

RANK FILTRATIONS IN ALGEBRAIC K-THEORY

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1. QUILLEN'S RANK FILTRATION

Let R be a (nonzero, commutative) ring, and let $\mathcal{F}(R) \subset \text{Mod}_R$ be the category of finitely generated free R -modules and R -linear homomorphisms. Let $\text{rank } M \in \mathbb{Z}_{\geq 0}$ denote the rank of any such R -module. We will soon use ‘ F ’ both for filtrations and for fields, so we introduce the abbreviation $\mathcal{C} := \mathcal{F}(R)$ and hereafter refer only to $\mathcal{C}$. The maximal subgroupoid $i\mathcal{C}$ of isomorphisms has classifying space, or nerve,

$$|i\mathcal{C}| \simeq \prod_{k \geq 0} BGL_k(R).$$

For each $k \geq 0$ let $F_k\mathcal{C}$ be the full subcategory generated by the modules M with $\text{rank } M \leq k$. We get an exhaustive space-level filtration

$$* = |iF_0\mathcal{C}| \subset \cdots \subset |iF_{k-1}\mathcal{C}| \subset |iF_k\mathcal{C}| \subset \cdots \subset |i\mathcal{C}|$$

with k -th associated graded

$$\frac{|iF_k\mathcal{C}|}{|iF_{k-1}\mathcal{C}|} = BGL_k(R)_+.$$

Direct sum of modules gives $|i\mathcal{C}|$ the structure of an $\mathbb{E}_\infty$ -space. Its group completion

$$|i\mathcal{C}| \xrightarrow{\gamma} K(R) \simeq \mathbb{Z} \times BGL_\infty(R)^+$$

defines the algebraic K -theory space, with homotopy groups $K_i(R) = \pi_i K(R)$ (for $i \geq 1$) the higher algebraic K -groups of R . Moreover,

$$K(R) \simeq \Omega^\infty \mathbb{K}(R)$$

is naturally the underlying infinite loop space of an algebraic K -theory spectrum $\mathbb{K}(R) = \{B^n K(R)\}_{n \geq 0}$.

Waldhausen's $S_\bullet$ -construction [Wal85] provides one way to construct the first delooping $BK(R)$: The category $\mathcal{C}$ becomes a category-with-cofibrations by declaring $M \rightarrow N$ to be a cofibration if it is splittably injective. We then write $M \twoheadrightarrow N$. For each $q \geq 0$ let $S_q\mathcal{C}$ be the category with objects sequences of cofibrations

$$0 = M_0 \twoheadrightarrow M_1 \twoheadrightarrow \cdots \twoheadrightarrow M_q,$$

together with coherent choices of subquotients M_j/M_i for all $0 \leq i \leq j \leq q$. With $[q] = \{0 < 1 < \cdots < q\}$, these are essentially functors $M: [q] \rightarrow \mathcal{C}$, and the morphisms in $S_q\mathcal{C}$ are the natural transformations between such functors. Let $iS_q\mathcal{C}$ be the subgroupoid of natural isomorphisms. The rule

$$S_\bullet\mathcal{C}: [q] \mapsto S_q\mathcal{C}$$

defines a simplicial category: For $0 < i \leq q$ the i -th face operator forgets M_i , while the 0-th face operator takes M above to

$$0 = M_1/M_1 \twoheadrightarrow M_2/M_1 \twoheadrightarrow \cdots \twoheadrightarrow M_q/M_1,$$

with the evident choices of subquotients. The degeneracy operators insert identity homomorphisms. The classifying space

$$BK(R) = |iS_\bullet \mathcal{C}|$$

then defines the level $n = 1$ term of $\mathbb{K}(R)$, and $K(R) = \Omega(BK(R))$ realizes the group completion of $|i\mathcal{C}|$. Note that $|iS_0 \mathcal{C}| = *$ and $iS_1 \mathcal{C} \cong i\mathcal{C}$, so there is a natural inclusion

$$\sigma: \Sigma|i\mathcal{C}| \longrightarrow BK(R)$$

with right adjoint the group completion map γ .

For each $k \geq 0$, let $F_k S_q \mathcal{C} \subset S_q \mathcal{C}$ be the full subcategory generated by the functors $M: [q] \rightarrow \mathcal{C}$ factoring through $F_k \mathcal{C}$, i.e., with rank $M_q \leq k$. Letting

$$F_k BK(R) = |iF_k S_\bullet \mathcal{C}|$$

we obtain an exhaustive filtration

$$* = F_0 BK(R) \subset \cdots \subset F_{k-1} BK(R) \subset F_k BK(R) \subset \cdots \subset BK(R).$$

Definition 1.1. Let $k \geq 1$. The set $\text{Sub}_*(R^k)$ of submodules $M \subset R^k$ with $M \hookrightarrow R^k$ a cofibration can be partially ordered by letting $M \leq N$ if $M \subset N$ with $M \hookrightarrow N$ a cofibration. Let $\text{Sub}(R^k) \subset \text{Sub}_*(R^k)$ be the partially ordered subset of proper and nontrivial submodules M , i.e., with $0 \neq M \neq R^k$.

The *Tits building* $T(R^k) = |\text{Sub}(R^k)|$ is the classifying space of the partially ordered set $\text{Sub}(R^k)$. If $R = F$ is a field, this is just the set of proper, nontrivial vector subspaces of F^k , partially ordered by inclusion. The action of $\text{GL}_k(R)$ on R^k specifies an action on $\text{Sub}(R^k)$ and $T(R^k)$.

For example, $T(R^1) = S^{-1} = \emptyset$, and $T(R^2) = P^1(R)$ is the projective line of rank 1 direct summands of R^2 .

Proposition 1.2 ([Qui73]). *The k -th filtration quotient*

$$\frac{F_k BK(R)}{F_{k-1} BK(R)} \simeq \Sigma^2 T(R^k)_{h\text{GL}_k(R)}$$

is equivalent to the $\text{GL}_k(R)$ -homotopy orbits of the double suspension of the Tits building $T(R^k)$.

Sketch proof. Each object $M: [q] \rightarrow F_k \mathcal{C}$ in $F_k S_q \mathcal{C} \setminus F_{k-1} S_q \mathcal{C}$ is isomorphic to one with $M_0 = 0$, $M_q = R^k$ and each $M_{i-1} \hookrightarrow M_i$ the inclusion of a submodule. This implies that $|iF_k S_\bullet \mathcal{C}|/|iF_{k-1} S_\bullet \mathcal{C}|$ is equivalent to the $\text{GL}_k(R)$ -homotopy orbits of the quotient of the nerve of $\text{Sub}_*(R^k)$ by the simplicial subset of chains $M_0 \subset \cdots \subset M_q$ where $M_0 \neq 0$ or $M_q \neq R^k$. The latter quotient is equivalent to the double suspension of the nerve of $\text{Sub}(R^k)$, i.e., $\Sigma^2 T(R^k)$. $\square$

When $R = F$ is a field, the Solomon–Tits theorem shows that $T(F^k) \simeq \bigvee S^{k-2}$, so that $\tilde{H}_*(T(F^k))$ is concentrated in degree $k - 2$. This defines the *Steinberg representation*

$$\text{St}_k(F) = \tilde{H}_{k-2}(T(F^k))$$

of $\text{GL}_k(F)$. (When F is a finite or local field, this is a previously known irreducible representation, characterized by its character.) For example, $\text{St}_1(F) = \mathbb{Z}$ and $\text{St}_2(F) = \ker(\epsilon: \mathbb{Z}\{P^1(F)\} \rightarrow \mathbb{Z})$. When R is a PID, such as $R = \mathbb{Z}$, with fraction field F , we have $T(R^k) \cong T(F^k)$, so the Solomon–Tits theorem still applies. We then have an isomorphism

$$\tilde{H}_{*+k}\left(\frac{F_k BK(R)}{F_{k-1} BK(R)}\right) \cong H_*(\text{GL}_k(R); \text{St}_k(R))$$

(with group homology at the right-hand side) and a spectral sequence

$$E_{k,\ell}^1 = H_\ell(\mathrm{GL}_k(R); \mathrm{St}_k(R)) \implies \tilde{H}_{k+\ell}(BK(R))$$

concentrated in the region $k \geq 1, \ell \geq 0$.

Corollary 1.3. *$F_k BK(R) \rightarrow BK(R)$ is k -connected, so $\pi_{i+1} F_k BK(R) \rightarrow K_i(R)$ is surjective for $i = k - 1$ and an isomorphism for $i \leq k - 2$.*

In combination with information about the group (co-)homology of arithmetic groups, this was used by Quillen to prove finite generation of the groups $K_i(R)$ for $R = \mathcal{O}_F$ the ring of integers in any number field.

2. THE SPECTRUM LEVEL RANK FILTRATION

Waldhausen's $S_\bullet$ -construction can be iterated: The category $S_q \mathcal{C}$ itself becomes a category-with-cofibrations by declaring a natural transformation $M \rightarrow N$ of functors $[q] \rightarrow \mathcal{C}$ to be a cofibration if in each commutative square

$$\begin{array}{ccc} M_{i-1} & \twoheadrightarrow & M_i \\ \downarrow & & \downarrow \\ N_{i-1} & \twoheadrightarrow & N_i \end{array}$$

the arrows $M_{i-1} \twoheadrightarrow N_{i-1}$ and $M_i \cup_{M_{i-1}} N_{i-1} \twoheadrightarrow N_i$ are cofibrations (which implies that $M_i \twoheadrightarrow N_i$ is a cofibration).

Hence one can form a bisimplicial category

$$S_\bullet S_\bullet \mathcal{C}: ([q_1], [q_2]) \mapsto S_{q_1} S_{q_2} \mathcal{C}.$$

The classifying space

$$B^2 K(R) = |i S_\bullet S_\bullet \mathcal{C}|$$

defines the level $n = 2$ term of $\mathbb{K}(R)$, and $BK(R) \simeq \Omega(B^2 K(R))$. Restriction to the diagonal simplicial category

$$S_\bullet^2 \mathcal{C}: [q] \mapsto S_q S_q \mathcal{C}$$

yields a homeomorphic classifying space. Iterating $n \geq 1$ times, we can therefore form

$$B^n K(R) = |i S_\bullet^n \mathcal{C}|.$$

Here an object of $S_q^n \mathcal{C}$ is a functor $M: [q]^n \rightarrow \mathcal{C}$, subject to vanishing and cofibrancy conditions, together with implicit coherent choices of subquotients. Since $B^n K(R) \simeq \Omega(B^{n+1} K(R))$, these spaces combine to define the Ω -spectrum $\mathbb{K}(R)$.

For each $k \geq 0$, let $F_k S_q^n \mathcal{C} \subset S_\bullet^n \mathcal{C}$ be the full subcategory on the $M: [q]^n \rightarrow \mathcal{C}$ factoring through $F_k \mathcal{C}$, i.e., with rank $M_{q,\dots,q} \leq k$. Letting

$$F_k B^n K(R) = |i F_k S_\bullet^n \mathcal{C}|$$

we obtain exhaustive filtrations

$$* = F_0 B^n K(R) \subset \cdots \subset F_{k-1} B^n K(R) \subset F_k B^n K(R) \subset \cdots \subset B^n K(R)$$

for all $n \geq 1$. These are compatible with the spectrum structure maps σ , hence combine to an exhaustive filtration of spectra

$$* = F_0 \mathbb{K}(R) \subset \cdots \subset F_{k-1} \mathbb{K}(R) \subset F_k \mathbb{K}(R) \subset \cdots \subset \mathbb{K}(R)$$

This is the spectrum-level rank filtration. Note that the $F_k \mathbb{K}(R)$ for $0 < k < \infty$ are hardly ever Ω -spectra. This has the consequence that there is more information in the spectrum-level rank filtration than the one at level $n = 1$ employed by Quillen.

Definition 2.1. Let $k \geq 1$. The *common basis complex* $\text{CB}(R^k)$ is the simplicial complex with vertices the proper, nontrivial submodules $M \subset R^k$ with $M \hookrightarrow R^k$ a cofibration. A set $\{M_0, \dots, M_q\}$ of vertices spans a simplex if and only if the M_i admit a common basis, i.e., if there is a basis $\mathcal{B}$ for R^k as a free R -module such that for each i there is a subset $\mathcal{B}_i \subset \mathcal{B}$ that is a basis for M_i .

Proposition 2.2 ([Rog92]). *The k -th spectrum level filtration quotient*

$$\frac{F_k \mathbb{K}(R)}{F_{k-1} \mathbb{K}(R)} \simeq \Sigma^\infty \Sigma \text{CB}(R^k)_{h\text{GL}_k(R)}$$

is equivalent to the $\text{GL}_k(R)$ -homotopy orbits of the suspension of the common basis complex $\text{CB}(R^k)$. In particular, $F_1 \mathbb{K}(R) = \Sigma^\infty B\text{GL}_1(R)_+$.

Sketch proof. Each object $M: [q]^n \rightarrow F_k \mathcal{C}$ in $F_k S_q^n \mathcal{C} \setminus F_{k-1} S_q^n \mathcal{C}$ is isomorphic to one with $M_{q, \dots, q} = R^k$ and each $M_{i_1, \dots, i_n} \hookrightarrow M_{j_1, \dots, j_n}$ the inclusion of a submodule. These define the non-basepoints of a based simplicial set $D^n(R^k)$ with $\text{GL}_k(R)$ -action, and

$$\frac{|i F_k S_\bullet^n \mathcal{C}|}{|i F_{k-1} S_\bullet^n \mathcal{C}|} \simeq D^n(R^k)_{h\text{GL}_k(R)}.$$

(In particular, $D^1(R^k) \simeq \Sigma^2 T(R^k)$.) The $D^n(R^k)$ are compatible with the spectrum structure maps σ , hence combine to a spectrum with $\text{GL}_k(R)$ -action $\mathbb{D}(R^k)$, called the k -th *stable building*, with $F_k \mathbb{K}(R)/F_{k-1} \mathbb{K}(R) \simeq \mathbb{D}(R^k)_{h\text{GL}_k(R)}$.

Say that a submodule $M \subset R^k$ is *axial* if it is spanned by a subset of the standard basis for R^k . Let $A_k^n \subset D^n(R^k)$ be the subspace with simplices $M: [q]^n \rightarrow \mathcal{C}$ where each $M_{i_1, \dots, i_n}$ is an axial submodule of R^k . These combine to a subspectrum $\mathbb{A}_k \subset \mathbb{D}(R^k)$, called the k -th *stable apartment*. In fact, $A_k^n \simeq S^{nk}$, so $\mathbb{A}_1 \simeq \mathbb{S}$ and $\mathbb{A}_k \simeq *$ for $k \geq 2$.

The translates $g\mathbb{A}_k \subset \mathbb{D}(R^k)$ cover $\mathbb{D}(R^k)$ as g ranges through $\text{GL}_k(R)$. Moreover, the finite intersections

$$g_0 \mathbb{A}_k \cap \dots \cap g_q \mathbb{A}_k$$

are either (stably) contractible or equivalent to $\mathbb{S}$. The case distinction can be made combinatorially explicit in terms of a partial ordering on $\{1, \dots, k\}$ defined by the $g_0, \dots, g_q$. It follows that $\mathbb{D}(R^k) \simeq \Sigma^\infty \Sigma X_k$ for a simplicial complex X_k with vertex set $\text{GL}_k(R)$, in which $\{g_0, \dots, g_q\}$ span a simplex if and only if $g_0 \mathbb{A}_k \cap \dots \cap g_q \mathbb{A}_k \simeq *$.

The simplicial complex X_k in turn admits a cover by contractible subcomplexes $C(M) \subset X$ indexed by the proper, nontrivial submodules $0 \neq M \subsetneq R^k$, and the intersections

$$C(M_0) \cap \dots \cap C(M_q)$$

are either contractible or empty, depending whether $\{M_0, \dots, M_q\}$ admit a common basis or not. Hence $X_k \simeq \text{CB}(R^k)$. $\square$

3. THE CONNECTIVITY CONJECTURE

Miller, Patzt and Wilson [MPW25] prove the following, using bar constructions for $\text{GL}(R)$ -sequences under Day convolution.

Theorem 3.1. *If $R = F$ is a field, then $\text{CB}(F^k) \simeq \bigvee S^{2k-3}$, so $\mathbb{D}(F^k) \simeq \bigvee \Sigma^{2k-2} \mathbb{S}$.*

I had confirmed this for R a local ring, or a Euclidean domain, when $k \leq 3$. It is still an open conjecture that it holds in this generality, also for $k > 3$. The homology of $\mathbb{D}(R^k)$ is known to be concentrated in the range $k-1 \leq * \leq 2k-2$, so the conjecture amounts to the claim that $\mathbb{D}(R^k)$ is $(2k-3)$ -connected, or that $\text{CB}(R^k)$ is $(2k-4)$ -connected. For example, $\pi_4(F_3 \mathbb{K}(\mathbb{Z})) \cong K_4(\mathbb{Z})$, and this reduced the proof that $K_4(\mathbb{Z}) = 0$ in [Rog00] to calculations with $\text{GL}_k(\mathbb{Z})$ for $k \leq 3$.

The following corollary was proved earlier by Galatius, Kupers and Randal-Williams [GKRW25], using $\mathbb{E}_\infty$ -cell structures.

Corollary 3.2. *If $R = F$ is a field, then $\mathbb{D}(F^k)_{h\mathrm{GL}_k(F)}$ is $(2k - 3)$ -connected, $F_k\mathbb{K}(F) \rightarrow \mathbb{K}(F)$ is $(2k - 1)$ -connected, and $\pi_i F_k\mathbb{K}(F) \rightarrow K_i(F)$ is surjective for $i = 2k - 1$ and an isomorphism for $i \leq 2k - 2$.*

When the connectivity conjecture holds for R , we define the *stable Steinberg representation* to be

$$\mathrm{St}_k^\infty(R) := \tilde{H}_{2k-3}(\mathrm{CB}(R^k)) = H_{2k-2}(\mathbb{D}(R^k))$$

with its natural $\mathrm{GL}_k(R)$ -action, so that

$$H_{*+2k-2}\left(\frac{F_k\mathbb{K}(R)}{F_{k-1}\mathbb{K}(R)}\right) \cong H_*(\mathrm{GL}_k(R); \mathrm{St}_k^\infty(R)).$$

In particular, $\mathrm{St}_1^\infty(R) = \mathbb{Z}$, while $\mathrm{St}_2^\infty(R)$ and $\mathrm{St}_3^\infty(R)$ can be made quite explicit.

Offsetting $F_k\mathbb{K}(R)$ to filtration $k - 1$, we obtain the (stable) rank spectral sequence

$$E_{k,\ell}^1(\mathrm{rank}) = H_{\ell-k}(\mathrm{GL}_{k+1}(R); \mathrm{St}_{k+1}^\infty(R)) \implies H_{k+\ell}(\mathbb{K}(R)),$$

concentrated in the region $k \geq 0$, $\ell \geq k$. For each $\ell \geq 0$ we obtain a complex

$$0 \leftarrow H_\ell(\mathrm{GL}_1(R); \mathbb{Z}) \xleftarrow{d_{1,\ell}^1} H_{\ell-1}(\mathrm{GL}_2(R); \mathrm{St}_2^\infty(R)) \xleftarrow{d_{2,\ell}^1} \dots \xleftarrow{d_{\ell,\ell}^1} H_0(\mathrm{GL}_{\ell+1}(R); \mathrm{St}_{\ell+1}^\infty(R)) \leftarrow 0$$

at vertical height ℓ , with (co-)homology groups

$$H_{\mathrm{rank}}^{\ell-k}(R; \mathbb{Z}(\ell)) := E_{k,\ell}^2(\mathrm{rank})$$

for $0 \leq k \leq \ell$. In other words,

$$H_{\mathrm{rank}}^s(R; \mathbb{Z}(w)) := E_{w-s,w}^2(\mathrm{rank})$$

for $w \geq 0$ and $0 \leq s \leq w$ is the (co-)homology of

$$H_{s+1}(\mathrm{GL}_{w-s}; \mathrm{St}_{w-s}^\infty) \leftarrow H_s(\mathrm{GL}_{w-s+1}; \mathrm{St}_{w-s+1}^\infty) \leftarrow H_{s-1}(\mathrm{GL}_{w-s+2}; \mathrm{St}_{w-s+2}^\infty),$$

where we have omitted ‘ (R) ’ after GL and St^∞ for typographical reasons. Here we think of $w = \ell$ as the weight, s as the cohomological degree, $w - s$ as the shifted filtration and $w - s + 1$ as the rank. In this notation, the stable rank spectral sequence has the form

$$E_2^{2w-s,s}(\mathrm{rank}) = H_{\mathrm{rank}}^s(R; \mathbb{Z}(w)) \implies H_{2w-s}(\mathbb{K}(R)).$$

In particular, this is zero outside the region $w \geq 0$, $0 \leq s \leq w$.

When $R = F$ is a field, there is a motivic filtration of $\mathbb{K}(F)$ with associated graded given by motivic cohomology. This leads to a spectral sequence

$$E_2^{2w-s,s}(\mathrm{mot}) = H_{\mathrm{mot}}^s(F; \mathbb{Z}(w)) \implies K_{2w-s}(F).$$

It is known that $H_{\mathrm{mot}}^s(F; \mathbb{Z}(w))$ is concentrated in the range $s \leq w$, and that the corresponding groups with finite coefficients vanish for $s < 0$, but it remains an important open problem whether the rationalized groups $H_{\mathrm{mot}}^s(F; \mathbb{Z}(w)) \otimes \mathbb{Q}$ are zero for $s < 0$. This is the Beilinson–Soulé vanishing conjecture.

Note that the Hurewicz homomorphism $K_*(R) = \pi_*(\mathbb{K}(R)) \rightarrow H_*(\mathbb{K}(R))$ is a rational equivalence. An optimistic conjecture is then that the stable rank and motivic spectral sequences agree rationally for fields, so that

$$H_{\mathrm{rank}}^s(F; \mathbb{Z}(w)) \otimes \mathbb{Q} \cong H_{\mathrm{mot}}^s(F; \mathbb{Z}(w)) \otimes \mathbb{Q}.$$

This would immediately imply the Beilinson–Soulé vanishing conjecture. It would also exhibit the height ℓ complex above as a ‘motivic complex’ calculating motivic

cohomology, as sought by Lichtenbaum and Goncharov, but only after rationalization. The optimistic conjecture is valid at the minimal rank $k = 1$ (or filtration $k = 0$) edge of the stable rank spectral sequence, corresponding to the maximal cohomological degree edge of the motivic spectral sequence, where motivic cohomology is given by Milnor K -theory.

Proposition 3.3. *When $R = F$ is a field, $H_{\text{mot}}^i(F; \mathbb{Z}(i)) \cong K_i^M(F)$ is rationally isomorphic to $H_{\text{rank}}^i(F; \mathbb{Z}(i)) = E_2^{i,i}(\text{rank}) = E_{0,i}^2(\text{rank})$, for each $i \geq 0$.*

Sketch proof. Chase the following diagram.

$$\begin{array}{ccccc} \Lambda^* \text{GL}_1(F) & \longrightarrow & K_*^M(F) & \longrightarrow & K_*(F) \\ \downarrow \cong_{\mathbb{Q}} & & \downarrow & & \downarrow \cong_{\mathbb{Q}} \\ H_*(\text{GL}_1(F)) & \longrightarrow & E_{0,*}^2(\text{rank}) & \longrightarrow & H_*(\mathbb{K}(F)) \end{array}$$

□

Nonetheless, recent work by Kupers, Rudenko, Sierra [KRS26a], [KRS26b] may show that the stable rank spectral sequence does not collapse rationally at E_2 , whereas the motivic spectral sequence is known to do so. I may return to this in a later talk.

In order to relate the stable rank filtration to the motivic filtration, it might be useful to extend the connectivity conjecture to the case when R is Henselian, i.e., a local ring for the Nisnevich topology, and then proceed by Nisnevich descent (as is satisfied by algebraic K -theory and motivic cohomology).

Is there an analog of (the stable rank filtration and) stable Steinberg representation for other reductive groups than GL_k ?

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