

EXPONENTIALS ON NON-SINGULAR SIMPLICIAL SETS

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ABSTRACT. A simplicial set is **non-singular** if the representing maps of its non-degenerate simplices are degree-wise injective. The category of simplicial sets has a **simplicial mapping set** X^K whose set of n -simplices are the simplicial maps $\Delta[n] \times K \rightarrow X$. We prove that X^K is non-singular whenever X is non-singular.

1. INTRODUCTION

There are times when one would like to know whether a category behaves similarly, in some sense, to the category of sets and functions. As an example, for homotopy-theoretical purpose the author would like to know whether the endofunctor $- \times \Delta[1]$ of non-singular simplicial sets preserves colimits. Here, $\Delta[1]$ denotes the standard 1-simplex.

Let $sSet$ denote the category of simplicial sets. The full subcategory $nsSet$ whose objects are the non-singular simplicial sets sits strictly between $sSet$ and the category of ordered simplicial complexes. Despite the fact that non-singular simplicial sets have a natural PL structure [WJR13, p. 126–127] they almost never appear in the literature, though they do play a role in the book *Spaces of PL Manifolds and Categories of Simple Maps* by Waldhausen, Jahren and Rognes [WJR13].

The endofunctor $(-)^K : sSet \rightarrow sSet$ is designed so that the Yoneda lemma makes it right adjoint to $- \times K$. Our main result is the following.

Theorem 1.1. *Let K be some simplicial set. Then X^K is non-singular whenever X is.*

Part of the author's interest in this result comes from the case when K non-singular. Then the restriction of $(-)^K$ to $nsSet$ corestricts to an endofunctor of non-singular simplicial sets. Moreover, $(-)^K$ viewed as a functor $nsSet \rightarrow nsSet$ is right adjoint to the endofunctor $- \times K$ of $nsSet$. This means that we can derive the following consequence of theorem 1.1.

Corollary 1.2. *Taking the product $- \times K : nsSet \rightarrow nsSet$ with a non-singular simplicial set K preserves colimits.*

In particular, taking the product $- \times \Delta[1]$ with an interval is a cocontinuous endofunctor of non-singular simplicial sets.

The case of the interval is not only of practical concern, but it is also the theoretical focus of this article as it is not hard to argue that theorem 1.1 follows from the following result.

Proposition 1.3. *The simplicial set $X^{\Delta[1]}$ is non-singular whenever X is.*

The proof of the latter result is the subject of section 4, whereas theorem 1.1 is derived from proposition 1.3 in section 3.

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In section 2, we will discuss applications of theorem 1.1 beyond corollary 1.2. We explain how theorem 1.1 follows from proposition 1.3 in section 3. Finally, the case of the interval is discussed section 4.

2. APPLICATIONS

The inclusion $U : nsSet \rightarrow sSet$ admits a left adjoint functor called desingularization [WJR13, Rem. 2.2.12., p. 39], which is denoted D . Note that the unit

$$\eta_X : X \rightarrow UDX$$

is degreewise surjective and that desingularization has the universal property that any simplicial map $f : X \rightarrow Y$ whose target Y is non-singular factors through the unit by a unique map $UDX \rightarrow Y$.

In general, we say that a full subcategory of some category is a **reflective subcategory** if the inclusion admits a left adjoint, which is then referred to as a **reflector**. Thus $nsSet$ is a reflective subcategory of $sSet$. Note that the word reflective is not quite standard terminology. For example, Mac Lane [ML98, §IV.3] Adámek and Rosický [AR15, p. 1306] do not include fullness as an assumption in their definition, although some other authors do. proposition 1.3 and its generalization theorem 1.1 has a noteworthy application and a couple of consequences.

?? establishes a model structure on $nsSet$ that is right-induced a la Thomason [Tho80] from $sSet$ equipped with the standard model structure due to Quillen [Qui67]. Moreover, the theorem says that (D, U) is a Quillen equivalence. proposition 1.3 is used as a technical ingredient in the proof of ??.

Another way to state theorem 1.1 is to say that the non-singular simplicial sets form an exponential ideal in $sSet$. The category of simplicial sets is cartesian closed and even a topos. Part of this is the fact that $(-)^K$ is right adjoint to $- \times K$. Here, the author has in mind the notions, terminology and notation from [ML98, §IV.6–§IV.10]. Note that the construction X^K is bifunctorial. A generalized result known as the parameter theorem ensures this [ML98, p. 102].

Corollary 2.1. *Desingularization preserves finite products.*

It seems that corollary 2.1 follows from Day’s reflection theorem [Day72, Thm. 1.2] and its corollary [Day72, Cor. 2.1]. Day’s reflection theorem concerns a more general setting, although he does refer to the condition that the *reflective subcategory is closed under exponentiation* [Day72, §0]. Another phrase that is used in the literature is that the non-singular simplicial sets form an *exponential ideal* in $sSet$, which is exactly the content of theorem 1.1.

In case one does not want to unravel the general form of Day’s reflection theorem, we provide the following elementary proof.

Proof of corollary 2.1. It is enough to consider two factors. Suppose X and Y simplicial sets.

Consider the map

$$Y \times X \xrightarrow{\eta_Y \times X} D(Y \times X).$$

Here, we omit the redundant symbol U for the inclusion functor. By theorem 1.1, the simplicial set $D(Y \times X)^X$ is non-singular, so we obtain a factorization

$$(1) \quad \begin{array}{ccc} Y & \xrightarrow{\eta_Y} & DY \\ & \searrow & \swarrow \text{dashed} \\ & D(Y \times X)^X & \end{array}$$

of the adjoint. Next, we switch the two factors of the adjoint

$$DY \times X \rightarrow D(Y \times X)$$

of the dashed map in (1) and factor the adjoint of the resulting map by means of the diagram

$$(2) \quad \begin{array}{ccc} X & \xrightarrow{\eta_X} & DX \\ & \searrow & \swarrow \text{dashed} \\ & D(X \times Y)^{DY} & \end{array}$$

in which the dashed map arises by theorem 1.1 as $D(X \times Y)^{DY}$ is non-singular.

By adjunction, we can combine (1) and (2) into the solid commutative diagram

$$(3) \quad \begin{array}{ccccc} X \times Y & \xrightarrow{id \times \eta_X} & X \times DY & & \\ \eta_Y \times x \searrow & & \swarrow & \eta_X \times id \searrow & \\ & D(X \times Y) & \xleftarrow{\quad} & DX \times DY & \\ & & \text{dashed} \xrightarrow{(D(pr_1), D(pr_2))} & & \end{array}$$

in which a dashed map arises because $DX \times DY$ is non-singular, being a product of non-singular simplicial sets. Indeed, the dashed map must be equal to the canonical map $(D(pr_1), D(pr_2))$ due to the universal property of desingularization.

Because the map $\eta_{X \times Y}$ is degreewise surjective and because (3) commutes, it follows immediately that

$$DX \times DY \rightarrow D(X \times Y)$$

is degreewise surjective.

Furthermore, by the universal property of desingularization, it follows that the composite

$$DX \times DY \rightarrow D(X \times Y) \xrightarrow{(D(pr_1), D(pr_2))} DX \times DY$$

is the identity. This implies that the first of the two maps of the composite is even degreewise injective, which implies that it is degreewise bijective and hence an isomorphism. In this way, we see that $(D(pr_1), D(pr_2))$ is degreewise bijective and hence an isomorphism. $\square$

Another consequence of theorem 1.1 is the following result.

Corollary 2.2. *The category of non-singular simplicial sets is cartesian closed.*

3. ARBITRARY EXPONENT

In this section we will prove theorem 1.1, assuming that proposition 1.3 holds. First we will point out that the latter result can be generalized fairly easily from the interval to the standard n -simplex, for all $n \geq 0$.

Lemma 3.1. *Suppose $n \geq 0$. The simplicial set $X^{\Delta[n]}$ is non-singular if X is.*

To verify lemma 3.1 we note that proposition 1.3 implies that $X^{\Delta[1]^n}$ is non-singular if X is. This is by induction on n , which is made possible by the exponential law $(X^K)^L \cong X^{L \times K}$, which holds because $sSet$ is cartesian closed.

Let $[n]$ denote the totally ordered set $\{0 < 1 < \dots < n\}$. Following [FP90, p. 132], we shall refer to an **operator** as a function $\alpha : [m] \rightarrow [n]$ such that $\alpha(i) \leq \alpha(j)$ if $i \leq j$. Observe that $\Delta[n]$ embeds in $\Delta[1]^n$ in such a way that

$\Delta[1]^n$ retracts onto $\Delta[n]$. The embedding i that we have in mind is induced by the operator

$$[n] \rightarrow [1]^n$$

given by

$$j \mapsto 1 \dots 10 \dots 0$$

where the string $1 \dots 10 \dots 0$ starts with j 1's and the rest are 0's. One can make a retraction $r : \Delta[1]^n \rightarrow \Delta[n]$ by taking the string $k_1 \dots k_n$ from $[1]^n$ and then finding the lowest index j such that $k_j = 0$. Then one defines an operator by the rule

$$k_1 \dots k_n \mapsto j - 1,$$

which induces the announced r . We get that the composite ri is the identity as this is true on the level of operators.

There are induced maps

$$X^{\Delta[n]} \xleftarrow{X^i} X^{\Delta[1]^n} \xleftarrow{X^r} X^{\Delta[n]}$$

such that the composite is equal to the identity. In other words, the simplicial set $X^{\Delta[n]}$ is identified with a simplicial subset of $X^{\Delta[1]^n}$, which is non-singular if X is. Hence, the simplicial set $X^{\Delta[n]}$ is non-singular if X is. This concludes our proof of lemma 3.1, given that proposition 1.3 holds.

By means of lemma 3.1, we can derive our main result.

Proof of theorem 1.1. Suppose K is some simplicial set and let X be non-singular. Let $\Delta \downarrow K$ denote the **simplex category**, meaning the category whose objects are the pairs (x, n) , where x is a simplex of K whose degree is n , and whose morphisms $(y, m) \rightarrow (x, n)$ are the pairs (x, α) with α an operator such that $y = x\alpha$.

The simplicial set K can be viewed as the colimit of the diagram

$$\Upsilon_K : \Delta \downarrow K \rightarrow sSet$$

that sends a simplex of degree n to the standard n -simplex $\Delta[n]$ [FP90, Lem. 4.2.1]. We explain that X^K is the limit of the composite

$$\Delta \downarrow K \xrightarrow{\Upsilon_K} sSet \xrightarrow{X^{(-)}} sSet,$$

denoted X^{Υ_K} , or in other words that the cone $\underline{X^K} \Rightarrow X^{\Upsilon_K}$ is universal.

Assume that $\underline{Z} \Rightarrow X^{\Upsilon_K}$ is a cone. Recall that $sSet$ is cartesian closed. Via the natural bijection

$$sSet(Z \times \Delta[n], X) \xrightarrow{\cong} sSet(Z, X^{\Delta[n]}),$$

we can consider the cocone $Z \times X^{\Upsilon_K} \Rightarrow \underline{X}$ illustrated in the diagram

$$\begin{array}{ccc} Z \times \Delta[m] & & \\ \downarrow id \times \alpha & \searrow id \times \bar{y} & \nearrow id \times \bar{x} \\ & Z \times K & \\ \downarrow id \times \alpha & \nearrow id \times \bar{x} & \\ Z \times \Delta[n] & & \end{array} \quad \begin{array}{c} \xrightarrow{\exists!} X \\ \xrightarrow{\exists!} X \end{array}$$

instead. Because $Z \times -$ is a cocontinuous endofunctor of simplicial sets, the simplicial set $Z \times K$ is the colimit of $Z \times \Upsilon_K$. Hence, there exists a (unique) map $Z \times K \rightarrow X$ that gives rise to a factorization of the cocone $Z \times X^{\Upsilon_K} \Rightarrow \underline{X}$. By adjointness, we obtain a map $Z \rightarrow X^K$ such that the given, arbitrary cone on X^{Υ_K} factors through $\underline{X^K} \Rightarrow X^{\Upsilon_K}$.

On the other hand, any map $Z \rightarrow X^K$ that gives rise to such a factorization corresponds to a map $Z \times K \rightarrow X$ that factors the cocone $Z \times \Upsilon_K \Rightarrow \underline{X}$ through

the universal cocone. However, there is only one map $Z \times K \rightarrow X$ of the latter type. By adjointness, the map $Z \rightarrow X^K$ is therefore unique.

The diagram $X^{\mathcal{I}K}$ is by lemma 3.1 a diagram whose objects are non-singular. Because $nsSet$ is a reflective subcategory of $sSet$, it follows that X^K is non-singular [AR15, p. 1306]. $\square$

In the proof of theorem 1.1, we used the non-trivial fact that a reflective subcategory inherits limits from its surrounding category, although we could have argued in more elementary terms.

According to Adámek and Rosický [AR15, p. 1306], the earliest proof that appears in the literature, of the inheritance of limits by reflective subcategories, is to be found in the works of H. Herrlich [Her68].

4. RIGIDITY OF THE PRISM

We give a proof that $X^{\Delta[1]}$ is non-singular whenever X is non-singular. This is the claim presented in proposition 1.3. An informal way of stating this result is to say that prisms on non-singular simplicial sets are very rigid. Recall that section 3 explains how to derive theorem 1.1 from proposition 1.3. Thus the work of this section finishes the proof of our main result.

For convenience, we introduce some terminology and notation before we present the proof. An injective operator is said to be a **face operator** and a surjective operator is said to be a **degeneracy operator**. Special face operators are the **elementary face operators** $\delta_i^n : [n-1] \rightarrow [n]$ that omit the index i and **vertex operators** $\varepsilon_i^n : [0] \rightarrow [n]$ that hit the indices i . Special degeneracy operators are the **elementary degeneracy operators** $\sigma_i^n : [n+1] \rightarrow [n]$ that send i and its successor $i+1$ to i . Frequently, we omit the upper index in the notation. Similar to the terminology in [WJR13], we will refer to $\delta_n^n \dots \delta_q^q : [q-1] \rightarrow [n]$, $0 < q \leq n$, as the **q -th front face** of $[n]$ and to $\delta_p^n \dots \delta_0^{n-p} : [n-(p+1)] \rightarrow [n]$, $0 \leq p < n$, as the **p -th back face** of $[n]$.

A face operator or degeneracy operator is **proper** if it is not the identity. Consider a simplicial set. A simplex y is a **(proper) face** of another simplex x if $y = x\mu$ for a (proper) face operator μ . Analogously, a simplex y is a **(proper) degeneracy** of another simplex x if $y = x\rho$ for a (proper) degeneracy operator ρ . A simplex is **degenerate** if it is a proper degeneracy of some simplex. Otherwise, it is said to be **non-degenerate**.

In the proof, we will use the Eilenberg-Zilber lemma [FP90, Thm. 4.2.3], which says that any simplex x of any simplicial set X is uniquely a degeneration $x = x^\sharp x^\flat$ of some non-degenerate simplex $x^\sharp$. We say that $x^\sharp$ is the **non-degenerate part** of x , following [WJR13], and that $x^\flat$ is the **degenerate part** of x . Note that x and $x^\sharp$ are objects in the category $\Delta \downarrow X$ while $x^\flat$ can be regarded as a morphism $x \rightarrow x^\sharp$. Thus the terminology is not perfect, however it is useful. According to the Yoneda lemma, the n -simplices x of a simplicial set X are in natural bijective correspondence $x \mapsto \bar{x}$ with the simplicial maps $\Delta[n] \rightarrow X$. The map $\bar{x}$ is the **representing map** of x . We say that a simplex is **embedded** if its representing map is degreewise injective.

Because of the new terminology, we get a shorter definition of *non-singular* in the second condition of lemma 4.1, below. Furthermore, there is another formulation that is useful in the proof of proposition 1.3, though a bit awkward. It is given as the third condition lemma 4.1

Lemma 4.1. *The following statements are equivalent.*

- (1) *The simplicial set X is non-singular.*
- (2) *Each non-degenerate simplex of X is embedded.*

- (3) *Each simplex of X is degenerate provided its vertices are not pairwise distinct.*

The equivalence of the second and third statement is somewhat refined by the next lemma.

Lemma 4.2. *Let X be a non-singular simplicial set and x some simplex with $z\varepsilon_k = z\varepsilon_l$. Then the degenerate part x^b of x factors uniquely through the degeneracy operator $\sigma_k \dots \sigma_{l-1}$.*

Proof. Write $\rho = \sigma_k \dots \sigma_{l-1}$. The uniqueness of a factorization of x^b through ρ is automatic as ρ is epic in Cat . It is the existence part that requires an argument.

Because X is non-singular it follows that the non-degenerate part $x^\sharp$ is embedded, which is the same as saying that its vertices are pairwise distinct. This means that $x^b(k) = x^b(l)$. As x^b is order-preserving, it follows that $x^b(j) = x^b(k)$ if $k \leq j \leq l$. Thus $\rho(i) = \rho(j)$ implies $x^b(i) = x^b(j)$. Take a section μ of ρ . We get that $x^b = (x^b\mu)\rho$. $\square$

lemma 4.2 will be used to break down the proof of proposition 1.3 into two parts.

If x is some simplex, say of degree n , whose degenerate part factors through the elementary degeneracy operator σ_k for some k with $0 \leq k < n$, then we will say that x **splits off** σ_k . In particular, if X is non-singular and if x is a simplex of X such that $x\varepsilon_k = x\varepsilon_{k+1}$, then x splits off σ_k according to lemma 4.2.

The canonical identification

$$N([n] \times [1]) \xrightarrow{\cong} \Delta[n] \times \Delta[1]$$

gives us a preferred set of generators of the prism $\Delta[n] \times \Delta[1]$, namely the $n+1$ non-degenerate $(n+1)$ -simplices

$$\gamma_j^{n+1} : [n+1] \rightarrow [n] \times [1],$$

$0 \leq j \leq n$, given by

$$\gamma_j^{n+1}(i) = \begin{cases} (i, 0), & 0 \leq i \leq j \\ (i-1, 1), & j < i \leq n. \end{cases}$$

Coming from the diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & (j, 1) & \longrightarrow & (j+1, 1) & \longrightarrow & \dots \longrightarrow (n, 1) \\ & & \uparrow & \nearrow & \uparrow & & \\ (0, 0) & \longrightarrow & \dots \longrightarrow (j, 0) & \longrightarrow & (j+1, 0) & \longrightarrow & \dots \end{array}$$

are the conditions

$$(4) \quad \gamma_j^{n+1} \delta_{j+1} = \gamma_{j+1}^{n+1} \delta_{j+1}$$

for $0 \leq j \leq n$. These conditions, which can be thought of glueing conditions for constructing the prism from $n+1$ copies of the standard $(n+1)$ -simplex, generate all relations that the generators satisfy.

We are done with the setup and are ready to prove proposition 1.3. Suppose X non-singular. Keep in mind the third and equivalent way to state this, as formulated in lemma 4.1. The proof is divided into two parts, the first of which is the following result.

Lemma 4.3. *Assume that Φ is an n -simplex of $X^{\Delta[1]}$ such that the k -th vertex and the l -th vertex are equal, for some k and some l with $0 \leq k < l \leq n$. Then*

$$\Phi\varepsilon_k = \Phi\varepsilon_{k+1} = \dots = \Phi\varepsilon_l.$$

The second part is lemma 4.4, where we prove that any given n -simplex Φ of $X^{\Delta[1]}$ is degenerate if it is such that the k -th vertex is equal to the $(k+1)$ -th vertex, for some k with $0 \leq k < n$.

Thus, by lemma 4.3 and lemma 4.4, any simplex of $X^{\Delta[1]}$ is degenerate provided its vertices are not pairwise distinct. Lemma lemma 4.1 then says that $X^{\Delta[1]}$ is non-singular. We can therefore conclude that proposition 1.3 holds when we have proven the two lemmas.

Proof of lemma 4.3. Suppose Φ an n -simplex of $X^{\Delta[1]}$ such that $\Phi\varepsilon_k = \Phi\varepsilon_l$ for some k and some l with $0 \leq k < l \leq n$. What is immediately noticeable is that the composite of Φ with the inclusion of the bottom of the prism is an n -simplex

$$x_0 = \Phi \circ (id, N\varepsilon_0)$$

of X whose k -th and l -th vertex are also equal. Doing something similar at the top of the prism we get a simplex $x_1 = \Phi \circ (id, N\varepsilon_1)$ with $x_1\varepsilon_k = x_1\varepsilon_l$.

From lemma 4.2 it follows that the degenerate part x_0^b of x_0 factors uniquely through $\sigma_k \dots \sigma_{l-1}$. Thus we can write

$$\begin{aligned} x_0 &= y_0 \sigma_k \dots \sigma_{l-1} \\ x_1 &= y_1 \sigma_k \dots \sigma_{l-1} \end{aligned}$$

for some $(k+n-l)$ -simplices y_0 and y_1 of X .

Suppose $k \leq j < l$. Writing x_0 and x_1 as degenerations indicates that the $(n+1)$ -simplices $\Phi(\gamma_{j+1}^{n+1})$ and $\Phi(\gamma_j^{n+1})$ of X must be degenerate. To answer how they are degenerate, form the left hand cartesian square in the following diagram.

$$\begin{array}{ccccc} \Delta[n+1] & \xrightarrow{\gamma_{j+1}^{n+1}} & \Delta[n] \times \Delta[1] & \xrightarrow{\Phi} & X \\ \uparrow & & \uparrow (id, N\varepsilon_0) & \nearrow x_0 & \uparrow y_0 \\ \Delta[j+1] & \longrightarrow & \Delta[n] & \xrightarrow{N(\sigma_k \dots \sigma_{l-1})} & \Delta[k+n-l] \end{array}$$

The canonical map $\Delta[j+1] \rightarrow \Delta[n+1]$ is then induced by the $(j+2)$ -th front face of $[n+1]$ and the canonical map $\Delta[j+1] \rightarrow \Delta[n]$ is induced by the $(j+2)$ -th front face of $[n]$.

The above implies that the j -th and the $(j+1)$ -th vertex of $\Phi(\gamma_{j+1}^{n+1})$ are equal. A similarly constructed diagram involving x_1 , y_1 , $(id, N\varepsilon_1)$ and $\Phi(\gamma_j^{n+1})$ shows that the $(j+1)$ -th and the $(j+2)$ -th vertex of $\Phi(\gamma_j^{n+1})$ are equal.

As a consequence of the previous paragraph, we will argue that the j -th and the $(j+1)$ -th vertex of the n -simplex Φ of $X^{\Delta[1]}$ are equal. They are the vertices of the 1-simplex

$$\Delta[1] \times \Delta[1] \xrightarrow{N\mu \times id} \Delta[n] \times \Delta[1] \xrightarrow{\Phi} X,$$

of $X^{\Delta[1]}$ where μ is given by $0 \mapsto j$ and $1 \mapsto j+1$.

We can view the vertices $\Phi\varepsilon_j$ and $\Phi\varepsilon_{j+1}$ of the simplex Φ of $X^{\Delta[1]}$ as 1-simplices of X . When we do, they fit into the commutative diagram

$$\begin{array}{ccccc} & \Delta[2] & \xleftarrow{\delta_0} & \Delta[1] & \\ \delta_1 \nearrow & & \searrow \gamma_1^2 & & \searrow \Phi\varepsilon_{j+1} \\ \Delta[1] & & \Delta[1] \times \Delta[1] & \xrightarrow{\Phi \circ (N\mu \times id)} & X \\ \delta_1 \searrow & \nwarrow \gamma_0^2 & \nwarrow & \nearrow \Phi\varepsilon_j & \\ & \Delta[2] & \xleftarrow{\delta_2} & \Delta[1] & \end{array}$$

that establishes $\Phi\varepsilon_{j+1}$ as a face of the 2-simplex

$$z_1 = \Phi \circ (N\mu \times 1) \circ \gamma_1^2$$

and $\Phi\varepsilon_j$ as a face of the 2-simplex

$$z_0 = \Phi \circ (N\mu \times 1) \circ \gamma_0^2,$$

in such a way that $z_1\delta_1 = z_0\delta_1$.

Recall that the j -th and the $(j+1)$ -th vertex of the simplex $\Phi(\gamma_{j+1}^n)$ of X are equal. This implies that

$$z_1 = w_1\sigma_1.$$

Similarly, the $(j+1)$ -st and the $(j+2)$ -nd vertex of $\Phi(\gamma_j^n)$ are equal, implying that $z_0 = w_0\sigma_0$. It follows that $\Phi\varepsilon_j = \Phi\varepsilon_{j+1}$ as δ_1 and δ_0 are sections of σ_0 and δ_1 and δ_2 are sections of σ_1 . $\square$

Lemma 4.4. *Let Φ be an n -simplex of $X^{\Delta[1]}$ such that the k -th vertex is equal to the $(k+1)$ -th vertex, for some k with $0 \leq k < n$. Then there is an $(n-1)$ -simplex Ψ such that $\Phi = \Psi\sigma_k$.*

Proof. For the purpose of constructing Ψ we apply $N\sigma_k \times id$ to the elements of the preferred set $\{\gamma_0^{n+1}, \dots, \gamma_n^{n+1}\}$ of generators of the prism. The result of the calculation is the set of equations

$$(N\sigma_k \times id)(\gamma_j^{n+1}) = \begin{cases} \gamma_j^n \sigma_{k+1}, & 0 \leq j \leq k \\ \gamma_{j-1}^n \sigma_k, & k < j \leq n. \end{cases}$$

Should Ψ exist, then it must therefore satisfy

$$\Phi(\gamma_j^{n+1}) = \begin{cases} \Psi(\gamma_j^n) \sigma_{k+1}, & 0 \leq j \leq k \\ \Psi(\gamma_{j-1}^n) \sigma_k, & k < j \leq n. \end{cases}$$

As δ_{k+1} is a section of both σ_k and σ_{k+1} we are lead to define a function

$$\psi : \{\gamma_0^n, \dots, \gamma_{n-1}^n\} \rightarrow X_n$$

by

$$\psi(\gamma_j^n) = \begin{cases} \Phi(\gamma_j^{n+1})\delta_{k+1}, & 0 \leq j \leq k \\ \Phi(\gamma_{j+1}^{n+1})\delta_{k+1}, & k < j < n \end{cases}$$

that specifies where Ψ sends the generators, if it exists.

Note the following regarding the definition of ψ . First, we have made the choices of the section δ_{k+1} of σ_{k+1} and the section δ_{k+1} of σ_k . These choices seem to make the argument below as simple as possible. Second, we have that

$$\psi(\gamma_k^n) = \Phi(\gamma_k^{n+1})\delta_{k+1} = \Phi(\gamma_k^{n+1}\delta_{k+1}) = \Phi(\gamma_{k+1}^{n+1})\delta_{k+1} = \Phi(\gamma_{k+1}^{n+1})\delta_{k+1}$$

due to (4). This ensures that there is some compatibility between the two clauses of the definition of ψ by cases. We take advantage of the equation below.

Crucially, the function ψ obeys the compatibility criterion

$$(5) \quad \psi(\gamma_j^n)\delta_{j+1} = \psi(\gamma_{j+1}^n)\delta_{j+1}$$

for $0 \leq j < n-1$, as we now explain. There are three cases. Either $j < k$, $j = k$ or $j > k$.

First, we verify (5) in the case when $j = k$. For this we use (4) and the general rule $\delta_i \delta_j = \delta_j \delta_{i-1}$ for $j < i$. We get that

$$\begin{aligned} \psi(\gamma_k^n) \delta_{k+1} &= (\Phi(\gamma_k^{n+1}) \delta_{k+1}) \delta_{k+1} \\ &= (\Phi(\gamma_{k+1}^{n+1}) \delta_{k+1}) \delta_{k+1} \\ &= \Phi(\gamma_{k+1}^{n+1}) (\delta_{k+1} \delta_{k+1}) \\ &= \Phi(\gamma_{k+1}^{n+1}) (\delta_{k+2} \delta_{k+1}) \\ &= (\Phi(\gamma_{k+1}^{n+1}) \delta_{k+2}) \delta_{k+1} \\ &= (\Phi(\gamma_{k+1}^{n+1} \delta_{k+2})) \delta_{k+1} \\ &= (\Phi(\gamma_{k+2}^{n+1} \delta_{k+2})) \delta_{k+1} \end{aligned}$$

and that

$$\begin{aligned} \psi(\gamma_{k+1}^n) \delta_{k+1} &= (\Phi(\gamma_{k+2}^{n+1}) \delta_{k+1}) \delta_{k+1} \\ &= \Phi(\gamma_{k+2}^{n+1}) (\delta_{k+1} \delta_{k+1}) \\ &= \Phi(\gamma_{k+2}^{n+1}) (\delta_{k+2} \delta_{k+1}), \end{aligned}$$

which confirms that (5) holds in the case when $j = k$.

Second, consider the case when $j < k$. We get that

$$\begin{aligned} \psi(\gamma_j^n) \delta_{j+1} &= (\Phi(\gamma_j^{n+1}) \delta_{k+1}) \delta_{j+1} \\ &= \Phi(\gamma_j^{n+1}) (\delta_{k+1} \delta_{j+1}) \\ &= \Phi(\gamma_j^{n+1}) (\delta_{j+1} \delta_k) \\ &= (\Phi(\gamma_j^{n+1}) \delta_{j+1}) \delta_k \\ &= (\Phi(\gamma_j^{n+1} \delta_{j+1})) \delta_k \\ &= (\Phi(\gamma_{j+1}^{n+1} \delta_{j+1})) \delta_k \end{aligned}$$

and that

$$\begin{aligned} \psi(\gamma_{j+1}^n) \delta_{j+1} &= (\Phi(\gamma_{j+1}^{n+1}) \delta_{k+1}) \delta_{j+1} \\ &= \Phi(\gamma_{j+1}^{n+1}) (\delta_{k+1} \delta_{j+1}) \\ &= \Phi(\gamma_{j+1}^{n+1}) (\delta_{j+1} \delta_k), \end{aligned}$$

which confirms that (5) holds in the case when $j < k$.

Third, consider the case when $j > k$. We get that

$$\begin{aligned} \psi(\gamma_j^n) \delta_{j+1} &= (\Phi(\gamma_{j+1}^{n+1}) \delta_{k+1}) \delta_{j+1} \\ &= \Phi(\gamma_{j+1}^{n+1}) (\delta_{k+1} \delta_{j+1}) \\ &= \Phi(\gamma_{j+1}^{n+1}) (\delta_{j+2} \delta_{k+1}) \\ &= (\Phi(\gamma_{j+1}^{n+1}) \delta_{j+2}) \delta_{k+1} \\ &= (\Phi(\gamma_{j+1}^{n+1} \delta_{j+2})) \delta_{k+1} \\ &= (\Phi(\gamma_{j+2}^{n+1} \delta_{j+2})) \delta_{k+1} \end{aligned}$$

and that

$$\begin{aligned} \psi(\gamma_{j+1}^n) \delta_{j+1} &= (\Phi(\gamma_{j+2}^{n+1}) \delta_{k+1}) \delta_{j+1} \\ &= \Phi(\gamma_{j+2}^{n+1}) (\delta_{k+1} \delta_{j+1}) \\ &= \Phi(\gamma_{j+2}^{n+1}) (\delta_{j+2} \delta_{k+1}). \end{aligned}$$

This confirms that (5) holds in the case when $j > k$ and concludes our verification of (5) for any j with $0 \leq j < n-1$.

We define $\Psi : \Delta[n-1] \times \Delta[1] \rightarrow X$ by letting

$$\Psi(\gamma_j^n \alpha) = \psi(\gamma_j^n) \alpha$$

for all j with $0 \leq j < n$. The map Ψ is well defined and a simplicial map as ψ satisfies the glueing condition (5). Thus it remains to argue that

$$(6) \quad \Phi = \Psi \circ (N\sigma_k \times id).$$

It suffices to check that the equation holds on the generators $\gamma_0^{n+1}, \dots, \gamma_n^{n+1}$ for the prism $\Delta[n] \times \Delta[1]$.

We use the calculation of $(N\sigma_k \times id)(\gamma_j^{n+1})$, $0 \leq j \leq n$, above. There are three cases. Either $0 \leq j \leq k$, $j = k + 1$ or $j > k + 1$.

If $0 \leq j \leq k$, then

$$\begin{aligned} \Psi \circ (N\sigma_k \times id)(\gamma_j^{n+1}) &= \Psi(\gamma_j^n \sigma_{k+1}) \\ &= \psi(\gamma_j^n) \sigma_{k+1} \\ &= (\Phi(\gamma_j^{n+1}) \delta_{k+1}) \sigma_{k+1} \\ &= \Phi(\gamma_j^{n+1}), \end{aligned}$$

which confirms (6) for the generators $\gamma_0^{n+1}, \dots, \gamma_k^{n+1}$. This is because the vertices of $\Phi(\gamma_j^{n+1})$ that are numbered $k + 1$ and $k + 2$ are equal. Thus the simplex splits off σ_{k+1} by lemma 4.2 as X is non-singular. Furthermore, δ_{k+1} is a section of σ_{k+1} .

Note that $\Phi(\gamma_j^{n+1})$ splits off σ_k when $j > k$. This is because the vertices of $\Phi(\gamma_j^{n+1})$ that are numbered k and $k + 1$ are equal. Thus the simplex splits off σ_k by lemma 4.2 as X is non-singular. Furthermore, δ_{k+1} is a section of σ_k .

Consider the case when $j = k + 1$. We get that

$$\begin{aligned} \Psi \circ (N\sigma_k \times id)(\gamma_{k+1}^{n+1}) &= \Psi(\gamma_k^n \sigma_k) \\ &= \psi(\gamma_k^n) \sigma_k \\ &= (\Phi(\gamma_k^{n+1}) \delta_{k+1}) \sigma_k \\ &= (\Phi(\gamma_{k+1}^{n+1}) \delta_{k+1}) \sigma_k \\ &= \Phi(\gamma_{k+1}^{n+1}), \end{aligned}$$

which confirms (6) for the generator γ_{k+1}^{n+1} .

Finally, we consider the case when $j > k + 1$. Then

$$\begin{aligned} \Psi \circ (N\sigma_k \times id)(\gamma_j^{n+1}) &= \Psi(\gamma_{j-1}^n \sigma_k) \\ &= \psi(\gamma_{j-1}^n) \sigma_k \\ &= (\Phi(\gamma_j^{n+1}) \delta_{k+1}) \sigma_k \\ &= \Phi(\gamma_j^{n+1}), \end{aligned}$$

which confirms (6) for the generators $\gamma_{k+2}^{n+1}, \dots, \gamma_n^{n+1}$. This concludes our verification of (6). Thus Φ is a degenerate simplex of $X^{\Delta[1]}$. $\square$

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