

# Group actions on homotopy spheres

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# Outline

Exotic spheres

Transformation groups

Involutions on spheres

Proof of sufficiency

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# Framed bordism

Theorem (Pontryagin (1936, 1950), Thom (1954))

$$\begin{aligned}\Omega_m^{fr} &\cong \pi_m^s \\ [M, \bar{f}] &\leftrightarrow \alpha\end{aligned}$$

- ▶ Closed, smooth  $M^m \subset \mathbb{R}^{m+n}$
- ▶ Normal framing  $\bar{f}: \nu^n \cong_s \epsilon^n$
- ▶ Collapse and projection maps

$$\alpha: S^{m+n} \rightarrow Th(\nu^n) \cong_s Th(\epsilon^n) \rightarrow S^n$$

- ▶ Framed cobordism  $W^{m+1}$  with  $\partial W \cong M - M'$ .

# The $h$ -cobordism theorem

## Theorem (Smale (1960))

$$(W; M, M') \text{ } h\text{-cobordism} \iff W \cong M \times [0, 1].$$

- ▶ Assumes  $m \geq 5$ ,  $\pi = \pi_1(M)$  trivial
- ▶  $h$ -cobordism means  $M \xrightarrow{\sim} W \xleftarrow{\sim} M'$
- ▶ Implies  $M \cong M'$
- ▶ Obstruction  $\tau \in Wh(\pi)$  in general;  $Wh(\mathbb{Z}/2) = 0$ .

# Homotopy spheres

## Theorem (Kervaire–Milnor (1963))

$$0 \rightarrow bP_{m+1} \rightarrow \Theta_m \rightarrow \pi_m^S / \text{Im } J_m \rightarrow \mathbb{Z}/2$$

- ▶  $\Theta_m \ni [\Sigma^m]$   $h$ -cobordism classes of homotopy  $m$ -spheres
- ▶  $bP_{m+1}$  subgroup of  $\Sigma^m = \partial P$  with  $P$  parallelizable
- ▶ Each homotopy  $m$ -sphere admits a normal framing
- ▶ Change choice of framing  $\leftrightarrow$  add element of  $\text{Im } J_m$ .

# Surgery obstruction groups

- Browder–Novikov–Sullivan–Wall surgery theory assigns to a degree 1 normal map  $(f, \bar{f}): (M^m, \nu_M) \rightarrow (X, \xi)$  an obstruction

$$\sigma(f, \bar{f}) \in L_m(\mathbb{Z}[\pi_1(X)], w_1(\xi))$$

to modifying  $M$  by surgeries until  $f$  becomes a (simple) homotopy equivalence.

- Milnor–Kervaire and Wall calculated

$$L_*(\mathbb{Z}) \cong (\mathbb{Z}, 0, \mathbb{Z}/2, 0, \dots)$$

$$L_*(\mathbb{Z}[\mathbb{Z}/2], +1) \cong (\mathbb{Z} \oplus \mathbb{Z}, 0, \mathbb{Z}/2, \mathbb{Z}/2, \dots)$$

$$L_*(\mathbb{Z}[\mathbb{Z}/2], -1) \cong (\mathbb{Z}/2, 0, \mathbb{Z}/2, 0, \dots)$$

- Hence  $L_{k+n}(\mathbb{Z}[\mathbb{Z}/2], (-1)^n) = 0$  if  $n$  is even and  $k + n \equiv 1 \pmod{4}$ , or if  $n$  is odd and  $k + n$  is odd.

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# Large and small actions

- ▶ The standard  $m$ -sphere  $S^m$  admits a nontrivial smooth  $O(m+1)$ -action;  $\dim O(m+1) = m^2/2 + m/2$ .
- ▶ (Wu-Yi Hsiang, 1967) Exotic  $m$ -spheres  $\Sigma^m$  (with  $m \geq 40$ ) admit effective smooth  $G$ -action only for  $\dim G \leq m^2/8 + 1$ .

## Question

*Does every homotopy  $m$ -sphere admit a nontrivial smooth circle action?*

## Theorem (Schultz (ca. 1985))

*Yes, if  $m \leq 30$ .*

# Circle/odd cyclic actions

## Theorem (Schultz (ca. 1985))

*Given*

- ▶  $(\Sigma^k, \bar{a})$  framed homotopy  $k$ -sphere
- ▶  $[\Sigma^k, \bar{a}] = \alpha \in (\pi_k^S)_{(p)}$ ,  $p$  odd prime
- ▶  $\beta \in R(\alpha) \subset (\pi_{k+n}^S)_{(p)}$  root invariant

*there exists*

- ▶  $(\Sigma^{k+n}, \bar{b})$  framed homotopy  $(k+n)$ -sphere
- ▶  $[\Sigma^{k+n}, \bar{b}] = \beta$

*such that:*

- ▶ If  $n$  is even, then  $\Sigma^{k+n}$  admits a smooth  $\mathbb{Z}/p$ -action with fixed set  $\Sigma^k$ .
- ▶ If  $n$  is odd, then  $\Sigma^{k+n}$  admits a smooth circle action with  $(k-1)$ -dimensional fixed set.

# Example

Consider

- ▶  $p$  odd,  $q = 2p - 2$ ,  $i \geq 1$
- ▶  $\alpha_i \in (\pi_k^s)_{(p)}$  with  $k = iq - 1$
- ▶  $\beta_i \in R(\alpha_i) \subset (\pi_{k+n}^s)_{(p)}$  with  $k + n = i(p + 1)q - 2p$ .

$\implies$  There exists a framed homotopy  $(k + n)$ -sphere  $(\Sigma^{k+n}, \bar{b})$  with a nontrivial smooth circle action and  $[\Sigma^{k+n}, \bar{b}] = \beta_i$ .

- ▶  $\beta_i \notin (\text{Im } J)_{k+n}$

$\implies$  This  $(k + n)$ -sphere is exotic.

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# The Mahowald root invariant at $p = 2$

- ▶  $\gamma \downarrow \mathbb{R}P^\infty$  tautological line bundle
- ▶  $P_{-n} = Th(-n\gamma)$  Thom spectrum
- ▶  $\alpha \in \pi_k^S = \pi_{k-1}(S^{-1})$ ,  $\beta \in \pi_{k+n}^S = \pi_{k-1}(S^{-n-1})$
- ▶  $R(\alpha) = \beta + \text{Im } t$  if  $0 \neq \text{pr}(\alpha) = j(\beta)$

$$\begin{array}{ccccccc}
 & & & \alpha & \pi_{k-1}(S^{-1}) & & \\
 & & & & \downarrow \cong_2^\wedge & & \\
 & & & & \pi_{k-1}(P_{-\infty}) & & \\
 & & & & \downarrow \text{pr} & & \\
 & & \beta & & & & \\
 \pi_k(P_{-n}) & \xrightarrow{t} & \pi_{k-1}(S^{-n-1}) & \xrightarrow{j} & \pi_{k-1}(P_{-n-1}) & \xrightarrow{p} & \pi_{k-1}(P_{-n})
 \end{array}$$

# Necessary condition for $\mathbb{Z}/2$ -action

## Theorem (Stolz (1988))

*Let  $\Sigma^k$  and  $\Sigma^{k+n}$  be homotopy spheres. If there is a smooth  $\mathbb{Z}/2$ -action on  $\Sigma^{k+n}$  with fixed points  $\Sigma^k$ , then there exist framings  $\bar{a}$  and  $\bar{b}$  such that  $\alpha = [\Sigma^k, \bar{a}]$  and  $\beta = [\Sigma^{k+n}, \bar{b}]$  satisfy*

$$\mathrm{pr}(\alpha) = j(\beta) \in \pi_{k-1}(P_{-n-1}).$$

## Remark

*Let  $\beta' \in R(\alpha) \subset \pi_{k+n'}^s$ . The conclusion is then equivalent to*

$$\begin{cases} \beta \in R(\alpha) & \text{if } n = n' \\ \beta \in \mathrm{Im} \, t & \text{if } n < n'. \end{cases}$$

# Sufficient condition for $\mathbb{Z}/2$ -action

## Theorem (Stolz (1988))

Let  $(\Sigma^k, \bar{a})$  and  $(\Sigma^{k+n}, \bar{b})$  be framed homotopy spheres, with bordism classes  $\alpha \in \pi_k^S$  and  $\beta \in \pi_{k+n}^S$ , such that

1.  $\text{pr}(\alpha) = j(\beta)$  in  $\pi_{k-1}(P_{-n-1})$
2.  $n \geq k + 1$  and  $L_{k+n}(\mathbb{Z}[\mathbb{Z}/2], (-1)^n) = 0$ .

Then there exists a smooth  $\mathbb{Z}/2$ -action on  $\Sigma^{k+n} \# \Sigma'$ , for some  $[\Sigma'] \in bP_{k+n+1}$ , with fixed point set  $\Sigma^k$ .

## Remark

Let  $\beta' \in R(\alpha) \subset \pi_{k+n'}^S$ . The first condition is then equivalent to

$$\begin{cases} \beta \in R(\alpha) & \text{if } n = n' \\ \beta \in \text{Im } t & \text{if } n < n'. \end{cases}$$

# Twisted normal bordism

## Lemma

$$\Omega_{k+n}(\mathbb{R}P^\infty; -n\gamma) \cong \pi_k(P_{-n})$$

is the bordism group of triples  $(M, f, \bar{f})$  where

- ▶  $M$  is a closed, smooth  $(k+n)$ -manifold
- ▶  $\nu_M$  is the stable normal bundle of  $M$
- ▶  $f: M \rightarrow \mathbb{R}P^\infty$  is a map
- ▶  $\bar{f}: \nu_M \oplus \epsilon^k \cong_s f^*(-n\gamma)$  is a stable isomorphism.

## Proof.

Use Pontryagin–Thom collapse map and functoriality

$$S^k \rightarrow Th(\nu_M \oplus \epsilon^k) \cong_s Th(f^*(-n\gamma)) \rightarrow Th(-n\gamma) = P_{-n}.$$



# Projective $n$ -space

For  $\text{pr}: S^{-1} \rightarrow P_{-\infty} \rightarrow P_{-n-1}$

$$\begin{aligned}\Omega_n(\mathbb{R}P^\infty; -(n+1)\gamma) &\cong \pi_{-1}(P_{-n-1}) \\ [\mathbb{R}P^n, f, \bar{f}] &\leftrightarrow [\text{pr}]\end{aligned}$$

where

- ▶  $f: \mathbb{R}P^n \rightarrow \mathbb{R}P^\infty$  is the inclusion
- ▶  $\bar{f}$  is negative of  $\tau_{\mathbb{R}P^n} \oplus \epsilon^1 \cong f^*(n+1)\gamma$ .

# Geometric interpretation of root invariant

Set  $P = \mathbb{R}P^\infty = B\mathbb{Z}/2$ , with base point inclusion  $b: * \rightarrow P$ .

$$\begin{array}{c}
 [M^k, \bar{a}] \quad \Omega_k^{fr} \\
 \downarrow \text{pr} \\
 \Omega_{k+n}(P; -n\gamma) \xrightarrow{t} \Omega_{k+n}^{fr} \xrightarrow{j} \Omega_{k+n}(P; -(n+1)\gamma) \xrightarrow{p} \Omega_{k+n-1}(P; -n\gamma)
 \end{array}$$

►  $\text{pr}: [M^k, \bar{a}] \mapsto [M^k \times \mathbb{R}P^n, f \circ \text{pr}_2, \bar{a} \times \bar{f}]$

►  $j: [N^{k+n}, \bar{b}] \mapsto [N^{k+n}, b, \bar{b}]$

$$\bar{b}: \nu_N \oplus \epsilon^{k-1} \cong \epsilon^{-n-1} = b^*(-(n+1)\gamma)$$

►  $R([M^k, \bar{a}]) = [N^{k+n}, \bar{b}] + \text{Im } t$  if

$$0 \neq [M^k \times \mathbb{R}P^n, f \circ \text{pr}_2, \bar{a} \times \bar{f}] = [N^{k+n}, b, \bar{b}].$$

# Equivariant framed bordism

Let  $\sigma = \mathbb{R}$  with the sign action.

## Lemma

$$\Omega_{k+n}(\mathbb{R}P^\infty; -n\gamma) \cong \Omega_{k+n\sigma}^{\mathbb{Z}/2}$$

is the bordism group of pairs  $(M, \bar{c})$  where

- ▶  $M^{k+n}$  has free  $\mathbb{Z}/2$ -action
- ▶  $\bar{c}: \tau_M \cong_s \epsilon^{k+n\sigma}$  is  $\mathbb{Z}/2$ -equivariant.

## Proof.

Pull  $f: M \rightarrow \mathbb{R}P^\infty$  back along  $\mathbb{Z}/2$ -cover  $S^\infty \rightarrow \mathbb{R}P^\infty$ . Forget structure map to  $S^\infty$ . Let  $\bar{c}$  be negative of pullback of  $\bar{f}$ . □

# Equivariant $n$ -sphere

The class  $[pr] \in \pi_{-1}(P_{-n-1})$  corresponds to

$$\begin{aligned}\Omega_n(\mathbb{R}P^\infty; -(n+1)\gamma) &\cong \Omega_{-1+(n+1)\sigma}^{\mathbb{Z}/2} \\ [\mathbb{R}P^n, f, \bar{f}] &\leftrightarrow [S((n+1)\sigma), \bar{c}]\end{aligned}$$

where

- ▶  $S((n+1)\sigma) = S^n$  with antipodal action
- ▶  $\bar{c}: \tau_{S((n+1)\sigma)} \oplus \epsilon^1 \cong \epsilon^{(n+1)\sigma}$  is  $\mathbb{Z}/2$ -equivariant.

# Equivariant interpretation of root invariant

$$\begin{array}{ccccccc}
 & & [M^k, -\bar{a}] & \Omega_k^{fr} & & & \\
 & & & \downarrow \text{pr} & & & \\
 & [N^{k+n}, -\bar{b}] & & & & & \\
 \Omega_{k+n\sigma}^{\mathbb{Z}/2} & \xrightarrow{t} & \Omega_{k+n}^{fr} & \xrightarrow{j} & \Omega_{k-1+(n+1)\sigma}^{\mathbb{Z}/2} & \xrightarrow{p} & \Omega_{k-1+n\sigma}^{\mathbb{Z}/2}
 \end{array}$$

- ▶  $\text{pr}: [M^k, -\bar{a}] \mapsto [M^k \times S((n+1)\sigma), \bar{a} \times \bar{c}]$
- ▶  $j: [N^{k+n}, -\bar{b}] \mapsto [N \sqcup N, \bar{b} \sqcup \bar{b}]$

$$\bar{b} \sqcup \bar{b}: \tau_{N \sqcup N} \cong_s \epsilon^{k-1} \oplus \epsilon^{(n+1)\sigma}$$

- ▶  $R([M^k, -\bar{a}]) = [N^{k+n}, -\bar{b}] + \text{Im } t$  if

$$0 \neq [M^k \times S((n+1)\sigma), \bar{a} \times \bar{c}] = [N^{k+n} \sqcup N^{k+n}, \bar{b} \sqcup \bar{b}].$$

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# Existence of $\mathbb{Z}/2$ -action

## Theorem (Stolz (1988))

Let  $(\Sigma^k, \bar{a})$  and  $(\Sigma^{k+n}, \bar{b})$  be framed homotopy spheres, with bordism classes  $\alpha \in \pi_k^S$  and  $\beta \in \pi_{k+n}^S$ , such that

1.  $\text{pr}(\alpha) = j(\beta)$  in  $\pi_{k-1}(P_{-n-1})$
2.  $n \geq k + 1$  and  $L_{k+n}(\mathbb{Z}[\mathbb{Z}/2], (-1)^n) = 0$ .

Then there exists a smooth  $\mathbb{Z}/2$ -action on  $\Sigma^{k+n} \# \Sigma'$ , for some  $[\Sigma'] \in bP_{k+n+1}$ , with fixed point set  $\Sigma^k$ .

## Remark

Let  $\beta' \in R(\alpha) \subset \pi_{k+n'}^S$ . The first condition is then equivalent to

$$\begin{cases} \beta \in R(\alpha) & \text{if } n = n', \\ \beta \in \text{Im } t & \text{if } n < n'. \end{cases}$$

## Proof (1/5)

1. Since  $\text{pr}(\alpha) = j(\beta)$  we can choose a  $\mathbb{Z}/2$ -equivariant  $k + (n + 1)\sigma$ -framed bordism

$$(W^{k+n+1}, \bar{w}): (\Sigma^k \times S((n+1)\sigma), \bar{a} \times \bar{c}) \sim (\Sigma^{k+n} \sqcup \Sigma^{k+n}, \bar{b} \sqcup \bar{b}).$$

2. Choose  $\mathbb{Z}/2$ -equivariant map  $s: W \rightarrow \sigma$ , transverse to 0, with
  - ▶  $s|_{\Sigma^k \times S((n+1)\sigma)}$  projecting to the last coordinate
  - ▶  $s|_{\Sigma^{k+n} \sqcup \Sigma^{k+n}}$  mapping to  $\{\pm 1\}$ .
3. Let  $V^{k+n} = s^{-1}(0)$  be  $k + n\sigma$ -framed. Then

$$\partial V = \Sigma^k \times S(n\sigma)$$

with  $k - 1 + n\sigma$ -framing  $\bar{a} \times \bar{c}'$ .

$$\sum^k x S(n\sigma) = dV$$

$$\sum^k x D_+(n\sigma)$$

$$\sum^k x S((n+1)\sigma)$$

$$W^{k+n+1}$$

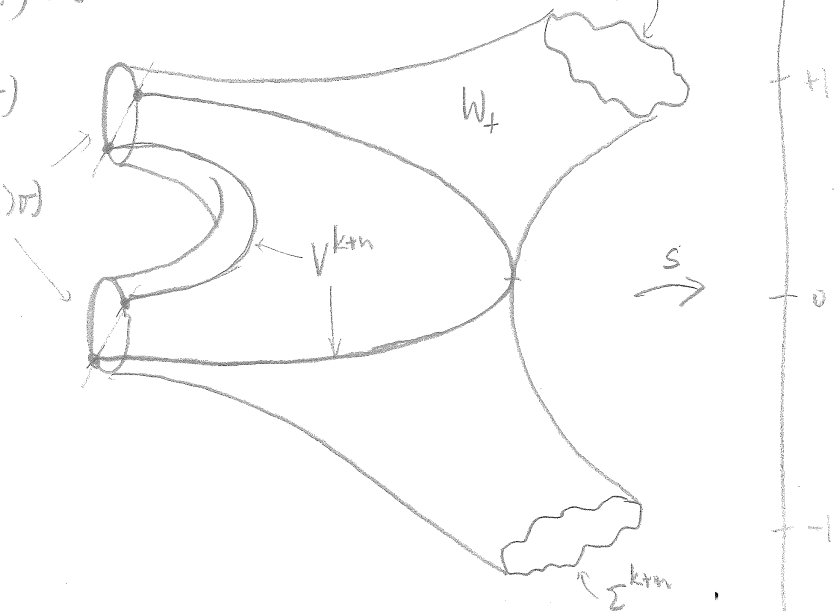
$$\Sigma^{k+n}$$

$$W_+$$

$$V^{k+n}$$

$$S$$

$$\Sigma^{k+n}$$



## Proof (2/5)

4. The orbit manifold  $\bar{V}^{k+n} = V/(\mathbb{Z}/2)$  is normally  $-n\gamma$ -twisted, with

$$\partial \bar{V} = \Sigma^k \times \mathbb{R}P^{n-1}.$$

5. Split  $\Sigma^k \cong D_0^k \cup_g D_1^k$  for some  $g: \partial D_0^k \cong \partial D_1^k$ .  
6. View  $\bar{V}$  as a  $-n\gamma$ -twisted cobordism

$$\bar{V}: D_0^k \times \mathbb{R}P^{n-1} \sim D_1^k \times \mathbb{R}P^{n-1}$$

relative to  $g \times \text{id}: \partial D_0^k \times \mathbb{R}P^{n-1} \cong \partial D_1^k \times \mathbb{R}P^{n-1}$ .

## Proof (3/5)

7. Kreck (1999) gives a version of surgery theory for “normal  $q$ -smoothings”, showing that surgery relative to the boundary can change  $\bar{V}$  to a relative  $h$ -cobordism  $\bar{V}_1$  if an obstruction in

$$L_{k+n}(\mathbb{Z}[\mathbb{Z}/2], (-1)^n) \quad (= 0)$$

vanishes. This assumes  $k + n \geq 5$  and  $n \geq k + 1$ .

8. Let  $\bar{W}_0: \bar{V} \sim \bar{V}_1 \text{ rel } \partial$  be the trace of this  $-n\gamma$ -twisted surgery, so that  $\mathbb{R}P^{n-1} \simeq \bar{V}_1$ .
9. The universal cover  $W_0: V \sim V_1 \text{ rel } \partial$  is the trace of a framed surgery, with  $S^{n-1} \simeq V_1$ .

## Proof (4/5)

10. Let

$$W_+^{n+k+1} \subset W \quad \text{and} \quad D_+(n\sigma) \subset S((n+1)\sigma)$$

be the parts mapping by  $s$  to  $[0, \infty) \subset \sigma$ . Then

$$W_+ : \Sigma^k \times D_+(n\sigma) \cup V \sim \Sigma^{k+n}$$

is a framed bordism.

11. The union of bordisms

$$(\text{id} \cup W_0) \cup W_+ : \Sigma^k \times D_+(n\sigma) \cup V_1 \sim \Sigma^{k+n}$$

is also a framed bordism.

12. The union

$$\begin{array}{ccc} \Sigma^k \times S(n\sigma) = \partial V_1 & \xrightarrow{\quad} & V_1 \simeq S^{n-1} \\ \downarrow & & \downarrow \\ \Sigma^k \times D_+(n\sigma) & \xrightarrow{\quad} & \Sigma^k \times D_+(n\sigma) \cup V_1 \end{array}$$

is a homotopy  $k + n$ -sphere.

## Proof (5/5)

13. The framed homotopy spheres  $\Sigma^k \times D_+(n\sigma) \cup V_1$  and  $\Sigma^{k+n}$  represent the same element in  $\pi_{k+n}^S$ .
14. By Kervaire–Milnor, there is an  $\Sigma'$  with  $[\Sigma'] \in bP_{k+n+1}$  and an  $h$ -cobordism from  $\Sigma^k \times D_+(n\sigma) \cup V_1$  to  $\Sigma^{k+n} \# \Sigma'$ .
15. By Smale, there is a diffeomorphism

$$h: \Sigma^k \times D_+(n\sigma) \cup V_1 \cong \Sigma^{k+n} \# \Sigma'.$$

16. The trivial action on  $\Sigma^k$ , the sign action on  $D_+(n\sigma) \cong D(n\sigma)$  and the free action on  $V_1$  combine to a smooth  $\mathbb{Z}/2$ -action on  $\Sigma^k \times D_+(n\sigma) \cup V_1$  with fixed points  $\Sigma^k$ . Conjugate by  $h$  to transport this action to  $\Sigma^{k+n} \# \Sigma'$ .



# Example

## Theorem (Behrens, Quigley)

*Each of the*

$$1, 28, 2, 992$$

*oriented homotopy spheres*

$$S^3, \Sigma^7, \Sigma^9, \Sigma^{11}$$

*that bound parallelizable manifolds arises, respectively, as the fixed points of some smooth involution on an exotic sphere of dimension*

$$9, 17, 21, 33$$

*that admits a framing with bordism class*

$$\nu^3 \in R(\eta^3), \eta\eta^* \in R(4\sigma), \nu\nu^* \in R(\eta^2\sigma), \eta\{q\} \in R(\eta^2\mu).$$

$$E_\infty \Longrightarrow \pi_*(S)_2^\wedge$$

