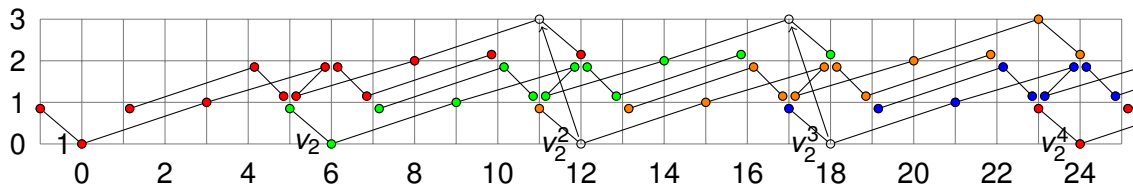


The Hahn–Raksit–Wilson Even Filtration

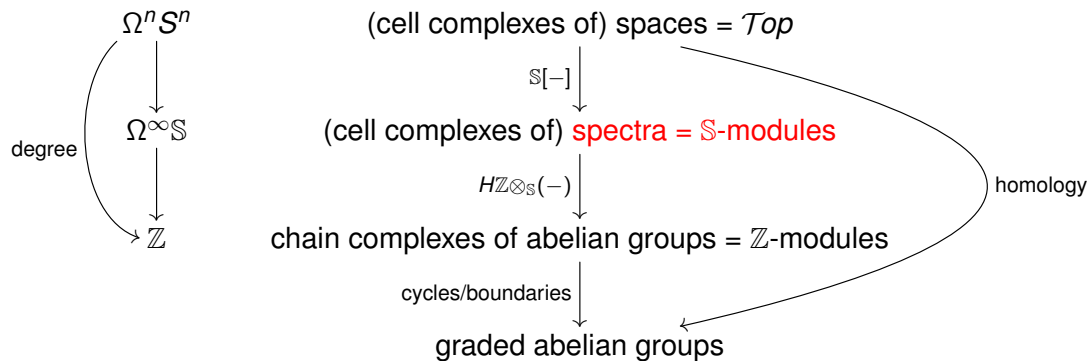
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Stable homotopy theory



Spectra

- ▶ $(\mathcal{S}p, \mathbb{S}, \otimes)$ is a stable symmetric monoidal ∞ -category.
- ▶ Abelian homotopy groups $\pi_* X = [\Sigma^* \mathbb{S}, X]$ for $* \in \mathbb{Z}$.
- ▶ $\mathrm{Ho}(\mathcal{S}p)$ triangulated by cofiber sequences.
- ▶ t-structure $(\tau_{\geq 0} \mathcal{S}p, \tau_{\leq 0} \mathcal{S}p)$ with X **connective** iff $\pi_* X = 0$ for $* < 0$.
- ▶ Equivalence

$$\mathcal{S}p^{\heartsuit} = \tau_{\geq 0} \mathcal{S}p \cap \tau_{\leq 0} \mathcal{S}p \xrightarrow{\pi_0} \mathcal{A}b = \text{abelian groups}$$

of abelian categories, taking HR in **heart** to R .

- ▶ Endomorphism rings

$$\begin{aligned} [\Sigma^* \mathbb{S}, \mathbb{S}] &= \pi_* \mathbb{S} = (\mathbb{Z}\{1\}, \mathbb{Z}/2\{\eta\}, \mathbb{Z}/2\{\eta^2\}, \dots) \\ [H\mathbb{F}_p, \Sigma^* H\mathbb{F}_p] &= \mathcal{A}^* = \mathbb{F}_p\{1, \beta, P^1, \dots\} \end{aligned}$$

are the **stable homotopy groups of spheres** and the **mod p Steenrod algebra**.

Nilpotence and periodicity

- ▶ Self map $p: \mathbb{S} \rightarrow \mathbb{S}$ is periodic. Cofiber $\mathbb{S}/p$ has type 1: it admits a periodic self map $v_1: \Sigma^{2p-2}\mathbb{S}/p \rightarrow \mathbb{S}/p$ for p an odd prime, and $v_1^4: \Sigma^8\mathbb{S}/2 \rightarrow \mathbb{S}/2$ for $p = 2$.
- ▶ **Hopkins–Smith (1998)**: Inductively, any p -local finite spectrum $F(n)$ of **type n** admits an essentially unique periodic self map

$$v_n^e: \Sigma^{(2p^n-2)e}F(n) \longrightarrow F(n),$$

and its cofiber $F(n)/(v_n^e)$ has **type $n + 1$** .

- ▶ For each $0 < n < \infty$, the type $\geq n$ spectra form a thick subcategory of $\mathcal{S}p_{(p)}^\omega$, and this is the complete list of proper nontrivial such.
- ▶ If $e = 1$ in each case, $F(n+1) = \mathbb{S}/(p, v_1, \dots, v_n) = V(n)$ is called a Smith–Toda complex.
- ▶ Example: $V(0) = \mathbb{S}/p$ for all p , $V(1) = \mathbb{S}/(p, v_1)$ for p odd.

$\mathbb{E}_1$ -rings and $\mathbb{E}_\infty$ -rings

- ▶ An $\mathbb{E}_1$ -ring is a spectrum A with a unital and associative pairing $\mu: A \otimes A \rightarrow A$.
- ▶ If μ is also commutative, then A is an $\mathbb{E}_\infty$ -ring.
- ▶ The Eilenberg–MacLane spectrum HR of an ordinary commutative ring R is an $\mathbb{E}_\infty$ -ring.
- ▶ The sphere spectrum $\mathbb{S}$ is an $\mathbb{E}_\infty$ -ring.
- ▶ The homology theory

$$A_*(X) = \pi_*(A \wedge X_+)$$

and cohomology theory

$$A^*(X) = \pi_{-*} \operatorname{Map}(X_+, A)$$

represented by an $\mathbb{E}_1$ -ring A have products, which are graded commutative (with power operations) if A is an $\mathbb{E}_\infty$ -ring.

Topological K -theory

- ▶ Isomorphism classes of real or complex vector bundles over a given base space X form a semiring under fiberwise direct sum and tensor product.
- ▶ Group completion leads to the multiplicative cohomology theories $KO^*(X)$ and $KU^*(X)$ called **real** and **complex** K -theory.
- ▶ Represented by $\mathbb{E}_\infty$ -rings KO and KU .
- ▶ Bott periodicity: $\pi_* KU = \mathbb{Z}[u^{\pm 1}]$ with $|u| = 2$ and

$$\pi_* KO = \frac{\mathbb{Z}[\eta, \alpha, \beta^{\pm 1}]}{(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)}$$

with $|\eta| = 1$, $|\alpha| = 4$, $|\beta| = 8$.

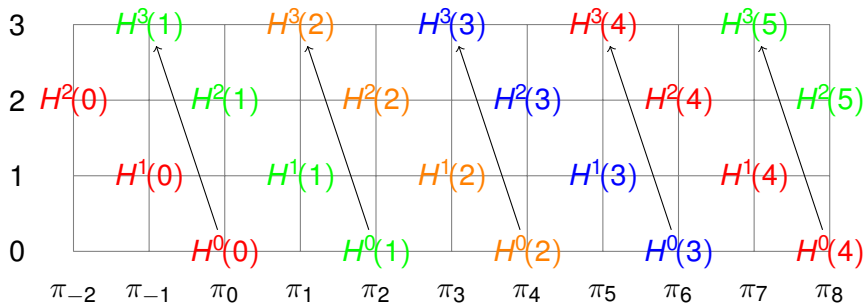
- ▶ Connective covers $ko = \tau_{\geq 0} KO$ and $ku = \tau_{\geq 0} KU$.
- ▶ ku , KU are **even spectra**, with $\pi_* = 0$ for $*$ odd, while ko , KO are not.

Atiyah–Hirzebruch spectral sequence

- Double-speed Whitehead filtration for ku , with $\pi_{2w}(ku) = \mathbb{Z}\{u^w\}$, gives Atiyah–Hirzebruch spectral sequence

$$E_2^{2w-s,s} = H^s(X; \mathbb{Z}\{u^w\}) \implies \pi_{2w-s} \text{Map}(X_+, ku)$$

in Adams grading



where $H^s(w) := H^s(X; \mathbb{Z}\{u^w\})$ and $\pi_n := \pi_n \text{Map}(X_+, ku)$.

Algebraic K -theory

- ▶ $Sp^{\omega, \simeq}$ is a space = ∞ -groupoid of finite spectra and π_* -isomorphisms.
- ▶ Making each cofiber sequence appear split produces a group completion $\Omega^\infty K(\mathbb{S})$ of this space.
- ▶ For any $\mathbb{E}_1$ -ring A , a group completion of finite (or perfect) A -modules and π_* -isomorphisms gives $\Omega^\infty K(A)$; here $K(A)$ is the **algebraic K -theory spectrum** of A .
- ▶ **Waldhausen (1979)**: $K(A)$ for $A = \mathbb{S}[\Omega X]$ controls parametrized classification and automorphism groups of high-dimensional manifolds.
- ▶ If A is an $\mathbb{E}_\infty$ -ring, then $K(A)$ is also an $\mathbb{E}_\infty$ -ring.
- ▶ **Suslin (1983)**: $V(0) \otimes K(\bar{F}) \simeq V(0) \otimes ku$ for fields F with $\text{char } F \neq p$.

Descent spectral sequence

- ▶ If $A \rightarrow B$ is a G -Galois extension of commutative rings, then $A = B^{hG}$ is the limit of a cosimplicial diagram

$$B \rightrightarrows B \otimes_A B \rightrightarrows \cdots \quad \text{isomorphic to} \quad B \rightrightarrows \prod_G B \rightrightarrows \cdots$$

- ▶ If the map from $K(A) = K(B^{hG})$ to the limit $K(B)^{hG}$ of

$$K(B) \rightrightarrows \prod_G K(B) \rightrightarrows \cdots$$

were an equivalence, then algebraic K -theory would satisfy **Galois descent**.

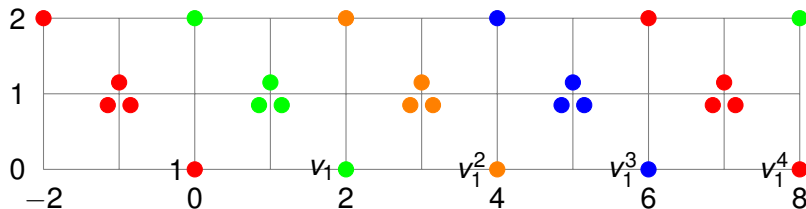
- ▶ With $G_F = \text{Gal}(\bar{F}/F)$ the absolute Galois group there is a descent spectral sequence

$$\begin{aligned} E_2^{2w-s,s} &= H_{\text{Gal}}^s(F; \mathbb{Z}/p(w)) := H_{\text{cont}}^s(G_F; \mathbb{Z}/p\{u^w\}) \\ &\implies V(0)_{2w-s} K(\bar{F})^{hG_F}. \end{aligned}$$

Example: The 2-adic number field $\mathbb{Q}_2$



$$H_{\text{Gal}}^s(\mathbb{Q}_2; \mathbb{Z}/2(w)) = \begin{cases} \mathbb{Z}/2 & \text{for } s \in \{0, 2\} \\ \mathbb{Q}_2^\times / (\mathbb{Q}_2^\times)^2 \cong (\mathbb{Z}/2)^3 & \text{for } s = 1 \\ 0 & \text{else.} \end{cases}$$



$$E_2^{2w-s,s} = H_{\text{Gal}}^s(\mathbb{Q}_2; \mathbb{Z}/2(w)) \implies V(0)_{2w-s} K(\bar{\mathbb{Q}}_2)^{hG_{\mathbb{Q}_2}}$$

► Does not quite converge to $V(0)_* K(\mathbb{Q}_2)$.

Motivic cohomology

- ▶ [Morel–Voevodsky \(2001\)](#): Define an $\mathbb{A}^1$ -homotopy type using the Nisnevich topology, intermediate between Zariski and étale.
- ▶ [Voevodsky et al.](#): Get a motivic spectral sequence

$$E_2^{2w-s,s} = H_{\text{mot}}^s(F; \mathbb{Z}/p(w)) \implies V(0)_{2w-s}K(F)$$

with

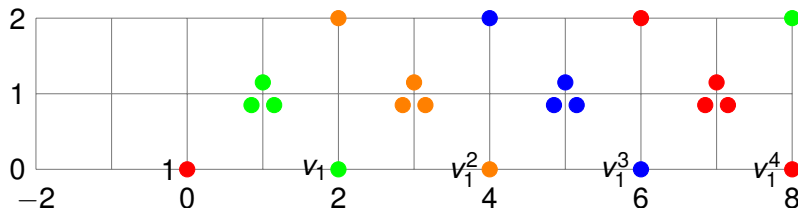
$$H_{\text{mot}}^s(F; \mathbb{Z}/p(w)) = \begin{cases} H_{\text{Gal}}^s(F; \mathbb{Z}/p(w)) & \text{for } 0 \leq s \leq w, \\ 0 & \text{else.} \end{cases}$$

- ▶ The fiber of $V(0) \otimes K(F) \rightarrow v_1^{-1}V(0) \otimes K(F)$ is often truncated from above, e.g. for F a local or global number field.
- ▶ Confirms conjectures by Lichtenbaum–Quillen, Milnor, Bloch–Kato and Beilinson–Lichtenbaum.

Example: The 2-adic number field $\mathbb{Q}_2$



$$H_{\text{mot}}^s(\mathbb{Q}_2; \mathbb{Z}/2(w)) = \begin{cases} H_{\text{Gal}}^s(\mathbb{Q}_2; \mathbb{Z}/2(w)) & \text{for } 0 \leq s \leq w \\ 0 & \text{else.} \end{cases}$$

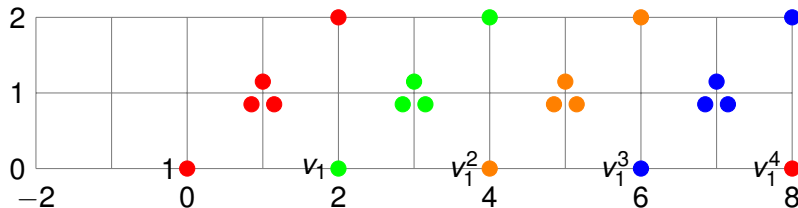


$$E_2^{2w-s,s} = H_{\text{mot}}^s(\mathbb{Q}_2; \mathbb{Z}/2(w)) \implies V(0)_{2w-s}K(\mathbb{Q}_2)$$

Example: The 2-adic number field $\mathbb{Q}_2$



$$H_{\text{mot}}^s(\mathbb{Q}_2; \mathbb{Z}/2(w)) = \mathbb{F}_2[v_1] \otimes \bigoplus_{s \geq 0} H_{\text{mot}}^s(\mathbb{Q}_2; \mathbb{Z}/2(s))$$



$$E_2^{2w-s,s} = H_{\text{mot}}^s(\mathbb{Q}_2; \mathbb{Z}/2(w)) \implies V(0)_{2w-s} K(\mathbb{Q}_2)$$

Height of $\mathbb{E}_\infty$ -rings

- ▶ Let $T(n) = (v_n^e)^{-1}F(n)$ be the mapping telescope.
- ▶ [Hahn \(2016\)](#): For any $\mathbb{E}_\infty$ -ring A , if $T(n)_*A = 0$, then $T(n+1)_*A = 0$.
- ▶ Definition: $\text{height}(A) = \max\{n \mid T(n)_*A \neq 0\}$.
- ▶ For F a local or global number field, HF has height 0.
- ▶ $K(F)$ then has height 1, by Lichtenbaum–Quillen.
- ▶ ku , ko and Adams summand $BP\langle 1 \rangle$ have height 1.
- ▶ Elliptic cohomology and tmf have height 2.

Algebraic K -theory of topological K -theory

- ▶ Ausoni–R. (2002): Calculate $V(1)_*K(BP\langle 1 \rangle)$ for $p \geq 5$;
- ▶ Shows that $K(BP\langle 1 \rangle)$ has height 2.
- ▶ Fiber of $V(1) \otimes K(BP\langle 1 \rangle) \rightarrow v_2^{-1}V(1) \otimes K(BP\langle 1 \rangle)$ is truncated.
- ▶ Proof uses Bökstedt–Hsiang–Madsen (1993) cyclotomic trace map $K \rightarrow TC$ from algebraic K -theory to topological cyclic homology, and Dundas–Goodwillie–McCarthy (1997) theorem.
- ▶ Nikolaus–Scholze (2018) presentation:

$$TC(A) = \text{eq}(\text{can}, \varphi: TC^-(A) \rightarrow TP(A))$$

with $TC^-(A) = THH(A)^{h\mathbb{T}}$ and $TP(A) = THH(A)^{t\mathbb{T}}$.

Redshift conjecture

- ▶ R. (Oberwolfach, 2000):

$$\text{height } K(A) = \text{height}(A) + 1$$

for $\mathbb{E}_\infty$ -rings (and “suitable” $\mathbb{E}_1$ -rings) A .

- ▶ Slogan: Algebraic K -theory ($\approx$ group completed passage to module categories) turns v_n -periodicity into v_{n+1} -periodicity.
- ▶ Lichtenbaum–Quillen for some A with $\text{height}(A) = 0$.
- ▶ Ausoni–R. (2002) for some A with $\text{height}(A) = 1$.
- ▶ Proven for all $\mathbb{E}_\infty$ -rings A by Land–Mathew–Meier–Tamme (2020), Clausen–Mathew–Naumann–Noel (2020), Yuan (2021) and Burklund–Schlank–Yuan (2022).

Telescope conjecture

- ▶ Comparable Bousfield localizations

$$M \rightarrow L_{T(n)}M \xrightarrow{\lambda_n} L_{K(n)}M$$

- ▶ Easy: λ_0 is an equivalence.
- ▶ [Mahowald \(1981\)](#), [Miller \(1981\)](#): λ_1 is an equivalence.
- ▶ Ravenel (Northwestern, 1977) telescope conjecture: λ_n equivalence for all n .
- ▶ [Burklund–Hahn–Levy–Schlank \(2023\)](#): This fails for all $n \geq 2$.
- ▶ Counterexample: Adams operation ψ^{p+1} generates a $\mathbb{Z}$ -action on $BP\langle 1 \rangle$, such that

$$\begin{aligned} L_{K(2)}K(BP\langle 1 \rangle^{h\mathbb{Z}}) &\simeq L_{K(2)}K(BP\langle 1 \rangle)^{h\mathbb{Z}} \\ L_{T(2)}K(BP\langle 1 \rangle^{h\mathbb{Z}}) &\not\simeq L_{T(2)}K(BP\langle 1 \rangle)^{h\mathbb{Z}}. \end{aligned}$$

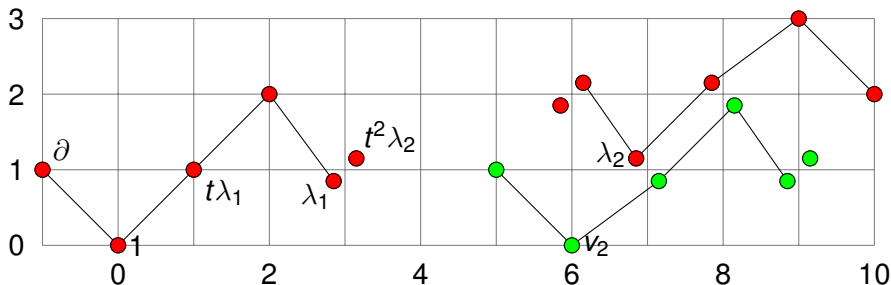
- ▶ Proof uses $T(2)_*K(BP\langle 1 \rangle) \neq 0$, from [Ausoni–R. \(2002\)](#).

Arithmetic cohomology of $\mathbb{E}_\infty$ -rings

- ▶ $v_2^{-1} V(1)_* K(BP\langle 1 \rangle)$ free $\mathbb{F}_p[v_2^{\pm 1}]$ -module on

$$\Lambda(\partial, \lambda_1, \lambda_2) \oplus \Lambda(\lambda_2) \{t^d \lambda_1\}_{d=1}^{p-1} \oplus \Lambda(\lambda_1) \{t^{dp} \lambda_2\}_{d=1}^{p-1}$$

- ▶ Apparent bigrading with generators in $(-1, 1)$, $(2p-1, 1)$, $(2p^2-1, 1)$, etc.
- ▶ Hypothetical case $p=2$, with $BP\langle 1 \rangle = ku$:



- ▶ R. (Wuppertal, 2005): Is this an étale cohomology for $\mathbb{E}_\infty$ -rings?
- ▶ R. (Seoul ICM, 2014): Is the corresponding free $\mathbb{F}_p[v_2]$ -module a motivic cohomology for $\mathbb{E}_\infty$ -rings?

Even filtration

- ▶ Even spectra B have double-speed Whitehead filtration $\tau_{\geq 2*} B$ with associated graded $\Sigma^{2*} H\pi_{2*} B$.
- ▶ If B is also an $\mathbb{E}_\infty$ -ring, then $\tau_{\geq 2*} B$ is an $\mathbb{E}_\infty$ -algebra in $\mathbb{Z}$ -filtered spectra and $\Sigma^{2*} H\pi_{2*} B$ is an $\mathbb{E}_\infty$ -algebra in $\mathbb{Z}$ -graded spectra.
- ▶ [Hahn–Raksit–Wilson \(2022\)](#) right Kan extend this filtration from even $\mathbb{E}_\infty$ -rings to all $\mathbb{E}_\infty$ -rings A .
- ▶ Define weight w term of **even filtration** to be

$$\mathrm{fil}_{\mathrm{ev}}^w A := \lim_{\substack{\mathbb{E}_\infty\text{-maps } A \rightarrow B \\ \text{with } B \text{ even}}} \tau_{\geq 2w} B.$$

- ▶ $\mathrm{fil}_{\mathrm{ev}}^* A$ is an $\mathbb{E}_\infty$ -algebra in filtered spectra.
- ▶ Cofiber sequence $\mathrm{fil}_{\mathrm{ev}}^{w+1} A \rightarrow \mathrm{fil}_{\mathrm{ev}}^w A \rightarrow \mathrm{gr}_{\mathrm{ev}}^w A$ defines weight w **associated graded** of even filtration.
- ▶ $\mathrm{gr}_{\mathrm{ev}}^* A$ is an $\mathbb{E}_\infty$ -algebra in graded spectra.

Evenly faithfully flat descent

- ▶ $\mathbb{E}_\infty$ -map $A \rightarrow B$ is **evenly faithfully flat** if for any $\mathbb{E}_\infty$ -map $A \rightarrow E$ with E even, the base change $B \otimes_A E$ is even and faithfully flat as an E -module.
- ▶ Cosimplicial resolution $C^\bullet(B/A): [q] \mapsto \overbrace{B \otimes_A \cdots \otimes_A B}^{1+q}$
- ▶ Descent: If $A \rightarrow B$ is evenly faithfully flat, then

$$\mathrm{fil}_{\mathrm{ev}}^* A \simeq \lim_{[q] \in \Delta} \mathrm{fil}_{\mathrm{ev}}^* C^q(B/A)$$

- ▶ If B is also even, then

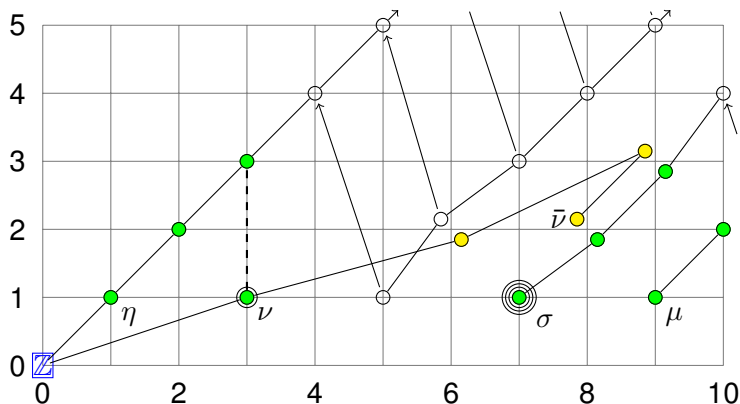
$$\mathrm{fil}_{\mathrm{ev}}^* A \simeq \lim_{[q] \in \Delta} \tau_{\geq 2*} C^q(B/A)$$

$$\mathrm{gr}_{\mathrm{ev}}^* A \simeq \lim_{[q] \in \Delta} \Sigma^{2*} H\pi_{2*} C^q(B/A)$$

Novikov spectral sequence

- ▶ MU complex bordism spectrum, with $\pi_* MU$ and $H_* MU$ both even.
- ▶ $\mathbb{S} \rightarrow MU$ is an evenly faithfully flat map to an even $\mathbb{E}_\infty$ -ring.
- ▶ $[q] \mapsto \pi_* C^q(MU/\mathbb{S})$ cobar complex for $(MU_* MU, MU_*)$
- ▶

$$E_2^{**} = \pi_* \operatorname{gr}_{\operatorname{ev}}^* \mathbb{S} = \operatorname{Ext}_{MU_* MU}^{*,*}(MU_*, MU_*) \implies \pi_* \mathbb{S}$$

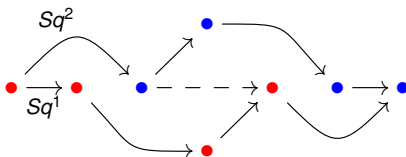


$\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}\text{-modules}$

- ▶ $A \mapsto \mathrm{gr}_{\mathrm{ev}}^* A$ takes values in the category of $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -modules.
- ▶ This has a t -structure with heart equivalent to even $MU_* MU$ -comodules.
- ▶ Get $\mathbb{E}_\infty$ -algebras in $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -modules $V(n)^*$ realizing $MU_*/(p, v_1, \dots, v_n)$, even if $V(n)$ does not exist in $\mathcal{S}p$.
- ▶ Cofiber sequence in $\mathrm{gr}_{\mathrm{ev}}^* \mathbb{S}$ -modules

$$\Sigma^{1,1} V(1)^* \xrightarrow{\eta} V(1)^* \longrightarrow \mathrm{gr}_{\mathrm{ev}}^* A(1) \longrightarrow \Sigma^{2,0} V(1)^* \quad (*)$$

where $A(1)$ is a spectrum with $H^* A(1) = \langle Sq^1, Sq^2 \rangle \subset \mathcal{A}^*$.



- ▶ No product structure on $A(1)$.

p -completely quasisyntomic rings

- ▶ Suppressing p -completion, bounded p -power torsion, etc.
- ▶ A commutative ring A is **quasisyntomic** if the cotangent complex $L_{A/\mathbb{Z}}$ has Tor-amplitude in homological degrees $[0, 1]$.
- ▶ Example: $A =$ quotient of a polynomial ring by a regular sequence.
- ▶ **Bhatt–Morrow–Scholze (2019)**: Each quasisyntomic A admits a quasisyntomic cover $A \rightarrow B$ with B quasiregular semiperfectoid.
- ▶ Each $C^q(B/A)$ is then also quasiregular semiperfectoid.

Trace invariants of quasiregular semiperfectoid rings

- ▶ Suppose $A \rightarrow B$ is a quasiregular semiperfectoid cover.
- ▶ $THH(A) \rightarrow THH(B)$ is evenly faithfully flat, with $THH(B)$ even.
- ▶ $TC^-(B) = THH(B)^{h\mathbb{T}}$ and $TP(B) = THH(B)^{t\mathbb{T}}$ are even.
- ▶ [Bhatt–Scholze \(2022\)](#):

$$TC(B) = \mathrm{eq}(\mathrm{can}, \varphi: TC^-(B) \rightarrow TP(B))$$

is even in “enough cases” (**odd vanishing conjecture**).

Prismatic and syntomic cohomology

- ▶ $\mathrm{fil}_{BMS}^w TP(A) = \lim_{[q] \in \Delta} \tau_{\geq 2w} TP(C^q(B/A))$
- ▶ The weight w **prismatic cohomology** of A is
 $\mathrm{gr}_{BMS}^w TP(A) = \lim_{[q] \in \Delta} \Sigma^{2w} H\pi_{2w} TP(C^q(B/A)).$
- ▶ $\mathrm{fil}_{BMS}^w TC(A) = \lim_{[q] \in \Delta} \tau_{\geq 2w} TC(C^q(B/A))$
- ▶ The weight w **syntomic cohomology** of A is
 $\mathrm{gr}_{BMS}^w TC(A) = \lim_{[q] \in \Delta} \Sigma^{2w} H\pi_{2w} TC(C^q(B/A)).$
- ▶ Get syntomic/motivic spectral sequence

$$E_2^{n,w} = \pi_n \mathrm{gr}_{BMS}^w TC(A) \implies \pi_n TC(A).$$

Motivic filtrations for $\mathbb{E}_\infty$ -rings

- ▶ Hahn–Raksit–Wilson (2022) define

$$\mathrm{fil}_{\mathrm{mot}}^w THH(A) := \mathrm{fil}_{\mathrm{ev}}^w THH(A) = \lim_{\substack{\text{maps } THH(A) \rightarrow B \\ \text{of } \mathbb{E}_\infty\text{-rings} \\ \text{with } B \text{ even}}} \tau_{\geq 2w} B$$



$$\mathrm{fil}_{\mathrm{mot}}^w TC^-(A) := \lim_{\substack{\text{maps } THH(A) \rightarrow B \\ \text{of } \mathbb{T}\text{-equivariant } \mathbb{E}_\infty\text{-rings} \\ \text{with } B \text{ even}}} \tau_{\geq 2w}(B^{h\mathbb{T}})$$



$$\mathrm{fil}_{\mathrm{mot}}^w TP(A) := \lim_{\substack{\text{maps } THH(A) \rightarrow B \\ \text{of } \mathbb{T}\text{-equivariant } \mathbb{E}_\infty\text{-rings} \\ \text{with } B \text{ even}}} \tau_{\geq 2w}(B^{t\mathbb{T}})$$

- ▶ $\mathrm{fil}_{\mathrm{mot}}^w TC(A) :=$

$$\lim_{\substack{\text{maps } THH(A) \rightarrow B \\ \text{of cyclotomic } \mathbb{E}_\infty\text{-rings} \\ \text{with } B \text{ even}}} \mathrm{eq}(\mathrm{can}, \varphi: \tau_{\geq 2w}(B^{h\mathbb{T}}) \rightarrow \tau_{\geq 2w}(B^{t\mathbb{T}}))$$

Prismatic and syntomic cohomology for $\mathbb{E}_\infty$ -rings

- ▶ Prismatic cohomology of A = homotopy groups of the spectrum

$$\mathrm{gr}_{\mathrm{mot}}^w TP(A) = \mathrm{fil}_{\mathrm{mot}}^w TP(A) / \mathrm{fil}_{\mathrm{mot}}^{w+1} TP(A)$$

- ▶ Syntomic cohomology of A = homotopy groups of the spectrum

$$\mathrm{gr}_{\mathrm{mot}}^w TC(A) = \mathrm{fil}_{\mathrm{mot}}^w TC(A) / \mathrm{fil}_{\mathrm{mot}}^{w+1} TC(A)$$

Chromatically quasisyntomic $\mathbb{E}_\infty$ -rings

- ▶ [Hahn–Raksit–Wilson \(2022\)](#): A connective $\mathbb{E}_\infty$ -ring A is **chromatically quasisyntomic** if MU_*A is even and is quasisyntomic as a ring.
- ▶ Examples: $\mathbb{S}$, MU , ko , ku , tmf and HR for quasisyntomic R .
- ▶ Theorem [HRW22]: For A chromatically quasisyntomic the motivic filtrations $\mathrm{fil}_{\mathrm{mot}}^\star(-)$ are exhaustive, and $\mathrm{fil}_{\mathrm{mot}}^W TC(A) = \mathrm{eq}(\mathrm{can}, \varphi: \mathrm{fil}_{\mathrm{mot}}^W TC^-(A) \rightarrow \mathrm{fil}_{\mathrm{mot}}^W TP(A))$.
- ▶ Get conditionally convergent **motivic spectral sequence**

$$E_2^{n, 2w-n} = \pi_n \mathrm{gr}_{\mathrm{mot}}^W TC(A) \implies \pi_n TC(A).$$

Real topological K -theory

- ▶ $H = H\mathbb{F}_2$ represents mod 2 (co-)homology.
- ▶ $H_*H = \mathcal{A}_* = \mathbb{F}_2[\zeta_k \mid k \geq 1]$ dual Steenrod algebra, $|\zeta_k| = 2^k - 1$.
- ▶ $H_*ko = \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_k \mid k \geq 3]$.
- ▶ $H_*THH(ko) = H_*ko \otimes \Lambda(\sigma\zeta_1^4, \sigma\zeta_2^2) \otimes \mathbb{F}_2[\sigma\zeta_3]$.
- ▶ $A(1)_*THH(ko) = \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu]$, $|\lambda'_1| = 5$, $|\lambda_2| = 7$, $|\mu| = 8$.

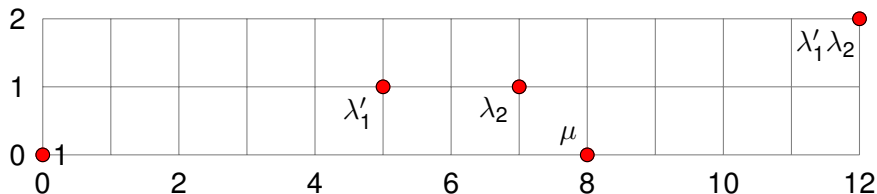
Motivic = even filtration of $THH(ko)$

- ▶ [Angelini-Knoll-Ausoni-R. \(2023\)](#): $THH(ko) \rightarrow THH(ku/MU)$ is an evenly faithfully flat map of $\mathbb{T}$ -equivariant $\mathbb{E}_\infty$ -rings with $THH(ku/MU)$ even.



$$A(1)_* \operatorname{gr}_{\operatorname{mot}}^* THH(ko) = \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu]$$

with $\|\lambda'_1\| = (5, 1)$, $\|\lambda_2\| = (7, 1)$, $\|\mu\| = (8, 0)$.



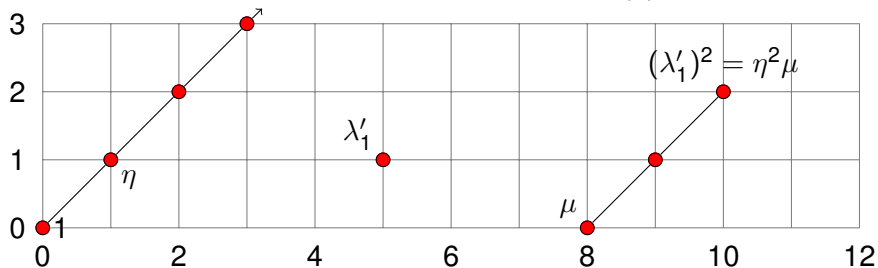
- ▶ Admits no cyclotomic product structure.

Multiplicative $V(1)^*$ -coefficients

- Use [Angeltveit–Hill–Lawson \(2010\)](#) to calculate

$$V(1)_*^* \operatorname{gr}_{\operatorname{mot}}^* THH(ko) = \frac{\mathbb{F}_2[\eta, \lambda'_1, \mu]}{(\eta \lambda'_1, (\lambda'_1)^2 = \eta^2 \mu)}$$

the η -Bockstein spectral sequence derived from (*).

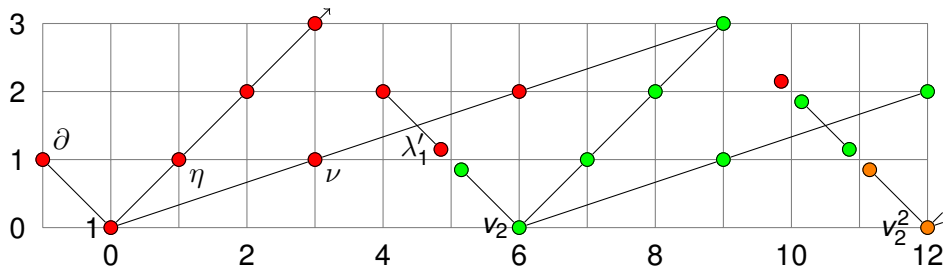


- Gain cyclotomic $\mathbb{E}_\infty$ -algebra structure, in $\operatorname{gr}_{\operatorname{ev}}^* \mathbb{S}$ -modules.

$V(1)^*$ -coefficient syntomic cohomology of ko

- ▶ Calculate $\mathbb{T}$ -Tate and $\mathbb{T}$ -homotopy fixed point algebra spectral sequences with $V(2)^*$ -coefficients
- ▶ Extract $V(2)^*_* \operatorname{gr}_{\operatorname{mot}}^* TC(ko)$ as equalizer
- ▶ Compare along $\mathbb{S} \rightarrow TC(ko)$ to resolve v_2 -Bockstein algebra spectral sequence converging to

$$V(1)^*_* \operatorname{gr}_{\operatorname{mot}}^* TC(ko) = \Lambda(\partial) \otimes \mathbb{F}_2[\eta, \nu, \lambda'_1, v_2]/(\sim)$$

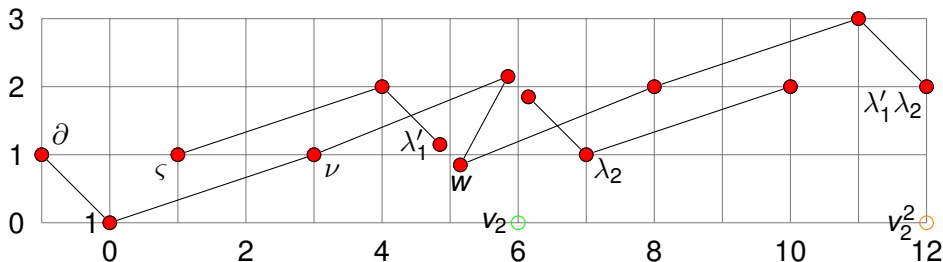


$A(1)$ -coefficient syntomic cohomology of ko

- Use $(*)$ to return to $A(1)$ -coefficients, obtaining

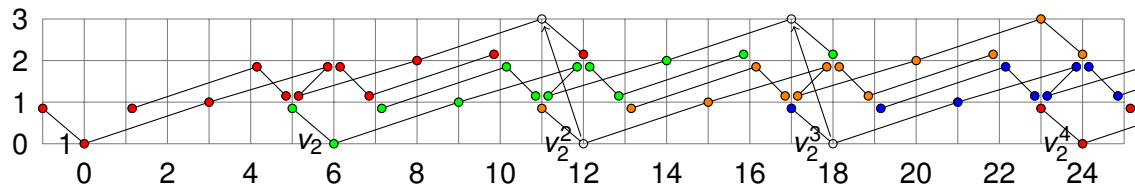
$$A(1)_* gr_{\text{mot}}^* TC(ko) =$$

$$\mathbb{F}_2[v_2] \otimes (\wedge(\partial, \lambda'_1)\{1, \lambda_2\} \oplus \mathbb{F}_2\{\varsigma, \nu, w, \nu^2, \nu w, \nu \lambda_2\})$$



Motivic spectral sequence

$$\blacktriangleright E_2^{**} = A(1)_* \operatorname{gr}_{\operatorname{mot}}^* TC(ko) \implies A(1)_* TC(ko)$$



$\blacktriangleright$ Comparison along $\mathbb{S} \rightarrow TC(ko)$ implies nonzero differentials

$$d_3(v_2^i) = v_2^{i-2} v^2 w$$

for $i \equiv 2, 3 \pmod{4}$.

$\blacktriangleright E_4 = E_\infty$ for degree reasons.

$\blacktriangleright$ Additive extensions depend on choice of spectrum $A(1)$.

Consequences

- ▶ $K(ko)$ has height 2 (redshift).
- ▶ $L_{T(2)}K(ko) \simeq L_{K(2)}K(ko)$ (telescope conjecture holds).
- ▶ $K(ko) \rightarrow L_2^f K(ko)$ has truncated fiber (higher Lichtenbaum–Quillen).
- ▶ There is no unital map $K(ko) \rightarrow tmf$.
- ▶ Apparent **arithmetic duality** in syntomic cohomology for ko and ku , similar to Poincaré duality for manifolds, and Poitou–Tate duality for Galois cohomology of number fields.