{-4} = (\lambda'_1)^2 \neq 0$, which can only happen if $\varphi_2(\lambda'_1)$ is detected by λ'_1 . Summarizing, we have an isomorphism

$$\overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})^{tC_2} \cong \mathbb{F}_2\{1, \eta, \eta^2, \lambda'_1\} \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 4}],$$

where φ_2 is given by $\varepsilon_2 \mapsto u_1 t^{-4}$, $\eta \mapsto \eta$, $\lambda'_1 \mapsto \lambda'_1$ and $\mu \mapsto t^{-4}$. The claim about localization then amounts to the isomorphism

$$\mu^{-1} \frac{\mathbb{F}_2[\eta, \lambda'_1, \mu]}{(\eta \lambda'_1, (\lambda'_1)^2 = \eta^2 \mu)} \cong \mathbb{F}_2\{1, \eta, \eta^2, \lambda'_1\} \otimes \mathbb{F}_2[\mu^{\pm 1}]. \quad \square$$

PROPOSITION 6.4. *There is an isomorphism*

$$(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})^{tC_2} \cong \Lambda(\lambda'_1)\{1, \lambda_2\} \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 4}],$$

where 1 , λ'_1 , λ_2 , $\lambda'_1 \lambda_2$ and $t^{\pm 4}$ are detected by classes with the same names. Under this correspondence, the cyclotomic structure map

$$\varphi_2: (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko}) \longrightarrow (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})^{tC_2}$$

is given by $\varepsilon_2 \mapsto u_1 t^{-4}$, $\lambda'_1 \mapsto \lambda'_1$, $\lambda_2 \mapsto \lambda_2$ and $\mu \mapsto t^{-4}$, and hence can be identified with the localization homomorphism

$$(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko}) \longrightarrow \mu^{-1}(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})$$

that inverts μ .

Proof. Naturality with respect to $i: \overline{V}(2) \rightarrow \overline{V}(2) \otimes \overline{C}\eta$ shows that u_1 is a permanent cycle in the spectral sequence (6.4). When combined with the differentials in (5.2), listed in Theorem 5.2,

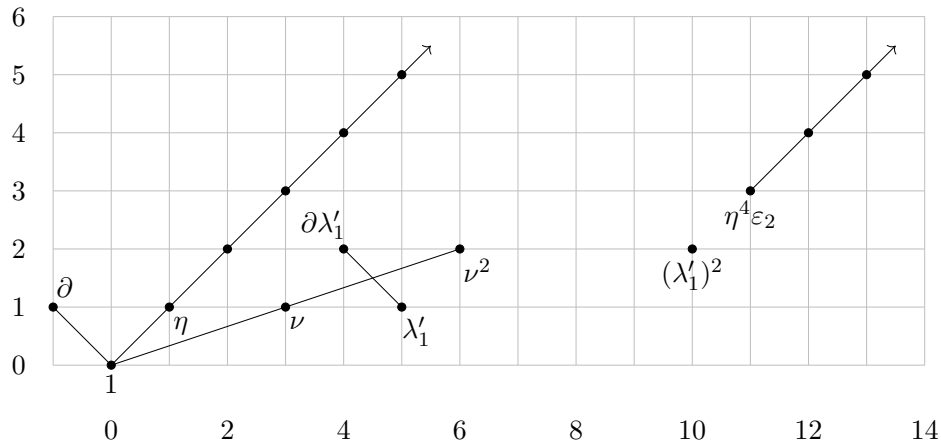


FIGURE 11. The bigraded $\mathbb{F}_2$ -algebra $\overline{V}(2)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko})$, with lines of slope -1 , 1 and $1/3$ indicating multiplication by ∂ , η and ν , respectively.

this shows that

$$\hat{E}^4(C_2) = \Lambda(\lambda'_1)\{1, \lambda_2\} \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 1}]$$

and

$$\hat{E}^\infty(C_2) = \Lambda(\lambda'_1)\{1, \lambda_2\} \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 4}].$$

The detection results then follow from those in Theorem 5.2. The evaluation of φ_2 on ε_2 , λ'_1 and μ follows from that in Proposition 6.3 by comparison along the same map i .

To show that $\varphi_2(\lambda_2)$ is detected by λ_2 , we note that by naturality with respect to $j_2: \overline{V}(2) \otimes \overline{C}\eta \rightarrow \Sigma^{7,-1}\overline{A}(1)$ it cannot be detected by $u_1 t^{-4}$. On the other hand, by naturality along $j: \overline{V}(2) \otimes \overline{C}\eta \rightarrow \Sigma^{2,0}\overline{V}(2)$ it is nonzero, since we have seen in the proof of Proposition 5.11 that $j: \lambda_2 \mapsto \Sigma^{2,0}\lambda'_1$. Hence $\varphi_2(\lambda_2)$ must be the class detected by λ_2 . $\square$

Remark 6.5. The computations in Theorems 5.1 and 5.2 and Propositions 6.3 and 6.4 are consistent with isomorphisms

$$\begin{aligned} \overline{V}(2)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TP}(\mathrm{ko}) &\cong \overline{V}(1)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2}, \\ (\overline{V}(2) \otimes \overline{C}\eta)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TP}(\mathrm{ko}) &\cong \overline{A}(1)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{ko})^{tC_2}, \end{aligned}$$

in analogy with [HRW22, Theorem 6.4.1].

THEOREM 6.6 (Syntomic cohomology modulo $(2, v_1, v_2)$ of ko). *We have an algebra isomorphism*

$$\overline{V}(2)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko}) \cong \mathbb{F}_2[\eta]\{1, \eta^4\varepsilon_2\} \oplus \mathbb{F}_2\{\partial, \nu, \lambda'_1, \partial\lambda'_1, \nu^2, (\lambda'_1)^2\},$$

with generators in bidegrees $\|\partial\| = (-1, 1)$, $\|\eta\| = (1, 1)$, $\|\nu\| = (3, 1)$, $\|\lambda'_1\| = (5, 1)$ and $\|\eta^4\varepsilon_2\| = (11, 3)$. For a view of the algebra structure of the right-hand side, see Figure 11.

Proof. To calculate the effect in $\overline{V}(2)$ -homotopy of $\mathrm{can}: \mathrm{TC}^-(\mathrm{ko}) \rightarrow \mathrm{TP}(\mathrm{ko})$, we use the map of spectral sequences from (6.1) to (5.1), described in Proposition 6.1 and Theorem 5.1, given at the E^2 -terms by inverting t . To calculate the effect of $\varphi_2^{h\mathbb{T}}: \mathrm{TC}^-(\mathrm{ko}) \rightarrow (\mathrm{THH}(\mathrm{ko})^{tC_2})^{h\mathbb{T}}$, we appeal to Corollary 3.4 and Proposition 6.3 to see that there is a μ -inverted $\mathbb{T}$ -homotopy fixed

point spectral sequence

$$\begin{aligned}\mu^{-1}E^2(\mathbb{T}) &= \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{THH}(\operatorname{ko})^{tC_2}[t] \\ &= \Lambda(\varepsilon_2) \otimes \mathbb{F}_2\{1, \eta, \eta^2, \lambda'_1\} \otimes \mathbb{F}_2[\mu^{\pm 1}] \otimes \mathbb{F}_2[t] \\ &\implies \overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* (\operatorname{THH}(\operatorname{ko})^{tC_2})^{h\mathbb{T}},\end{aligned}\tag{6.5}$$

and $\varphi_2^{h\mathbb{T}}$ is calculated by the map of spectral sequences from (6.1) to (6.5) that is given at the E^2 -terms by inverting μ . The differentials

$$d^2(\varepsilon_2) = t\mu, \quad d^2(\mu) = t\eta\mu$$

carry over from the proof of Proposition 6.1, leaving

$$\mu^{-1}E^4(\mathbb{T}) = \mu^{-1}E^\infty(\mathbb{T}) = \mathbb{F}_2\{1, \eta, \eta^2, \lambda'_1\} \otimes \mathbb{F}_2[\bar{\mu}^{\pm 1}],$$

concentrated on the vertical axis. As before, $\bar{\mu} = \mu + \eta\varepsilon_2$.

As recalled in Definition 3.1, we know *a priori* that the natural map

$$G: \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TP}(\operatorname{ko}) \xrightarrow{\cong} \operatorname{gr}_{\operatorname{mot}}^* (\operatorname{THH}(\operatorname{ko})^{tC_2})^{h\mathbb{T}}$$

is an equivalence. The $\overline{V}(2)$ -homotopy isomorphism

$$\mathbb{F}_2\{1, \eta, \eta^2, \lambda'_1\} \otimes \mathbb{F}_2[t^{\pm 4}] \xrightarrow{\cong} \mathbb{F}_2\{1, \eta, \eta^2, \lambda'_1\} \otimes \mathbb{F}_2[\bar{\mu}^{\pm 1}]$$

induced by the equivalence G can then only be given by $\eta \mapsto \eta$, $\lambda'_1 \mapsto \lambda'_1$ and $t^{\pm 4} \mapsto \bar{\mu}^{\mp 1}$.

We claim that the map $\operatorname{can} - \varphi$ (which is short for $G \circ \operatorname{can} - \varphi_2^{h\mathbb{T}}$) induces isomorphisms

$$\mathbb{F}_2[t^4]\{t^4\} \otimes \mathbb{F}_2\{1, t^2\lambda'_1, (t^2\lambda'_1)^2, \lambda'_1\} \xrightarrow{\cong} \mathbb{F}_2[\bar{\mu}^{-1}]\{\bar{\mu}^{-1}\} \otimes \mathbb{F}_2\{1, \eta, \eta^2, \lambda'_1\},\tag{6.6}$$

$$\mathbb{F}_2[\bar{\mu}]\{\bar{\mu}\} \otimes \mathbb{F}_2\{1, \eta, \eta^2, \lambda'_1\} \xrightarrow{\cong} \mathbb{F}_2[\bar{\mu}]\{\bar{\mu}\} \otimes \mathbb{F}_2\{1, \eta, \eta^2, \lambda'_1\},\tag{6.7}$$

$$\mathbb{F}_2\{t^2\lambda'_1, (t^2\lambda'_1)^2\} \xrightarrow{\cong} \mathbb{F}_2\{\eta, \eta^2\}\tag{6.8}$$

and the zero homomorphism

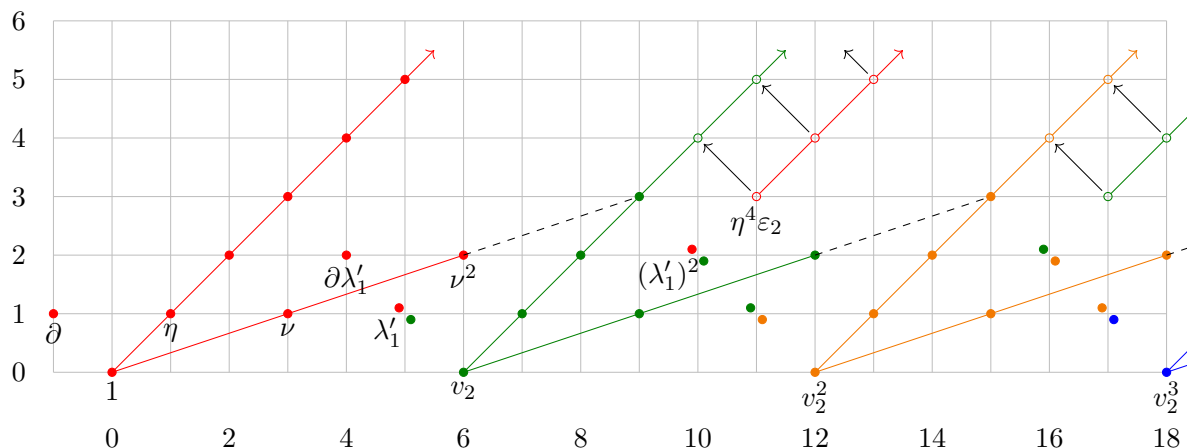
$$\mathbb{F}_2[\eta]\{1, \eta^4\varepsilon_2\} \oplus \mathbb{F}_2\{t\lambda'_1, \lambda'_1, (t\lambda'_1)^2, (\lambda'_1)^2\} \xrightarrow{0} \mathbb{F}_2\{1, \lambda'_1\}.\tag{6.9}$$

The isomorphism (6.6) occurs in horizontal degrees where inverting t (or t^4) is an isomorphism and $\varphi_2^{h\mathbb{T}}$ is zero. The isomorphism (6.7) occurs in vertical degrees where inverting μ (or $\bar{\mu}$) is an isomorphism and can is zero. The isomorphism (6.8) uses that $t^2\lambda'_1$ and $(t^2\lambda'_1)^2$ in (6.1) map to η and η^2 , which are detected in (5.1), but map to zero in (6.5). The homomorphisms $G \circ \operatorname{can}$ and $\varphi_2^{h\mathbb{T}}$ agree on classes coming from $\overline{V}(2)_* \operatorname{gr}_{\operatorname{ev}}^* \mathbb{S}$, such as 1 , η and ν , and hence their difference is zero on $\mathbb{F}_2[\eta]\{1\}$ and $\mathbb{F}_2\{t\lambda'_1, (t\lambda'_1)^2\}$. Both $G \circ \operatorname{can}(\lambda'_1)$ and $\varphi_2^{h\mathbb{T}}(\lambda'_1)$ are detected by λ'_1 , and hence agree in $\overline{V}(2)$ -homotopy, since there are no other classes in the same total degree, which implies that $G \circ \operatorname{can} - \varphi_2^{h\mathbb{T}}$ is zero on λ'_1 and its square. Both $G \circ \operatorname{can}$ and $\varphi_2^{h\mathbb{T}}$ take $\eta^4\varepsilon_2 = \eta^3\bar{\mu}$ to zero, so their difference is zero on $\mathbb{F}_2[\eta]\{\eta^4\varepsilon_2\}$.

Hence we have an isomorphism

$$\overline{V}(2)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TC}(\operatorname{ko}) \cong \mathbb{F}_2[\eta]\{1, \eta^4\varepsilon_2\} \oplus \mathbb{F}_2\{\partial, \partial\lambda'_1, t\lambda'_1, \lambda'_1, (t\lambda'_1)^2, (\lambda'_1)^2\}.$$

The classes $t\lambda'_1$ and $(t\lambda'_1)^2$ detect ν and ν^2 , respectively. Regarding the multiplicative structure, the subgroup $\mathbb{F}_2\{\partial, \partial\lambda'_1\}$ is the image of the target of (6.9) under the connecting homomorphism ∂ . The quotient algebra by this square-zero ideal maps isomorphically to the source of (6.9). It has an associated graded that is isomorphic to a subalgebra of the E^∞ -term displayed in Figure 9. Due to the sparsity of the situation, the only possible hidden multiplicative extension would be from η times λ'_1 to $(t\lambda'_1)^2$, but in fact $\eta\lambda'_1 = 0$, as we show in Proposition 6.7 below. $\square$


 FIGURE 12 (colour online). The v_2 -Bockstein $E_1 \Rightarrow \overline{V}(1)_* \text{gr}_{\text{mot}}^* \text{TC}(\text{ko})$.

Next, we compute the v_2 -Bockstein spectral sequence

$$E_1 = \overline{V}(2)_* \text{gr}_{\text{mot}}^* \text{TC}(\text{ko}) [v_2] \Rightarrow \overline{V}(1)_* \text{gr}_{\text{mot}}^* \text{TC}(\text{ko}). \quad (6.10)$$

PROPOSITION 6.7. *In the spectral sequence (6.10), there is a d_1 -differential*

$$d_1(\eta^4 \varepsilon_2) = v_2 \eta^4,$$

together with its various η - and v_2 -power multiples. This produces an algebra isomorphism

$$\overline{V}(1)_* \text{gr}_{\text{mot}}^* \text{TC}(\text{ko}) \cong \frac{\Lambda(\partial) \otimes \mathbb{F}_2[\eta, \nu, \lambda'_1, v_2]}{(\partial\eta, \partial\nu, \eta\nu, \eta\lambda'_1, \nu\lambda'_1, \nu^3 = v_2\eta^3 = \partial(\lambda'_1)^2, (\lambda'_1)^3 = c \cdot v_2^2\eta^3)},$$

where $c \in \mathbb{F}_2$ (and we have not resolved this indeterminacy).

Proof. The unit map $\mathbb{S} \rightarrow \text{TC}(\text{ko})$ induces a map of v_2 -Bockstein spectral sequences, from

$$E_1 = \text{Ext}_{\text{BP}_* \text{BP}}(\text{BP}_*, \text{BP}_*/I_3) [v_2] \Rightarrow \text{Ext}_{\text{BP}_* \text{BP}}(\text{BP}_*, \text{BP}_*/I_2) \quad (6.11)$$

shown in Figure 4 to (6.10) shown in Figure 12. Since $v_2^2 h_{10}^4 = 0$ in the abutment of the former, we must have that $v_2^2 \eta^4$ is a boundary in the latter. Considering bidegrees and v_2 -powers, this can only happen if $d_1(v_2 \eta^4 \varepsilon_2) = v_2^2 \eta^4$. Hence $d_1(v_2^i \eta^j \varepsilon_2) = v_2^{i+1} \eta^j$ for all $i \geq 0$ and $j \geq 4$, as claimed. There is no room for other v_2 -Bockstein differentials, so $E_2 = E_\infty$ in (6.10).

The relation $v_2 h_{11}^3 = v_2^2 h_{10}^3$ in the abutment of (6.11) (see (4.2)) also implies that $v_2 \nu^3 = v_2^2 \eta^3$ in the abutment of (6.10). Hence we have hidden ν -extensions from $v_2^i \nu^2$ to $v_2^{i+1} \eta^3$ for all $i \geq 0$, as shown by dashed lines of slope 1/3 in Figure 12.

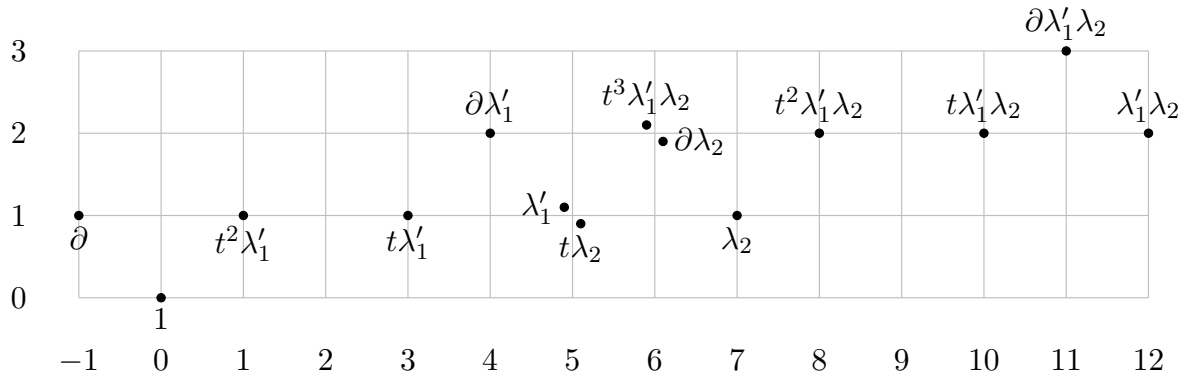
The products ∂^2 and $\partial\eta$ lie in trivial groups. The well-known relation $\eta\nu = 0$ implies the vanishing of $\partial\nu$, $\eta\lambda'_1$ and $\nu\lambda'_1$. We postpone the proof that $\partial(\lambda'_1)^2$ is equal to $\nu^3 = v_2\eta^3$ to Remark 6.10. We have not determined whether or not $(\lambda'_1)^3 \in \mathbb{F}_2\{v_2^2\eta^3\}$ is zero. $\square$

PROPOSITION 6.8. *We have an isomorphism*

$$\overline{A}(1)_* \text{gr}_{\text{mot}}^* \text{TC}(\text{ko}) \cong \mathbb{F}_2[v_2] \otimes (\Lambda(\partial, \lambda'_1)\{1, \lambda_2\} \oplus \mathbb{F}_2\{t^2\lambda'_1, t\lambda'_1, t\lambda_2, t^3\lambda'_1\lambda_2, t^2\lambda'_1\lambda_2, t\lambda'_1\lambda_2\})$$

of finitely generated and free $\mathbb{F}_2[v_2]$ -modules, where $\|v_2\| = (6, 0)$, $\|\partial\| = (-1, 1)$, $\|\lambda'_1\| = (5, 1)$, $\|\lambda_2\| = (7, 1)$ and $\|t\| = (-2, 0)$. See Figure 13.

Proof. This proof is similar to that of Theorem 6.6, to which we refer for a more elaborate review of some of the notations. To calculate the effect of can in $\overline{V}(2) \otimes \overline{C}\eta$ -homotopy, we use the map of


 FIGURE 13. The $\mathbb{F}_2[v_2]$ -basis for $\overline{A}(1)_* \text{gr}_{\text{mot}}^* \text{TC}(\text{ko})$.

spectral sequences from (6.2) to (5.2), described in Proposition 6.2 and Theorem 5.2, given at the E^2 -terms by inverting t . To calculate the effect of $\varphi_2^{h\mathbb{T}}$, we use Corollary 2.36 and Proposition 6.4 to see that there is a μ -inverted $\mathbb{T}$ -homotopy fixed point spectral sequence

$$\begin{aligned} \mu^{-1} E^2(\mathbb{T}) &= (\overline{V}(2) \otimes \overline{C}\eta)_* \text{gr}_{\text{mot}}^* \text{THH}(\text{ko})^{tC_2} [t] \\ &= \Lambda(\varepsilon_2) \otimes \Lambda(\lambda'_1) \{1, \lambda_2\} \otimes \mathbb{F}_2[\mu^{\pm 1}] \otimes \mathbb{F}_2[t] \\ &\implies (\overline{V}(2) \otimes \overline{C}\eta)_* \text{gr}_{\text{mot}}^* (\text{THH}(\text{ko})^{tC_2})^{h\mathbb{T}}, \end{aligned} \quad (6.12)$$

and $\varphi_2^{h\mathbb{T}}$ is given by the map of spectral sequences from (6.2) to (6.12) that is given at the E^2 -terms by inverting μ . The differential $d^2(\varepsilon_2) = t\mu$ carries over from the proof of Proposition 6.2, leaving

$$\mu^{-1} E^4(\mathbb{T}) = \mu^{-1} E^\infty(\mathbb{T}) = \Lambda(\lambda'_1) \{1, \lambda_2\} \otimes \mathbb{F}_2[\mu^{\pm 1}]$$

concentrated on the vertical axis. The $\overline{V}(2) \otimes \overline{C}\eta$ -homotopy isomorphism

$$\Lambda(\lambda'_1) \{1, \lambda_2\} \otimes \mathbb{F}_2[t^{\pm 4}] \xrightarrow{\cong} \Lambda(\lambda'_1) \{1, \lambda_2\} \otimes \mathbb{F}_2[\mu^{\pm 1}]$$

induced by the equivalence G must thus be given by $\lambda'_1 \mapsto \lambda'_1$, $\lambda_2 \mapsto \lambda_2$ and $t^{\pm 4} \mapsto \mu^{\mp 1}$.

The map $G \circ \text{can} - \varphi_2^{h\mathbb{T}}$ induces isomorphisms

$$\begin{aligned} \mathbb{F}_2[t^4] \{t^4\} \otimes \Lambda(\lambda'_1) \{1, \lambda_2\} &\xrightarrow{\cong} \Lambda(\lambda'_1) \{1, \lambda_2\} \otimes \mathbb{F}_2[\mu^{-1}] \{\mu^{-1}\}, \\ \mathbb{F}_2[\mu] \{\mu\} \otimes \Lambda(\lambda'_1) \{1, \lambda_2\} &\xrightarrow{\cong} \Lambda(\lambda'_1) \{1, \lambda_2\} \otimes \mathbb{F}_2[\mu] \{\mu\} \end{aligned}$$

and the zero homomorphism

$$\Lambda(\lambda'_1) \{1, \lambda_2\} \oplus \mathbb{F}_2 \{t^2 \lambda'_1, t \lambda'_1, t \lambda_2, t^3 \lambda'_1 \lambda_2, t^2 \lambda'_1 \lambda_2, t \lambda'_1 \lambda_2\} \xrightarrow{0} \Lambda(\lambda'_1) \{1, \lambda_2\},$$

by the same arguments as in the proof of Theorem 6.6. Hence we have an isomorphism

$$(\overline{V}(2) \otimes \overline{C}\eta)_* \text{gr}_{\text{mot}}^* \text{TC}(\text{ko}) \cong \Lambda(\partial, \lambda'_1) \{1, \lambda_2\} \oplus \mathbb{F}_2 \{t^2 \lambda'_1, t \lambda'_1, t \lambda_2, t^3 \lambda'_1 \lambda_2, t^2 \lambda'_1 \lambda_2, t \lambda'_1 \lambda_2\}.$$

There is no room for differentials in the v_2 -Bockstein spectral sequence

$$E_1 = (\overline{V}(2) \otimes \overline{C}\eta)_* \text{gr}_{\text{mot}}^* \text{TC}(\text{ko}) [v_2] \implies \overline{A}(1)_* \text{gr}_{\text{mot}}^* \text{TC}(\text{ko}). \quad \square$$

LEMMA 6.9. *The unit map*

$$\pi_* \overline{A}(1) \longrightarrow \overline{A}(1)_* \text{gr}_{\text{mot}}^* \text{TC}(\text{ko})$$

sends the classes 1 , h_{11} , w , $h_{11}^2 = h_{10}w$, $h_{11}w$ and h_{11}^2w to classes that are detected by 1 , $t\lambda'_1$, $t\lambda_2$, $t^3\lambda'_1\lambda_2 \bmod \partial\lambda_2$, $t^2\lambda'_1\lambda_2$ and $\partial\lambda'_1\lambda_2$, respectively. The product $h_{11}\lambda_2$ is detected by $t\lambda'_1\lambda_2$.

Proof. The $\overline{V}(1)$ -module cofiber sequence (2.5) induces a long exact sequence

$$\cdots \xrightarrow{\eta} \overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TC}(\operatorname{ko}) \xrightarrow{i} \overline{A}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TC}(\operatorname{ko}) \xrightarrow{j} \Sigma^{2,0} \overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TC}(\operatorname{ko}) \xrightarrow{\eta} \cdots$$

of $\mathbb{F}_2[v_2]$ -modules (see Figures 12 and 13). Having chosen $\lambda'_1 \in \overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TC}(\operatorname{ko})$, we choose $\lambda'_1, \lambda_2 \in \overline{A}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TC}(\operatorname{ko})$ so that $i(\lambda'_1) = \lambda'_1$ and $j(\lambda_2) = \Sigma^{2,0} \lambda'_1$. By exactness, i is then given by

$$\begin{aligned} 1 &\mapsto 1, \\ \partial &\mapsto \partial, \\ \nu &\mapsto t\lambda'_1, \\ \lambda'_1 &\mapsto \lambda'_1, \\ \partial\lambda'_1 &\mapsto \partial\lambda'_1, \\ \nu^2 &\mapsto t^3\lambda'_1\lambda_2 \bmod \partial\lambda_2, \\ (\lambda'_1)^2 &\mapsto t\lambda'_1\lambda_2 \bmod v_2\partial\lambda'_1, \end{aligned}$$

while j is given by

$$\begin{aligned} t^2\lambda'_1 &\mapsto \Sigma^{2,0}\partial, \\ t\lambda_2 &\mapsto \Sigma^{2,0}\nu, \\ \lambda_2 &\mapsto \Sigma^{2,0}\lambda'_1, \\ \partial\lambda_2 &\mapsto \Sigma^{2,0}\partial\lambda'_1, \\ t^2\lambda'_1\lambda_2 &\mapsto \Sigma^{2,0}\nu^2, \\ \lambda'_1\lambda_2 &\mapsto \Sigma^{2,0}(\lambda'_1)^2, \\ \partial\lambda'_1\lambda_2 &\mapsto \Sigma^{2,0}\nu^3. \end{aligned}$$

The formulas for i imply the claims for $1, \nu = h_{11}$ and $\nu^2 = h_{11}^2$. We know from Corollary 4.5 that w is detected by $t\lambda_2$, so the formulas for j imply the claims for $\nu w = h_{11}w$ and $\nu^2 w = h_{11}^2 w$.

The $\overline{V}(1)$ -module action on $\overline{A}(1)$ shows that $\nu \cdot \lambda_2 = h_{11}\lambda_2$ is detected by $t\lambda'_1 \cdot \lambda_2$, since the latter product is nonzero in $\overline{A}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TC}^-(\operatorname{ko})$. $\square$

Remark 6.10. We can now complete the unfinished business in the proof of Proposition 6.7. Since $\nu^2 w$ is detected by $\partial\lambda'_1\lambda_2$, and j maps w to $\Sigma^{2,0}\nu$ and λ_2 to $\Sigma^{2,0}\lambda'_1$, it follows that $\Sigma^{2,0}\nu^3$ is detected by $\Sigma^{2,0}\partial(\lambda'_1)^2$, so $\partial(\lambda'_1)^2$ is equal to $\nu^3 = v_2\eta^3$ in $\overline{V}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TC}(\operatorname{ko})$.

LEMMA 6.11. *Let $\varsigma \in \overline{A}(1)_* \operatorname{gr}_{\operatorname{mot}}^* \operatorname{TC}(\operatorname{ko})$ be the class in bidegree $(1, 1)$ detected by $t^2\lambda'_1$. Then $\varsigma\nu$ is the class in bidegree $(4, 2)$ detected by $\partial\lambda'_1$.*

Proof. By [BHM93, Theorem 5.17] and [Rog02, Corollary 1.21] there is a 2-complete equivalence $\operatorname{TC}(\mathbb{S}) \simeq \mathbb{S} \oplus \Sigma\operatorname{CP}_{-1}^\infty$, and by [BM94, Proposition 10.9] and [Dun97, Main Theorem] the 3-connected map $\mathbb{S} \rightarrow \operatorname{ko}$ induces a 4-connected map $\operatorname{TC}(\mathbb{S}) \rightarrow \operatorname{TC}(\operatorname{ko})$. For each $i \geq -1$ let $\Sigma\beta_i \in H_{2i+1}(\Sigma\operatorname{CP}_{-1}^\infty)$ denote the generator. The Atiyah–Hirzebruch spectral sequence

$$E^2 = H_*(\Sigma\operatorname{CP}_{-1}^\infty; \pi_* A(1)) \Longrightarrow A(1)_*(\Sigma\operatorname{CP}_{-1}^\infty)$$

has nonzero differentials $d^4(\Sigma\beta_1) = \nu\Sigma\beta_{-1}$ and $d^6(\Sigma\beta_2) = w\Sigma\beta_{-1}$. This follows from [Mos68, Proposition 5.2, Proposition 5.4], using that $w \in \langle \nu, \eta, \iota \rangle$ in $\pi_* A(1)$, where ι is the class of $\mathbb{S} \rightarrow A(1)$. Hence

$$A(1)_* \operatorname{TC}(\mathbb{S}) \cong \mathbb{F}_2\{\Sigma\beta_{-1}, \iota, \Sigma\beta_0, \nu\iota, \nu\Sigma\beta_0\}$$

in stems $-1 \leq * \leq 4$, mapping isomorphically to

$$A(1)_* \mathrm{TC}(\mathrm{ko}) \cong \mathbb{F}_2\{\partial, 1, t^2\lambda'_1, t\lambda'_1, \partial\lambda'_1\}$$

in this range. It follows that $\Sigma\beta_0$ maps to the class ς detected by $t^2\lambda'_1$ and $\nu\Sigma\beta_0$ to the class detected by $\partial\lambda'_1$, which must therefore be equal to $\varsigma\nu$. $\square$

THEOREM 6.12 (Syntomic cohomology modulo $(2, \eta, v_1)$ of ko). *We have an isomorphism*

$$\begin{aligned} \overline{A}(1)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko}) &\cong \mathbb{F}_2[v_2] \otimes (\mathbb{F}_2\{1, \partial, \nu, w, \nu^2 = \eta w, \nu w, \lambda'_1 \lambda_2, \nu^2 w = \partial\lambda'_1 \lambda_2\} \\ &\quad \oplus \mathbb{F}_2\{\varsigma, \lambda'_1, \varsigma\nu = \partial\lambda'_1\} \oplus \mathbb{F}_2\{\lambda_2, \partial\lambda_2, \nu\lambda_2\}) \end{aligned}$$

of $\overline{V}(1)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko})$ -modules, where the (stem, motivic filtration) bidegrees and detecting classes of the $\mathbb{F}_2[v_2]$ -module generators are as in Table 1 (see also Figure 1).

Proof. This summarizes the results of Proposition 6.8 and Lemmas 6.9 and 6.11. The lift of $t^3\lambda'_1\lambda_2$ over $\pi: \mathrm{TC}(\mathrm{ko}) \rightarrow \mathrm{TC}^-(\mathrm{ko})$ is only defined modulo $\partial\lambda_2$ in the image under $\partial: \Sigma^{-1} \mathrm{TP}(\mathrm{ko}) \rightarrow \mathrm{TC}(\mathrm{ko})$, but the image of ν^2 specifies one such choice of lift. $\square$

7. Topological cyclic homology and algebraic K -theory

We now use the motivic spectral sequence

$$E_2 = \overline{A}(1)_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko}) \implies A(1)_* \mathrm{TC}(\mathrm{ko}) \quad (7.1)$$

to compute the $A(1)$ -homotopy of the topological cyclic homology of ko . The E_2 -term, given in Theorem 6.12, is concentrated in even total degrees and motivic filtrations $0 \leq * \leq 3$, so the only possibly nonzero differentials are

$$d_3(v_2^i) \in \mathbb{F}_2\{v_2^{i-2}\nu^2 w\}$$

for $i \geq 2$. We show that some, but not all, of these differentials are nonzero. This contrasts with the motivic spectral sequence converging to $V(1)_* \mathrm{TC}(\ell)$ at odd primes p , which has been shown to collapse at the E_2 -term by Hahn, Raksit and Wilson in [HRW22, Corollary 1.4.3].

NOTATION 7.1. We equip $A(1) \otimes \mathrm{TC}(\mathrm{ko})$ with the relative convolution filtration

$$\mathrm{fil}_{\mathrm{mot}}^*(A(1) \otimes \mathrm{TC}(\mathrm{ko})) := \mathrm{fil}_{\mathrm{ev}}^* A(1) \otimes_{\mathrm{fil}_{\mathrm{ev}}^* \mathbb{S}} \mathrm{fil}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko}),$$

with associated graded $\mathrm{gr}_{\mathrm{mot}}^*(A(1) \otimes \mathrm{TC}(\mathrm{ko})) \simeq \overline{A}(1) \otimes \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathrm{ko})$. This filtration is complete and exhaustive, since $\mathrm{fil}_{\mathrm{ev}}^* A(1)$ is a finite cell $\mathrm{fil}_{\mathrm{ev}}^* \mathbb{S}$ -module. The motivic spectral sequence (7.1) is the associated spectral sequence, converging to $\pi_*(A(1) \otimes \mathrm{TC}(\mathrm{ko})) = A(1)_* \mathrm{TC}(\mathrm{ko})$.

We write

$$\begin{aligned} \mathrm{Fil}_{\mathrm{mot}}^w A(1)_* \mathrm{TC}(\mathrm{ko}) &= \mathrm{im}(\pi_* \mathrm{fil}_{\mathrm{mot}}^w(A(1) \otimes \mathrm{TC}(\mathrm{ko})) \longrightarrow A(1)_* \mathrm{TC}(\mathrm{ko})), \\ \mathrm{Gr}_{\mathrm{mot}}^w A(1)_* \mathrm{TC}(\mathrm{ko}) &= \mathrm{Fil}_{\mathrm{mot}}^w A(1)_* \mathrm{TC}(\mathrm{ko}) / \mathrm{Fil}_{\mathrm{mot}}^{w+1} A(1)_* \mathrm{TC}(\mathrm{ko}) \end{aligned}$$

for the induced (algebraic) filtration on $A(1)_* \mathrm{TC}(\mathrm{ko})$ and its associated graded, so that

$$E_\infty \cong \mathrm{Gr}_{\mathrm{mot}}^* A(1)_* \mathrm{TC}(\mathrm{ko}).$$

In each stem n the E_∞ -term contains at most two nonzero groups, in motivic filtrations $s \in \{0, 2\}$ or $s \in \{1, 3\}$, according to the parity of n .

Bhattacharya, Egger and Mahowald [BEM17, Main Theorem] have proved that for each version of $A(1)$, there exists a v_2^{32} self-map $\Sigma^{192} A(1) \rightarrow A(1)$. We have noted in Lemma 4.6 that $\mathrm{id}: A(1) \rightarrow A(1)$ has additive exponent 4. Hence there is a natural $\mathbb{Z}/4[v_2^{32}]$ -module structure

on (7.1) and its abutment. This factors through a finitely generated and free $\mathbb{F}_2[v_2^4]$ -module structure on the associated graded.

THEOREM 7.2. *The motivic spectral sequence (7.1) has nonzero differentials*

$$d_3(v_2^i) = v_2^{i-2}\nu^2w$$

for $i \equiv 2, 3 \pmod{4}$. The remaining differentials are zero. Hence

$$\begin{aligned} \mathrm{Gr}_{\mathrm{mot}}^* A(1)_* \mathrm{TC}(\mathrm{ko}) &= \mathbb{F}_2\{v_2^i \mid i \equiv 0, 1 \pmod{4}\} \\ &\quad \oplus \mathbb{F}_2[v_2]\{\partial, \varsigma, \nu, \lambda'_1, w, \lambda_2\} \\ &\quad \oplus \mathbb{F}_2[v_2]\{\varsigma\nu, \nu^2, \partial\lambda_2, \nu w, \nu\lambda_2, \lambda'_1\lambda_2\} \\ &\quad \oplus \mathbb{F}_2\{v_2^j\nu^2w \mid j \equiv 2, 3 \pmod{4}\} \end{aligned}$$

is a finitely generated and free $\mathbb{F}_2[v_2^4]$ -module of rank 52. Here, $|\partial| = -1$, $|\varsigma| = 1$, $|\nu| = 3$, $|\lambda'_1| = |w| = 5$, $|v_2| = 6$ and $|\lambda_2| = 7$.

Proof. The unit map $\mathbb{S} \rightarrow \mathrm{TC}(\mathrm{ko})$ induces a map from the Novikov spectral sequence for $A(1)$, as discussed in Lemma 4.2, to the motivic spectral sequence (7.1). By Lemma 6.9, this map of E_2 -terms sends v_2^i to v_2^i and $v_2^i h_{11}^2 w$ to $v_2^i \partial \lambda'_1 \lambda_2$ for each $i \geq 0$. Since $d_3(1) = d_3(v_2) = 0$, $d_3(v_2^2) = h_{11}^2 w$ and $d_3(v_2^3) = v_2 h_{11}^2 w$ in the Novikov spectral sequence, we must have $d_3(1) = d_3(v_2) = 0$, $d_3(v_2^2) = \partial \lambda'_1 \lambda_2$ and $d_3(v_2^3) = v_2 \partial \lambda'_1 \lambda_2$ in the motivic spectral sequence. This handles the cases $0 \leq i < 4$. By Corollary 4.7, we know that these d_3 -differentials propagate v_2^4 -periodically, as claimed.

It follows that all classes in motivic filtrations 1 and 2 survive to E_∞ . In filtrations 0 and 3, only the classes v_2^i with $0 \leq i \equiv 0, 1 \pmod{4}$ and $v_2^{i-2}\nu^2w$ with $2 \leq i \equiv 0, 1 \pmod{4}$ survive. Setting $0 \leq j = i - 2$ gives the asserted formula. $\square$

Remark 7.3. The additive extensions

$$0 \rightarrow E_\infty^{n,s+2} \rightarrow A(1)_n \mathrm{TC}(\mathrm{ko}) \rightarrow E_\infty^{n,s} \rightarrow 0$$

(with $s=0$ for n even, $s=1$ for n odd) are sometimes nontrivial. For example, we see from Figure 3 that $2 \cdot v_2 = \nu^2$ in $\pi_6 A(1)[ij]$ if (and only if) $[ij] \in \{[10], [11]\}$, which implies that $A(1)_6 \mathrm{TC}(\mathrm{ko}) \cong \mathbb{Z}/4\{v_2\} \oplus \mathbb{Z}/2\{\partial\lambda_2\}$ in these two cases. We have not carried out a complete analysis of these extension problems.

Theorem 7.2 allows us to determine the $A(1)$ -homotopy of the algebraic K -theory spectra of ko and $\mathrm{ko}_2^\wedge$. We begin with the 2-complete case.

THEOREM 7.4. *There is an exact sequence of $\mathbb{Z}/4[v_2^{32}]$ -modules*

$$0 \rightarrow \Sigma^1 \mathbb{F}_2 \oplus \Sigma^3 \mathbb{F}_2 \rightarrow A(1)_* K(\mathrm{ko}_2^\wedge) \xrightarrow{\mathrm{trc}} A(1)_* \mathrm{TC}(\mathrm{ko}) \rightarrow \mathbb{F}_2\{\partial, \varsigma\} \rightarrow 0,$$

with $|\partial| = -1$ and $|\varsigma| = 1$.

Proof. Let $\mathbb{Z}_2 = \pi_0(\mathrm{ko}_2^\wedge)$ denote the 2-adic integers. By [HM97, Theorem D] and [Dun97, Main Theorem] (cf. [DGM13, Theorem 7.3.1.8]) applied to the 1-connected $\mathbb{E}_\infty$ -ring map $\mathrm{ko}_2^\wedge \rightarrow H\mathbb{Z}_2$ there is a cofiber sequence

$$K(\mathrm{ko}_2^\wedge)_2^\wedge \xrightarrow{\mathrm{trc}} \mathrm{TC}(\mathrm{ko})_2^\wedge \xrightarrow{p} \Sigma^{-1} H\mathbb{Z}_2.$$

The associated long exact sequence in $A(1)$ -homotopy breaks up into four-term exact sequences, as above.

In more detail, the 3-connected map $A(1) \rightarrow H = H\mathbb{F}_2$ identifies $A(1)_*H\mathbb{Z}_2$ with

$$\mathbb{F}_2\{1, \xi_1^2, \bar{\xi}_2, \xi_1^2\bar{\xi}_2\} \subset H_*H\mathbb{Z}_2 \subset \mathcal{A}^\vee.$$

By [BM94, Proposition 10.9], $K(\mathrm{ko}_2^\wedge) \rightarrow K(\mathbb{Z}_2)$ is 2-connected, where $K_0(\mathbb{Z}_2) = \mathbb{Z}$ and $K_1(\mathbb{Z}_2) = \mathbb{Z}_2^\times$, so that $A(1)_0 K(\mathrm{ko}_2^\wedge) \cong A(1)_0 K(\mathbb{Z}_2) = \mathbb{Z}/2$ and

$$A(1)_1 K(\mathrm{ko}_2^\wedge) \cong A(1)_1 K(\mathbb{Z}_2) = \mathbb{Z}_2^\times / (\pm(\mathbb{Z}_2^\times)^2) \cong \mathbb{Z}/2,$$

generated by any $u \in \mathbb{Z}_2^\times$ congruent to 3 or 5 modulo 8. This uses that $\eta \in \pi_1(\mathbb{S})$ maps to $-1 \in \mathbb{Z}_2^\times \cong K_1(\mathbb{Z}_2)$. By exactness, we know $p: \partial \mapsto \Sigma^{-1}1$ and $p: \varsigma \mapsto \Sigma^{-1}\xi_1^2$. Multiplication by ν acts trivially on $H\mathbb{Z}_2$, so $p: \varsigma\nu \mapsto 0$ does not hit $\Sigma^{-1}\xi_1^2\bar{\xi}_2$. There is no class in degree 2 that p could map to $\Sigma^{-1}\bar{\xi}_2$. Hence these two classes instead appear as $\Sigma^{-2}\bar{\xi}_2$ and $\Sigma^{-2}\xi_1^2\bar{\xi}_2$ in $A(1)_* K(\mathrm{ko}_2^\wedge)$, in degrees 1 and 3, respectively. $\square$

The proof in the integral case relies on the proven Lichtenbaum–Quillen conjecture for $\mathbb{Z}[1/2]$ (cf. [Voe03, RW00]).

THEOREM 7.5. *There is an exact sequence of $\mathbb{Z}/4[v_2^{32}]$ -modules*

$$0 \rightarrow \Sigma^3\mathbb{F}_2 \rightarrow A(1)_* K(\mathrm{ko}) \xrightarrow{\mathrm{trc}} A(1)_* \mathrm{TC}(\mathrm{ko}) \rightarrow \mathbb{F}_2\{\partial, \varsigma\} \rightarrow 0,$$

with $|\partial| = -1$ and $|\varsigma| = 1$.

Proof. By [Rog02, Theorem 3.13] there are two cofiber sequences

$$\begin{aligned} K(\mathrm{ko})_2^\wedge &\xrightarrow{\mathrm{trc}} \mathrm{TC}(\mathrm{ko})_2^\wedge \xrightarrow{q} X, \\ \Sigma^{-2}\mathrm{ku}_2^\wedge &\xrightarrow{\delta} \Sigma^4\mathrm{ko}_2^\wedge \rightarrow X \end{aligned}$$

with equivalent third terms. Passing to $A(1)$ -homotopy, the second cofiber sequence ensures that $A(1)_*X = \mathbb{F}_2\{x_{-1}, x_1, x_4\}$, where $|x_i| = i$. The long exact sequence associated to the first cofiber sequence then breaks up into four-term exact sequences, as shown.

This time, the details are as follows. The 3-connected $\mathbb{E}_\infty$ -ring map $\mathbb{S} \rightarrow \mathrm{ko}$ induces a 4-connected map $K(\mathbb{S}) \rightarrow K(\mathrm{ko})$, where

$$K(\mathbb{S}) \simeq \mathbb{S} \oplus \mathrm{Wh}^{\mathrm{Diff}}(*).$$

Here, $\mathrm{Wh}^{\mathrm{Diff}}(*)$ is 2-connected with $\pi_3 \mathrm{Wh}^{\mathrm{Diff}}(*) = \mathbb{Z}/2$ (cf. [Rog02, Theorem 5.8]). Hence we have $A(1)_0 K(\mathrm{ko}) \cong A(1)_0 K(\mathbb{S}) = \mathbb{Z}/2\{1\}$, $A(1)_1 K(\mathrm{ko}) \cong A(1)_1 K(\mathbb{S}) = 0$, $A(1)_2 K(\mathrm{ko}) \cong A(1)_2 K(\mathbb{S}) = 0$ and $A(1)_3 K(\mathrm{ko}) \cong A(1)_3 K(\mathbb{S}) = \mathbb{Z}/2\{\nu\} \oplus \mathbb{Z}/2$. By exactness, we know $q: \partial \mapsto x_{-1}$ and $q: \varsigma \mapsto x_1$, while x_4 must contribute to $A(1)_3 K(\mathrm{ko})$ and cannot be in the image of q . (It follows that $\nu x_1 = 0 \neq x_4$.) $\square$

COROLLARY 7.6. *The unit map $\mathbb{S} \rightarrow \mathrm{tmf}$ does not factor through $K(\mathrm{ko})$.*

Proof. In fact, the unit map $A(1) \rightarrow A(1) \otimes \mathrm{tmf}$ does not factor through $A(1) \otimes K(\mathrm{ko})$, since $\pi_{20}A(1) \rightarrow A(1)_{20}(\mathrm{tmf}) \cong (\mathbb{Z}/2)^2$ is surjective, as can be seen using Bruner’s `ext` program or from [Pha22, Figure 16], while

$$A(1)_{20} K(\mathrm{ko}) \cong A(1)_{20} \mathrm{TC}(\mathrm{ko}) \cong \mathbb{Z}/2\{v_2^2\nu w\}$$

by Theorems 7.2 and 7.5 (cf. Figure 1). $\square$

The proof by Hahn, Raksit and Wilson [HRW22] of the height 2 telescope conjecture for $\mathrm{TC}(\mathrm{ku})$ can be adapted to prove the corresponding statement for $\mathrm{TC}(\mathrm{ko})$, using our Proposition 2.14 and Theorem 2.27. However, as has kindly been pointed out to us by Ishan

Levy, this is also a direct consequence of the descent result of Clausen, Mathew, Naumann and Noel [CMN⁺20], as we summarize below.

THEOREM 7.7. *For each $X \in \{K(ko), K(ko_2^\wedge), TC(ko)\}$ the canonical map $L_2^f X \rightarrow L_2 X$ is an equivalence. In other words, these spectra all satisfy the height 2 telescope conjecture (at the prime 2).*

Proof. According to [Dun97, HM97] and [RW00], there are equivalences

$$L_{T(2)} K(ku) \simeq L_{T(2)} K(ku_2^\wedge) \simeq L_{T(2)} TC(ku).$$

By [HRW22, Theorem 6.6.4], $L_2^f TC(ku) \simeq L_2 TC(ku)$, which by [Hov95, Corollary 2.2] implies that $L_{T(2)} TC(ku) \simeq L_{K(2)} TC(ku)$ is $K(2)$ -local. Applying descent [CMN⁺20, Theorem 1.8, Theorem 1.10] along $ko \rightarrow ku$ or $ko_2^\wedge \rightarrow ku_2^\wedge$, for $E = K$ or TC , it follows that $L_{T(2)} K(ko)$, $L_{T(2)} K(ko_2^\wedge)$ and $L_{T(2)} TC(ko)$ are all limits of $K(2)$ -local spectra, and hence are $K(2)$ -local. In particular, there is an equivalence $L_{T(2)} X \simeq L_{K(2)} X$ in each case. Standard telescopic and chromatic fracture squares [DFH⁺14, Proposition 6.2.2], and the known validity of the height 1 telescope conjecture [Mah81], [Bou79, Proposition 4.2], then imply that $L_2^f X \simeq L_2 X$ in each case. $\square$

Our calculations also show that $TC(ko)$ has fp-type 2 in the sense of Mahowald and Rezk [MR99], with the following consequence.

THEOREM 7.8. *For $X \in \{K(ko), K(ko_2^\wedge), TC(ko)\}$ and $Y \in \{X_{(2)}, X_2^\wedge\}$, the canonical map $Y \rightarrow L_2^f Y$ is an equivalence in all sufficiently large degrees.*

Proof. Theorems 7.5, 7.4 and 7.2 show, respectively, that $(A(1)/(v_2^{32}))_* X_2^\wedge$ is finite for each of the three choices for X . This implies that $X_2^\wedge$ has fp-type 2 in the sense of [MR99, p. 5], by [MR99, Proposition 3.2]. According to [MR99, Theorem 8.2], this implies that the Brown–Comenetz dual spectrum $IC_2^f X_2^\wedge$ is bounded below and, consequently, that the fiber $C_2^f X_2^\wedge$ of the map $X_2^\wedge \rightarrow L_2^f X_2^\wedge$ is bounded above (cf. [HW22, Theorem 3.1.3]). Using the pullback square

$$\begin{array}{ccc} X_{(2)} & \longrightarrow & X_2^\wedge \\ \downarrow & & \downarrow \\ X_{(2)}[1/2] & \longrightarrow & X_2^\wedge[1/2] \end{array}$$

and the fact that $X_{(2)}[1/2]$ and $X_2^\wedge[1/2]$ are L_2^f -local, it also follows that $X_{(2)} \rightarrow L_2^f X_{(2)}$ is an equivalence in all sufficiently large degrees. $\square$

ACKNOWLEDGEMENTS

The first author would like to thank Jeremy Hahn and Dylan Wilson for helpful conversations during the course of this project. We thank Robert Bruner for freely sharing his minimal resolution calculator `ext`, which is used in § 4. We thank Ishan Levy for pointing out the proof by descent of Theorem 7.7. We also thank the anonymous referees for useful comments, which have improved the paper.

CONFLICTS OF INTEREST

None.

FINANCIAL SUPPORT

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement no. 101034255. ■

JOURNAL INFORMATION

Compositio Mathematica is owned by the Foundation Compositio Mathematica and published by the London Mathematical Society in partnership with Cambridge University Press. All surplus income from the publication of *Compositio Mathematica* is returned to mathematics and higher education through the charitable activities of the Foundation, the London Mathematical Society and Cambridge University Press.

REFERENCES

- AAC⁺25 G. Angelini-Knoll, C. Ausoni, D. L. Culver, E. Höning and J. Rognes, *Algebraic K-theory of elliptic cohomology*, *Geom. Topol.* **29** (2025), 619–686.
- AHL10 V. Angeltveit, M. A. Hill and T. Lawson, *Topological Hochschild homology of ℓ and ko* , *Amer. J. Math.* **132** (2010), 297–330.
- AR05 V. Angeltveit and J. Rognes, *Hopf algebra structure on topological Hochschild homology*, *Algebr. Geom. Topol.* **5** (2005), 1223–1290.
- AR02 C. Ausoni and J. Rognes, *Algebraic K-theory of topological K-theory*, *Acta Math.* **188** (2002), 1–39.
- AR08 C. Ausoni and J. Rognes, *The chromatic red-shift in algebraic K-theory*, *Enseign. Math.* **54** (2008), 13–15.
- BMS19 B. Bhatt, M. Morrow and P. Scholze, *Topological Hochschild homology and integral p-adic Hodge theory*, *Publ. Math. Inst. Hautes Études Sci.* **129** (2019), 199–310.
- BEM17 P. Bhattacharya, P. Egger and M. Mahowald, *On the periodic v_2 self-map of A_1* , *Algebr. Geom. Topol.* **17** (2017), 657–692.
- Boa99 J. M. Boardman, *Conditionally convergent spectral sequences*, in *Homotopy invariant algebraic structures. A conference in honor of J. Michael Boardman. AMS special session on homotopy theory*, Baltimore, MD, 7–10 January 1998 (American Mathematical Society, Providence, RI, 1999), 49–84.
- BBL⁺14 M. Bökstedt, R. R. Bruner, S. Lunøe-Nielsen and J. Rognes, *On cyclic fixed points of spectra*, *Math. Z.* **276** (2014), 81–91.
- BHM93 M. Bökstedt, W. C. Hsiang and I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, *Invent. Math.* **111** (1993), 465–539.
- BM94 M. Bökstedt and I. Madsen, *Topological cyclic homology of the integers*, *Astérisque* **226** (1994), 57–143, *K-theory* (Strasbourg, 1992).
- BM95 M. Bökstedt and I. Madsen, *Algebraic K-theory of local number fields: The unramified case*, in *Prospects in topology*, Princeton, NJ, 1994, *Annals of Mathematics Studies*, vol. 138 (Princeton University Press, Princeton, NJ, 1995), 28–57.
- Bou79 A. K. Bousfield, *The localization of spectra with respect to homology*, *Topology* **18** (1979), 257–281.
- Bru93 R. R. Bruner, *Ext in the nineties*, in *Algebraic topology*, Oaxtepec, 1991, *Contemporary Mathematics*, vol. 146 (American Mathematical Society, Providence, RI, 1993), 71–90.
- BG95 R. Bruner and J. Greenlees, *The Bredon–Löffler conjecture*, *Exp. Math.* **4** (1995), 289–297.

- BMM⁺86 R. R. Bruner, J. P. May, J. E. McClure and M. Steinberger, *H_∞ ring spectra and their applications*, Lecture Notes in Mathematics, vol. 1176 (Springer-Verlag, Berlin, 1986).
- BR05 R. R. Bruner and J. Rognes, *Differentials in the homological homotopy fixed point spectral sequence*, *Algebr. Geom. Topol.* **5** (2005), 653–690.
- BR21 R. R. Bruner and J. Rognes, *The cohomology of the mod 2 Steenrod algebra*, Preprint (2021), [arXiv:2109.13117](https://arxiv.org/abs/2109.13117).
- BR22 R. R. Bruner and J. Rognes, *The Adams spectral sequence for the image-of- J spectrum*, *Trans. Amer. Math. Soc.* **375** (2022), 5803–5827.
- BHL⁺23 R. Burklund, J. Hahn, I. Levy and T. M. Schlank, *K -theoretic counterexamples to Ravenel’s telescope conjecture*, Preprint (2023), [arXiv:2310.17459](https://arxiv.org/abs/2310.17459).
- BSY22 R. Burklund, T. M. Schlank and A. Yuan, *The chromatic Nullstellensatz*, *Ann. of Math.*, to appear. Preprint (2022), [arXiv:2207.09929](https://arxiv.org/abs/2207.09929).
- CE56 H. Cartan and S. Eilenberg, *Homological algebra* (Princeton University Press, Princeton, NJ, 1956).
- CMN⁺24 D. Clausen, A. Mathew, N. Naumann and J. Noel, *Descent and vanishing in chromatic algebraic K -theory via group actions*, *Ann. Sci. Éc. Norm. Supér. (4)* **57** (2024), 1135–1190.
- CMN⁺20 D. Clausen, A. Mathew, N. Naumann and J. Noel, *Descent in algebraic K -theory and a conjecture of Ausoni–Rognes*, *J. Eur. Math. Soc. (JEMS)* **22** (2020), 1149–1200.
- DM81 D. M. Davis and M. Mahowald, *v_1 - and v_2 -periodicity in stable homotopy theory*, *Amer. J. Math.* **103** (1981), 615–659.
- DFH⁺14 C. L. Douglas, J. Francis, A. G. Henriques and M. A. Hill (eds), *Topological modular forms*, *Mathematical Surveys and Monographs*, vol. 201 (American Mathematical Society, Providence, RI, 2014).
- Dun97 B. I. Dundas, *Relative K -theory and topological cyclic homology*, *Acta Math.* **179** (1997), 223–242.
- DGM13 B. I. Dundas, T. G. Goodwillie and R. McCarthy, *The local structure of algebraic K -theory*, *Algebra and Applications*, vol. 18 (Springer-Verlag London, London, 2013).
- DLR22 B. I. Dundas, A. Lindenstrauss and B. Richter, *Corrigendum: Towards an understanding of ramified extensions of structured ring spectra*, *Math. Proc. Cambridge Philos. Soc.* **172** (2022), 247–248.
- GIK⁺22 B. Gheorghe, D. C. Isaksen, A. Krause and N. Ricka, *$\mathbb{C}$ -motivic modular forms*, *J. Eur. Math. Soc. (JEMS)* **24** (2022), 3597–3628.
- GWX21 B. Gheorghe, G. Wang and Z. Xu, *The special fiber of the motivic deformation of the stable homotopy category is algebraic*, *Acta Math.* **226** (2021), 319–407.
- Gre87 J. P. C. Greenlees, *Representing Tate cohomology of G -spaces*, *Proc. Edinb. Math. Soc.* **30** (1987), 435–443.
- GM95 J. P. C. Greenlees and J. P. May, *Generalized Tate cohomology*, *Mem. Amer. Math. Soc.* **113** (1995), 543.
- HRW22 J. Hahn, A. Raksit and D. Wilson, *A motivic filtration on the topological cyclic homology of commutative ring spectra*, *Ann. of Math.*, to appear. Preprint (2022), [arXiv:2206.11208](https://arxiv.org/abs/2206.11208).
- HW22 J. Hahn and D. Wilson, *Redshift and multiplication for truncated Brown–Peterson spectra*, *Ann. of Math. (2)* **196** (2022), 1277–1351.
- HY20 J. Hahn and A. Yuan, *Exotic multiplications on periodic complex bordism*, *J. Topol.* **13** (2020), 1839–1852.

- HR24 A. Hedenlund and J. Rognes, *A multiplicative Tate spectral sequence for compact Lie group actions*, Mem. Amer. Math. Soc. **1468** (2024), doi:[10.1090/memo/1468](https://doi.org/10.1090/memo/1468).
- Hes96 L. Hesselholt, *On the p -typical curves in Quillen's K -theory*, Acta Math. **177** (1996), 1–53.
- HM97 L. Hesselholt and I. Madsen, *On the K -theory of finite algebras over Witt vectors of perfect fields*, Topology **36** (1997), 29–101.
- HN20 L. Hesselholt and T. Nikolaus, *Topological cyclic homology*, in *Handbook of homotopy theory*, CRC Press/Chapman Hall Handbooks in Mathematics Series (CRC Press, Boca Raton, FL, 2020), 619–656.
- Hov95 M. Hovey, *Bousfield localization functors and Hopkins' chromatic splitting conjecture*, in *The Čech centennial (Boston, MA, 1993)*, Contemporary Mathematics, vol. 181 (American Mathematical Society, Providence, RI, 1995), 225–250.
- Isa19 D. C. Isaksen, *Stable stems*, Mem. Amer. Math. Soc. **262** (2019), 1269.
- Koc73 S. O. Kochman, *Homology of the classical groups over the Dyer–Lashof algebra*, Trans. Amer. Math. Soc. **185** (1973), 83–136.
- LMM⁺24 M. Land, A. Mathew, L. Meier and G. Tamme, *Purity in chromatically localized algebraic K -theory*, J. Amer. Math. Soc. **37** (2024), 1011–1040.
- Lan73 P. S. Landweber, *Annihilator ideals and primitive elements in complex bordism*, Illinois J. Math. **17** (1973), 273–284.
- Lur17 J. Lurie, *Higher algebra* (2017), <https://www.math.ias.edu/lurie/papers/HA.pdf>.
- Mad94 I. Madsen, *The cyclotomic trace in algebraic K -theory*, in *First European congress of mathematics*, vol. II, Paris, 1992, Progress in Mathematics, vol. 120 (Birkhäuser, Basel, 1994), 213–241.
- Mah81 M. Mahowald, *bo-resolutions*, Pacific J. Math. **92** (1981), 365–383.
- MR99 M. Mahowald and C. Rezk, *Brown–Comenetz duality and the Adams spectral sequence*, Amer. J. Math. **121** (1999), 1153–1177.
- MNN17 A. Mathew, N. Naumann and J. Noel, *Nilpotence and descent in equivariant stable homotopy theory*, Adv. Math. **305** (2017), 994–1084.
- MS93 J. E. McClure and R. E. Staffeldt, *On the topological Hochschild homology of bu . I*, Amer. J. Math. **115** (1993), 1–45.
- Mos68 R. E. Mosher, *Some stable homotopy of complex projective space*, Topology **7** (1968), 179–193.
- NS18 T. Nikolaus and P. Scholze, *On topological cyclic homology*, Acta Math. **221** (2018), 203–409.
- Pha22 V.-C. Pham, *On homotopy groups of $E_C^{hG_{24}} \wedge A_1$* , Algebr. Geom. Topol. **22** (2022), 3855–3938.
- Pst23 P. Pstragowski, *Synthetic spectra and the cellular motivic category*, Invent. Math. **232** (2023), 553–681.
- Qui70 D. Quillen, *On the (co-)homology of commutative rings*, in *Applications of categorical algebra*, Proceedings of Symposia in Pure Mathematics, vol. XVII (American Mathematical Society, Providence, RI, 1970), 65–87.
- Rav86 D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121 (Academic Press, Orlando, FL, 1986).
- Rog00 J. Rognes, *Algebraic K -theory of finitely presented ring spectra*, Oberwolfach talk (2000), <https://folk.uio.no/rognes/papers/red-shift.pdf>.
- Rog02 J. Rognes, *Two-primary algebraic K -theory of pointed spaces*, Topology **41** (2002), 873–926.

- Rog20 J. Rognes, *The circle action on topological Hochschild homology of complex cobordism and the Brown–Peterson spectrum*, J. Topol. **13** (2020), 939–968.
- RW00 J. Rognes and C. Weibel, *Two-primary algebraic K -theory of rings of integers in number fields*, J. Amer. Math. Soc. **13** (2000), 1–54, with appendix A by M. Kolster.
- Voe03 V. Voevodsky, *Motivic cohomology with $\mathbf{Z}/2$ -coefficients*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 59–104.
- Yua26 A. Yuan, *Examples of chromatic redshift in algebraic K -theory*, J. Eur. Math. Soc. (JEMS) **28** (2026), 3453–3473.

Gabriel Angelini-Knoll gja39@case.edu

Department of Mathematics, Applied Mathematics and Statistics, Case Western Reserve University, 2145 Adelbert Rd, Cleveland, OH 44106, USA

Christian Ausoni ausoni@math.univ-paris13.fr

Laboratoire de Géométrie, Analyse et Applications, LAGA, UMR 7539, Université Sorbonne Paris Nord, CNRS, Villetaneuse 93430, France

John Rognes rogn@math.uio.no

Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, NO-0316 Oslo, Norway