TWO-PRIMARY ALGEBRAIC K-THEORY OF CURVES OVER FINITE FIELDS

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ABSTRACT. We apply the mod 2^∞ Bloch–Lichtenbaum spectral sequence to describe the two-completed algebraic K-theory of a curve X over a finite field k of odd characteristic. When the curve is smooth and affine or projective, the even K-groups are given in terms of invariants of the Frobenius action on the Iwasawa module of the curve.

INTRODUCTION

Let X be any curve over a finite field k of odd characteristic. In Proposition 1.8 of this note we compute the 2-adic algebraic K-theory of X in terms of its étale cohomology, up to an extension question.

Suppose further that X is connected, smooth and either projective or affine. In Propositions 2.7 and 2.8 we rewrite the étale cohomology of X in terms of the Frobenius action on groups of units and Picard groups related to X. In Theorems 3.5 and 3.6 we express the 2-adic algebraic K-theory of X in terms of the Frobenius action on the Iwasawa module of X.

These results are applications of Voevodsky's proof of the Milnor conjecture, and the Bloch–Lichtenbaum spectral sequence, together with extensions to finite coefficients and positive characteristics from [RW] and [Wb]. Related results for varieties are expected in a paper [Ka] by B. Kahn, that compares algebraic K-theory to étale K-theory.

1. ALGEBRAIC CURVES

Let $k = \mathbb{F}_q$ be a finite field, let X be an algebraic curve defined over k, and let $K = k(X)$ be the function field on X. So K has transcendence degree 1 over k. As our focus is on X rather than on k, we may and will assume that k is algebraically closed in K.

Throughout this paper, let $\ell = 2$ be the even prime, and suppose that k has odd characteristic $p = \mathrm{char}(k) \neq \ell$. There is a natural mod ℓ^∞ Bloch–Lichtenbaum spectral sequence converging to the mod ℓ^∞ algebraic K-theory of K:

$$E_2^{s,t} = H^{s-t}_{\text{ét}}(K; \mathbb{Q}_\ell/\mathbb{Z}_\ell(-t)) \Longrightarrow K_{-s-t}(K; \mathbb{Q}_\ell/\mathbb{Z}_\ell) \tag{1.1}$$

Here $t \leq s \leq 0$. See [RW, §1]. The identification of the E_2-term is obtained by functorially replacing K by a field L of characteristic zero, with naturally

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isomorphic ℓ-adic étale cohomology groups and ℓ-adic algebraic K-groups as K, and considering the mod ℓ^∞ Bloch–Lichtenbaum spectral sequence for L. See [Wb].

In some more detail, this is done as follows. First pass to the perfect closure K^p of $K = K_0$, by adjoining the p-th roots of all elements of K_i to obtain K_{i+1}, and iterate countably often. Then form the Witt ring $R = W(K^p)$. This is a complete discrete valuation ring of characteristic zero, with residue field K^p. Let L_0 be the quotient field of $R = R_0$. Adjoin the ℓ-th roots of a uniformizing parameter π in R_i to obtain R_{i+1}, with quotient field L_{i+1}, and iterate countably often. Let R_∞ be the union of the R_i, with quotient field L_∞. The Galois group of K^p over K is a pro-p group, and acts on R_∞ and L_∞. Let L be the invariant field for the latter Galois action. Then L has the asserted properties.

The ℓ-adic cohomological dimension of k is 1, so the ℓ-adic cohomological dimension of K is at most 2. Hence $H^n_{\text{ét}}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) = 0$ for all $n \geq 3$. Thus the spectral sequence above collapses at the E_2-term.

The edge map $K_{2i}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^0_{\text{ét}}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ of the mod ℓ^∞ Bloch–Lichtenbaum spectral sequence, and naturality with respect to the composite map $\operatorname{Spec}(K) \to X \to \operatorname{Spec}(k)$, gives us the following commutative square:

$$
\begin{tikzcd}
K_{2i}(k;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r] \arrow[d, "\cong"', "\text{edge}"] & K_{2i}(X;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r] & K_{2i}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[d, "\text{edge}"] \\
H^0_{\text{ét}}(k;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \arrow[r, "\cong"] & H^0_{\text{ét}}(X;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \arrow[r, "\cong"] & H^0_{\text{ét}}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))
\end{tikzcd}
\tag{1.2}
$$

The left hand map is an isomorphism since k has cohomological dimension 1, and the lower maps are isomorphisms because k is algebraically closed in K. Hence the edge map from $K_{2i}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is split surjective. This proves the following:

Proposition 1.3. *Let $\ell = 2$. For $m \geq 1$ we have*

$$K_m(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong \begin{cases} H^1_{\text{ét}}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) & \textit{for } m = 2i-1, \\ H^2_{\text{ét}}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \oplus H^0_{\text{ét}}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i-1)) & \textit{for } m = 2i-2. \end{cases}$$

Definition 1.4. Let $\tilde{K}_{2i}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong H^2_{\text{ét}}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i+1))$ be the kernel of the split surjection

$$K_{2i}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\text{edge}} H^0_{\text{ét}}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)),$$

and let $\tilde{K}_{2i}(X;\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ be the kernel of the composite split surjection

$$K_{2i}(X;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to K_{2i}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\text{edge}} H^0_{\text{ét}}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \cong H^0_{\text{ét}}(X;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)).$$

The localization sequence in étale cohomology for the curve X contains the following exact sequence:

(1.5)

$$0 \to H^1_{\text{ét}}(X;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \to H^1_{\text{ét}}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \xrightarrow{\partial^e} \bigoplus_{x\in X_1} H^0_{\text{ét}}(k(x);\mathbb{Q}_\ell/\mathbb{Z}_\ell(i-1))$$
$$\to H^2_{\text{ét}}(X;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \to H^2_{\text{ét}}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$$

Here X_1 is the set of closed points x of X, and $k(x)$ is the residue field at such a point x. The last map is surjective for $i \neq 1$. We will map the exact sequence above to the corresponding exact localization sequence in algebraic K-theory for X:

$$(1.6) \quad 0 \to K_{2i-1}(X;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to K_{2i-1}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\partial} \bigoplus_{x\in X_1} K_{2i-2}(k(x);\mathbb{Q}_\ell/\mathbb{Z}_\ell)$$
$$\to K_{2i-2}(X;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to K_{2i-2}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0$$

Let us first show that we can choose isomorphisms $\phi_x\colon H^0_{\text{ét}}(k(x);\mathbb{Q}_\ell/\mathbb{Z}_\ell(i-1)) \cong K_{2i-2}(k(x);\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ for all $x \in X_1$ that are compatible with the isomorphism $K_{2i-1}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong H^1_{\text{ét}}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ from (1.3) under the boundary maps ∂ and ∂^e.

For each closed point $x \in X_1$ we may form the Henselization K^h_x of K, which admits a natural map $K \to K^h_x$. This is a complete discrete valuation ring with residue field $k(x)$. There are corresponding localization sequences in étale cohomology and algebraic K-theory for K^h_x, in which the connecting maps

$$\partial^e_x\colon H^1_{\text{ét}}(K^h_x;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \xrightarrow{\cong} H^0_{\text{ét}}(k(x);\mathbb{Q}_\ell/\mathbb{Z}_\ell(i-1))$$

and

$$\partial_x\colon K_{2i-1}(K^h_x;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\cong} K_{2i-2}(k(x);\mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

are isomorphisms. This follows by rigidity for étale cohomology and algebraic K-theory. The following lemma is then a consequence of naturality of the localizations sequences.

Lemma 1.7. *The boundary map ∂^e in the étale cohomology localization sequence factors as the composite*

$$H^1_{\text{ét}}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \to \bigoplus_{x\in X_1} H^1_{\text{ét}}(K^h_x;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \xrightarrow[\cong]{\oplus_x \partial^e_x} \bigoplus_{x\in X_1} H^0_{\text{ét}}(k(x);\mathbb{Q}_\ell/\mathbb{Z}_\ell(i-1)).$$

Likewise the boundary map ∂ in the algebraic K-theory localization sequence factors as the composite

$$K_{2i-1}(K;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \bigoplus_{x\in X_1} K_{2i-1}(K^h_x;\mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow[\cong]{\oplus_x \partial_x} \bigoplus_{x\in X_1} K_{2i-2}(k(x);\mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

The left hand maps are induced by the embeddings $K \to K^h_x$.

For each $x \in X_1$ the isomorphism

$$\phi_x\colon H^0_{\text{ét}}(k(x);\mathbb{Q}_\ell/\mathbb{Z}_\ell(i-1)) \to K_{2i-2}(k(x);\mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

is now defined to correspond to the isomorphism

$$H^1_{\text{ét}}(K^h_x;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \cong K_{2i-1}(K^h_x;\mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

of (1.3) via the isomorphisms ∂^e_x and ∂_x. Then by naturality of the Bloch–Lichtenbaum spectral sequence (1.1) with respect to the field embeddings $K \to K^h_x$ we obtain an isomorphism

$$\oplus_x \phi_x \colon \bigoplus_{x \in X_1} H^0_{\text{ét}}(k(x); \mathbb{Q}_\ell/\mathbb{Z}_\ell(i-1)) \xrightarrow{\cong} \bigoplus_{x \in X_1} K_{2i-2}(k(x); \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

which is compatible with the boundary maps ∂^e and ∂.

Hence the induced map of kernels $\ker(\partial^e) \to \ker(\partial)$ is an isomorphism

$$H^1_{\text{ét}}(X; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \cong K_{2i-1}(X; \mathbb{Q}_\ell/\mathbb{Z}_\ell)\,.$$

Likewise the induced map of cokernels is an isomorphism $\operatorname{cok}(\partial^e) \cong \operatorname{cok}(\partial)$. The group $\tilde{K}_{2i-2}(X; \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is an extension of $\tilde{K}_{2i-2}(K; \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong H^2_{\text{ét}}(K; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ by $\operatorname{cok}(\partial) \cong \operatorname{cok}(\partial^e)$, due to (1.2), (1.4) and (1.6). By (1.5) also $H^2_{\text{ét}}(X; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ is an extension of the same groups when $i \neq 1$. Without asserting that the two extensions of $H^2_{\text{ét}}(K; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ agree, we have proven:

Proposition 1.8. *Let $\ell = 2$ and let X be a curve over a finite field k of characteristic $p \neq \ell$. For $m \geq 1$ there are isomorphisms*

$$K_m(X; \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong \begin{cases} H^1_{\text{ét}}(X; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) & \textit{for } m = 2i-1, \\ H^2_{\text{ét}}(X; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \oplus H^0_{\text{ét}}(X; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i-1)) & \textit{for } m = 2i-2, \end{cases}$$

up to extensions *within* $H^2_{\text{ét}}(X; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$.

2. Smooth curves

Now suppose the curve X/k is connected, smooth and projective, of genus g. We adopt some notations from [DM]: Let S be the finite set of points at infinity of X. Let $X' = X \setminus S$ be the corresponding smooth affine curve, and write $X' = \operatorname{Spec}(\mathcal{O})$. Let $\bar{k}$ be an algebraic closure of k, let $\bar{X} = X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\bar{k})$, and let $\bar{X}' = \bar{X} \setminus \bar{S}$ where $\bar{S}$ is the set of points in $\bar{X}$ above S. Also write $\bar{X}' = \operatorname{Spec}(\bar{\mathcal{O}})$.

We refer to the summary in [DM, §2] and [Mi, Ch.III] for the following facts about étale cohomology of curves.

The Picard group $\operatorname{Pic}(X) = H^1_{\text{ét}}(X; \mathbb{G}_m)$ admits a split surjective degree map to $\mathbb{Z}$, hence sits in the split extension

$$0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \to 0 \tag{2.1}$$

with $\operatorname{Pic}^0(X)$ finite. Passing to the algebraic closure, $\operatorname{Pic}^0(\bar{X})$ is ℓ-divisible with ℓ-torsion subgroup $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2g}$. In the affine case, $\operatorname{Pic}(X')$ is finite and $\operatorname{Pic}(\bar{X}')$ is ℓ-divisible with ℓ-torsion subgroup $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2g}$. We write $A\{\ell\}$ for the ℓ-torsion subgroup of an Abelian group A. As in [DM, 3.7] we obtain

$$H^n_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \cong \begin{cases} \mathbb{Q}_\ell/\mathbb{Z}_\ell(i) & \text{for } n = 0, \\ \operatorname{Pic}^0(\bar{X})\{\ell\}(i-1) & \text{for } n = 1, \\ \mathbb{Q}_\ell/\mathbb{Z}_\ell(i-1) & \text{for } n = 2, \\ 0 & \text{for } n \geq 3, \end{cases} \tag{2.2}$$

in the projective case, and

$$H^n_{\text{ét}}(\bar{X}'; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \cong \begin{cases} \mathbb{Q}_\ell/\mathbb{Z}_\ell(i) & \text{for } n = 0, \\ E(i-1) & \text{for } n = 1, \\ 0 & \text{for } n \geq 2, \end{cases} \tag{2.3}$$

in the affine case. Here E is an extension

$$0 \to \bar{\mathcal{O}}^\times \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to E \to \operatorname{Pic}(\bar{X}')\{\ell\} \to 0 \tag{2.4}$$

which is a finite sum of copies of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$. (We will retain the notation E for this extension.)

In particular each cohomology group $H^n_{\text{ét}}(\bar{X}; \mathbb{Z}/\ell^\nu(i))$ is finite, so $H^n_{\text{ét}}(\bar{X}; \mathbb{Z}_\ell(i))$ is a finitely generated $\mathbb{Z}_\ell$-module. The same assertions hold for each cohomology group of $\bar{X}'$, as well as for X and X' by Galois descent.

Since each group $H^n_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ is a direct sum of copies of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$, the sequence

$$0 \to H^n_{\text{ét}}(\bar{X}; \mathbb{Z}_\ell(i)) \to H^n_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell(i)) \to H^n_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \to 0 \tag{2.5}$$

is short exact for all n and i. This also holds for the cohomology of $\bar{X}'$.

The Galois group $\operatorname{Gal}(\bar{X}/X) \cong \operatorname{Gal}(\bar{k}/k)$ is topologically generated by the (geometric) Frobenius automorphism $\phi = \phi_q \colon z \mapsto z^q$. Let ϕ^* be the (algebraic) Frobenius map acting on $H^n_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell(0))$ induced by ϕ^{-1}. The action of ϕ^* on $H^n_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell(i))$ is scaled by a factor q^{-i} in comparison to that on $H^n_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell(0))$.

By a theorem of A. Weil [Wl], the eigenvalues of ϕ^* acting on $H^n_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell(0))$ have absolute value $q^{n/2}$, since X is smooth and projective. Thus 1 is only an eigenvalue of ϕ^* acting on $H^n_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell(i))$ when $i = n/2$, and therefore $\phi^* - 1$ maps $H^n_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell(i))$ isomorphically to itself as long as $i \neq n/2$. Since X is a curve the only exceptions are $(n, i) = (0, 0)$ and $(2, 1)$, which only affect $K_0(X)$.

By (2.5) it follows that $\phi^* - 1$ maps $H^n_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ surjectively to itself, whenever $i \neq n/2$. As in [DM, 3.12] we can apply descent to go from $\bar{X}$ to X. We find that when $i \neq n/2$, the coinvariants of ϕ^* acting on $H^n_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ are zero, while the invariants of ϕ^* acting on $H^n_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ are finite, and are identified with $H^n_{\text{ét}}(X; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$.

Definition 2.6. For an Abelian group A with a self-map $\phi^* \colon A \to A$ let $A^\phi = \ker(\phi^* - 1)$ denote the subgroup of invariants under the action by ϕ^*, and let $A_\phi = \operatorname{cok}(\phi^* - 1)$ denote the quotient group of coinvariants under the action.

Let $w_i(k)$ denote the maximal power ℓ^ν of ℓ such that the Galois group of k with all ℓ^ν-th roots of unity adjoined, over k, has exponent dividing i. Then $H^0_{\text{ét}}(k; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \cong \mathbb{Z}/w_i(k)$. (We allow $w_i(k) = \ell^\infty$, when $\mathbb{Z}/\ell^\infty \cong \mathbb{Q}_\ell/\mathbb{Z}_\ell$.)

Proposition 2.7. *Let X/k be a connected smooth projective curve, as above. For $(n, i) \neq (0, 0)$ or $(2, 1)$ we have*

$$H^n_{\text{ét}}(X; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \cong \begin{cases} \mathbb{Z}/w_i(k) & \textit{for } n = 0, \\ \operatorname{Pic}^0(\bar{X})\{\ell\}(i-1)^\phi & \textit{for } n = 1, \\ \mathbb{Z}/w_{i-1}(k) & \textit{for } n = 2, \\ 0 & \textit{for } n \geq 3. \end{cases}$$

These groups are finite.

We now turn to the affine case.

Proposition 2.8. *Let X'/k be a connected smooth affine curve, as above. For $(n,i) \neq (0,0)$ or $(2,1)$ we have $H^0_{ét}(X';\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \cong \mathbb{Z}/w_i(k)$, $H^n_{ét}(X';\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) = 0$ for $n \geq 2$, and there is an exact sequence*

$$\begin{aligned} 0 \to (\bar{\mathcal{O}}^\times \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)(i-1)^\phi \to H^1_{ét}(X';\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \\ \to \mathrm{Pic}(\bar{X}')\{\ell\}(i-1)^\phi \to (\bar{\mathcal{O}}^\times \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)(i-1)_\phi \to 0\,. \end{aligned}$$

These cohomology groups are finite, except for $(n,i) = (1,1)$.

Proof. We know that $H^3_{ét}(X';\mathbb{Z}_\ell(i))$ vanishes since each $H^2_{ét}(X';\mathbb{Z}/\ell^\nu(i))$ is finite. Hence there is a surjection $H^2_{ét}(X';\mathbb{Q}_\ell(i)) \to H^2_{ét}(X';\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ and we deduce that $H^2_{ét}(X';\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ must be divisible. But it is also finite for $i \neq 1$, due to the surjection

$$H^2_{ét}(X;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \to H^2_{ét}(X';\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$$

from the localization sequence for $X' \to X$. Hence $H^2_{ét}(X';\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) = 0$ for $i \neq 1$.

By descent, the invariants and coinvariants of the action of ϕ^* on

$$H^1_{ét}(\bar{X}';\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) = E(i-1)$$

are identified with $H^1_{ét}(X';\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ and $H^2_{ét}(X';\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$, respectively. Here E is the extension from (2.4). We have just seen that the coinvariants vanish for $i \neq 1$, hence the invariants are finite in these cases. The kernel-cokernel sequence for $\phi^* - 1$ acting on the extension $E(i-1)$ now simplifies to the asserted exact sequence. □

3. Iwasawa modules

Let $k_\infty \subset \bar{k}$ be obtained by adjoining all ℓ^ν-th roots of unity in $\bar{k}$ to k, for all $\nu \geq 1$. Let $X_\infty = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k_\infty)$ and similarly for $X'_\infty = \mathrm{Spec}(\mathcal{O}_\infty)$.

Let $\hat{X}$ be the maximal unramified pro–Galois pro-ℓ Abelian cover of X_∞. The *Iwasawa module* M_X is the Galois group of $\hat{X}$ over X_∞. Hence the Pontryagin dual $M_X^\# = \mathrm{Hom}(M_X, \mathbb{Q}/\mathbb{Z})$ is identified with $H^1_{ét}(X_\infty;\mathbb{Q}_\ell/\mathbb{Z}_\ell(0))$. By descent over $\mathrm{Gal}(\bar{X}/X_\infty) \cong \mathrm{Gal}(\bar{k}/k_\infty)$ we have

$$H^n_{ét}(X_\infty;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \cong \begin{cases} \mathbb{Q}_\ell/\mathbb{Z}_\ell(i) & \text{for } n=0, \\ M_X^\#(i) & \text{for } n=1, \\ \mathbb{Q}_\ell/\mathbb{Z}_\ell(i-1) & \text{for } n=2, \\ 0 & \text{for } n \geq 3. \end{cases} \tag{3.1}$$

Here $M_X^\#$ is a retract of $\mathrm{Pic}^0(\bar{X})\{\ell\}$. Thus it has the form $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^r$ for some $0 \leq r \leq 2g$. Dually M_X is a free $\mathbb{Z}_\ell$-module of rank r, and the action of ϕ^* on $M_X^\#(i)$ has zero coinvariants and finite invariants. Thus

$$H^n_{ét}(X;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \cong \begin{cases} \mathbb{Z}/w_i(k) & \text{for } n=0, \\ M_X^\#(i)^\phi & \text{for } n=1, \\ \mathbb{Z}/w_{i-1}(k) & \text{for } n=2, \\ 0 & \text{for } n \geq 3. \end{cases} \tag{3.2}$$

Likewise let $\hat{X}'$ be the maximal unramified pro–Galois pro-ℓ Abelian cover of X'_∞. The Iwasawa module $M_{X'}$ is the Galois group of $\hat{X}'$ over X'_∞. Hence the Pontryagin dual $M^{\#}_{X'} = \mathrm{Hom}(M_{X'}, \mathbb{Q}/\mathbb{Z})$ is identified with $H^1_{\text{ét}}(X'_\infty; \mathbb{Q}_\ell/\mathbb{Z}_\ell(0))$. By descent over $\mathrm{Gal}(\bar{X}'/X'_\infty) \cong \mathrm{Gal}(\bar{k}/k_\infty)$ we have

$$H^n_{\text{ét}}(X'_\infty; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \cong \begin{cases} \mathbb{Q}_\ell/\mathbb{Z}_\ell(i) & \text{for } n = 0, \\ M^{\#}_{X'}(i) & \text{for } n = 1, \\ 0 & \text{for } n \geq 2. \end{cases} \tag{3.3}$$

Here $M^{\#}_{X'}$ is a retract of the extension E of (2.4), hence is of the form $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{r'}$ for some finite r', and dually M_X is a free $\mathbb{Z}_\ell$-module of rank r'. As noted in the proof of (2.8) the action of ϕ^* on $E(i-1)$ has zero coinvariants, so the action of ϕ^* on $M^{\#}_{X'}(i)$ has zero coinvariants and finite invariants. Thus

$$H^n_{\text{ét}}(X'; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \cong \begin{cases} \mathbb{Z}/w_i(k) & \text{for } n = 0, \\ M^{\#}_{X'}(i)^\phi & \text{for } n = 1, \\ 0 & \text{for } n \geq 2. \end{cases} \tag{3.4}$$

Recall that $K_{2i-1}(k) \cong \mathbb{Z}/(q^i - 1)$ when k has q elements [Qu]. Let the notation $A \cong_{(\ell)} B$ mean that A and B are isomorphic modulo odd finite groups.

Theorem 3.5. *Let X be a connected smooth projective curve over a finite field k of characteristic $\neq \ell = 2$, with Iwasawa module M_X. When $m \geq 1$ we have*

$$K_m(X) \cong_{(\ell)} \begin{cases} K_m(k) \oplus K_m(k) & \textit{for } m = 2i-1 \textit{ odd}, \\ H^2_{\text{ét}}(X; \mathbb{Z}_\ell(i)) & \textit{for } m = 2i-2 \textit{ even}. \end{cases}$$

There are isomorphisms

$$H^2_{\text{ét}}(X; \mathbb{Z}_\ell(i)) \cong \mathrm{Pic}^0(\bar{X})\{\ell\}(i-1)^\phi \cong M^{\#}_X(i)^\phi .$$

Proof. From 2.8 and the localization sequence we deduce that $H^2_{\text{ét}}(K; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) = 0$ for $i \neq 1$. Hence there is no ambiguity about the extension in 1.8 for X smooth. The theorem follows from (1.8), (2.7), (3.2), the universal coefficient theorem, and finiteness of the groups $K_m(X)$ [Gr]. □

Theorem 3.6. *Let $X' = \mathrm{Spec}(\mathcal{O})$ be a connected smooth affine curve over a finite field k of characteristic $\neq \ell = 2$. When $m \geq 2$ we have*

$$K_m(X') \cong_{(\ell)} \begin{cases} K_m(k) & \textit{for } m = 2i-1 \textit{ odd}, \\ H^2_{\text{ét}}(X'; \mathbb{Z}_\ell(i)) & \textit{for } m = 2i-2 \textit{ even}. \end{cases}$$

There is an exact sequence

$$\begin{aligned} 0 \to (\bar{\mathcal{O}}^\times \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)(i-1)^\phi &\to H^2_{\text{ét}}(X'; \mathbb{Z}_\ell(i)) \\ &\to \mathrm{Pic}(\bar{X}')\{\ell\}(i-1)^\phi \to (\bar{\mathcal{O}}^\times \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)(i-1)_\phi \to 0 \end{aligned}$$

and an isomorphism $H^2_{\text{ét}}(X'; \mathbb{Z}_\ell(i)) \cong M^{\#}_{X'}(i)^\phi$.

Proof. Again $H^2_{\text{ét}}(K; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) = 0$ for $i \neq 1$, so we can use (1.8), (2.8) and (3.4), as in the previous proof. We omit the case $m = 1$, and in fact $K_1(X') \cong \mathcal{O}^\times$ is typically not finite. □

References

[DM] W.G. Dwyer and S.A. Mitchell, *On the K-theory spectrum of a smooth curve over a finite field*, Topology **36** (1997), 899–929.

[Gr] D.R. Grayson, *Finite generation of K-groups of a curve over a finite field. (After Daniel Quillen)*, Algebraic K-theory, Proc. Conf., Oberwolfach 1980, Part I, Lect. Notes Math., vol. 966, Springer, 1982, pp. 69–90.

[Ka] B. Kahn, *K-theory of semi-local rings with finite coefficients and étale cohomology*, UIUC K-theory preprint server, no. 177 (1997).

[Mi] J.S. Milne, *Etale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, 1980.

[Qu] D. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math., II. Ser. **96** (1972), 552–586.

[RW] J. Rognes and C. Weibel, *Two-primary algebraic K-theory of rings of integers in number fields*, UIUC K-theory preprint server, no. 220 (1997).

[Wb] C. Weibel, *Lecture 9. Bloch's higher Chow groups*, ICTP proceedings, School on Algebraic K-theory and its applications, Trieste, 1997.

[Wl] A. Weil, *Sur les courbes algébriques et les variétés qui s'en déduisent*, Hermann, Paris, 1948.

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