

GROUP COMPLETION OF $GL_n(\mathbb{N}_0)$

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INTRODUCTION

For $1 \leq n \leq \infty$, let $GL_n(\mathbb{N}_0) \subset GL_n(\mathbb{Z})$ be the multiplicative monoid of non-negative integer matrices with determinant ± 1 , and let $SL_n(\mathbb{N}_0) \subset SL_n(\mathbb{Z})$ be the submonoid of non-negative integer matrices with determinant $+1$.

For $1 \leq i, j \leq n$ and $i \neq j$ let $e_{ij} \in SL_n(\mathbb{N}_0)$ be the elementary matrix with (i, j) -th entry $+1$. We identify the n -th symmetric group with the subgroup $\Sigma_n \subset GL_n(\mathbb{N}_0)$ of permutation matrices. These are precisely the invertible elements in $GL_n(\mathbb{N}_0)$.

We are interested in $GL_n(\mathbb{N}_0)$, because the algebraic K -theory of 2-vector bundles is represented by the space

$$K(\mathcal{V}) = \mathbb{Z} \times BGL_\infty(\mathcal{V})^+ \simeq \Omega B\left(\coprod_{n \geq 0} BGL_n(\mathcal{V})\right)$$

where $\mathcal{V}$ denotes the bipermutative category of (finite-dimensional) complex vector spaces, and $GL_n(\mathcal{V})$ is the monoidal category of matrices of complex vector spaces whose dimension matrix has determinant ± 1 . Hence there is a monoidal functor $GL_n(\mathcal{V}) \rightarrow GL_n(\mathbb{N}_0)$ and an induced map $BGL_n(\mathcal{V}) \rightarrow BGL_n(\mathbb{N}_0)$.

This leads us to study the classifying space $BGL_n(\mathbb{N}_0)$, and in particular, its fundamental group

$$\pi_1 BGL_n(\mathbb{N}_0) = \text{Gr } GL_n(\mathbb{N}_0)$$

given by the group completion of the monoid $GL_n(\mathbb{N}_0)$. By the universal property of the group completion, the monoid homomorphism $GL_n(\mathbb{N}_0) \rightarrow GL_n(\mathbb{Z})$ extends to a unique group homomorphism $\text{Gr } GL_n(\mathbb{N}_0) \rightarrow GL_n(\mathbb{Z})$. How close is this to an isomorphism?

RANKS $n \leq 2$

The case $n = 1$ is trivial, since $GL_1(\mathbb{N}_0) \cong \text{Gr } GL_1(\mathbb{N}_0)$ are both trivial, while $GL_1(\mathbb{Z}) = \{\pm 1\}$ is non-trivial. However, $SL_1(\mathbb{N}_0) \cong \text{Gr } SL_1(\mathbb{N}_0) \cong SL_1(\mathbb{Z})$.

For $n = 2$, the inclusion $\Sigma_2 \subset GL_2(\mathbb{N}_0)$ splits the extension

$$\begin{array}{ccccccc} & & \Sigma_2 & & & & \\ & & \downarrow & \searrow \cong & & & \\ 1 & \longrightarrow & SL_2(\mathbb{N}_0) & \longrightarrow & GL_2(\mathbb{N}_0) & \xrightarrow{\det} & \{\pm 1\} \longrightarrow 1. \end{array}$$

The transposition $\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ acts on $SL_2(\mathbb{N}_0)$ by conjugation, taking $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$. Hence we can write $GL_2(\mathbb{N}_0) = \Sigma_2 \ltimes SL_2(\mathbb{N}_0)$ as a semi-direct product. Consider the elementary matrices $e_{12} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $e_{21} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

I learned the fact that e_{12} and e_{21} generate $SL_2(\mathbb{N}_0)$ from Thomas Kragh and Sverre Lunøe-Nielsen. It is not much harder to prove that these generators satisfy no relations.

Theorem. *For $n = 2$,*

$$SL_2(\mathbb{N}_0) \cong \langle e_{12}, e_{21} \rangle^{mon} \cong \mathbb{N}_0 * \mathbb{N}_0$$

is the free monoid generated by e_{12} and e_{21} . Hence

$$\text{Gr } SL_2(\mathbb{N}_0) \cong \langle e_{12}, e_{21} \rangle^{gp} \cong \mathbb{Z} * \mathbb{Z}$$

is the free group generated by e_{12} and e_{21} , and $BSL_2(\mathbb{N}_0) \simeq S^1 \vee S^1$.

Furthermore,

$$GL_2(\mathbb{N}_0) \cong \Sigma_2 \ltimes \langle e_{12}, e_{21} \rangle^{mon} \cong \Sigma_2 \ltimes (\mathbb{N}_0 * \mathbb{N}_0)$$

where the generator $\sigma \in \Sigma_2$ acts by interchanging e_{12} and e_{21} . Hence

$$\text{Gr } GL_2(\mathbb{N}_0) \cong \Sigma_2 \ltimes \langle e_{12}, e_{21} \rangle^{gp} \cong \Sigma_2 \ltimes (\mathbb{Z} * \mathbb{Z})$$

and $BGL_2(\mathbb{N}_0) \simeq E\Sigma_2 \times_{\Sigma_2} (S^1 \vee S^1)$, where $\sigma \in \Sigma_2$ interchanges the two copies of S^1 .

Proof. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{N}_0)$, we write $\|A\| = a + b + c + d$. Suppose that A is different from the identity matrix, so that $\|A\| > 2$. We claim that either $(a, b) \geq (c, d)$ or $(a, b) \leq (c, d)$ in the product partial ordering on $\mathbb{N}_0^2$, but not both. Assuming this, we can write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a-c & b-d \\ c & d \end{bmatrix}$$

or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c-a & d-b \end{bmatrix}$$

in $SL_2(\mathbb{N}_0)$, in the two respective cases. In either case, we can uniquely write $A = e_{12}B$ or $A = e_{21}B$ for some $B \in SL_2(\mathbb{N}_0)$, with $\|B\| < \|A\|$. Hence, by induction on $\|A\|$ it follows that A can be uniquely factored as a product of copies of e_{12} 's and e_{21} 's. In other words, the canonical monoid homomorphism $\langle e_{12}, e_{21} \rangle^{mon} \rightarrow SL_2(\mathbb{N}_0)$ is an isomorphism.

The other claims follow easily from this fact.

It remains to prove the claim. We cannot have both $(a, b) \leq (c, d)$ and $(a, b) \geq (c, d)$, since then $(a, b) = (c, d)$ which contradicts $ad - bc = 1$. If $a < c$ and $b > d$ then $ad < bc$, which also contradicts $ad - bc = 1$. The remaining possibility is $a > c$ and $b < d$. If $bc = 0$ then $ad = 1$, so $a = d = 1$ and $b = c = 0$, which makes A the identity matrix. Otherwise $b, c \geq 1$ so $a, d \geq 2$ and $ad > cd > cb = bc$ in $\mathbb{N}_0$. This contradicts $ad - bc = 1$. $\square$

Remark. Geometrically, we are considering two vectors (a, b) and (c, d) in $\mathbb{N}_0^2 \subset \mathbb{R}^2$. When $a > c$ and $b < d$, the parallelogram spanned by (a, b) and (c, d) must contain the rectangle with vertices (a, b) , (a, d) , (c, d) and (c, b) . Since the parallelogram has unit area, it must be equal to this rectangle. Hence $(b, c) = (0, 0)$, and A must be the identity matrix.

THE STEINBERG GROUP

Note that $e_{12}e_{21}^{-1}e_{12} = e_{21}^{-1}e_{12}e_{21}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has order 4 in $SL_2(\mathbb{Z})$. The Steinberg group $St_2(\mathbb{Z})$ can be defined by the presentation

$$St_2(\mathbb{Z}) = \langle x_{12}, x_{21} \mid x_{12}x_{21}^{-1}x_{12} = x_{21}^{-1}x_{12}x_{21}^{-1} \rangle^{gp}.$$

It is the universal central extension

$$1 \rightarrow \ker(\pi) \rightarrow St_2(\mathbb{Z}) \xrightarrow{\pi} SL_2(\mathbb{Z}) \rightarrow 1$$

of $SL_2(\mathbb{Z})$, where π takes x_{ij} to e_{ij} , and the kernel $\ker(\pi) \cong \mathbb{Z}$ is the infinite cyclic group generated by $(x_{12}x_{21}^{-1}x_{12})^4$. For a proof, see [Milnor71, Theorem 10.5]. Hence $SL_2(\mathbb{Z})$ has a presentation given by that for the Steinberg group, together with the relation $(x_{12}x_{21}^{-1}x_{12})^4 = 1$.

An alternate presentation is given by letting $\omega = x_{12}x_{21}^{-1}x_{12}$ and $\zeta = x_{12}x_{21}^{-1}$. Note that $x_{12} = \zeta^{-1}\omega$ and $x_{21} = \zeta^{-2}\omega$. Then

$$St_2(\mathbb{Z}) = \langle \omega, \zeta \mid \omega^2 = \zeta^3 \rangle^{gp} \cong \mathbb{Z} *_\mathbb{Z} \mathbb{Z}$$

is the pushout in groups (amalgamated product) of $\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z}$, and

$$SL_2(\mathbb{Z}) = \langle \omega, \zeta \mid \omega^2 = \zeta^3, \omega^4 = 1 \rangle^{gp} \cong C_4 *_{C_2} C_6$$

is the pushout in groups of $C_4 \xleftarrow{2} C_2 \xrightarrow{3} C_6$. Here $C_m \cong \mathbb{Z}/m$ is the cyclic group of order m . The relation $\zeta^6 = 1$ follows easily from the other two relations.

((Can also write $GL_2(\mathbb{Z}) = D_4 *_{D_2} D_6$ where $D_m = \Sigma_2 \ltimes C_m$ is the dihedral group of order $2m$.)

There is also a pullback diagram

$$\begin{array}{ccc} St_2(\mathbb{Z}) & \longrightarrow & \widetilde{SL_2(\mathbb{R})} \\ \pi \downarrow & & \downarrow \pi \\ SL_2(\mathbb{Z}) & \longrightarrow & SL_2(\mathbb{R}) \end{array}$$

where $\widetilde{SL_2(\mathbb{R})} \simeq *$ is the universal cover of $SL_2(\mathbb{R}) \simeq SO(2)$.

The linear path $t \mapsto e_{ij}^t$ from the identity to e_{ij} in $SL_2(\mathbb{R})$, for $0 \leq t \leq 1$, lifts uniquely to a path from the identity in $\widetilde{SL_2(\mathbb{R})}$. We can map $x_{ij} \in St_2(\mathbb{Z})$ to the endpoint of the lifted path, since the two paths

$$t \mapsto e_{12}^t e_{21}^{-t} e_{12}^t = \begin{bmatrix} 1-t^2 & 2t-t^3 \\ -t & 1-t^2 \end{bmatrix}$$

and

$$t \mapsto e_{21}^{-t} e_{12}^t e_{21}^{-t} = \begin{bmatrix} 1-t^2 & t \\ -2t+t^3 & 1-t^2 \end{bmatrix}$$

are homotopic in $SL_2(\mathbb{R})$. Sending $e_{ij} \in SL_2(\mathbb{N}_0)$ to $x_{ij} \in St_2(\mathbb{Z})$ then determines the factorization

$$SL_2(\mathbb{N}_0) \rightarrow \text{Gr } SL_2(\mathbb{N}_0) \rightarrow St_2(\mathbb{Z}) \xrightarrow{\pi} SL_2(\mathbb{Z})$$

of the usual monoid inclusion. Here $\mathrm{Gr} SL_2(\mathbb{N}_0) \cong \langle e_{12}, e_{21} \rangle^{gp} \cong \langle \omega, \zeta \rangle^{gp}$, with quotients $St_2(\mathbb{Z})$ and $SL_2(\mathbb{Z})$ obtained by cumulatively adding the relations $\omega^2 = \zeta^3$ and $\omega^4 = 1$.

The Steinberg group $St_2(\mathbb{Z})$ is isomorphic to the knot group $\pi_1(S^3 \setminus K)$ of the trefoil knot K . The following argument, due to Quillen, is discussed in [Milnor71, pp. 83–85].

The composite

$$\widetilde{SL_2(\mathbb{R})} \rightarrow SL_2(\mathbb{R}) \rightarrow SL_2(\mathbb{R})/SL_2(\mathbb{Z})$$

exhibits $\widetilde{SL_2(\mathbb{R})}$ as the universal cover of $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$, so that

$$\pi_1(SL_2(\mathbb{R})/SL_2(\mathbb{Z})) \cong St_2(\mathbb{Z}).$$

We can identify $GL_2(\mathbb{R})/GL_2(\mathbb{Z})$ with the space of lattices $L \subset \mathbb{R}^2 \cong \mathbb{C}$, i.e., discrete subgroups $L \subset \mathbb{R}^2$ with $L \cong \mathbb{Z}^2$, since $GL_2(\mathbb{R})$ acts transitively on such lattices, and $GL_2(\mathbb{Z})$ stabilizes $\mathbb{Z}^2 \subset \mathbb{R}^2$. The subspace $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ corresponds to the lattices where each fundamental domain has area 1.

For each lattice L , the Weierstraß $\wp$ -function is defined by the series

$$\wp(z) = \wp(z; L) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in L} \left(\frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right).$$

It is a meromorphic function of $z \in \mathbb{C}$, having a double pole with residue 0 at each lattice point $\lambda \in L$. Letting

$$\begin{aligned} g_2 &= g_2(L) = 60 \sum_{0 \neq \lambda \in L} \frac{1}{\lambda^4} \\ g_3 &= g_3(L) = 140 \sum_{0 \neq \lambda \in L} \frac{1}{\lambda^6} \end{aligned}$$

the $\wp$ -function satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

where the discriminant

$$\Delta = \Delta(L) = g_2^3 - 27g_3^2$$

of the cubic polynomial is nonzero. The correspondence $L \leftrightarrow (g_2(L), g_3(L))$ induces a diffeomorphism

$$GL_2(\mathbb{R})/GL_2(\mathbb{Z}) \cong \mathbb{C}^2 \setminus V,$$

where $V = \{(g_2, g_3) \mid g_2^3 = 27g_3^2\}$. Let $K = S^3 \cap V$, where $S^3 \subset \mathbb{C}^2$ is the unit sphere. Then K is a trefoil knot, and there are homotopy equivalences

$$SL_2(\mathbb{R})/SL_2(\mathbb{Z}) \simeq GL_2(\mathbb{R})/GL_2(\mathbb{Z}) \cong \mathbb{C}^2 \setminus V \simeq S^3 \setminus K$$

so that

$$St_2(\mathbb{Z}) \cong \pi_1(SL_2(\mathbb{R})/SL_2(\mathbb{Z})) \cong \pi_1(S^3 \setminus K)$$

is the knot group of the trefoil knot.

RANKS $n \geq 3$

The elementary matrices e_{ij} in $SL_n(\mathbb{Z})$, for $1 \leq i, j \leq n$, $i \neq j$, satisfy the following commutation relations, where also $1 \leq k, \ell \leq n$:

$$\begin{cases} [e_{ij}, e_{j\ell}] = e_{i\ell} & \text{for } i \neq \ell, \\ [e_{ij}, e_{k\ell}] = 1 & \text{for } j \neq k, i \neq \ell. \end{cases}$$

Here $[g, h] = ghg^{-1}h^{-1}$ is the usual commutator. For $n \geq 3$, the Steinberg group $St_n(\mathbb{Z})$ is defined by the presentation

$$St_n(\mathbb{Z}) = \langle x_{ij} \mid [x_{ij}, x_{j\ell}] = x_{i\ell}, [x_{ij}, x_{k\ell}] = 1 \rangle^{gp}$$

where $1 \leq i, j, k, \ell \leq n$, $i \neq j$, $i \neq \ell$ and $j \neq k$. It is the universal central extension

$$1 \rightarrow \ker(\pi) \rightarrow St_n(\mathbb{Z}) \xrightarrow{\pi} SL_n(\mathbb{Z}) \rightarrow 1$$

of $SL_n(\mathbb{Z})$, where π takes x_{ij} to e_{ij} , and the kernel $\ker(\pi) \cong \mathbb{Z}/2$ is the group of order two generated by $(x_{12}x_{21}^{-1}x_{12})^4$. For a proof, see [Milnor71, Theorem 10.1]. Hence $SL_n(\mathbb{Z})$ has a presentation given by that for $St_n(\mathbb{Z})$, together with the relation $(x_{12}x_{21}^{-1}x_{12})^4 = 1$.

There is also a pullback diagram

$$\begin{array}{ccc} St_n(\mathbb{Z}) & \longrightarrow & \widetilde{SL_n(\mathbb{R})} \\ \pi \downarrow & & \downarrow \pi \\ SL_n(\mathbb{Z}) & \longrightarrow & SL_n(\mathbb{R}) \end{array}$$

where $\widetilde{SL_n(\mathbb{R})} \simeq Spin(n)$ is the universal (double) cover of $SL_n(\mathbb{R}) \simeq SO(n)$.

Proposition. *For $n \geq 3$, the surjection π factors*

$$St_n(\mathbb{Z}) \rightarrow \text{Gr } SL_n(\mathbb{N}_0) \rightarrow SL_n(\mathbb{Z})$$

so that $\text{Gr } SL_n(\mathbb{N}_0) \rightarrow SL_n(\mathbb{Z})$ and $\text{Gr } GL_n(\mathbb{N}_0) \rightarrow GL_n(\mathbb{Z})$ are surjective.

To prove this, we define Steinberg monoids that are closely related to the Steinberg groups.

We can rewrite the defining relations for $St_n(\mathbb{Z})$ as

$$\begin{cases} x_{ij}x_{j\ell} = x_{i\ell}x_{j\ell}x_{ij} & \text{for } i \neq \ell, \\ x_{ij}x_{k\ell} = x_{k\ell}x_{ij} & \text{for } j \neq k, i \neq \ell. \end{cases}$$

We define the Steinberg monoid $St_n(\mathbb{N}_0)$ by the presentation

$$St_n(\mathbb{N}_0) = \langle x_{ij} \mid x_{ij}x_{j\ell} = x_{i\ell}x_{j\ell}x_{ij}, x_{ij}x_{k\ell} = x_{k\ell}x_{ij} \rangle^{mon}$$

where $1 \leq i, j \leq n$, $i \neq j$ in the generating set, $1 \leq \ell \leq n$, $i \neq j \neq \ell \neq i$ in the first set of relations, and $1 \leq k, \ell \leq n$, $i \neq j \neq k \neq \ell \neq i$ in the second set of relations.

Proof of Proposition. There is a monoid homomorphism $\pi: St_n(\mathbb{N}_0) \rightarrow SL_n(\mathbb{N}_0)$ that takes x_{ij} to e_{ij} , and we can identify the group completion $\text{Gr } St_n(\mathbb{N}_0)$ with the Steinberg group $St_n(\mathbb{Z})$. We get a commutative diagram:

$$\begin{array}{ccccc} St_n(\mathbb{N}_0) & \xrightarrow{\gamma} & \text{Gr } St_n(\mathbb{N}_0) & \xrightarrow{\cong} & St_n(\mathbb{Z}) \\ \pi \downarrow & & \downarrow & & \downarrow \pi \\ SL_n(\mathbb{N}_0) & \xrightarrow{\gamma} & \text{Gr } SL_n(\mathbb{N}_0) & \longrightarrow & SL_n(\mathbb{Z}) \end{array}$$

where the horizontal composites are the standard inclusions. Hence the surjection $\pi: St_n(\mathbb{Z}) \rightarrow SL_n(\mathbb{Z})$ factors as

$$St_n(\mathbb{Z}) \cong \text{Gr } St_n(\mathbb{N}_0) \rightarrow \text{Gr } SL_n(\mathbb{N}_0) \rightarrow SL_n(\mathbb{Z}),$$

so the right hand map must be surjective. The permutation matrices $\Sigma_n \subset GL_n(\mathbb{N}_0)$ together with $SL_n(\mathbb{Z})$ generate $GL_n(\mathbb{Z})$, so the result above also implies that $\text{Gr } GL_n(\mathbb{N}_0) \rightarrow GL_n(\mathbb{Z})$ is surjective. $\square$

Remark. If $St_n(\mathbb{Z}) \rightarrow \text{Gr } SL_n(\mathbb{N}_0)$ is surjective, for some $n \geq 3$, then either $\text{Gr } SL_n(\mathbb{N}_0) \cong St_n(\mathbb{Z})$ or $\text{Gr } SL_n(\mathbb{N}_0) \cong SL_n(\mathbb{Z})$.

Lemma. *The elementary matrices e_{ij} and Σ_n do not generate $GL_n(\mathbb{N}_0)$ as a monoid for $n \geq 3$.*

Proof. The matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

lies in $SL_3(\mathbb{N}_0)$, but cannot be written as $A = e_{ij}B$ for any matrix $B \in SL_3(\mathbb{N}_0)$. Hence $SL_3(\mathbb{N}_0)$ is not generated by elementary matrices. It does not help to add permutation matrices, since these can all be collected at the right end of a word. The same example, suitably stabilized, works for $n \geq 4$. $\square$

STABILIZATION

Let K_n be the kernel of the surjection $\text{Gr } GL_n(\mathbb{N}_0) \rightarrow GL_n(\mathbb{Z})$ for $n \geq 2$. The stabilization homomorphism $i: GL_n(\mathbb{Z}) \rightarrow GL_{n+1}(\mathbb{Z})$ taking g to $\begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}$ induces a commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & K_n & \longrightarrow & \text{Gr } GL_n(\mathbb{N}_0) & \longrightarrow & GL_n(\mathbb{Z}) \longrightarrow 1 \\ & & \downarrow i & & \downarrow i & & \downarrow i \\ 1 & \longrightarrow & K_{n+1} & \longrightarrow & \text{Gr } GL_{n+1}(\mathbb{N}_0) & \longrightarrow & GL_{n+1}(\mathbb{Z}) \longrightarrow 1 \end{array}$$

Proposition. *Each stabilization homomorphism $i: K_n \rightarrow K_{n+1}$ is trivial. Hence the natural map*

$$\text{Gr } GL_\infty(\mathbb{N}_0) \xrightarrow{\cong} GL_\infty(\mathbb{Z})$$

is an isomorphism.

This is a consequence of a stronger topological result that appeared in [BDR04]. The following Lemma is a direct application of Quillen's Theorem B.

Lemma. *Let $f: M \rightarrow G$ be a monoid homomorphism, with G a group. We write mg for $f(m) \cdot g$. Let $Q = B(*, M, G)$ be the translation category for M acting on G , with objects $g \in G$ and morphisms $[m]g \in M \times G$ from g to mg . Then the diagram*

$$Q = B(*, M, G) \rightarrow BM \xrightarrow{Bf} BG$$

is a homotopy fiber sequence.

Proof. View f as a functor $BM \rightarrow BG$ of one-object categories. The right fiber of f at $*$ is then the category $Q = B(*, M, G)$ with objects $g \in G$ and morphisms $[m]g \in M \times G$ from g to $f(m) \cdot g = mg$.

$$\begin{array}{ccccc} & & \lceil & \text{---} & \text{---} & * & \xrightarrow{m} & * \\ & & \downarrow & & & \downarrow f & & \downarrow f \\ * & \xrightarrow{h} & * & \xrightarrow{g} & * & \xrightarrow{f(m)} & * & \end{array}$$

For each $h \in G$, viewed as a morphism $* \rightarrow *$ in G , the pullback functor $Q \rightarrow Q$ takes g to gh and $[m]g$ to $[m]gh$. It is an isomorphism, hence a homotopy equivalence, since G is a group. Then by [Quillen73, Theorem B], the diagram

$$\begin{array}{ccc} B(*, M, G) & \longrightarrow & BM \\ \downarrow & & \downarrow Bf \\ B(*, G, G) & \longrightarrow & BG \end{array}$$

is homotopy cartesian. Here $EG = B(*, G, G)$ is the right fiber of the identity of G at $*$, which is contractible. The claim follows. $\square$

Proposition. *Let $Q_n = B(*, GL_n(\mathbb{N}_0), GL_n(\mathbb{Z}))$, so that $\pi_1(Q_n) = K_n$. The stabilization map $i: Q_n \rightarrow Q_{n+1}$ is null-homotopic, so that $i: K_n \rightarrow K_{n+1}$ is the trivial homomorphism. Hence $Q_\infty = B(*, GL_\infty(\mathbb{N}_0), GL_\infty(\mathbb{Z}))$ is contractible, and*

$$BGL_\infty(\mathbb{N}_0) \xrightarrow{\sim} BGL_\infty(\mathbb{Z})$$

is a weak equivalence.

Proof. We prove that the functor $i: Q_n \rightarrow Q_{n+1}$ is related by a chain of natural transformations

$$i \leftarrow j \rightarrow k$$

to a constant functor. The middle functor j will have the form

$$j(g) = \begin{bmatrix} g & v(g) \\ 0 & 1 \end{bmatrix}$$

for a suitably chosen function $v: GL_n(\mathbb{Z}) \rightarrow \mathbb{N}_0^n$, while the constant functor k will have the form

$$k(g) = \begin{bmatrix} I_n & 0 \\ -w & 1 \end{bmatrix}$$

where $w = [1, \dots, 1]$.

For each morphism $[m]g: mg \rightarrow g$ in Q_n , the diagram

$$\begin{array}{ccccc} i(mg) & \longleftarrow & j(mg) & \longrightarrow & k(mg) \\ \downarrow & & \downarrow & & \downarrow \\ i(g) & \longleftarrow & j(g) & \longrightarrow & k(g) \end{array}$$

must take the form

$$\begin{array}{ccccc} \begin{bmatrix} mg & 0 \\ 0 & 1 \end{bmatrix} & \xleftarrow{\begin{bmatrix} I_n & v(mg) \\ 0 & 1 \end{bmatrix}} & \begin{bmatrix} mg & v(mg) \\ 0 & 1 \end{bmatrix} & \xrightarrow{\begin{bmatrix} mg+v(mg)w & v(mg) \\ w & 1 \end{bmatrix}} & \begin{bmatrix} I_n & 0 \\ -w & 1 \end{bmatrix} \\ \downarrow \begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix} & & \downarrow \begin{bmatrix} m & v(mg)-mv(g) \\ 0 & 1 \end{bmatrix} & & \downarrow \begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} & \xleftarrow{\begin{bmatrix} I_n & v(g) \\ 0 & 1 \end{bmatrix}} & \begin{bmatrix} g & v(g) \\ 0 & 1 \end{bmatrix} & \xrightarrow{\begin{bmatrix} g+v(g)w & v(g) \\ w & 1 \end{bmatrix}} & \begin{bmatrix} I_n & 0 \\ -w & 1 \end{bmatrix} \end{array}$$

where each arrow $[m']g': m'g' \rightarrow g'$ is simply labelled m' . For j to define a functor, and for $j \rightarrow k$ to be a natural transformation, we must have

$$v(mg) \geq mv(g)$$

for all morphisms $[m]g: mg \rightarrow g$ in Q_n , and

$$g + v(g)w \geq 0$$

for all objects g in Q_n .

((Check inductive beginning!))

We define $v(h)$ inductively for each $h \in GL_n(\mathbb{Z})$, assuming that $v(g)$ has been defined for all $g \in GL_n(\mathbb{Z})$ such that $h = mg$ with $m \in GL_n(\mathbb{N}_0) \setminus \Sigma_n$. There are only finitely many such pairs $[m]g$, so we can find a $y \in \mathbb{N}_0$ with $[y, \dots, y]^T \geq mv(g)$ for all $[m]g$ with $mg = h$ with $m \notin \Sigma_n$. After possibly increasing y , we may also be sure that $g + v(g)w \geq 0$, and then we define $v(h) = [y, \dots, y]^T$. Since $\sigma v(h) = v(h)$ for $\sigma \in \Sigma_n$, we then have $v(mg) \geq mv(g)$ for all $[m]g$ with $mg = h$.

□