EXTENDED ESSAY

Mathematics

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ON TORUS KNOTS AND THE COMPOSITION OF MATHEMATICAL KNOTS

by

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I have written this essay on a typewriter belonging to the Department of Mathematics of the University of Oslo. I gratefully acknowledge this loan.

CONTENTS

CHAPTER ONE : INTRODUCTION.

This is an essay about tame mathematical knots. Interest is concentrated on the knots on the torus surface, the torus knots, and also on a way to combine knots, similar to the process of tying two real knots together. This leads us on to prime knots, one of which is the trivial knot. Considering the trivial knot as a special torus knot, a primary aim of the essay is to generalize the primeness property of the trivial knot towards also including torus knots.

In general we will here call the composition of two or more knots their tying. Using the algebraic center of a knot group we then demonstrate that no two nontrivial torus knots can be combined to give a new torus knot. Actually, although it is not explictly stated, we show that any knots with trivial knot group center, or nontrivial torus knots, never yield torus knots when tied together. Further, it is the content of a result by Burde and Zieschang (1966) that all nontrivial knots are of one of these kinds. Thus the essay demonstrates, assuming this result, that all torus knots are prime knots.

The level of difficulty is kept simple. The intent of this work has been to use quite elementary tools and techniques and to apply these in an original way so as to prove fairly interesting results. So the otherwise very useful constructions like Seifert surfaces, the elementary ideals, the Alexander polynomials and so on are all omitted and unused.

Space constraints has caused most of the early proofs and constructions to be omitted, but only if they appear in the references. The final size of the essay has thus been influenced by the desire to include the preliminary definitions leading up to what is really new in this essay. References are given with their year of publication bracketed.

All ambiguities in notation should be covered below: $\subset$ does not exclude equality. Equivalence, equality or isomorphism is generally denoted $=$, except from isomorphism with 'standard' groups like the infinite additive group of integers, Z, where $\simeq$ is used. The unit element of a group is denoted 1, and a finite presentation of a group, and the group itself, will be written

$$\{x_1,\ldots,x_n \mid r_1,\ldots,r_m\}$$

for generators $x_1,x_2,\ldots,x_n$ and relators $r_1,r_2,\ldots,r_m$. Sets are denoted similarly, but no confusion should arise. (p,q) means the g.c.d. of p and q, and $\pi_1(X)$ is the fundamental group of a topological space X. For groups A and B, with an amalgamating subgroup C, $A \underset{C}{*} B$ means their amalgamated product. Suzuki (1982) contains all that is needed on amalgamated products. Finally, indexation below and above any operation like $*,\cup,\cap$ or $+$ denotes repetition in the standard way.

Essentially, all proved results are numbered, but there are also three unproved results which are named instead. These are taken from other sources for reasons expounded upon when encountered.

CHAPTER TWO : BASIC CONCEPTS.

2.1.

Mathematical knots have been defined so as to capture the properties of twisting and entanglement most people associate with knots. A full discussion of what knots should be represented as, which knots should be considered the same and which should be different, how to generalize the theory of knots, and so on, is not included here. Time and space has however been devoted to this in several of the references (Armstrong 1979, Crowell and Fox 1963, Rolfsen 1976). Instead merely the essential results are recited here for the sake of completeness.

2.2.

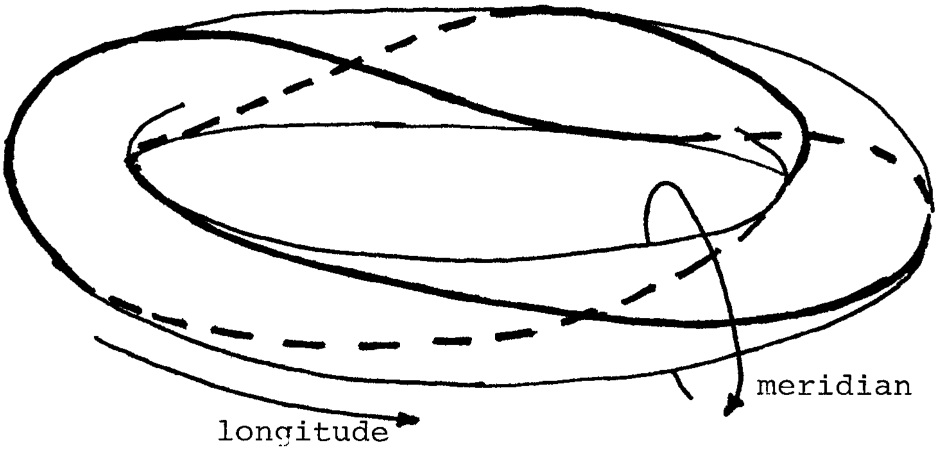
For our purposes a <u>knot</u> is a homeomorphic image of S^1 in S^3. Two knots are <u>equivalent</u> if there is a homeomorphism of S^3 onto itself taking one onto the other. This is an equivalence relation, and the equivalence class of a knot is called its <u>knot type</u>. One can often substitute R^3, or an open or closed ball which is a neighbourhood of the knot, for S^3 as the 'containing space'. Also the distinction between a knot and its knot type needs not always be too strict. The subset

$$\{(x,y,z)\,|\,x^2+y^2=1,z=0\}$$

of R^3 included in S^3 is the <u>trivial knot</u>, denoted by k_t. A knot is <u>tame</u> if it is equivalent to a polygonal knot, which is one consisting of a finite number of straight line-segments. Otherwise a knot is said to be wild. Wild knots can be horribly badly embedded in S^3, so hereafter we consider tame knots only. To be able to calculate anything with knots we define the knot group. For a knot k it is $\pi_1(S^3-k)$, and is denoted by G(k). As for knot types, we can reasonably freely use another containing space than S^3 without changing the knot group. For tame knots, the group is finitely presentable. Ways to obtain one such presentation, the Wirtinger presentation, are explained in Armstrong (1979), Crowell and Fox (1963), Rolfsen (1976) and Stillwell (1980). We assume the result here, and use the convention that the generators used correspond to loops winding once round the oriented knot like right-hand screws. We have $G(k_t)=\{x|\}\cong Z$, the infinite cyclic group.

2.3.

We now introduce the family of knots which will receive most of the attention in this essay. They are the torus knots, lying on the standard torus surface $T^2=S^1\times S^1$. For relatively prime integers p,q, the p,q-torus knot is the one winding p times longitudinally and q times meridinally round the torus surface, and it is denoted $t_{p,q}$. $t_{p,q}=t_{-p,q}=t_{q,p}$ and so on, and $t_{p,1}=k_t$ for nonzero p. Three distinct but precise definitions are given in Crowell and Fox (1963), Rolfsen (1976) and Stillwell (1980). Below is pictured the knot $t_{2,3}$ - the trefoil.



Rolfsen (1976) and Stillwell (1980) show that $G(t_{p,q})=\{x,y|x^p=y^q\}$ for all relatively prime p,q. Picking a base point in $T^2-t_{p,q}$, x corresponds to a loop winding once around the inside, and y to a loop winding once around the outside of the torus surface.

2.4.

The following calculation will be essential later. Take $p,q>0$, and let $G=G(t_{p,q})$, $C=Z(G)$ (the center). Then $G=\{x,y|x^p=y^q\}$. So x^p ($=y^q$) is a power of both the generators x and y. It thus commutes with both, and therefore with the whole of G. The same goes for $(x^p)^n$ for all integers n, so $L=\{x^p|\} \subset C$. L becomes a normal subgroup of G, and for $p,q>1$

$$G/L=\{x|x^p=1\}*\{y|y^q=1\}$$

must have trivial center. Now any onto homomorphism $h:X\to Y$ must map the center of X into the center of Y. For if $c\in Z(X)$, then for all h(x) in Y, x∈X, we have

$$h(c)\cdot h(x)=h(cx)=h(xc)=h(x)\cdot h(c)$$

so since h is onto $h(c)\in Z(Y)$. If we take h as the canonical homomorphism $h:G\to G/L$ this yields $h(C)\subset\{1\}$, so $C\subset L$. Thus $C=L=\{x^p|\}=\{y^q|\}$. For p=1 or q=1 this result is trivial.

Lemma 2.1.

The knot group of the p,q-torus knot is $\{x,y|x^p=y^q\}$, which has infinite cyclic center $\{x^p|\}=\{y^q|\}$.

2.5.

We will later need the order of a(G/C) where a is the abelianizing map, when $G=G(t_{p,q})$, C=Z(G) and (p,q)=1. We can assume p,q>0. Now

$$G/C=\{x|x^p=1\}*\{y|y^q=1\}.$$

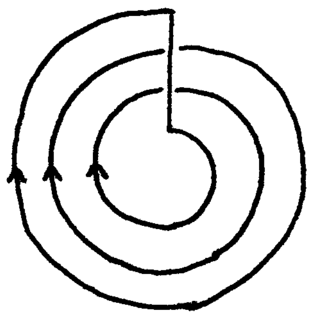
The groups in the product have p, respectively q, elements, so all the elements of a(G/C) can be written in the form x^iy^j for $0\leq i<p$, $0\leq j<q$, and the order of the group must be $p\cdot q$. An easy consequence of this is that there are infinitely many distinct torus knot types.

CHAPTER THREE : ON TORUS KNOTS AND THEIR MERIDIANS.

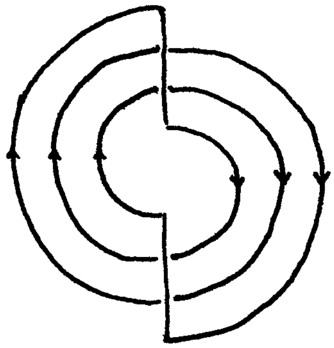
3.1.

We draw a couple of torus knots:

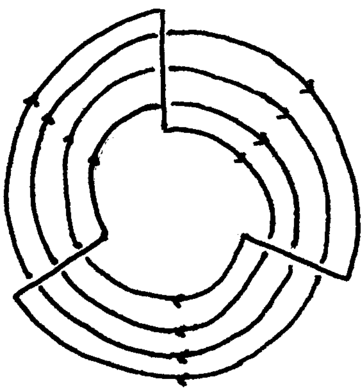
$t_{3,1}=k_t$



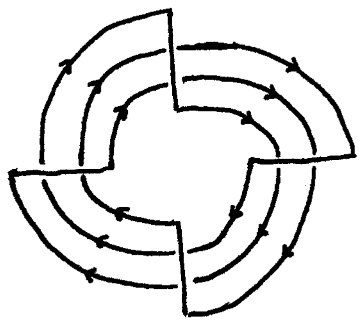
$t_{3,2}$



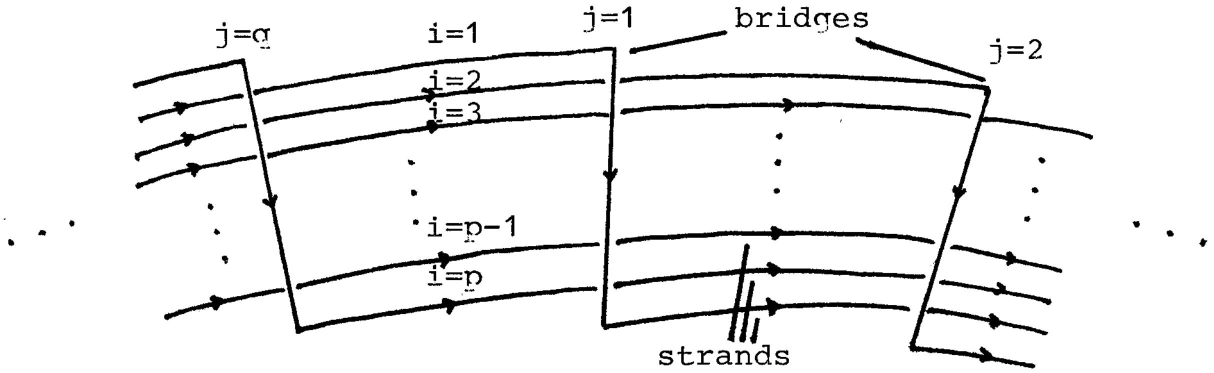
$t_{4,3}=t_{3,4}$



$t_{3,4}$



It is clear that we can find a diagram for the general p,q-torus knot (assuming p,q>0) as below:



As for the Wirtinger presentation we pick a loop for each overpass to give a generator $x_{i,j}$ for the knot group. The first index is the number of the strand, counting inwards at the left of the bridge, which the second index denotes the number of modulo q. This gives rise to a set of generators $\{x_{i,j} | 1 \leq i \leq p,\ 1 \leq j \leq q\}$ with $x_{1,j} = x_{p,j+1}$. For the Wirtinger presentation, the generators actually employed are the set $\{x_{i,j} | 2 \leq i \leq p,\ 1 \leq j \leq q\}$, and for each of the i,j we obtain the relation:

$$x_{i,j} \cdot x_{1,j} = x_{1,j} \cdot x_{i-1,j+1}.$$

For convenience we now add on the generators $x_{i,j}$, $1 \leq j \leq q$, and the relations:

$$x_{1,j} = x_{p,j+1}$$

for $1 \leq j \leq q$, to obtain a presentation for $G(t_{p,q})$. Let us call this presentation the bridge-strand presentation for the knot group. Note that whenever there is any difficulty with indexes in the first set of relations, the last set of relations compensates for this. The expression for the bridge-strand presentation is contained in the next lemma.

Lemma 3.1.

The p,q-torus knot group has a presentation

$$G(t_{p,q}) = \{x_{i,j},\ 1 \leq i \leq p,\ 1 \leq j \leq q \mid x_{i,j} \cdot x_{1,j} = x_{1,j} \cdot x_{i-1,j+1},\ 2 \leq i \leq p,\ 1 \leq j \leq q\ ;\ x_{1,j} = x_{p,j+1},\ 1 \leq j \leq q\}$$

where the second index is modulo q.

Next follows the major result of this chapter. Using the above presentation we determine an exact expression for a meridian of the p,q-torus knot in the group $\{x,y|x^p=y^q\}$. A meridian of a knot is defined in Neuwirth (1965). If we thicken a knot slightly to obtain a nice tube containing the knot, a meridian is roughly a loop on the surface of this thin tube, which also is homotopically trivial in the tube.

3.2.

We take the oriented p,q-torus knot for some relatively prime integers p,q. The knot group $G=G(t_{p,q})$ then has a bridge-strand presentation as given in Lemma 3.1., and a (simpler) presentation as the amalgamated product $\{x,y|x^p=y^q\}$. The generators x and y of the latter are expressible in terms of the $x_{i,j}$ and conversely. For one obvious choice of base point we have

$$x=x_{1,1}\cdot x_{1,2}\cdot x_{1,3}\cdot\ldots\cdot x_{1,q}\quad,$$
$$y=x_{p,1}\cdot x_{p-1,1}\cdot x_{p-2,1}\cdot\ldots\cdot x_{1,1}.$$

x then represents a loop going under each of the q bridges around the knot in turn, and y corresponds to a loop winding around all of the p strands to the left of the first bridge, meridinally in the sense of a right-hand screw. The geometrical situation ensures that $x^p=y^q$, which also is quite easy to show explictly, e.g. using the same technique as used directly below.

We turn to finding a meridian of $t_{p,q}$ in terms of x and y. Each of the $x_{i,j}$, $1\leq i\leq p$, $1\leq j\leq q$ are meridians. In particular we claim that there are integers k,l such that

$$x_{1,1}=x^l y^k.$$

Since $(p,q)=1$ we can find k,l such that $pk+ql=1$. We have assumed $p,q>0$, so one of the k,l must be non-positive. Adding any multiple of q to k and deducting the same multiple of p from l does not alter the value of pk+ql, so we can assume that k is positive and l is negative. Then:

$$\begin{aligned}
y^k &= y^{k-1}\cdot(x_{p,1}\cdot x_{p-1,1}\cdot\ldots\cdot x_{2,1}\cdot x_{1,1})\\
&= y^{k-1}\cdot(x_{p,1}\cdot x_{p-1,1}\cdot\ldots\cdot x_{1,1}\cdot x_{1,2})\\
&= y^{k-1}\cdot(x_{1,1}\cdot x_{p-1,2}\cdot\ldots\cdot x_{1,2})\\
&= x_{1,1}\cdot x_{p-1,2}\cdot\ldots\cdot x_{1,2}\cdot(x_{p,2}\cdot\ldots\cdot x_{1,2})^{k-1}\\
&= x_{1,1}\cdot x_{1,2}\cdot x_{p-2,3}\cdot\ldots\cdot x_{1,3}\cdot(x_{p,3}\cdot\ldots\cdot x_{1,3})^{k-1}\\
&= x_{1,1}\cdot\ldots\cdot x_{1,pk-1}\cdot x_{1,pk}.
\end{aligned}$$

But $pk-1=q(-l)$, so

$$\begin{aligned}
y^k &= x_{1,1}\cdot\ldots\cdot x_{1,pk-1}\cdot x_{1,pk}\\
&= x_{1,1}\cdot\ldots\cdot x_{1,q(-l)}\cdot x_{1,1}\\
&= (x_{1,1}\cdot\ldots\cdot x_{1,q})^{(-l)}\cdot x_{1,1}\\
y^k &= x^{-l}\cdot x_{1,1}
\end{aligned}$$

or

$$\underline{x_{1,1}=x^l y^k.}$$

Which of course is independent of the particular choice of k and l, as long as $pk+ql=1$.

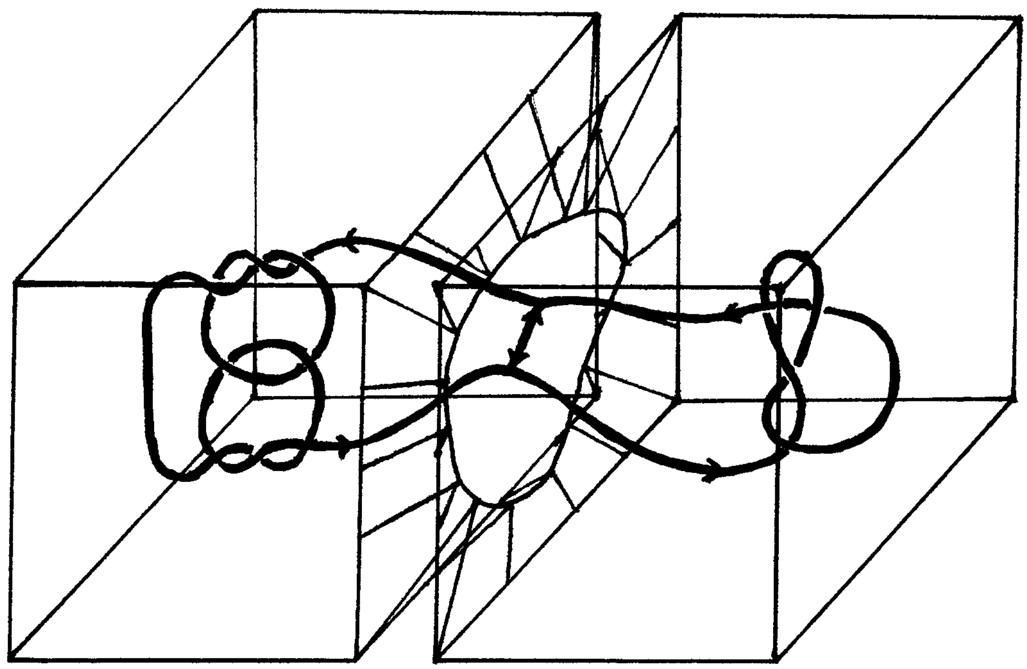
Lemma 3.2.

For the torus knot $t_{p,q}$, the element $x^j y^i$, where $pi+qj=1$, represents a meridian of the knot in the knot group $\{x,y \mid x^p=y^q\}$.

CHAPTER FOUR : COMPOSITION OF KNOTS.

4.1.

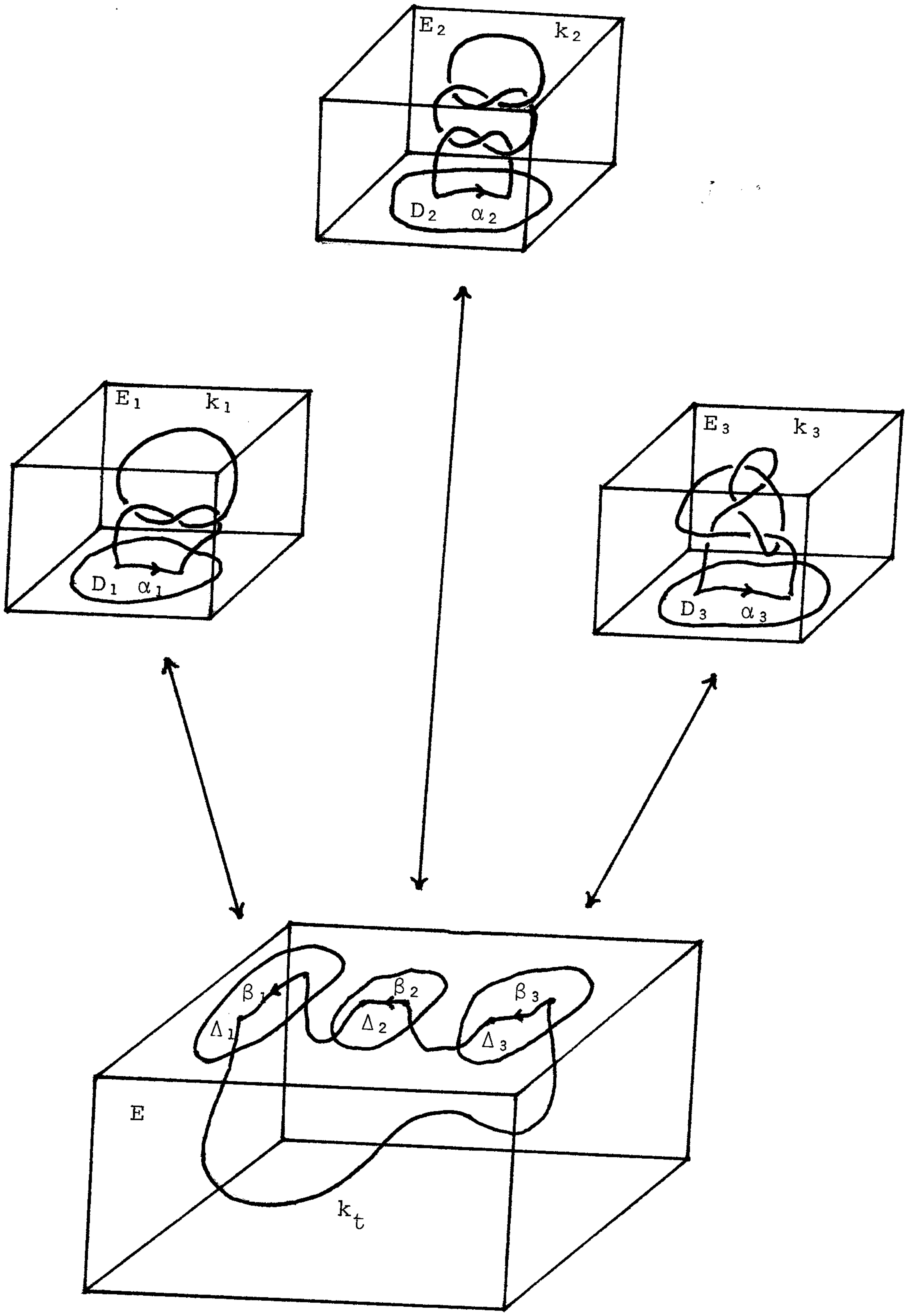
We now take a look at composition of knots. The most intuitive way to combine two knots is to split each open at one point each and connect the loose ends of one in a nice way to the corresponding ends of the other. Precise constructions are found in Neuwirth (1965) and Rolfsen (1976). The below drawing may indicate how they do it:



4.2.

Instead of dwelling upon this, we show how to tie several knots together.

Take n oriented knots $k_1, k_2, \ldots, k_n$ in S^3. We find a closed


E₂
k₂
D₂
α₂
E₁
k₁
D₁
α₁
E₃
k₃
D₃
α₃
β₁
β₂
β₃
Δ₁
Δ₂
Δ₃
E
kₜ

ball E_i containing k_i for each i, and assume that all the E_i are disjoint. On the surface of each E_i we can assume that we have a closed disc D_i and an arc α_i from k_i in the interior of this disc. Orientations inherit from S^3, and are assumed to fit that of the k_i. Finally we take a closed ball E disjoint from all of the E_i. On E we find n disjoint closed discs $\Delta_1, \Delta_2, \ldots, \Delta_n$, and n arcs $\beta_1, \beta_2, \ldots, \beta_n$ in the respective interiors as before. We join the endpoints of each β_i to β_{i-1} and β_{i+1} by arcs in the interior of E or a Δ_i, in such a way that the result is the trivial knot k_t in E. We orient E, k_t, the Δ_i and the β_i as before. Then for each i we identify by opposing orientations the pair (D_i, α_i) with the corresponding pair (Δ_i, β_i). Now remove from $\bigcup_{i=1}^{n} k_i$ the identified interiors of the arcs α_i, β_i. The result is a knot k lying in $E \cup (\bigcup_{i=1}^{n} E_i)$, which we extend by adding on some space to obtain S^3. The resultant knot is then called the <u>tying of the knots $k_1, k_2, \ldots, k_n$</u>, and we write:

$$k = \mathop{+}_{i=1}^{n} k_i$$

As for the combination of two knots, the knot type of k is well defined whenever all the k_i are given orientations. Also we can show that the tying of two knots, written k_1+k_2, as defined in the above references, is equivalent to $\mathop{+}_{i=1}^{2} k_i$ as defined here. So the definitions are properly compatible. In fact the operation + becomes commutative and associative, By a suitable deformation retraction we obtain :

$\pi_1(S^3-k_i') = \pi_1(S^3-(k_i+k_t)) = \pi_1(S^3-k_i)$ for i=1,...,n.

Furthermore

$$\bigcup_{i=1}^{n} k_i' = k_t \cup \left(\bigcup_{j=1}^{n} E_j\right)$$

$$\bigcap_{i=1}^{n} k_i' = \mathop{+}_{i=1}^{n} k_i$$

so $\pi_1(S^3-\bigcup_{i=1}^{n} k_i') = G(k_t) = \{t|\}$ for a generator t of $G(k_t)$.

Also $\pi_1(\bigcap_{i=1}^{n} k_i') = \pi_1(\mathop{+}_{i=1}^{n} k_i)$. We now apply van Kampen's

theorem (n-1) times to get:

$$\begin{aligned}
G(\mathop{+}_{i=1}^{n} k_i) &= \pi_1(S^3-\mathop{+}_{i=1}^{n} k_i) = \pi_1(S^3-\bigcap_{i=1}^{n} k_i') = \pi_1(\bigcup_{i=1}^{n}(S^3-k_i')) \\
&= \pi_1(\bigcup_{i=1}^{n-1}(S^3-k_i')) \mathop{\ast}_{\pi_1(\bigcup_{i=1}^{n-1}(S^3-k_i')\cap(S^3-k_n'))} \pi_1(S^3-k_n') \\
&= \pi_1(\bigcup_{i=1}^{n-1}(S^3-k_i')) \mathop{\ast}_{\pi_1(S^3-k_t)} \pi_1(S^3-k_n') \\
&= \pi_1(\bigcup_{i=1}^{n-1}(S^3-k_i')) \mathop{\ast}_{\{t|\}} \pi_1(S^3-k_n') \\
&= \mathop{\ast}_{\substack{\{t|\} \\ i=1}}^{n} \pi_1(S^3-k_i') = \mathop{\ast}_{\substack{\{t|\} \\ i=1}}^{n} \pi_1(S^3-k_i),
\end{aligned}$$

or

$$G(\mathop{+}_{i=1}^{n} k_i) = \mathop{\ast}_{\substack{\{t|\} \\ i=1}}^{n} G(k_i).$$

For a choice of t we find meridians $m_1, m_2, \ldots, m_n$ of the respective knots, winding in the opposite directions of what

t corresponds to around each of the respective $\alpha_i = \beta_i$. The amalgamating maps then take a power of t onto the same power of the equivalence classes of $m_1, \ldots, m_n$ in $G(k_1), \ldots, G(k_n)$ as appropriate.

If we change the orientations of any of the k_i, and thus the m_i, the knot group of $\overset{n}{\underset{i=1}{+}} k_i$ is only changed up to isomorphism. So we can state:

<u>Theorem 4.1.</u>

For knots $k_1, k_2, \ldots, k_n$, $G(\overset{n}{\underset{i=1}{+}} k_i) = \overset{n}{\underset{\substack{\{t|\} \\ i=1}}{*}} G(k_i)$ where

t is identified with elements corresponding to a meridian for each of the knots k_i.

4.4.

In particular, if $n=2$, $G(k_1) = \{x_1, \ldots, x_{n1} | r_1, \ldots, r_{m1}\}$ and $G(k_2) = \{y_1, \ldots, y_{n2} | s_1, \ldots, s_{m2}\}$, then

$G(k_1 + k_2) = \{x_1, \ldots, x_{n1}, y_1, \ldots, y_{n2} | r_1, \ldots, r_{m1}, s_1, \ldots, s_{m2}, R\}$

where R is a relation $x_i^\delta = y_j^\varepsilon$ for any $i \in \{1, \ldots, n1\}$, $j \in \{1, \ldots, n2\}$ and $\delta, \varepsilon = \pm 1$.

For any two torus knots $t_{p,q}$ and $t_{r,s}$ we can find integers i, j, k, l such that $pi + qj = rk + sl = 1$, and $x^j y^i$ and $z^l w^k$ in $G(t_{p,q}) = \{x, y | x^p = y^q\}$ and $G(t_{r,s}) = \{z, w | z^r = w^s\}$ represent meridians. The theorem then gives us

$$G(t_{p,q} + t_{r,s}) = \{x, y, z, w | x^p = y^q, z^r = w^s, x^j y^i = z^l w^k\}.$$

To conclude the chapter we show an essentially obvious lemma.

<u>Lemma 4.1.</u>

For knots $k_1, k_2, \ldots, k_n$ and $1 \leq m \leq n$, $G(\overset{m}{\underset{i=1}{+}} k_i)$ is isomorphic to a subgroup of $G(\overset{n}{\underset{i=1}{+}} k_i)$.

$$G(\overset{m}{\underset{i=1}{+}} k_i) = \underset{\substack{\{t|\} \\ i=1}}{\overset{m}{*}} G(k_i) \subset (\underset{\substack{\{t|\} \\ i=1}}{\overset{m}{*}} G(k_i)) \underset{\{t|\}}{*} (\underset{\substack{\{t|\} \\ i=m+1}}{\overset{n}{*}} G(k_i))$$

$$= \underset{\substack{\{t|\} \\ i=1}}{\overset{n}{*}} G(k_i) = G(\overset{n}{\underset{i=1}{+}} k_i)$$

by a property of the amalgamated free product of groups.

CHAPTER FIVE : SUBGROUPS, AND THE CENTER IN PARTICULAR.

5.1.

In the last chapter prime knots were mentioned. The trivial knot is one such. This result is not trivial to prove, so we will here cheat a little and use a theorem which we do not attempt to prove. The difficult part of it requires Dehn's lemma and the placement of certain discs in S^3 which we have not developed any basis for studying. The theorem is:

The Unknotting Theorem. A tame knot is trivial if and only if its knot group is infinite cyclic.

We refer to Rolfsen (1976) for the two results on this page. The theorem above is really beyond the kind of tools we want to employ here, but we do not really need this result in the development of our theory either. Instead it will serve as an inspiration for what is to come, and also it bestows some geometrical insight which may help to give a little perspective to what is happening in the next section. Using it we prove:

The Basic Untying Lemma. The tying of two knots is trivial if and only if both are trivial.

Proof: The tying of two trivial knots is of course trivial. Conversely assume $k_1+k_2=k_t$. Then $G(k_1+k_2)=G(k_t)\simeq Z$, so by lemma 4.1. $G(k_i) \subset Z$ for i=1,2. No knot group equals {1}, so $G(k_i) \simeq Z$ for i=1,2. By the unknotting theorem it follows that $k_1=k_2=k_t$ as required. The statement that no knot group equals {1} also follows from lemma 4.1.

An untying theorem appearing in the next chapter will also prove that certain knots cannot be tied together to give the trivial knot, without the use of the unknotting theorem. Meanwhile we turn to pondering upon what we actually just proved.

5.2.

In this section the trivial group is that of the trivial knot. We can then rewrite the last proof, now using the unknotting theorem at the very start: Given two nontrivial knots, their knot groups are nontrivial, and being subgroups of the tying's knot group, the latter is nontrivial. Thus the tying is nontrivial.

It is the property of not being infinite cyclic which translates through to the tying's group. This is a very general assumption. It is natural to try to impose stronger restrictions in order to arrive at similar results holding more information. One way to do this is to study certain subgroups of the various knot groups. If we can find a subgroup exhibited by a whole class of knots, and further demonstrate that some or all tyings lack this subgroup, no tyings of this kind can possibly be of the first kind.

5.3.

So we look for subgroups which do not carry over to tyings. We have considered infinite cyclic ones. The order of a finite subgroup would be a powerful characteristic, but there appears to be none. Abelian groups are easy to handle, but abelianizing a knot group always yields the infinite cyclic group, so we cannot find any tyings at all without that property. The kernel of the abelianizing map, the commutator subgroup, is easily seen to consist of those elements in the knot group where the sum of the exponents in a representing word from the Wirtinger presentation is zero. A lot of work on such subgroups is outlined in Neuwirth (1965), and some results are found in the other references on knot theory too. The center $C=Z(G(k))$ of a knot group will however be the most useful for us. We have already worked it out for torus knots, and found:

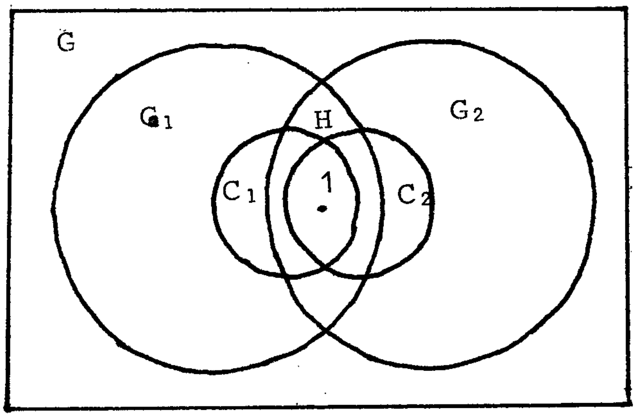
$$Z(G(t_{p,q}))=\{x^p|\}\simeq Z.$$

Having an infinite cyclic center is therefore characteristic for torus knots. In the spirit of 5.2. we consequently devote the rest of the chapter to see what happens to a group's center under amalgamation.

5.4.

For two knots k_1,k_2 (any orientations will do) set $k=k_1+k_2$, $G_i=G(k_i)$, $G=G(k)$, $C_i=Z(G_i)$ and $C=Z(G)$ for $i=1,2$. Treating all groups as subgroups of G (lemma 4.1.), the amalgamating subgroup $H=\{t|\}$ equals

$$G_1 \cap G_2 = H.$$



To have $c \in C$ we must have $cg=gc$ for all $g \in G$. In particular we can require $cg=gc$ for all $g \in G_1$. The only such elements in G_1 lie in C_1. It is easy to see that no elements in $G-G_1$ commute with those in G_1-H. Thus if the latter in nonempty we can conclude that $C \subset C_1$. If $G_1=H$ then $G=G_2$, so $C=C_2$. Similarly for G_2 and C_2 we obtain $C \subset C_2$ if $G_2-H \neq \emptyset$. Else $C=C_1$. Of course, if $c \in C_1 \cap C_2$ then c commutes with all generators of G in particular, and all of G in general. Thus this implies that $C_1 \cap C_2 \subset C$.

<u>Lemma 5.1.</u>

If $G_1 \neq H$ and $G_2 \neq H$ then $Z(G)=Z(G_1) \cap Z(G_2)$. If any $G_i=H$ then $Z(G)=Z(G_j)$ where $i \neq j$.

Corollary 5.1.

If $G_1 \neq H$ and $G_2 \neq H$ then $Z(G) \subset H$.

The argument immediately generalizes to multiple tyings. We take knots $k_1, k_2, \ldots, k_n$, $k = \overset{n}{\underset{i=1}{+}} k_i$ and groups G_i, C_i, G and C as before, but for $i=1,2,\ldots,n$. Let $H=\{t|\}$ be the common amalgamating subgroup. As subgroups of G, for $i \neq j$ we then have:

$$G_i \cap G_j = H$$

If $G_i \neq H$ then $C \subset C_i$ as before for all i. If $G_i = H$ then

$$Z(G(\overset{n}{\underset{j=1}{+}} k_j)) = Z(G(\overset{n}{\underset{j \neq i}{+}} k_j)) \quad \text{if } n>1.$$

Again if $c \in \bigcap_{i=1}^{n} C_i$ then c commutes with all of G, so $c \in C$. Thus we obtain:

Lemma 5.2.

If at least one $G_i \neq H$, then $Z(G) = \bigcap_{\substack{i=1 \\ G_i \neq H}}^{n} Z(G_i)$.

Corollary 5.2.

If at least two $G_i \neq H$, then $Z(G) \subset H$.

CHAPTER SIX : STRONGER RESULTS ON UNTYING.

6.1.

Combining the past results we show:

<u>Theorem 6.1.</u>

The tying of two nontrivial torus knots is not a torus knot.

An immediate corollary is:

<u>Corollary 6.1.</u>

The tying of two nontrivial torus knots is not the trivial knot.

Proof of theorem:

We take two nontrivial torus knots $t_{p,q}$, $t_{r,s}$ as previously. Then $(p,q)=(r,s)=1$, and we can assume that $1<p,q,r,s$. Set $k_1=t_{p,q}$, $k_2=t_{r,s}$, $k=k_1+k_2$ and other notation as before. As in section 2.5. we can show that $a(G_i/C_i)$ have pq and rs elements for $i=1,2$, so no $G_i=H$ as $a(H/Z(H))=\{1\}$ with but 1 element. Thus by lemma 5.1.

$$C = C_1 \cap C_2 \subset H.$$

But $C_1=\{x^p|\}$ and $C_2=\{z^r|\}$. Furthermore we have integers i,j,k,l such that $pi+qj=rk+sl=1$, which gives us:

$$H = \{t|\} = \{x^j y^i|\} = \{z^l w^k|\}.$$

If there are any $c\neq 1$ in C, then there must be nonzero integers m,n such that

$$(x^p)^m = c = t^n = (x^j y^i)^n.$$

We can consider both $(x^p)^m$ and $(x^j y^i)^n$ as elements of G_1. And it will be useful to look at G_1 as the free product with amalgamation below:

$$G_1 = \{x|\} \underset{\{\ell|\}}{*} \{y|\}$$

of $A=\{x|\}$ and $B=\{y|\}$ where $\ell=x^p$ and $\ell=y^q$ define the amalgamation. Considering A and B as subgroups of G_1, $(x^p)^m \in A\cap B$. For $n\neq 0$ the only way $(x^j y^i)^n$ can lie in $A\cap B$ is that $q|i$ or $p|j$. But if $q|i$ then

$$q|pi \text{ and } q|qj \text{ so } q|pi+qj \text{ so } q|1,$$

and if $p|j$ then

$$p|qj \text{ and } p|pi \text{ so } p|pi+qj \text{ and } p|1.$$

Both of these are impossible for $1<p,q$. Therefore if m and n are nonzero, $(x^p)^m \neq (x^j y^i)^n$, so c cannot equal both and C must be trivial:

$$Z(G(t_{p,q}+t_{r,s}))=\{1\}.$$

Since this is not infinite cyclic,it is not the center of a torus knot group. Which completes the proof.

6.2.

For multiple tyings we can weaken the assumption, at least apparently:

Theorem 6.2.

The tying of any n knots $k_1, k_2, \ldots, k_n$ of which at least two are nontrivial torus knots, is not a torus knot, nor is it trivial.

We see that torus knots resist being split up into smaller knots.

Proof of theorem:

We have the knots $k_1, k_2, \ldots, k_n$, n>1, and assume that k_1 and k_2 are nontrivial torus knots. Then $G_1 \neq H$ and $G_2 \neq H$, so by lemma 5.2. we get:

$$Z(G) = \bigcap_{\substack{i=1 \\ G_i \neq H}}^{n} Z(G_i) \subset Z(G_1) \cap Z(G_2) = \{1\}.$$

whence $k = \overset{n}{\underset{i=1}{+}} k_i$ is neither trivial nor a torus knot. Which is what we wanted to prove.

6.3.

With the techniques available we do not get much further. There is however a result, of fairly new origin, about centers of knot groups. Applied on theorem 6.2. it gives us a somewhat neater result. The result referred to is:

The Knot Group Center Theorem. A knot group has nontrivial center if and only if the knot is a torus knot.

A proof is given by Burde and Zieschang (1966), but is much too difficult to be presented here, and relies upon several other results of a similar nature. Using this theorem we can however now prove that all torus knots are prime knots, which theorem 6.1. was a partial proof of. So we state:

Theorem 6.3.

All torus knots are prime knots.

Proof of theorem:

Assume that we have knots k_1 and k_2 such that $k_1+k_2=t_{p,q}$ for some relatively prime p,q. If any of the k_i for i=1,2 are not torus knots, Burde and Zieschang's result shows that $C_i=\{1\}$. It follows that for this $i, G_i \neq H$, since if $G_i=H$ then $C_i=Z(H)=H\neq\{1\}$. So by our lemma 5.2.:

$$C = \bigcap_{\substack{j=1 \\ G_j \neq H}}^{2} C_j \subset C_i = \{1\}$$

and $C=\{1\}$ contradicting $C=Z(G(t_{p,q}))\simeq Z$.

Thus both of the k_i must be torus knots. If both are non-trivial it follows from theorem 6.1. that k_1+k_2 is not a torus knot, so at most 1 of the k_i for i=1,2 are nontrivial. Thus $t_{p,q}$ is always a prime knot. Which completes the proof.

An equivalent restatement of the theorem is:

Corollary 6.2.

Any tying of two or more nontrivial knots is never a torus knot, and in particular not a trivial knot.

The last results also show that there are infinitely many distinct prime knot types. Which concludes this treatment.

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