K-Theory 7: 175–200, 1992.

Approximating $K_*(\mathbb{Z})$ through Degree Five

JOHN ROGNES
Matematisk Institutt, Universitetet i Oslo, Postboks 1053, Blindern, N-0316 Oslo, Norway. e-mail: rognes @ math.uio.no.

(Received: February 1992)

Abstract. We approximate the K-theory spectrum of the integers using a spectrum level rank filtration. By means of a certain poset spectral sequence, we explicitly compute the first three subquotients of this filtration. Assuming a conjecture about the filtration's rate of convergence, we conclude that $K_4(\mathbb{Z}) = 0$ and $K_5(\mathbb{Z})$ is a copy of $\mathbb{Z}$ (the Borel summand) plus two-torsion of order at most eight.

Key words. Rank filtration, poset spectral sequence, K-theory of the integers.

1. Introduction

This paper will show that, assuming a connectivity conjecture, the higher algebraic K-group $K_4(\mathbb{Z})$ is zero, while $K_5(\mathbb{Z})$ is the free Borel summand plus two-torsion of order at most eight. The proof is by computation, utilizing constructions made in the author's paper [14]. We first present the main computational result and explain how to determine $K_4(\mathbb{Z})$ and the bound on $K_5(\mathbb{Z})$ from it. Next, we review the requisite constructions. Thereafter, we give more detailed statements of our results, and finally present the calculations.

This work continues the fundamental computations from [1, 8, and 9] of the higher algebraic K-groups of the rational integers.

Let $\mathbf{K}\mathbb{Z}$ denote the K-theory spectrum of the integers, and let $F_k\mathbf{K}\mathbb{Z}$ denote the kth stage of the spectrum level rank filtration of $\mathbf{K}\mathbb{Z}$ as defined in Section 2.

THEOREM 1.1. $H_*(F_3\mathbf{K}\mathbb{Z}) \cong (\mathbb{Z}, 0, 0, \mathbb{Z}/2, 0, \mathbb{Z} + (\# | 4), \ldots)$.

Here $\# | 4$ denotes an elementary Abelian two-group of rank one or two.

We have made the following connectivity conjecture, which these computations are compatible with

CONJECTURE 1.2. *$F_k\mathbf{K}\mathbb{Z} \to F_{k+1}\mathbf{K}\mathbb{Z}$ is $(2k-1)$-connected for $k \geqslant 0$.*

From this follows our main theorem:

THEOREM 1.3. *Assuming the connectivity Conjecture above, $K_4(\mathbb{Z}) = 0$ and $K_5(\mathbb{Z}) \cong \mathbb{Z} + (\# | 8)$.*

Here $\# | 8$ denotes a two-group of order at most eight.

We can also describe the order forty-eight elements in $K_3(\mathbb{Z})$ [8], and the integral Borel generator in $K_5(\mathbb{Z})$ [1]. See 2.1, 5.2, and 6.3 for notation. Duality refers to the bases given there.

THEOREM 1.4. *The homology class in $H_2(P_1)$ dual to $p_5 \in H^3(P_1)$ in the size one subquotient in the second subquotient of the rank filtration lifts to the nonzero class in $H_3(\mathbf{K}\mathbb{Z})$, which lifts further to an order* 48 *class in $K_3(\mathbb{Z})$.*

Assuming the connectivity conjecture, 24 *times the homology Thom class $U \in H_3(SP_{1,12})$ in the size two subquotient of the third subquotient of the rank filtration lifts to twice the generator of the free summand in $H_5(\mathbf{K}\mathbb{Z})$, which lifts further to an integral generator of the Borel summand in $K_5(\mathbb{Z})$.*

We will review the rank filtration and poset spectral sequences in Section 2, list the spectrum homology results in Section 3, prove Theorems 1.1, 1.3, and 1.4 in Section 4, and give the group homological calculations in the last three sections.

The results are compatible with the Lichtenbaum–Quillen conjectures, also at the prime two [5], which predict that $K_4(\mathbb{Z}) = 0$ and $K_5(\mathbb{Z}) \cong \mathbb{Z}$.

Our computations prove Conjecture 1.2 for $k \leqslant 2$. In [14], we more generally constructed a spectrum level rank filtration $\{F_k\mathbf{K}R\}_k$ of the free K-theory spectrum $\mathbf{K}R$ for all rings satisfying the *strong invariant dimension property* [10], i.e. there only exists split injections $R^i \hookrightarrow R^j$ when $i \leqslant j$. This includes all commutative rings. We conjectured in [14, 12.3] that $F_k\mathbf{K}R \to F_{k+1}\mathbf{K}R$ would be $(2k-1)$-connected for local rings or Euclidean domains R, when $k \geqslant 0$. Analogous computations to those of this paper, but easier, prove this form of the conjecture for the cases $R = \mathbb{F}_2, \mathbb{F}_3, \mathbb{R}$ and $\mathbb{C}$, when $k \leqslant 2$.

In principle, these methods extend to higher subquotients of the rank filtration, but it might be desirable to further structure the work using the tensor product on $\mathbf{K}\mathbb{Z}$ before proceeding, and to work with $\mathbf{K}\mathbb{Q}$ to benefit from better homological stability results for fields.

2. The Poset Spectral Sequence

This section reviews the construction of the spectrum level rank filtration, the poset filtration on each subquotient of this rank filtration, and its associated poset spectral sequence. We introduce notation for parabolic groups and other matrix groups which we will need. Finally, we consider some useful generalizations of the poset spectral sequence.

Recall that the nth space of the K-theory spectrum $\mathbf{K}\mathbb{Z}$ can be defined as the diagonal of the nerve of the n-multisimplicial category $wS^n_\cdot\mathscr{P}(\mathbb{Z})$. Here $\mathscr{P}(\mathbb{Z})$ is the category of finitely generated projective $\mathbb{Z}$-modules and $S^n_\cdot$ is Waldhausen's $S_\cdot$-construction iterated n times [15]. The kth stage $F_k\mathbf{K}\mathbb{Z}$ of the spectrum level rank filtration of $\mathbf{K}\mathbb{Z}$ is the (pre-)spectrum with nth space $F_k wS^n_\cdot\mathscr{P}(\mathbb{Z})$ given as the subcomplex of the nerve of $wS^n_\cdot\mathscr{P}(\mathbb{Z})$ consisting of the simplices which are diagrams in $\mathscr{P}(\mathbb{Z})$ involving only $\mathbb{Z}$-modules of rank k or less.

The poset spectral sequence computes, for a given rank k, the spectrum homology of the subquotient spectrum $F_k\mathbf{K}\mathbb{Z}/F_{k-1}\mathbf{K}\mathbb{Z}$. The nth space of this (pre-) spectrum has simplices which are chains of isomorphic diagrams on some n-dimensional cubical indexing categories. To each simplex we can associate k distinguished unordered *sites*, or objects in the cubical indexing category [14, §5]. These inherit a partial ordering from the product ordering on this n-dimensional cube, in the weak sense that two distinct sites may be mutually less than each other. Hence, to each simplex we may associate an isomorphism class of partial orderings on the k-element set $\{1, \ldots, k\}$. Such partial orderings may, in turn, be ordered by their *sizes*, where the size of a poset is defined as its number of components plus its number of equivalence classes, less two [14, §7]. This gives rise to a filtration on the nth space of $F_k\mathbf{K}\mathbb{Z}/F_{k-1}\mathbf{K}\mathbb{Z}$, and as n tends to infinity the associated homology spectral sequence stabilizes. The colimit is the poset spectral sequence which converges to $H_*(F_k\mathbf{K}\mathbb{Z}/F_{k-1}\mathbf{K}\mathbb{Z})$. It is located in the first quadrant, and is of homological type.

We shall describe the poset spectral sequence's E^1-term shortly, after making two further remarks on the geometry of the situation, and introducing some notation.

Embedded inside the nth space of the kth subquotient of the rank filtration is a complex called the n-dimensional *building* of $\mathbb{Z}^k$, denoted $D^n(\mathbb{Z}^k)$. It consists of the simplices which are isomorphism chains of length zero, i.e. single cubical diagrams. We may assume that the final module occurring in such a diagram is precisely $\mathbb{Z}^k$, and that all the other modules are included in it as submodules. $D^n(\mathbb{Z}^k)$ is a based $\mathrm{GL}_k\mathbb{Z}$-space, and its (reduced) homotopy orbit space, or Borel construction, is precisely the nth space of the kth subquotient of the rank filtration [14, §3]. The buildings assemble into a spectrum, $\mathbf{D}(\mathbb{Z}^k)$.

Call the submodules of $\mathbb{Z}^k$ generated by a subset of the standard basis for $\mathbb{Z}^k$ the *axial* submodules. The standard *apartment* is the subcomplex of the n-dimensional building with simplices the diagrams consisting of axial submodules. It is homeomorphic to an nk-sphere [14, §4], so for $k \geqslant 2$ it is stably contractible as n tends to infinity.

Notation 2.1. For $I \subseteq \{1, \ldots, k\}$, let $P_I \subseteq \mathrm{GL}_k\mathbb{Z}$ denote the parabolic subgroup consisting of matrices (g_{ij}) such that $g_{ij} = 0$ if $j \in I$ and $i \notin I$. Similarly, let $P_{I,J} = P_I \cap P_J$, etc. We omit curly parentheses in these subscripts. If we wish to emphasize the size k of the matrices in question, we will write $P_I(k)$. SP_I will denote the subgroup of P_I consisting of matrices with determinant one. For example, $P_1(2)$ and $P_{1,12}(3)$ are the upper triangular 2×2 and 3×3 matrices.

View the symmetric group Σ_k as embedded in $\mathrm{GL}_k\mathbb{Z}$ as the permutation matrices, and let $T_k \subset \mathrm{GL}_k\mathbb{Z}$ denote the diagonal torus consisting of matrices which are zero off the diagonal. Σ_k normalizes T_k and we set $D_k = \Sigma_k \cdot T_k$. ST_k and SD_k are the determinant one subgroups. D_2 is the dihedral group of order eight and SD_3 the symmetric group on four letters.

Let $C_k \subseteq \mathrm{GL}_k\mathbb{Z}$ denote the center. It consists of plus and minus the identity matrix.

Let W_k denote the integral rank $(k-1)!$ representation of Σ_k dual to the free abelian group generated by the iterated Lie brackets of weight one in each generator in the free Lie algebra on k generators, with Σ_k permuting these generators. Explicitly, W_2 is the rank one sign representation of Σ_2, while W_3 is a Σ_3-representation of rank two with

$$\phi = (12) \text{ acting by } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \psi = (123) \text{ acting by } \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

We also view the W_k as D_k- and SD_k-representations through the natural homomorphisms $SD_k \to D_k \to \Sigma_k$.

Then the poset spectral sequence for the second subquotient of K-theory has E^1-term:

(GL$_2\mathbb{Z}$ PSS)

$$H_*(\mathrm{GL}_2\mathbb{Z}) \xleftarrow{d^1} H_*(P_1) \xleftarrow{d^1} H_*(D_2; W_2)$$

(group homology), and converges to the spectrum homology $H_*(F_2\mathbf{K}\mathbb{Z}/F_1\mathbf{K}\mathbb{Z})$. The d^1-differential $H_*(P_1) \to H_*(\mathrm{GL}_2\mathbb{Z})$ is induced by the inclusion of groups. The other d^1-differential factors as $H_*(D_2; W_2) \to H_*(T_2) \to H_*(P_1)$, where the right-hand map is induced by the group inclusion and the left-hand map occurs in the long exact sequence of D_2-homology coming from the short exact coefficient sequence $W_2 \to \mathbb{Z}\Sigma_2 \to \mathbb{Z}$, using Shapiro's lemma. See [14, 12.4 and 15.2] for a proof.

Similarly, the poset spectral sequence for the third subquotient of K-theory has E^1-term:

(GL$_3\mathbb{Z}$ PSS)

$$\begin{aligned} H_*\mathrm{GL}_3\mathbb{Z} \xleftarrow{d^1} H_*P_1 \oplus H_*P_{12} &\xleftarrow{d^1} H_*P_{1,12} \oplus H_*P_{12,3} \\ &\xleftarrow{d^1} H_*P_{1,12,3} \xleftarrow{d^1} H_*(D_3; W_3) \end{aligned}$$

converging to $H_*(F_3\mathbf{K}\mathbb{Z}/F_2\mathbf{K}\mathbb{Z})$. Up to sign, the two leftmost differentials are induced by inclusions of groups, together with (right) conjugation by $\psi^2 = (132)$ in the case of $P_{12,3} \hookrightarrow P_1$. The third differential $d^1\colon E^1_{3*} \to E^1_{2*}$ maps to a direct sum. We describe its two summands separately: The homomorphism to $H_*(P_{12,3})$ is induced by inclusion. The homomorphism to $H_*(P_{1,12})$ is the alternating signed sum of three maps induced by inclusion followed by (right) conjugation by respectively $e = (\,)$, $\phi\psi = (23)$ and $\psi^2 = (132)$. The right-most differential is harder to describe, and we shall use naturality to handle it. These descriptions follow easily from [14, 15.3].

By alluding to naturality, we now get to various variant poset spectral sequences. For subgroups of $\mathrm{GL}_k\mathbb{Z}$, such as $\mathrm{SL}_k\mathbb{Z}$, which act transitively on all submodule configurations [14, §8] we may consider the corresponding homotopy orbit space of $D^n(\mathbb{Z}^k)$ as a subcomplex of the $\mathrm{GL}_k\mathbb{Z}$-homotopy orbit space of $D^n(\mathbb{Z}^k)$. This subcomplex inherits a poset filtration and a spectral sequence as indicated above, which naturally maps to the $\mathrm{GL}_k\mathbb{Z}$ poset spectral sequence. Our interest is in the case of $\mathrm{SL}_3\mathbb{Z}$. In view of the splitting $\mathrm{GL}_3\mathbb{Z} \cong \mathrm{SL}_3\mathbb{Z} \times C_3$, with $C_3 \cong \mathbb{Z}/2$ generated by minus the identity matrix, we obtain the following $\mathrm{SL}_3\mathbb{Z}$ poset spectral sequence which splits off the $\mathrm{GL}_3\mathbb{Z}$ poset spectral sequence:

($\mathrm{SL}_3\mathbb{Z}$ PSS)

$$H_*(\mathrm{SL}_3\mathbb{Z}) \xleftarrow{d^1} H_*(SP_1) \oplus H_*(SP_{12}) \xleftarrow{d^1} H_*(SP_{1,12}) \oplus H_*(SP_{12,3}) \xleftarrow{d^1} H_*(SP_{1,12,3}) \xleftarrow{d^1} H_*(SD_3; W_3)$$

It converges to the spectrum homology of $\mathbf{D}(\mathbb{Z}^3)/h\mathrm{SL}_3\mathbb{Z} = \{D^n(\mathbb{Z}^3)/h\mathrm{SL}_3\mathbb{Z}\}$. As the center acts trivially by conjugation on the n-dimensional buildings, we can recover $F_3\mathbf{K}\mathbb{Z}/F_2\mathbf{K}\mathbb{Z}$ as $\mathbf{D}(\mathbb{Z}^3)/h\mathrm{SL}_3\mathbb{Z} \wedge BC_{3+}$. Here $X/hG = X \wedge_G EG_+$ denotes the homotopy orbit space.

In a different direction, we may consider subgroups of $\mathrm{GL}_k\mathbb{Z}$ containing Σ_k which leave the axial submodules invariant. The largest such is D_k, but we also consider SD_k. These groups act on the standard apartment, and the corresponding homotopy orbit spaces inherit a poset filtration and spectral sequence. As the standard apartment is stably contractible for $k \geqslant 2$, so is its homotopy orbit space, and these spectral sequences will abut to zero. We will use

(D_2 PSS)

$$H_*(D_2) \xleftarrow{d_1} H_*(T_2) \xleftarrow{d_1} H_*(D_2; W_2)$$

and

(SD_3 PSS)

$$H_*SD_3 \xleftarrow{d_1} H_*S(\mathbb{Z}/2 \times D_2) \oplus H_*S(D_2 \times \mathbb{Z}/2) \xleftarrow{d_1} H_*ST_3 \oplus H_*S(D_2 \times \mathbb{Z}/2) \xleftarrow{d_1} H_*ST_3 \xleftarrow{d_1} H_*(SD_3; W_3)$$

As is clear from their construction, the groups occurring in these spectral sequences are the intersections of the parabolic subgroups of the $\mathrm{GL}_k\mathbb{Z}$ poset spectral sequence with the group in question, i.e. D_2 or SD_3. For instance, $P_1(3) \cap SD_3 = S(\mathbb{Z}/2 \times D_2)$.

3. The Results

PROPOSITION 3.1.

$$F_1\mathbf{K}\mathbb{Z} \cong \Sigma^\infty B\mathbb{Z}/2_+, \text{ so } H_*(F_1\mathbf{K}\mathbb{Z}) \cong (\mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}/2, 0, \mathbb{Z}/2, \ldots).$$

Proof. This is [14, 15.1].

THEOREM 3.2. (1) *Localized at the prime two:*

$$H_*(F_2\mathbf{K}\mathbb{Z}/F_1\mathbf{K}\mathbb{Z}) \cong (0, 0, \mathbb{Z}/2, \mathbb{Z}/2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^3, (\mathbb{Z}/2)^3, \ldots)$$

There are nonzero classes in degrees three and four dual to $p_5 \in H^3(P_1)$ *and the sum of the duals of* $p_1 p_3$ *and* $p_2 p_3$ *in* $H^4(P_1)$*, respectively.*

(2) *Modulo two-torsion:*

$$H_*(F_2\mathbf{K}\mathbb{Z}/F_1\mathbf{K}\mathbb{Z}) \cong (0, 0, 0, \mathbb{Z}/3, 0, 0, 0, \mathbb{Z}/3, 0, \ldots)$$

PROPOSITION 3.3. *The connecting maps* $\partial\colon H_{2i}(F_2\mathbf{K}\mathbb{Z}/F_1\mathbf{K}\mathbb{Z}) \to H_{2i-1}(F_1\mathbf{K}\mathbb{Z})$ *are onto for all* $i \geqslant 1$.

Let $\#n$ denote a group with n elements.

THEOREM 3.4. (1) *Localized at the prime two,* $F_3\mathbf{K}\mathbb{Z}/F_2\mathbf{K}\mathbb{Z}$ *is four-connected,* $H_5(F_3\mathbf{K}\mathbb{Z}/F_2\mathbf{K}\mathbb{Z}) \cong \mathbb{Z}$ *is generated by* $2U + V \in H_3(SP_{1,12})$*, where* U *is a Thom class of infinite order and* V *a class of order two, and* $H_6(F_3\mathbf{K}\mathbb{Z}/F_2\mathbf{K}\mathbb{Z}) \cong \mathbb{Z}/2$.

(2) *Modulo two-torsion*

$$H_*(F_3\mathbf{K}\mathbb{Z}/F_2\mathbf{K}\mathbb{Z}) \cong (0, 0, 0, 0, \mathbb{Z}/3, \mathbb{Z}, 0, 0, \#9, 0, \ldots)$$

and $H_5(F_3\mathbf{K}\mathbb{Z}/F_2\mathbf{K}\mathbb{Z}) \cong \mathbb{Z}$ *is generated by* $3U \in H_3(SP_{1,12})$*, where* U *is as above.*

PROPOSITION 3.5. (1) *The connecting map*

$$\partial\colon H_5(F_3\mathbf{K}\mathbb{Z}/F_2\mathbf{K}\mathbb{Z}) \to H_4(F_2\mathbf{K}\mathbb{Z}/F_1\mathbf{K}\mathbb{Z})$$

is nonzero, hitting the sum of the classes dual to $p_1 p_3$ *and* $p_2 p_3$ *in* $E^{\infty}_{1,3}$ *in the* $\mathrm{GL}_2\mathbb{Z}$ *PSS.*

(2) *Modulo two-torsion, the connecting maps*

$$\partial\colon H_{4i}(F_3\mathbf{K}\mathbb{Z}/F_2\mathbf{K}\mathbb{Z}) \to H_{4i-1}(F_2\mathbf{K}\mathbb{Z}/F_1\mathbf{K}\mathbb{Z})$$

are onto for all $i \geqslant 1$.

4. The Proofs

Proof of Theorems 1.1 and 1.4. Let us assemble the beginnings of the spectrum level rank filtration of $\mathbf{K}\mathbb{Z}$ into two spectrum homology spectral sequences as below; one localized at two, the other away from two:

$(H_*(\mathbf{K}\mathbb{Z}; \mathbb{Z}_{(2)}))$

$$\begin{array}{ccccc}
0 & & (\mathbb{Z}/2)^3 & & ? \\
\mathbb{Z}/2 & \leftarrow & (\mathbb{Z}/2)^3 & & ? \\
0 & & (\mathbb{Z}/2)^2 & & \mathbb{Z}/2 \\
\mathbb{Z}/2 & \leftarrow & (\mathbb{Z}/2)^2 & \leftarrow & \mathbb{Z}_{(2)} \\
0 & & \mathbb{Z}/2 & & 0 \\
\mathbb{Z}/2 & \leftarrow & \mathbb{Z}/2 & & 0 \\
\mathbb{Z}_{(2)} & & 0 & & 0
\end{array}$$

and

$(H_*(\mathbf{K}\mathbb{Z}; \mathbb{Z}[\frac{1}{2}]))$

$$
\begin{array}{ccccc}
0 & \mathbb{Z}/3 & \leftarrow & \#9 \\
0 & 0 & & 0 \\
0 & 0 & & 0 \\
0 & 0 & & \mathbb{Z}[\frac{1}{2}] \\
0 & \mathbb{Z}/3 & \leftarrow & \mathbb{Z}/3 \\
0 & 0 & & 0 \\
\mathbb{Z}[\frac{1}{2}] & 0 & & 0
\end{array}
$$

The connectivity conjecture can be seen as a potential vanishing line in these spectral sequences.

There may be a connecting map, i.e. a d^1-differential, $E^1_{2,4} \to E^1_{1,4}$ in the two-primary case, so the integral E^∞-term begins as follows:

$$
\begin{array}{ccc}
0 & ? & ? \\
0 & \#|4 & ? \\
0 & 0 & \mathbb{Z} \\
0 & \mathbb{Z}/2 & 0 \\
0 & 0 & 0 \\
\mathbb{Z} & 0 & 0
\end{array}
$$

This proves Theorem 1.1. Twice the $\mathbb{Z}$-generator in $E^1_{2,3}$ survives to E^∞, so the second part of Theorem 1.4 follows from Theorem 3.4. The first part of Theorem 1.4 is contained in Theorem 3.2. □

Recall Bökstedt's construction of a spectrum $\mathbf{JK}\mathbb{Z}_2^\wedge$ related to two-primary étale K-theory, receiving a spectrum map f from $\mathbf{K}\mathbb{Z}_2^\wedge$ [3]. Here $\mathbf{X}_2^\wedge$ denotes the Bousfield–Kan completion of $\mathbf{X}$ at two. We can define $\mathbf{JK}\mathbb{Z}_2^\wedge$ as the homotopy fiber of the composite map of spectra

$$\mathbf{ko} \xrightarrow{\psi^3 - 1} \mathbf{bspin} \xrightarrow{c} \mathbf{bsu}$$

from connective real K-theory to three-connected complex K-theory, with ψ^3 the Adams operation, and c complexification. Then $\mathbf{JK}\mathbb{Z}_2^\wedge$ is a covering of the pullback in the diagram of two-completed algebraic K-theory spectra induced by a square of ring homomorphisms:

$$
\begin{array}{ccc}
\mathbb{Z} & \longrightarrow & \mathbb{R} \\
\downarrow & & \downarrow \\
\mathbb{Z}_3^\wedge & \longrightarrow & \mathbb{C}
\end{array}
$$

and f is the canonical map from $\mathbf{K}\mathbb{Z}$. Bökstedt proves, using étale methods, that $\Omega\Omega^\infty\mathbf{JK}\mathbb{Z}_2^\wedge$ splits off $\Omega\Omega^\infty\mathbf{K}\mathbb{Z}_2^\wedge$ *as a space*, with $\Omega\Omega^\infty f$ as the retraction. Hence, the spherical classes of $H_*(\mathbf{JK}\mathbb{Z}_2^\wedge)$ split off those of $H_*(\mathbf{K}\mathbb{Z})$.

Proof of Theorem 1.3. We will compare the Atiyah–Hirzebruch spectral sequences (AH-SS) $E^2 = H_*(\mathbf{X}; \pi_*\mathbf{S}) \Rightarrow \pi_*(\mathbf{X})$ for the spectra $F_3\mathbf{K}\mathbb{Z}$, $\mathbf{K}\mathbb{Z}$ and $\mathbf{JK}\mathbb{Z}_2^\wedge$.

From [13] and [8], we know that the spectrum homology of $\mathbf{K}\mathbb{Z}$ begins $H_*(\mathbf{K}\mathbb{Z}) \cong (\mathbb{Z}, 0, 0, \mathbb{Z}/2, \ldots)$, and there are no differentials in the AH-SS for $\mathbf{K}\mathbb{Z}$ originating in total degree four or below. Let $\lambda \in H_3(\mathbf{K}\mathbb{Z})$ and $\eta \in \pi_1(\mathbf{S})$ denote the nonzero elements.

By the assumed connectivity Conjecture 1.2, the inclusion map $F_3\mathbf{K}\mathbb{Z} \to \mathbf{K}\mathbb{Z}$ induces a five-connected map $H_*(F_3\mathbf{K}\mathbb{Z}) \to H_*(\mathbf{K}\mathbb{Z})$. So $H_4(\mathbf{K}\mathbb{Z}) = 0$ and the free summand in $H_5(F_3\mathbf{K}\mathbb{Z})$ must map isomorphically to a free summand in $H_5(\mathbf{K}\mathbb{Z})$ (cf. [1]).

From the description above, it follows that

$$H_*(\mathbf{JK}\mathbb{Z}_2^\wedge) \cong (\mathbb{Z}_2^\wedge, 0, 0, \mathbb{Z}/2, 0, \mathbb{Z}_2^\wedge, \ldots),$$

and

$$\pi_*(\mathbf{JK}\mathbb{Z}_2^\wedge) \cong (\mathbb{Z}_2^\wedge, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/16, 0, \mathbb{Z}_2^\wedge, \ldots).$$

In particular, there are d^2-differentials hitting the nonzero classes in bidegrees $(3, 1)$, $(3, 2)$ and $(4, 1)$ in the AH-SS for $\mathbf{JK}\mathbb{Z}_2^\wedge$.

Since the $\mathbb{Z}/2$ in $H_3(\mathbf{JK}\mathbb{Z}_2^\wedge) = E^2_{3,0}$ survives to E^∞, it is a spherical class, and thus splits off $H_3(\mathbf{K}\mathbb{Z})$. It follows that f is five-connected, and by the relative Hurewicz theorem $H_5(\mathbf{K}\mathbb{Z}) \to H_5(\mathbf{JK}\mathbb{Z}_2^\wedge) \cong \mathbb{Z}$ is onto. The latter observation is due to Marcel Bökstedt.

Then there is a nonzero differential $d^2\colon E^2_{5,0} \to E^2_{3,1}$ in the AH-SS for $\mathbf{K}\mathbb{Z}$, factoring through $H_5(f)$, the d^2 for $\mathbf{JK}\mathbb{Z}_2^\wedge$, and $H_3(f)^{-1}$. It is necessarily zero on torsion, and takes a generator of a free summand to $\eta\lambda$.

Using the $\pi_*(\mathbf{S})$-module structure on the AH-SS, η times this differential must hit $\eta^2\lambda$ which generates $E^2_{3,2} \cong \mathbb{Z}/2$. However, we do not know if the differential ending in bidegree $(4, 1)$ in the AH-SS for $\mathbf{JK}\mathbb{Z}_2^\wedge$ lifts to one for $\mathbf{K}\mathbb{Z}$.

In conclusion, the contributions to $\pi_*(F_3\mathbf{K}\mathbb{Z})$ through degree five are: $\pi_*(\mathbf{S})$, the nontrivial extension $\pi_3(\mathbf{S}) \to K_3(\mathbb{Z}) \to H_3(\mathbf{K}\mathbb{Z})$, some possibly surviving two-torsion in $E^2_{4,1} = H_3(\mathbf{K}\mathbb{Z}) * \pi_1(\mathbf{S}) \cong \mathbb{Z}/2$, and twice the $\mathbb{Z}$-generator plus some possible two-torsion remaining in $E^2_{5,0} = H_5(F_3\mathbf{K}\mathbb{Z})$. Theorem 1.3 follows. □

5. Computations at the Second Stage

Let us settle on the following notation:

NOTATION 5.1. In $\mathrm{GL}_2\mathbb{Z}$ let

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ generate } D_2,$$

$$x = sr = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad y = rs = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

generate T_2, s and

$$t = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

generate $\mathrm{SL}_2\mathbb{Z}$, and

$$x, y \text{ and } c = t^{-1}s^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

generate P_1. s and t are of order four and six, with $s^2 = t^3$.

Let σ and ρ in $H^1(D_2; \mathbb{Z}/2)$ be dual to s and r, ξ and η in $H^1(T_2; \mathbb{Z}/2)$ be dual to x and y, and ξ, η and γ in $H^1(P_1; \mathbb{Z}/2)$ be dual to x, y and c.

Let $\omega \in H^2(D_2; \mathbb{Z}/2)$ be the restriction of the second Stiefel–Whitney class from $H^2(\mathrm{GL}_2\mathbb{R}; \mathbb{Z}/2)$.

The following cohomology algebras will be useful:

LEMMA 5.2.

(1) $H^*(T_2) \cong \mathbb{Z}[t_1, t_2, t_3]/t_3^2 = t_1 t_2(t_1 + t_2)$

where the generators (t_1, t_2, t_3) *are of degrees* $(2, 2, 3)$ *and of orders* $(2, 2, 2)$ *respectively. Reduced modulo two,* $t_1 = \xi^2$, $t_2 = \eta^2$ *and* $t_3 = \xi\eta(\xi + \eta)$.

(2) $H^*(\mathrm{GL}_2\mathbb{Z})_{(2)} \cong H^*(D_2) \cong \mathbb{Z}[x_1, x_2, x_3, x_4]/x_1^2 = x_1 x_2, x_3^2 = x_2 x_4$

where the generators (x_1, x_2, x_3, x_4) *are of degrees* $(2, 2, 3, 4)$ *and of orders* $(2, 2, 2, 4)$ *respectively. Reduced modulo two,* $x_1 = \sigma^2$, $x_2 = \rho^2$, $x_3 = \rho\omega$ *and* $x_4 = \omega^2$.

(3) $H^*(P_1) \cong \mathbb{Z}[p_1, p_2, p_3, p_4, p_5]/p_3^2 = (p_1 + p_2)p_3, p_3 p_4 = p_3 p_5 = (p_1 + p_2)p_5, p_4^2 = p_1 p_2(p_1 + p_2), p_4 p_5 = p_5^2 = p_1 p_2 p_3,$

where the generators $(p_1, \ldots, p_5)$ *are of degrees* $(2, 2, 2, 3, 3)$ *and of orders* $(2, 2, 2, 2, 2)$, *respectively. Reduced modulo two,*

$$p_1 = \xi^2, \qquad p_2 = \eta^2, \qquad p_3 = \gamma^2, \qquad p_4 = \xi\eta(\xi + \eta) \quad \textit{and} \quad p_5 = \xi\eta\gamma.$$

Proof (note the implicit relations $2 \cdot t_1 = 0$, etc.). First compute the $\mathbb{Z}/2$-cohomology algebras. The case T_2 is easy, the case D_2 is given in [6, p. 322] ($\sigma^2 = \sigma\rho$), and for P_1 use the Lyndon–Hochschild–Serre spectral sequence (LHS-SS) for the split extension $\langle c \rangle = \mathbb{Z} \to P_1 \to T_2$ which collapses at $E_2 = \mathbb{Z}/2[\xi, \eta] \otimes E(\gamma)$. The algebra structure ($\gamma^2 = \xi\gamma + \eta\gamma$) will be forced by the result for D_2 and the form of the algebra homomorphism induced by $P_1 \subset \mathrm{GL}_2\mathbb{Z}$ (see Lemma 5.3).

Next extract the integral information using a Bockstein spectral sequence [2]. Let β_{2^n} denote the Bockstein differentials killing elements of integral order 2^n ($n \geqslant 1$). Then the β_2 differentials square all degree one generators. The remaining nonzero differentials can be determined from the computations in [7], and are $\beta_2(\omega) = \rho\omega$ and $\beta_4(\sigma\omega) = \omega^2$, both in the case of $H^*(D_2)$.

The two-primary identification of $H^*(D_2)$ and $H^*(\mathrm{GL}_2\mathbb{Z})$ comes about by comparing the extensions $\langle s\rangle = \mathbb{Z}/4 \to D_2 \to \mathbb{Z}/2 = \langle r\rangle$ and $\langle s, t\rangle = \mathrm{SL}_2\mathbb{Z} \to \mathrm{GL}_2\mathbb{Z} \to \mathbb{Z}/2$, where the inclusion $\mathbb{Z}/4 \hookrightarrow \mathrm{SL}_2\mathbb{Z} \cong \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$ induces a homology isomorphism at two.

The algebra relations given follow from those in $\mathbb{Z}/2$-cohomology. An induction argument on the growth of the order of $H^*(P_1)$ shows that they form a complete set. It is easy to check that $H^*(T_2)$, $H^*(D_2)$ and $H^*(P_1)$ contain no odd torsion. □

LEMMA 5.3. *The inclusion $P_1 \subset \mathrm{GL}_2\mathbb{Z}$ induces the algebra homomorphism given by*

$$x_1 \mapsto p_1 + p_2 + p_3, \qquad x_2 \mapsto p_1 + p_2, \qquad x_3 \mapsto p_4 \quad \text{and} \quad x_4 \mapsto p_1 p_2$$

on integral cohomology. With $\mathbb{Z}/2$-coefficients, the homomorphism is

$$\sigma \mapsto \xi + \eta + \gamma, \qquad \rho \mapsto \xi + \eta \quad \text{and} \quad \omega \mapsto \xi\eta.$$

Proof. Most of this follows from expressing the generators x, y and c in s, r and t. $\omega \in H^2(D_2; \mathbb{Z}/2)$ comes from $w_2 \in H^2(\mathrm{GL}_2\mathbb{R}; \mathbb{Z}/2)$, and its restriction to P_1 factors through the real parabolic Borel subgroup $P_1\mathbb{R} \subset \mathrm{GL}_2\mathbb{R}$ consisting of upper-triangular real matrices. The inclusion $T_2 \subset P_1\mathbb{R}$ is a homotopy equivalence, so the restriction of ω to P_1 is the same as that coming from $P_1\mathbb{R} \simeq T_2$, namely $\xi\eta$. Jørgen Tornehave supplied this argument. □

Staying with induced homomorphisms, we gather together:

LEMMA 5.4. *The inclusion $k_1: T_2 \hookrightarrow D_2$ induces $x_1 \mapsto t_1 + t_2$, $x_2 \mapsto t_1 + t_2$, $x_3 \mapsto t_3$ and $x_4 \mapsto t_1 t_2$ on cohomology.*

The injection $k_2: T_2 \hookrightarrow D_2$ taking x to s^2 and y to r induces $x_1 \mapsto 0$, $x_2 \mapsto t_2$, $x_3 \mapsto t_3$ and $x_4 \mapsto t_1^2 + t_1 t_2$.

Proof. This is easiest with $\mathbb{Z}/2$-coefficients, where the classes in degree one are determined by what group elements they detect. The image of the second Stiefel–Whitney class is determined by its restrictions to the three order two subgroups of T_2, which can be found by the Cartan formula. □

The last groups in the $\mathrm{GL}_2\mathbb{Z}$ PSS are found as follows:

LEMMA 5.5. *The LHS-SS for $H_*(D_2; W_2)$ using the split extension $\mathbb{Z}/4 \to D_2 \to \mathbb{Z}/2$ collapses at E^2, and splits, i.e. there are no extensions in the E^∞-term. Thus*

$$H_*(D_2; W_2) \cong (\mathbb{Z}/2, \mathbb{Z}/2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^3, (\mathbb{Z}/2)^3, \ldots)$$

Proof. Actually, we can compute $H_*(D_2; W_2)$ directly using Hamada's resolution [7]. Alternatively, we can determine it up to extensions from the long exact sequence linking this group to $H_*(D_2)$ and $H_*(T_2)$, which appears in another guise as the D_2 PSS abutting to zero. Comparing with the computation of $H_*(D_2; W_2 \otimes \mathbb{Z}/2) \cong H_*(D_2; \mathbb{Z}/2)$ above, it follows that there are no extensions, for any cyclic summands of order four or more in $H_*(D_2; W_2)$ would give too small a rank for $H_*(D_2; W_2 \otimes \mathbb{Z}/2)$. Lastly, we can find the E^2-term via computing $H_*(\mathbb{Z}/4; W_2)$ with the periodic resolution, and the collapsing and splitting results follow in view of the preceding remarks. □

We may now describe the complete two-primary behavior of the $\mathrm{GL}_2\mathbb{Z}$ PSS.

Proof of Theorem 3.2. (1). We will compare the D_2 PSS with the $\mathrm{GL}_2\mathbb{Z}$ PSS:

$$\begin{array}{ccccc} H_*(D_2) & \xleftarrow{d^1} & H_*(T_2) & \xleftarrow{d^1} & H_*(D_2; W_2) \\ \downarrow{\cong} & & \downarrow & & \| \\ H_*(\mathrm{GL}_2\mathbb{Z})_{(2)} & \xleftarrow{d^1} & H_*(P_1) & \xleftarrow{d^1} & H_*(D_2; W_2) \end{array}$$

Lemmas 5.2 and 5.5 give the E^1-term of the $\mathrm{GL}_2\mathbb{Z}$ PSS. The left d^1-differential is given by Lemma 5.3, while the (graded) rank of the right d^1-differential agrees with that for the D_2 PSS as $H_*(T_2) \to H_*(P_1)$ is a split injection. This, in turn, agrees with the rank of the kernel of $H_*(T_2) \to H_*(D_2)$ by exactness in the D_2 PSS, which is found using Lemma 5.4:

$$\begin{array}{ccccc} (\mathbb{Z}/2)^4 & \xleftarrow{4} & (\mathbb{Z}/2)^7 & \xleftarrow{2} & (\mathbb{Z}/2)^3 \\ (\mathbb{Z}/2)^2 & \xleftarrow{2} & (\mathbb{Z}/2)^4 & \xleftarrow{1} & (\mathbb{Z}/2)^3 \\ (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/4 & \xleftarrow{3} & (\mathbb{Z}/2)^5 & \xleftarrow{1} & (\mathbb{Z}/2)^2 \\ \mathbb{Z}/2 & \xleftarrow{1} & (\mathbb{Z}/2)^2 & \xleftarrow{0} & (\mathbb{Z}/2)^2 \\ (\mathbb{Z}/2)^2 & \xleftarrow{2} & (\mathbb{Z}/2)^3 & \xleftarrow{1} & \mathbb{Z}/2 \\ \mathbb{Z}_{(2)} & \xleftarrow{1} & \mathbb{Z}_{(2)} & \xleftarrow{0} & \mathbb{Z}/2 \end{array}$$

Here each morphism is superscripted by its rank, i.e. the rank of its image.

This gives the following E^2-term, where the classes in the zeroth column correspond to the cohomology classes $2x_4, 2x_4^2, \ldots$:

$$\begin{array}{ccc} 0 & \mathbb{Z}/2 & \mathbb{Z}/2 \\ 0 & \mathbb{Z}/2 & (\mathbb{Z}/2)^2 \\ \mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 \\ 0 & \mathbb{Z}/2 & (\mathbb{Z}/2)^2 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbb{Z}/2 \end{array}$$

By exactness, these classes are hit by d^2-differentials in the D_2 PSS, and by naturality, the same happens in the $\mathrm{GL}_2\mathbb{Z}$ PSS.

This leaves us with the E^∞-term:

$$\begin{array}{ccc} 0 & \mathbb{Z}/2 & \mathbb{Z}/2 \\ 0 & \mathbb{Z}/2 & (\mathbb{Z}/2)^2 \\ 0 & \mathbb{Z}/2 & \mathbb{Z}/2 \\ 0 & \mathbb{Z}/2 & \mathbb{Z}/2 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbb{Z}/2 \end{array}$$

giving $H_*(F_2\mathbf{K}\mathbb{Z}/F_1\mathbf{K}\mathbb{Z}; \mathbb{Z}_{(2)}) \cong (0, 0, \mathbb{Z}/2, \mathbb{Z}/2, \#4, \#4, \ldots)$.

The classes surviving in $E^\infty_{1,2}$ and $E^\infty_{1,3}$ are easily seen to be dual to $p_5 \in H^3(P_1)$ and the sum of the duals of $p_1 p_3$ and $p_2 p_3$, respectively. All the groups and ranks involved grow linearly with period dividing four, so this also applies to the abutment. A similar computation with $\mathbb{Z}/2$-coefficients gives $H_n(F_2\mathbf{K}\mathbb{Z}/F_1\mathbf{K}\mathbb{Z};\mathbb{Z}/2) \cong (\mathbb{Z}/2)^{n-1}$ for $n \geqslant 1$, so all homology groups occurring are actually elementary abelian. □

The odd primary case is much easier:

Proof of Theorem 3.2 (2). Away from the prime two $H_*(P_1) \cong H_*(1)$, as is seen from the LHS-SS for $\mathbb{Z} \to P_1 \to T_2$, and $H_*(D_2; W_2) = 0$ since D_2 is a two-group. As for $H_*(\mathrm{GL}_2\mathbb{Z})$, the inclusion $\langle t^2, b\rangle = \Sigma_3 \subset \mathrm{GL}_2\mathbb{Z}$ induces a homology isomorphism away from two. This follows from the map of extensions from $\mathbb{Z}/3 \to \Sigma_3 \to \mathbb{Z}/2$ to $\mathrm{SL}_2\mathbb{Z} \to \mathrm{GL}_2\mathbb{Z} \to \mathbb{Z}/2$ taking

$$\langle t^2\rangle = \mathbb{Z}/3 \hookrightarrow \mathrm{SL}_2\mathbb{Z} \cong \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6,$$

which induces an isomorphism away from two. Hence, the E^1-term for $H_*(F_2\mathbf{K}\mathbb{Z}/F_1\mathbf{K}\mathbb{Z})$ modulo two-torsion looks like:

$$H_*(\Sigma_3) \xleftarrow{d^1} H_*(1) \xleftarrow{d^1} 0$$

Thus

$$H_*\left(F_2\mathbf{K}\mathbb{Z}/F_1\mathbf{K}\mathbb{Z};\mathbb{Z}\left[\frac{1}{2}\right]\right) \cong \tilde{H}_*\left(\Sigma_3;\mathbb{Z}\left[\frac{1}{2}\right]\right)$$
$$\cong (0,0,0,\mathbb{Z}/3,0,0,0,\mathbb{Z}/3,0,\ldots).$$ □

Proof of Proposition 3.3. First prove the case $i = 1$. This would follow from the connectivity conjecture, but to make Theorem 1.1 independent of the conjecture we provide a chain-level computation below. In the $\mathrm{GL}_2\mathbb{Z}$ PSS,

$$d^2\colon \ker d^1 \cong H_*(D_2)/\mathrm{im}\, H_*(T_2) \to \operatorname{coker} d^1 \cong H_*(\mathrm{GL}_2\mathbb{Z})/\mathrm{im}\, H_*(P_1)$$

has kernel $E^\infty_{2,*-1} = \ker d^2 \cong H_*(P_1)/\mathrm{im}\, H_*(T_2)$. The connecting map ∂ extends to $H_*(T_2) \to H_*(P_1) \to H_*(\mathbb{Z}/2)$ where the composite is zero. Dually, by inspection of the $S_\bullet$-construction, $\partial^*\colon H^*(\mathbb{Z}/2) \cong \mathbb{Z}[u]/2u \to H^*(P_1)$ is a sum of algebra homomorphisms. The case $i = 1$ shows that $\partial^*(u) = p_3$; whence $\partial^*(u^i) = p_3^i \neq 0$ in general.

Let $\mathscr{E}$ denote the category (with cofibrations and weak equivalences) of finite based sets. Its K-theory spectrum $\mathbf{K}\mathscr{E}$ is mapped to $\mathbf{K}\mathbb{Z}$ by the functor taking a finite set to the free $\mathbb{Z}$-module generated by its non-base point elements. The image in $D^n(\mathbb{Z}^k)$ is precisely the standard apartment ([14, §4]). $\mathbf{K}\mathscr{E}$ admits a rank filtration and poset filtrations on the subquotients compatible with the map to $\mathbf{K}\mathbb{Z}$.

Let F_1 and F_2 denote the nth spaces of $F_1\mathbf{K}\mathbb{Z}$ and $F_2\mathbf{K}\mathbb{Z}$, for n suitably large. We express the degrees of chains and homology groups relative to n. Let $\Phi_0 \subset \Phi_1 \subset \Phi_2 = F_2/F_1$ denote the poset filtration, and let $\tilde{\Phi}_i$ be the preimage of Φ_i in F_2. Simplices in F_1 involve a single line (free rank one $\mathbb{Z}$-module), Φ_0 includes simplices involving a single plane (free rank two $\mathbb{Z}$-module), Φ_1 adjoins configurations of a line contained in a plane, and Φ_2 adjoins pairs of lines. Similarly, write $F_i\mathscr{E}$, $\Phi_i\mathscr{E}$ and $\tilde{\Phi}_i\mathscr{E}$ for the corresponding constructions in $\mathbf{K}\mathscr{E}$. There are splittings $\tilde{\Phi}_0 \simeq F_1 \vee \Phi_0$ and $\tilde{\Phi}_0\mathscr{E} \simeq F_1\mathscr{E} \vee \Phi_0\mathscr{E}$, and stably $F_2\mathscr{E}/F_1\mathscr{E} \simeq *$.

Start with the one-cycle $b\colon \mathbb{Z}^2 \xrightarrow{r} \mathbb{Z}^2$ generating $H_1\Phi_0\mathscr{E}$ which lifts to $\tilde{b} \in H_1\tilde{\Phi}_0\mathscr{E}$ by the splitting. As $F_1\mathscr{E} \to F_2\mathscr{E}$ is a stable equivalence, the image of $\tilde{b}$ in $F_2\mathscr{E}$ is null-homologous; say $\partial\tilde{a} = \tilde{b}$, and let $\tilde{a} \mapsto a$ under $F_2\mathscr{E} \to F_2\mathscr{E}/F_1\mathscr{E}$. Then a is a two-chain in $F_2\mathscr{E}/F_1\mathscr{E}$ with $\partial a = b$. The image of a in $\Phi_2\mathscr{E}/\Phi_1\mathscr{E}$ generates, and its image in the K-theory of $\mathbb{Z}$ represents the nonzero class in $E^{\infty}_{2,0}$ of the $\mathrm{GL}_2\mathbb{Z}$ PSS. Our aim is to compute the connecting map on this class.

We should consider a lifting of a to F_2 with boundary contained in F_1 (this is possible as the class of a survives to E^{∞}). Modulo odd torsion, b is exact when included into Φ_0, as $r = t \cdot cx$, so the desired lifting of a is obtained by adjoining to $\tilde{a}$ a lifting of the class of cx in Φ_1/Φ_0. Call the resulting two-chain a'. $\partial a'$ is a cycle in F_1. By the splitting, $\partial\tilde{a} = \tilde{b}$ makes no contribution. The chain

$$cx = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

represents an automorphism of the flag $\mathbb{Z}^1 \hookrightarrow \mathbb{Z}^2$, and its boundary contributes to F_1 by the restrictions of this automorphism to the lines $\mathbb{Z}^1$ and $\mathbb{Z}^2/\mathbb{Z}^1$, i.e. the diagonal entries. Hence one maps to the generator, and the other to zero. Added together, the connecting map ∂ takes the generator a onto the generator of $H_1(F_1\mathbf{K}\mathbb{Z})$. $\square$

6. Two-Primary Computations at the Third Stage

The computations at the third stage are somewhat more involved, especially in the two-primary case. We present the calculation at two of the spectrum homology of the third subquotient of the rank filtration, using the $\mathrm{SL}_3\mathbb{Z}$ poset spectral sequence. Thereafter, we compute the connecting map from the third to the second subquotient. In the next section, we will do the same at odd primes.

In some more detail, the computation of the E^1-term of the $\mathrm{SL}_3\mathbb{Z}$ PSS at the prime two is accomplished by using Soulé's computation [11] of $H^*(\mathrm{SL}_3\mathbb{Z})$ and LHS-SS techniques for the various parabolic subgroups occurring. These methods suffice to describe most of the d^1-differentials needed, except $d^1\colon E^1_{2,*} \to E^1_{1,*}$. For these differentials we use a description of the relevant parabolic subgroups as isometry groups of a tessellation of the plane (crystallographic groups) to compute their $\mathbb{Z}/2$-cohomology algebras, from which the ranks of the differentials can be estimated. This suffices to describe the $E^2 = E^{\infty}$-term in the range we are studying.

We need further notation for 3×3 matrices, which is the default size for matrices in this Section:

Notation 6.1. We carry over the notation from 5.1 for matrices in $\mathrm{GL}_2\mathbb{Z}$ to $\mathrm{SL}_3\mathbb{Z}$ by means of a fixed embedding

$$g \mapsto \begin{pmatrix} \det(g) & 0 \\ 0 & g \end{pmatrix}.$$

In addition we shall use the matrices

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We also use an involution on $\mathrm{SL}_3\mathbb{Z}$ given by $g \mapsto \bar{g} = (c_{(13)}g^T)^{-1}$ which first reflects the matrix about a line perpendicular to the main diagonal, and then inverts the matrix. Here c_g denotes (left) conjugation $h \mapsto ghg^{-1}$, while we use c^g to denote (right) conjugation $h \mapsto g^{-1}hg$. The involution $g \mapsto \bar{g}$ takes SP_1 to SP_{12}, and $SP_{1,2}$ to $SP_{1,12,13}$. For instance, $x = (\bar{s})^2$ and $y = \overline{rs}$.

Let Di_n denote the dihedral group with $2n$ elements. We view Di_2 as embedded in $Di_4 = D_2$ as $\langle s^2, r\rangle$, and Di_6 as embedded in $\mathrm{GL}_2\mathbb{Z}$ as $\langle t, r\rangle$. Thus, $Di_2 \subset Di_6$.

We let $\mathbb{Z}_{\det}$ and $\mathbb{Z}^2_{\mathrm{std}}$ denote the determinant and standard representations of $\mathrm{GL}_2\mathbb{Z}$, or any of its subgroups. For a group on given generators let $\mathbb{Z}_{\mathrm{sgn}(x,\ldots,y)}$ denote the representation on $\mathbb{Z}$ where the subset $\{x, \ldots, y\}$ of generators act by reversing sign.

Let us begin with extending some of the group homology computations from the preceding Section to twisted coefficients.

LEMMA 6.2. *The following homology groups eventually grow linearly with period dividing four*:

$$\begin{aligned}
H_*(T_2; \mathbb{Z}_{\det}) &\cong (\mathbb{Z}/2, \mathbb{Z}/2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^3, (\mathbb{Z}/2)^3, \ldots),\\
H_*(P_1(2); \mathbb{Z}_{\det}) &\cong (\mathbb{Z}/2, \mathbb{Z} + \mathbb{Z}/2, (\mathbb{Z}/2)^4, (\mathbb{Z}/2)^3, (\mathbb{Z}/2)^6, (\mathbb{Z}/2)^5, (\mathbb{Z}/2)^8, \ldots),\\
H_*(Di_4; \mathbb{Z}_{\det}) &\cong (\mathbb{Z}/2, \mathbb{Z}/4, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^3, \mathbb{Z}/4 + (\mathbb{Z}/2)^2, \ldots),\\
H_*(P_1(2); \mathbb{Z}^2_{\mathrm{std}}) &\cong (\mathbb{Z}/2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^3, (\mathbb{Z}/2)^4, (\mathbb{Z}/2)^5, (\mathbb{Z}/2)^6, \ldots),\\
H_*(Di_2; \mathbb{Z}^2_{\mathrm{std}}) &\cong (\mathbb{Z}/2, 0, \mathbb{Z}/2, 0, \mathbb{Z}/2, 0, \ldots),\\
H_*(Di_4; \mathbb{Z}^2_{\mathrm{std}}) &\cong (\mathbb{Z}/2, \mathbb{Z}/2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^3, (\mathbb{Z}/2)^3, \ldots),\\
H_*(Di_6; \mathbb{Z}^2_{\mathrm{std}})_{(2)} &\cong (0, 0, 0, 0, 0, 0, \ldots)
\end{aligned}$$

and

$$\begin{aligned}
H_*(\mathrm{GL}_2\mathbb{Z}; \mathbb{Z}_{\det})_{(2)} &\cong H_*(D_2; \mathbb{Z}_{\det}) = H_*(Di_4; \mathbb{Z}_{\det}),\\
H_*(\mathrm{GL}_2\mathbb{Z}; \mathbb{Z}^2_{\mathrm{std}})_{(2)} &\cong (0, \mathbb{Z}/2, \mathbb{Z}/2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^3, \ldots).
\end{aligned}$$

Proof. By 'eventually growing linearly with period four' we mean that, beginning at the end of the list given, H_{n+4} is abstractly the sum of H_n and some fixed group.

In all but the last two cases, these groups are found using the LHS-SS for one of the extensions $C_2 \to T_2 \to \mathbb{Z}/2$, $\mathbb{Z} \to P_1(2) \to T_2$ or $\mathbb{Z}/n \to Di_n \to \mathbb{Z}/2$. The E^2-terms are found using the periodic resolutions for C_2 or $\mathbb{Z}/n$, or the short resolution for $\mathbb{Z}$. Each spectral sequence can be seen to collapse at $E^2 = E^\infty$ using that these extensions are split, and Hamada's resolution [7] in the case of $Di_4 = D_2$.

The last two cases follow from the amalgamation $\mathrm{GL}_2\mathbb{Z} \cong Di_4 *_{Di_2} Di_6$, by noting that the inclusions induce an isomorphism $H_*(Di_2; \mathbb{Z}_{\det}) \to H_*(Di_6; \mathbb{Z}_{\det})_{(2)}$ and an injection $H_*(Di_2; \mathbb{Z}^2_{\mathrm{std}}) \hookrightarrow H_*(Di_4; \mathbb{Z}^2_{\mathrm{std}})$. □

We will study $SP_{1,12}$ using an extension $\check{\mathbb{Z}}^3 \to SP_{1,12} \to ST_3$. Here $\check{\mathbb{Z}}^3 = \langle a, b, c \rangle$ fits into the extensions $\langle b \rangle = \mathbb{Z} \to \check{\mathbb{Z}}^3 \to \mathbb{Z}^2 = \langle a, c \rangle$, $\langle a, b \rangle = \mathbb{Z}^2 \to \check{\mathbb{Z}}^3 \to \mathbb{Z} = \langle c \rangle$, and similarly with a and c interchanged. $\check{\mathbb{Z}}^3$ acts freely and cocompactly on the contractible three-manifold $\check{\mathbb{R}}^3$ consisting of upper-triangular real matrices with ones on the diagonal. The orbit space $B\check{\mathbb{Z}}^3$ is compact and orientable (admitting the seventh geometry in Thurston's list [12]). By Poincaré duality and using $b = [a, c]$, the one possible d^2-differential in the LHS-SS for the first extension is an isomorphism. Let a and c generate $\mathbb{Z}_a$ and $\mathbb{Z}_c$ and let their Poincaré duals generate $\mathbb{Z}_{bc}$ and $\mathbb{Z}_{ab}$ (when dualized back to homology).

LEMMA 6.3. *$H_*(\check{\mathbb{Z}}^3) \cong (\mathbb{Z}, \mathbb{Z}_a + \mathbb{Z}_c, \mathbb{Z}_{ab} + \mathbb{Z}_{bc}, \mathbb{Z}_{abc})$. As ST_3-modules from the split extension*

$$\check{\mathbb{Z}}^3 \to SP_{1,12} \to ST_3$$

$\mathbb{Z}_a \cong \mathbb{Z}_{bc} \cong \mathbb{Z}_{\mathrm{sgn}(y)}$, $\mathbb{Z}_c \cong \mathbb{Z}_{ab} \cong \mathbb{Z}_{\mathrm{sgn}(x,y)}$ and $\mathbb{Z} \cong \mathbb{Z}_{abc}$ is the trivial module. In particular the orientation class $U = [B\check{\mathbb{Z}}^3] \in H_3(\check{\mathbb{Z}}^3)$ is a generator, and is invariant under the ST_3-action. □

Recall from [11, Theorem 4]:

THEOREM 6.4. (Soulé)

$$H_*(\mathrm{SL}_3\mathbb{Z}) \cong (\mathbb{Z}, 0, (\mathbb{Z}/2)^2, (\mathbb{Z}/4)^2 + (\mathbb{Z}/3)^2, \mathbb{Z}/2, (\mathbb{Z}/2)^4, (\mathbb{Z}/2)^3, \ldots).$$

Soulé gives the whole cohomology algebra; we have merely listed these groups for concreteness.

Some other groups in the $\mathrm{SL}_3\mathbb{Z}$ PSS have already been computed. Clearly $\mathrm{GL}_2\mathbb{Z} \cong SP_{12,3}$ and $P_1(2) \cong SP_{1,12,3}$ through the fixed embedding followed by the involution.

From [11, Proposition 2(iii)] we also recall:

PROPOSITION 6.5. *$H^*(SD_3)_{(2)} \cong \mathbb{Z}[y_1, y_2, y_3]/y_1^4 + y_1 y_2^2 + y_1^2 y_3 = 0$ where the generators (y_1, y_2, y_3) are of degrees $(2, 3, 4)$ and of orders $(2, 2, 4)$ respectively.*

Here we have used $SD_3 \cong \Sigma_4$. From this we may set up the E^1-term of the SD_3 PSS, except that we do not know the fourth column $H_*(SD_3; W_3)$. These groups may presumably be found using a stable element argument as in [4, XII Theorem 10.1]. Instead, we have used that the SD_3 PSS abuts to zero and computed its

d^1-differentials by methods similar to those we will use for the $SL_3\mathbb{Z}$ PSS below, to conclude:

LEMMA 6.6.

$$H_*(SD_3; W_3)_{(2)} \cong (0, \mathbb{Z}/2, 0, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, (\mathbb{Z}/2)^2, \ldots)$$

and

$$d^1 : E^1_{4,*} = H_*(SD_3; W_3)_{(2)} \to E^1_{3,*} = H_*(ST_3)$$

in the SD_3 PSS is an injection in this range.

Proof. We omit the calculation, as it is rather similar to what follows, but simpler. We only point out that $D_2 \cong S(\mathbb{Z}/2 \times D_2)$ and $T_2 \cong ST_3$ by the fixed embedding. It turns out that $E^2 = 0$ in this range. □

View $\mathbb{Z}^2$ as embedded in $\mathrm{SL}_3\mathbb{Z}$ as $\langle a, b\rangle$ or $\langle b, c\rangle$ as appropriate.

LEMMA 6.7. *The LHS-SS for the split extension $\mathbb{Z}^2 \to SP_{1,12,13} \to ST_3 \cong T_2$ collapses at E^2 and splits. Similarly for $\mathbb{Z}^2 \to SP_{1,2} \to ST_3$:*

$$(E^\infty)\quad \begin{array}{cccccc} \mathbb{Z}/2 & \mathbb{Z}/2 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^3 & (\mathbb{Z}/2)^3 \\ (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^4 & (\mathbb{Z}/2)^4 & (\mathbb{Z}/2)^6 & (\mathbb{Z}/2)^6 \\ \mathbb{Z} & (\mathbb{Z}/2)^2 & \mathbb{Z}/2 & (\mathbb{Z}/2)^3 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^4 \end{array}$$

So $H_*(SP_{1,12,13}) \cong (\mathbb{Z}, (\mathbb{Z}/2)^4, (\mathbb{Z}/2)^4, (\mathbb{Z}/2)^8, (\mathbb{Z}/2)^8, (\mathbb{Z}/2)^{12}, \ldots)$.

Proof. $SP_{1,12,3}$ and $SP_{1,2,13}$ split off $SP_{1,12,13}$, and force the collapse and splitting. □

LEMMA 6.8. *The LHS-SS for the split extension $\mathbb{Z}^2 \to SP_{1,12} \to SP_{1,2,23} \cong P_1(2)$ collapses at E^2 and splits:*

$$(E^\infty)\quad \begin{array}{cccccc} \mathbb{Z}/2 & \mathbb{Z}+\mathbb{Z}/2 & (\mathbb{Z}/2)^4 & (\mathbb{Z}/2)^3 & (\mathbb{Z}/2)^6 & (\mathbb{Z}/2)^5 \\ \mathbb{Z}/2 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^3 & (\mathbb{Z}/2)^4 & (\mathbb{Z}/2)^5 & (\mathbb{Z}/2)^6 \\ \mathbb{Z} & (\mathbb{Z}/2)^3 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^5 & (\mathbb{Z}/2)^4 & (\mathbb{Z}/2)^7 \end{array}$$

Similarly for $\mathbb{Z}^3 \to SP_{1,12} \to ST_3$:

$$(E^\infty)\quad \begin{array}{cccccc} \mathbb{Z} & (\mathbb{Z}/2)^2 & \mathbb{Z}/2 & (\mathbb{Z}/2)^3 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^4 \\ (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^4 & (\mathbb{Z}/2)^4 & (\mathbb{Z}/2)^6 & (\mathbb{Z}/2)^6 \\ (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^4 & (\mathbb{Z}/2)^4 & (\mathbb{Z}/2)^6 & (\mathbb{Z}/2)^6 \\ \mathbb{Z} & (\mathbb{Z}/2)^2 & \mathbb{Z}/2 & (\mathbb{Z}/2)^3 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^4 \end{array}$$

So $H_*(SP_{1,12}) \cong (\mathbb{Z}, (\mathbb{Z}/2)^4, (\mathbb{Z}/2)^5, \mathbb{Z} + (\mathbb{Z}/2)^9, (\mathbb{Z}/2)^{12}, (\mathbb{Z}/2)^{15}, \ldots)$.

Proof. To find the E^2-terms use Lemma 6.2. The conjugation action of $SP_{1,2,23}$ makes $H_1(\mathbb{Z}^2)$ abstractly isomorphic to $\mathbb{Z}^2_{\mathrm{std}}$ and $H_2(\mathbb{Z}^2)$ isomorphic to $\mathbb{Z}_{\det}$.

The inclusions $SP_{1,12,13} \hookrightarrow SP_{1,12}$ and $SP_{1,2} \hookrightarrow SP_{1,12}$ map onto the zeroth through second rows of the latter spectral sequence, so by Lemma 6.7 no differentials may originate in these rows. We claim that the orientation class U generating $E^2_{0,3} \cong \mathbb{Z}$ survives to E^∞, from which it follows (by considering the $H^*(ST_3)$-module structure in the cohomological case) that no differentials originate in the third row either. Thus $E^2 = E^\infty$. To prove the claim we compare $BSP_{1,12}$ with the total space of an orientable sphere bundle over BST_3. View the unit cube $[0,1]^3$ as a fundamental domain in $\check{\mathbb{R}}^3$ and construct a ST_3-equivariant map $B\check{\mathbb{Z}}^3 \to S^3$ by collapsing its boundary to a point. By inspection, ST_3 acts orientably on S^3, and U maps to the orientation class of S^3, which by the Thom isomorphism survives to the homology of the total space. Thus U survives in $H_*(SP_{1,12})$.

For the splitting results, there can be no extensions in the zeroth through second row in the latter spectral sequence by the same comparisons we used to show collapsing, and Lemma 6.7. Also there can be no extensions from the third to zeroth or first row by the splittings used in the proof of Lemma 6.7. Lastly, there can be no extensions from the third to second row as these groups appear without extensions in the second row of the former spectral sequence (and for $\mathbb{Z}^2 \to SP_{1,12} \to SP_{1,12,3}$). □

Note. For the remainder of this Section, all homology groups are given modulo odd torsion.

LEMMA 6.9. *The LHS-SS for the split extension* $\mathbb{Z}^2 \to SP_1 \to SP_{1,23} \cong \mathrm{GL}_2\mathbb{Z}$ *collapses at* E^2 *and splits*:

$$
(E^\infty) \quad
\begin{matrix}
\mathbb{Z}/2 & \mathbb{Z}/4 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^3 & \mathbb{Z}/4 + (\mathbb{Z}/2)^2 \\
0 & \mathbb{Z}/2 & \mathbb{Z}/2 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^3 \\
\mathbb{Z} & (\mathbb{Z}/2)^2 & \mathbb{Z}/2 & \mathbb{Z}/4 + (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^4
\end{matrix}
$$

So

$$H_*(SP_1) \cong (\mathbb{Z}, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^3, (\mathbb{Z}/4)^2 + (\mathbb{Z}/2)^3, (\mathbb{Z}/2)^6, (\mathbb{Z}/2)^8, \ldots).$$

Proof using Lemma 6.16. The E^2-term is computed using Lemma 6.2. To prove collapsing and splitting we will compare with the crystallographic computation of $H^*(SP_1; \mathbb{Z}/2)$ in Lemma 6.16. The E^2-term above poses an upper bound on the (graded) rank of the group $H_*(SP_1)$, which agrees with the lower bound given there. Any differentials or extensions would have reduced this rank, and can therefore not occur. □

We are now able to list the E^1-term for the $\mathrm{SL}_3\mathbb{Z}$ PSS:

$$
\begin{matrix}
2^3 & \xleftarrow{3} & 2^9 \oplus 2^9 & & & & & & \\
2^4 & \xleftarrow{4} & 2^8 \oplus 2^8 & \xleftarrow{12} & 2^{15} \oplus 2^4 & \xleftarrow{6} & 2^7 & \xleftarrow{1} & 2 \\
2 & \xleftarrow{1} & 2^6 \oplus 2^6 & \xleftarrow{11} & 2^{12} \oplus 2^2 & \xleftarrow{3} & 2^4 & \xleftarrow{1} & 2 \\
4^2 & \xleftarrow{2.0} & 4^2 + 2^3 \oplus 4^2 + 2^3 & \xleftarrow{2.6} & \mathbb{Z} + 2^9 \oplus 4 + 2^2 & \xleftarrow{4} & 2^5 & \xleftarrow{1} & 2 \\
2^2 & \xleftarrow{2} & 2^3 \oplus 2^3 & \xleftarrow{4} & 2^5 \oplus 2 & \xleftarrow{2} & 2^2 & \xleftarrow{0} & 0 \\
0 & \xleftarrow{0} & 2^2 \oplus 2^2 & \xleftarrow{4} & 2^4 \oplus 2^2 & \xleftarrow{2} & 2^3 & \xleftarrow{1} & 2 \\
\mathbb{Z} & \xleftarrow{1} & \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{1} & \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{1} & \mathbb{Z} & \xleftarrow{0} & 0
\end{matrix}
$$

We have not yet computed the d^1-differentials given here, but include them for convenience. Here '$4^n + 2^m$' is short for a group $(\mathbb{Z}/4)^n + (\mathbb{Z}/2)^m$, a differential superscript 'n' means the image has rank n, while a superscript '$n.m$' means the image is isomorphic to $(\mathbb{Z}/4)^n + (\mathbb{Z}/2)^m$. We use '+' to denote direct sum within a homology group, and '$\oplus$' to separate summands of the PSS.

Let d_s^1 denote the differential $d_s^1: E^1_{s,*} \to E^1_{s-1,*}$.

LEMMA 6.10. (1) *d_1^1 is surjective.* (2) *d_4^1 is injective (through homological degree seven).*

Proof. In view of [11, Theorem 4(ii) and Proposition 3(i)] the cohomology of $\mathrm{SL}_3\mathbb{Z}$ is detected on the subgroups

$$S(\mathbb{Z}/2 \times D_2) \subset SD_3 = [O] \quad \text{and} \quad [M'P] \cong Di_4,$$

adapting Soulé's notation somewhat. These are subconjugate in $\mathrm{SL}_3\mathbb{Z}$ to SP_1 ($c^{a^T}[M'P] \subset SP_1$), so $H^*(\mathrm{SL}_3\mathbb{Z}) \hookrightarrow H^*(SP_1)$ injects and part (1) of the Lemma follows. Part (2) is immediate from Lemma 6.6 and the splitting of ST_3 off $SP_{1,12,3}$. □

LEMMA 6.11. *The image of the differential d_3^1 is isomorphic to*

$$(\mathbb{Z}, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^4, (\mathbb{Z}/2)^3, (\mathbb{Z}/2)^6, \ldots).$$

Proof. Compute the dual differential in cohomology:

$$d_1^{3,*}: H^*(SP_{1,12}) \oplus H^*(SP_{12,3}) \to H^*(SP_{1,12,3}).$$

On the second summand, the map is given by Lemma 5.3. On the first summand, the map is the signed sum $(c^e)^* - (c^{\phi\psi})^* + (c^{\psi^2})^*$ followed by restriction over $i: SP_{1,12,3} \to SP_{1,12}$. We choose a (possibly redundant) set of algebra generators $t_1, t_2, t_3, p_3, \bar{p}_3, p_5, \bar{p}_5, q_3, \bar{q}_3, q_5, \bar{q}_5$, and U for $H^*(SP_{1,12})$ in bidegrees $(2,0)$, $(2,0), (3,0), (1,1), (1,1), (2,1), (2,1), (1,2), (1,2), (2,2), (2,2)$ and $(0,3)$ in the LHS-SS for the extension $\tilde{\mathbb{Z}}^3 \to SP_{1,12} \to ST_3$; compare Lemma 6.8. We may assume that the t's, p's and $\bar{p}$'s restrict to the generators of Lemma 5.2 by the obvious restrictions. Also let $(\bar{x}_1, \ldots, \bar{x}_4)$ denote the generators for $H^*(SP_{12,3})$, and $(\bar{p}_1, \ldots, \bar{p}_5)$ the generators for $H^*(SP_{1,12,3})$. In particular, $\bar{p}_1$, $\bar{p}_2$ and $\bar{p}_3$ are the Bockstein images of classes detecting $\bar{x} = xy$, $\bar{y} = y$ and $\bar{c} = a^{-1}$.

Then by Lemma 5.3

$$\bar{x}_1 \mapsto \bar{p}_1 + \bar{p}_2 + \bar{p}_3, \qquad \bar{x}_2 \mapsto \bar{p}_1 + \bar{p}_2, \qquad \bar{x}_3 \mapsto \bar{p}_4 \quad \text{and} \quad \bar{x}_4 \mapsto \bar{p}_1 \bar{p}_2.$$

By explicit calculation with $\mathbb{Z}/2$-coefficients, we find:

$$(c^e i)^*: (t_1, t_2, t_3, p_3, \bar{p}_3, p_5, \bar{p}_5) \mapsto (\bar{p}_1, \bar{p}_1 + \bar{p}_2, \bar{p}_4, 0, \bar{p}_3, 0, \bar{p}_5),$$
$$(c^{\phi\psi} i)^*: (t_1, t_2, t_3, p_3, \bar{p}_3, p_5, \bar{p}_5) \mapsto (\bar{p}_1 + \bar{p}_2, \bar{p}_1, \bar{p}_4, 0, 0, 0, 0),$$
$$(c^{\psi^2} i)^*: (t_1, t_2, t_3, p_3, \bar{p}_3, p_5, \bar{p}_5) \mapsto (\bar{p}_2, \bar{p}_1, \bar{p}_4, \bar{p}_3, 0, \bar{p}_5, 0).$$

The other generators map to zero.

Granted this, it is straightforward to list generators and maps for d_1^3 in low degrees, giving the image stated above. □

We shall now consider cocompact group actions of our parabolic groups on the plane. We obtain a lower bound for the rank of d_2^1 by computing the composite $H_*(SP_{1,12,13};\mathbb{Z}/2)\oplus H_*(SP_{1,2};\mathbb{Z}/2)\to H_*(SP_{1,12};\mathbb{Z}/2)\to H_*(SP_1;\mathbb{Z}/2)$. The corresponding map into $H_*(SP_{12};\mathbb{Z}/2)$ will be obtained by applying the involution.

Notation 6.12. Let

$$X=(0,0),\qquad Y=(1/2,0),\qquad Z=(1/2,1/2),$$
$$U=(0,1/2),\qquad V=(1/3,1/3)\quad\text{and}\quad W=(1,0)$$

be points in the xy-plane. There is an action of SP_1 on this plane defined as follows: a and b translate by unit vectors in the x and y-directions, r reflects in the line $x+y=0$, s rotates counterclockwise by $\pi/2$ radians, and t applies the matrix $\left(\begin{smallmatrix}1&1\\-1&0\end{smallmatrix}\right)$. Let $\mathbb{Z}^2\rtimes Di_n$ denote the subgroup of SP_1 corresponding to our embeddings $Di_n\hookrightarrow \mathrm{GL}_2\mathbb{Z}$ for $n=2,4,6$.

Let $[X]$ denote the isotropy group at X for the $\mathbb{Z}^2\rtimes Di_4$ action on the plane at X, and similarly for the other letters. Let $[XY]$ denote the isotropy of the edge from X to Y, etc. Let $[X]^*$, $[X]^\vee$ and $[X]^\circ$ denote the isotropy groups for the $\mathbb{Z}^2\rtimes Di_2$, $\mathbb{Z}^2\rtimes Di_6$ and $SP_{1,12,13}$ actions, respectively. Further, let $[X]'$, etc., denote the subgroups of $SP_{1,2}$ corresponding to $[X]^\circ$ under the involution. We shall use these accents throughout the following calculations to separate the cases.

The top square of the following diagram is a pullback square due to the amalgamation $\mathrm{GL}_2\mathbb{Z}\cong Di_4 *_{Di_2} Di_6$:

$$\begin{array}{ccc}
H^*(\mathbb{Z}^2\rtimes Di_6;\mathbb{Z}/2)^\vee & \longrightarrow & H^*(\mathbb{Z}^2\rtimes Di_2;\mathbb{Z}/2)^* \\
\uparrow & & \uparrow \\
H^*(SP_1;\mathbb{Z}/2) & \longrightarrow & H^*(\mathbb{Z}^2\rtimes Di_4;\mathbb{Z}/2) \\
\downarrow & & \downarrow \\
H^*(SP_{1,2};\mathbb{Z}/2)' & & H^*(SP_{1,12,13};\mathbb{Z}/2)^\circ
\end{array}\qquad (*)$$

LEMMA 6.13. (1) *$\triangle XYZ$ is a fundamental domain for the action of $\mathbb{Z}^2\rtimes Di_4$ on the plane,*

$$[X]=\langle s,r\rangle\cong D_2,\qquad [Y]=\langle sr,ars\rangle\cong T_2,\qquad [Z]=\langle as,abr\rangle\cong D_2,$$
$$[XY]=\langle sr\rangle\cong\mathbb{Z}/2,\qquad [XZ]=\langle rs\rangle\cong\mathbb{Z}/2,\qquad [YZ]=\langle ars\rangle\cong\mathbb{Z}/2.$$

(2) *$\square XYZU$ is a fundamental domain for the action of $SP_{1,12,13}$ on the plane,*

$$[X]^\circ=\langle sr,rs\rangle\cong T_2,\qquad [Y]^\circ=\langle sr,ars\rangle\cong T_2,\qquad [Z]^\circ=\langle bsr,ars\rangle\cong T_2,$$
$$[U]^\circ=\langle rs,bsr\rangle\cong T_2,\qquad [XY]^\circ=\langle sr\rangle\cong\mathbb{Z}/2,\qquad [YZ]^\circ=\langle ars\rangle\cong\mathbb{Z}/2,$$
$$[UZ]^\circ=\langle bsr\rangle\cong\mathbb{Z}/2,\qquad [XU]^\circ=\langle rs\rangle\cong\mathbb{Z}/2.$$

(3) *$\triangle XYV$ is a fundamental domain for the action of $\mathbb{Z}^2 \rtimes Di_6$ on the plane,*

$$[X]^\vee = \langle t, r\rangle \cong Di_6, \qquad [Y]^\vee = \langle rt^2, at^3\rangle \cong T_2, \qquad [V]^\vee = \langle brt, bt^2\rangle \cong \Sigma_3,$$
$$[XY]^\vee = \langle rt^2\rangle \cong \mathbb{Z}/2, \qquad [XV]^\vee = \langle rt^3\rangle \cong \mathbb{Z}/2, \qquad [YV]^\vee = \langle art^5\rangle \cong \mathbb{Z}/2.$$

(4) *$\triangle XWZ$ is a fundamental domain for the action of $\mathbb{Z}^2 \rtimes Di_2$ on the plane,*

$$[X]^* = \langle s^2, r\rangle \cong T_2, \qquad [Y]^* = \langle as^2\rangle \cong \mathbb{Z}/2, \qquad [Z]^* = \langle abr, rs^2\rangle \cong T_2,$$
$$([W]^* = c_a[X]^*), \qquad [XY]^* = 1, \qquad [XZ]^* = \langle rs^2\rangle \cong \mathbb{Z}/2,$$
$$[ZW]^* = \langle abr\rangle \cong \mathbb{Z}/2.$$

(5) *Under the involution, the isotropy subgroups of $SP_{1,12,13}$ map to the following subgroups of $SP_{1,2}$:*

$$[X]' = \langle s^2, rs\rangle \subset [X], \qquad [Y]' = \langle s^2, strs\rangle \sim \langle s^2, rs^2\rangle \subset [X],$$
$$[Z]' = \langle b^{-1}s^2, strs\rangle \sim \langle abs^2, rs^2\rangle \subset [Z],$$
$$[U]' = \langle rs, b^{-1}s^2\rangle \sim \langle sr, as^2\rangle = [Y].$$

Here '$\sim$' denotes (right) conjugation in SP_1 by $srtrs^{-1}$, $b^{-1}srtrs^{-1}$ and $b^{-1}s$, respectively in the three cases. □

In the Lemma below we have named the generators of the $\mathbb{Z}/2$-cohomology algebras of the groups above as follows: For a dihedral group of order eight like $[X]$, with given group generators $\langle s, r\rangle$ with s of order four, take cohomology generators x_1, x_2, x_3 corresponding to σ, ρ and ω in Notation 5.1;

$$H^*([X]; \mathbb{Z}/2) = \mathbb{Z}/2[x_1, x_2, x_3]/x_1^2 = x_1x_2.$$

For a Klein four group (or dihedral group of order 12) like $[Y]$, with given group generators $\langle x, y\rangle$, take cohomology generators y_1 and y_2 corresponding to ξ and η,

$$H^*([Y]; \mathbb{Z}/2) = \mathbb{Z}/2[y_1, y_2].$$

For a symmetric group on three letters like $[V]^\vee$ call the cohomology generator $v_1^\vee$. Use analogous notation for the other letters and accents.

LEMMA 6.14.

(1) $$H^*(\mathbb{Z}^2 \rtimes Di_4; \mathbb{Z}/2) \cong \mathbb{Z}/2[g_1, g_2, g_3, g_4, g_5]/g_1^2 = g_1g_2,$$
$$g_1g_3 = g_2g_3, \qquad g_3g_4 = 0, \qquad g_1g_5 = g_3g_5, \qquad g_4g_5 = 0$$

with generators

$$g_1 = x_1 + y_1 + y_2 + z_1, \qquad g_2 = x_2 + y_1 + y_2 + z_2,$$
$$g_3 = y_2 + z_1, \qquad g_4 = x_3 \quad \text{and} \quad g_5 = z_3$$

in degrees $(1, 1, 1, 2, 2)$. *g_1, g_2 and g_3 detect s, r and a, respectively.*

(2) $$H^*(SP_{1,12,13}; \mathbb{Z}/2) \cong \mathbb{Z}/2[g_1^\circ, g_2^\circ, g_3^\circ, g_4^\circ]/g_1^\circ g_3^\circ = g_2^\circ g_4^\circ = 0$$

with generators

$$g_1^\circ = x_1^\circ + y_1^\circ, \qquad g_2^\circ = y_2^\circ + z_2^\circ, \qquad g_3^\circ = z_1^\circ + u_2^\circ \quad \textit{and} \quad g_4^\circ = x_2^\circ + u_1^\circ$$

all in degree one.

(3) $H^*(\mathbb{Z}^2 \rtimes Di_6; \mathbb{Z}/2) \cong \mathbb{Z}/2[g_1^\vee, g_2^\vee, g_3^\vee]$

(no relations) with generators

$$g_1^\vee = x_1^\vee + y_2^\vee + v_1^\vee, \qquad g_2^\vee = x_2^\vee + y_1^\vee + v_1^\vee \quad \textit{and} \quad g_3^\vee = y_1^\vee y_2^\vee + (y_2^\vee)^2$$

in degrees $(1, 1, 2)$. $g_1^\vee$ *and* $g_2^\vee$ *detect* t *and* r *respectively.*

(4) $H^*(\mathbb{Z}^2 \rtimes Di_2; \mathbb{Z}/2) \cong \mathbb{Z}/2[g_1^*, g_2^*, g_3^*, g_4^*]/g_1^* g_3^* = (g_3^*)^2,$

$$g_2^* g_3^* = 0, \qquad g_3^* g_4^* = 0, \qquad g_1^*(g_1^* + g_2^*)g_4^* = (g_4^*)^2$$

with generators

$$g_1^* = x_1^* + y_1^* + z_2^*, \qquad g_2^* = x_2^* + z_1^* + z_2^*,$$
$$g_3^* = y_1^* \quad \textit{and} \quad g_4^* = z_1^* z_2^*$$

in degrees $(1, 1, 1, 2)$. g_1^*, g_2^* *and* g_3^* *detect* s^2, r *and* a *respectively.*

(5) $H^*(SP_{1,2}; \mathbb{Z}/2) \cong \mathbb{Z}/2[g_1', g_2', g_3', g_4']/g_1' g_3' = g_2' g_4' = 0$

with generators

$$g_1' = x_1' + y_1', \qquad g_2' = y_2' + z_2',$$
$$g_3' = z_1' + u_2' \quad \textit{and} \quad g_4' = x_2' + u_1'$$

all in degree one.

Proof. We do case (1). By comparing with Lemma 6.2, all differentials in the spectral sequence associated with the cellular filtration on $\triangle XYZ$ have maximal rank, so there is an exact sequence

$$\begin{aligned}0 &\to H^*(\mathbb{Z}^2 \rtimes Di_4; \mathbb{Z}/2) \to H^*([X]; \mathbb{Z}/2) \oplus H^*([Y]; \mathbb{Z}/2) \oplus H^*([Z]; \mathbb{Z}/2)\\ &\to H^*([XY]; \mathbb{Z}/2) \oplus H^*([XZ]; \mathbb{Z}/2) \oplus H^*([YZ]; \mathbb{Z}/2) \to H^*(1; \mathbb{Z}/2) \to 0\end{aligned}$$

where the maps are easily computed. The structure of $H^*(\mathbb{Z}^2 \rtimes Di_4; \mathbb{Z}/2)$ follows. □

LEMMA 6.15.

(1) $SP_{1,12,13} \to \mathbb{Z}^2 \rtimes Di_4$ *induces*

$$g_1 \mapsto g_1^\circ + g_2^\circ + g_3^\circ + g_4^\circ, \qquad g_2 \mapsto g_1^\circ + g_2^\circ + g_3^\circ + g_4^\circ,$$
$$g_3 \mapsto g_2^\circ + g_3^\circ, \qquad g_4 \mapsto g_1^\circ g_4^\circ \quad \textit{and} \quad g_5 \mapsto g_2^\circ g_3^\circ.$$

(2) $\mathbb{Z}^2 \rtimes Di_2 \to \mathbb{Z}^2 \rtimes Di_6$ *induces an injection*

$$g_1^\vee \mapsto g_1^*, \qquad g_2^\vee \mapsto g_2^* \quad \textit{and} \quad g_3^\vee \mapsto (g_3^*)^2 + g_4^*$$

with cokernel of rank one in every odd degree.

(3) $\mathbb{Z}^2 \rtimes Di_2 \to \mathbb{Z}^2 \rtimes Di_4$ *induces*

$$g_1 \mapsto 0, \quad g_2 \mapsto g_2^*, \quad g_3 \mapsto g_3^*,$$
$$g_4 \mapsto (g_1^*)^2 + g_1^* g_2^* + (g_3^*)^2 + g_4^* \quad \text{and} \quad g_5 \mapsto g_4^*.$$

Proof. The homomorphisms are found by detecting what happens on the isotropy subgroups, using the exact sequence(s) of the previous proof. Case (2) makes use of the observation that $c_{bt}[Y]^\vee = [Z]^*$. Lemma 5.4 becomes useful at this stage. We omit the details. □

We may now use the pullback square in (*) to determine the $\mathbb{Z}/2$-cohomology of SP_1. As the map in Lemma 6.15 (2) is an injection with cokernel of rank one in odd degrees, so is

$$H^*(SP_1; \mathbb{Z}/2) \hookrightarrow H^*(\mathbb{Z}^2 \rtimes Di_4; \mathbb{Z}/2).$$

The Bockstein operations on $H^*(SP_1; \mathbb{Z}/2)$ are determined by those on the isotropy subgroups, and we recover the order two elements in $H^*(SP_1)$ as the kernel of β_2.

LEMMA 6.16.

$$H^*(SP_1; \mathbb{Z}/2) \cong (\mathbb{Z}/2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^5, (\mathbb{Z}/2)^8, (\mathbb{Z}/2)^{11}, (\mathbb{Z}/2)^{14}, \ldots)$$

(growing linearly) is generated by $g_1, g_2, g_4, g_5 + g_3^2$ *and* $g_1(g_3)^i$ *for* $i \geqslant 1$. □

LEMMA 6.17. $SP_{1,2} \to SP_1$ *induces*

$$g_1 \mapsto g_4', \quad g_2 \mapsto g_2' + g_4', \quad g_4 \mapsto (g_1')^2 + g_1' g_2' + g_1' g_4',$$
$$g_5 + g_3^2 \mapsto g_2' g_3' + (g_3')^2 \quad \text{and} \quad g_1(g_3)^i \mapsto (g_3')^i g_4' \quad \text{for } i \geqslant 1.$$

Proof. The image of each class in $H^*(SP_1; \mathbb{Z}/2)$ in $H^*(SP_{1,2}; \mathbb{Z}/2)$ is determined by the restrictions to $[X]'$, $[Y]'$, $[Z]'$ and $[U]'$. These can be found using the subconjugacies listed in Lemma 6.13 (5). □

This information suffices to write down the composite

$$H^*(SP_1) \oplus H^*(SP_{12}) \to H^*(SP_{1,12}) \to H^*(SP_{1,12,13}) \oplus H^*(SP_{1,2})$$

on the order two elements. On the first summand, the map is determined by Lemma 6.15 (1) and Lemma 6.17, using the Bockstein structure. The involution takes the basis for $H^*(SP_1)$ to a basis for $H^*(SP_{12})$ and interchanges the bases for $H^*(SP_{1,12,13})$ and $H^*(SP_{1,2})$, while respecting the map. Therefore the matrix form of the map on the second summand follows directly from that of the first. The combined matrix will have the form $\left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$.

We have computed these matrices through cohomological degree six, finding:

LEMMA 6.18. *Restricted to the order two elements, the composite*

$$H^*(SP_1) \oplus H^*(SP_{12}) \to H^*(SP_{1,12}) \to H^*(SP_{1,12,13}) \oplus H^*(SP_{1,2})$$

has ranks (4, 4, 8, 11, 12) *in cohomological degrees two through six.* □

PROPOSITION 6.19. *In the* $\mathrm{SL}_3\mathbb{Z}$ *PSS:*

(1) *Through homological degree five,* d_2^1 *maps onto the order two elements in* $\ker d_1^1$.

(2) *In homological degree three, the Thom class* $U \in H_3(SP_{1,12})$ *and a* $\mathbb{Z}/4$*-generator in* $H_3(SP_{12,3})$ *map to independent elements of order four in* $E_{1,3}^1$.

(3) *There exists an order two element* $V \in H_3(SP_{1,12})$ *such that* $d_2^1(2U) = d_2^1(V)$. *It is the image under inclusion of the dual of* $(g_2^\circ)^2(g_3^\circ)^2$ *in* $H_3(SP_{1,12,13})$.

Proof. Part (1) is the outcome of the preceding discussion.

Part (2) is proved by considering the map from the first spectral sequence in Lemma 6.8 to that in Lemma 6.9. The map induced on the second row is

$$H_*(P_1(2); \mathbb{Z}_{\det}) \to H_*(\mathrm{GL}_2\mathbb{Z}; \mathbb{Z}_{\det}),$$

which in degree one takes the class of c which generates a free summand in $H_1(P_1(2); \mathbb{Z}_{\det})$ to the class of $t^{-1}s^{-1}$, which modulo three-torsion equals the class of s^{-1} in $H_1(D_2; \mathbb{Z}_{\det})$, of order four. On the other hand, the subconjugation $SP_{12,3} \hookrightarrow SP_1$ maps isomorphically to the zeroth row of the spectral sequence in Lemma 6.9, converging to $H_*(SP_1)$.

The existence in part (3) follows from part (1), but we need the more precise definition of the class V. Twice the order four classes in $H_3(SP_1)$ are dual to g_4^2 and $g_5^2 + g_3^4$, as is seen from the Bockstein structure. The latter class is the one hit from $H_1(\mathrm{GL}_2\mathbb{Z}; \mathbb{Z}_{\det})$ (cf. the description of g_5 in Lemma 6.14 (1)). From the matrix form of the map

$$H^4(SP_1; \mathbb{Z}/2) \oplus H^4(SP_{12}; \mathbb{Z}/2) \to H^4(SP_{1,12,13}; \mathbb{Z}/2) \oplus H^4(SP_{1,2}; \mathbb{Z}/2),$$

we may see that the dual of $(g_2^\circ)^2(g_3^\circ)^2$ is mapped diagonally to the duals of $g_5^2 + g_3^4$ and $\bar{g}_5^2 + \bar{g}_3^4$, respectively. Thus, V is precisely the image in $H_3(SP_{1,12})$ of the dual of $(g_2^\circ)^2(g_3^\circ)^2$. □

Proof of Theorem 3.4. (1). The outcome of the computations above is that through total degree six in the $\mathrm{SL}_3\mathbb{Z}$ PSS, the only group surviving to E^2 (and also to E^∞) is the $\mathbb{Z}$-summand generated by $2U + V$ in bidegree (2, 3). Hence,

$$H_*(\mathbf{D}(\mathbb{Z}^3)/h\mathrm{SL}_3\mathbb{Z}) \cong (0, 0, 0, 0, 0, \mathbb{Z}, 0, \ldots),$$

and using

$$F_3\mathbf{K}\mathbb{Z}/F_2\mathbf{K}\mathbb{Z} \cong \mathbf{D}(\mathbb{Z}^3)/h\mathrm{SL}_3\mathbb{Z} \wedge BC_{3+}$$

from Section 2, the Theorem follows. □

Proof of Proposition 3.5 (1). By inspection of the $S_\bullet$-construction, the connecting map $\partial: H_5(F_3\mathbf{K}\mathbb{Z}/F_2\mathbf{K}\mathbb{Z}) \to H_4(F_2\mathbf{K}\mathbb{Z}/F_1\mathbf{K}\mathbb{Z})$ is a restriction of a homomorphism

$(\pi_{12} + \pi_{23})_*: H_3(SP_{1,12}) \to H_3(P_1(2))$. Here π_{ij} projects $SP_{1,12} \to P_1(2)$ to the elements in the ith or jth rows and columns. We are interested in the image of the class $2U + V \in H_3(SP_{1,12})$, where U is the Thom class and V is the image of the dual to $(g_2^\circ)^2(g_3^\circ)^2$.

π_{12} and π_{23} take U to zero by Lemma 6.8.

$(g_2^\circ)^2(g_3^\circ)^2$ is detected on $[Z]^\circ$, and by computing the maps $\pi_{ij}: [Z]^\circ \to P_1(2)$ we find that the dual of $(g_2^\circ)^2(g_3^\circ)^2$ maps, respectively, to the sum of the duals of p_1p_2, p_1p_3 and p_2p_3, and to the dual of p_1p_2 in $H_3(P_1(2))$. Thus, $\partial(2U + V)$ is the sum of the duals of p_1p_3 and p_2p_3 in $H_3(P_1(2))$, which is in $\ker d^1: H_3(P_1(2)) \to H_3(\mathrm{GL}_2\mathbb{Z})$ but not in $\operatorname{im} d^1: H_3(D_2; W_2) \to H_3(P_1(2))$. In short, this class is nonzero in $E^\infty_{1,3}$ and thus in $H_4(F_2\mathbf{K}\mathbb{Z}/F_1\mathbf{K}\mathbb{Z})$. □

7. Odd Primary Computations at the Third Stage

We compute the $\mathrm{SL}_3\mathbb{Z}$ PSS at odd primes. In view of the splitting $\mathrm{GL}_3\mathbb{Z} = \mathrm{SL}_3\mathbb{Z} \times C_3$, this is the same as the $\mathrm{GL}_3\mathbb{Z}$ PSS. In this section, all homology groups are given modulo two-torsion.

LEMMA 7.1.

$$H_*(\mathrm{SL}_3\mathbb{Z}) \cong (\mathbb{Z}, 0, 0, (\mathbb{Z}/3)^2, 0, 0, 0, (\mathbb{Z}/3)^2, \ldots),$$
$$H_*(SP_1) \cong H_*(SP_{12}) \cong (\mathbb{Z}, 0, 0, (\mathbb{Z}/3)^2, 0, 0, 0, (\mathbb{Z}/3)^2, \ldots),$$
$$H_*(SP_{12,3}) \cong (\mathbb{Z}, 0, 0, \mathbb{Z}/3, 0, 0, 0, \mathbb{Z}/3, \ldots).$$

The subconjugation $SP_{12,3} \hookrightarrow SP_1$ induces an injection.

$$H_*(SP_{1,12}) \cong (\mathbb{Z}, 0, 0, \mathbb{Z}, 0, 0, 0, 0, \ldots),$$

$H_3(SP_{1,12})$ is generated by the Thom class U, which is mapped to a class of order three in $H_3(SP_1)$, independent of the image of $H_*(SP_{12,3})$.

$$H_*(SP_{1,12,3}) \cong H_*(1).$$
$$H_*(SD_3; W_3) \cong (\mathbb{Z}/3, 0, 0, 0, \mathbb{Z}/3, 0, 0, 0, \ldots).$$

All these groups are eventually repeating with period four.

Proof. The result for $H_*(\mathrm{SL}_3\mathbb{Z})$ is from [11].

SP_1 and SP_{12} are isomorphic under the involution. The LHS-SS for the split extension $\langle a, b\rangle = \mathbb{Z}^2 \to SP_1 \to \mathrm{GL}_2\mathbb{Z}$ collapses at E^2 and splits, as the first row is zero. The zeroth row is isomorphic to $H_*(\mathrm{GL}_2\mathbb{Z})$ and the second row is isomorphic to $H_*(\mathrm{GL}_2\mathbb{Z}; \mathbb{Z}_{\det})$, both of which are easily computed.

For $H_*(SP_{1,12})$ we use the split extension $\langle a, b\rangle = \mathbb{Z}^2 \to SP_{1,12} \to P_1(2)$ which maps to the one used for SP_1. The LHS-SS collapses at E^2 and splits, in view of the $P_1(2)$-homology computations in Lemmas 5.2 and 6.2. In bidegree (1, 2) the inclusion $SP_{1,12} \subset SP_1$ induces a surjection $H_1(P_1(2); \mathbb{Z}_{\det}) \twoheadrightarrow H_1(\mathrm{GL}_2\mathbb{Z}; \mathbb{Z}_{\det})$ as the class of c generating $\mathbb{Z}$ maps to the class of $t^{-1}s^{-1}$, which is of order precisely three.

The results for $P_{12,3}$ and $P_{1,12,3}$ are obvious.

To compute $H_*(SD_3; W_3)$ we use the SD_3 PSS abutting to zero. Clearly, its only differentials $d^4\colon E^4_{4,4i} \to E^4_{0,4i+3}$ are isomorphisms. □

LEMMA 7.2. *In the* $\mathrm{SL}_3\mathbb{Z}$ *PSS* $d^1_1\colon E^1_{1,*} \to E^1_{0,*}$ *is surjective.*

Proof. By [11, Proposition 1] the odd primary cohomology of $\mathrm{SL}_3\mathbb{Z}$ is detected on certain subgroups $[O] = SD_3$ and $[Q] \cong Di_6$. $[Q]$ is contained in SP_1, and the cohomology of $[O]$ is detected on $\langle rs^2 = -\phi, \psi\rangle \cong \Sigma_3$, which is subconjugate to SP_1 ($c^{(ab)^T}\langle -\phi, \psi\rangle \subset SP_1$). Thus $H^*(\mathrm{SL}_3\mathbb{Z}) \to H^*(SP_1)$ is an injection and the Lemma follows. □

Proof of Theorem 3.4 (2). The $\mathrm{SL}_3\mathbb{Z}$ PSS appears as

$$\begin{array}{cccccccc}
(\mathbb{Z}/3)^2 & \xleftarrow{2} & (\mathbb{Z}/3)^4 & \xleftarrow{1} & \mathbb{Z}/3 & & 0 & 0 \\
0 & & 0 & & 0 & & 0 & 0 \\
0 & & 0 & & 0 & & 0 & 0 \\
0 & & 0 & & 0 & & 0 & \mathbb{Z}/3 \\
(\mathbb{Z}/3)^2 & \xleftarrow{2} & (\mathbb{Z}/3)^4 & \xleftarrow{2} & \mathbb{Z}/3 \oplus \mathbb{Z} & & 0 & 0 \\
0 & & 0 & & 0 & & 0 & 0 \\
0 & & 0 & & 0 & & 0 & 0 \\
\mathbb{Z} & \xleftarrow{1} & \mathbb{Z}^2 & \xleftarrow{1} & \mathbb{Z}^2 & \xleftarrow{1} & \mathbb{Z} & \mathbb{Z}/3
\end{array}$$

Hence $H_*(F_3\mathbf{K}\mathbb{Z}/F_2\mathbf{K}\mathbb{Z}) \cong (0, 0, 0, 0, \mathbb{Z}/3, \mathbb{Z}, 0, 0, \#9, 0, \ldots)$. □

Proof of Proposition 3.5 (2). This is similar to the proof of Proposition 3.3. Analogous splittings and $\mathbf{K}\mathscr{E}$-theoretic equivalences appear, and a three-chain in the bar resolution for SP_1 replaces the class cx. We omit the details. □

Acknowledgement

I thank Marcel Bökstedt and Jørgen Tornehave for helpful conversations on étale K-theory and crystallographic methods. This work was done during an exciting year at Århus University.

References

1. Borel, A.: Stable real cohomology of arithmetic groups, *Ann. Sci. École Norm. Sup.* (*4*) 7 (1974), 235–272.
2. Browder, W.: Torsion in H-spaces, *Ann. of Math.* **74** (1961), 24–51.
3. Bökstedt, M.: The rational homotopy type of $\Omega\,\mathrm{Wh}^{\mathrm{Diff}}(*)$, in I. Madsen and B. Oliver (eds), *Algebraic Topology. Aarhus 1982. Proceedings*, Lecture Notes in Math. 1051, Springer-Verlag, New York, 1984, pp. 25–37.
4. Cartan, H. and Eilenberg, S.: *Homological Algebra*, Princeton University Press, 1956.
5. Dwyer, W. and Friedlander, E.: Conjectural calculations of general linear group homology, in, S. J. Block *et al.* (eds.), *Applications of Algebraic K-theory to Algebraic Geometry and Number Theory*, Contemporary Math., vol. 55 part 1, Amer. Math. Soc., 1986, pp. 135–147.

6. Fiedorowicz, Z. and Priddy, S. B.: Homology of classical groups over finite fields and their associated infinite loop spaces, *Lecture Notes in Math.* **674**, Springer-Verlag, New York, 1978.
7. Hamada, S.: On a free resolution of a dihedral group, *Tohuku Math. J.* **15** (1963), 212–219.
8. Lee, R. and Szczarba, R. H.: The group $K_3(\mathbb{Z})$ is cyclic of order forty-eight, *Ann. of Math.* (2) **104** (1) (1976), 31–60.
9. Lee, R. and Szczarba, R. H.: On the torsion in $K_4(\mathbb{Z})$ and $K_5(\mathbb{Z})$, *Duke Math. J.* **45** (1978), 101–129.
10. Mitchell, S. A.: The rank filtration of Quillen's Q-construction, in S. M. Salamon, B. Steer and W. A. Sutherland (eds.), *Advances in Homotopy Theory*, London Math. Soc. Lect. Note Series, vol. 139, Cambridge Univ. Press, 1989, pp. 127–142.
11. Soulé, C.: The cohomology of $\mathrm{SL}_3(\mathbb{Z})$, *Topology* **17** (1978), 1–22.
12. Thurston, W. P.: Three dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Amer. Math. Soc.* **6**(3) (1982), 357–381.
13. Quillen, D.: Letter from Quillen to Milnor on $\mathrm{Im}(\pi_i O \to \pi_i^s \to K_i\mathbb{Z})$, *Algebraic K-theory*, (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976), Lecture Notes in Math. **551**, Springer-Verlag, New York, 1976, pp. 182–188.
14. Rognes, J.: A spectrum level rank filtration in algebraic K-theory, *Topology* (to appear).
15. Waldhausen, F.: Algebraic K-theory of spaces, in A. Ranicki, N. Levitt and F. Quinn, (eds.), *Algebraic and Geometric Topology. Proceedings, Rutgers 1983*, Lecture Notes in Math. **1126**, Springer-Verlag, New York, 1985, pp. 318–419.