

Localization sequences for logarithmic topological cyclic homology

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Abstract

We introduce the notion of an $\mathbb{E}_k$ -ring with pre-logarithmic structure, define logarithmic topological Hochschild homology and logarithmic topological cyclic homology in this context, and establish localization sequences for these theories. Our approach is based on Thom R -algebras. It extends and strengthens our earlier work on the subject in several regards. Our examples include the fraction field of topological K -theory, the existence of which was suggested by calculations by Ausoni and the first author. To illustrate the computational accessibility of $\log \mathrm{THH}$ and $\log \mathrm{TC}$, we determine these for non-negative even periodic sphere spectra, with their canonical prelogarithmic structures.

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1 | INTRODUCTION

Trace maps from algebraic K -theory to topological Hochschild homology (THH) and topological cyclic homology (TC) provide powerful tools for the subtle computations of algebraic K -theory of discrete rings or structured ring spectra. Partly triggered by work of Nikolaus–Scholze [49] and Bhatt–Morrow–Scholze [12], these methods have seen a boost in recent years, witnessed, for example, by important results of Hahn–Wilson [31] and Burklund–Schlank–Yuan [23] on chromatic redshift, of Burklund–Hahn–Levy–Schlank on Ravenel’s telescope conjecture [22], and of Antieau–Krause–Nikolaus on the algebraic K -theory of $\mathbb{Z}/p^n$ [5].

However, there are interesting examples of discrete rings or structured ring spectra for which trace maps do not give good approximations to algebraic K -theory. This can be explained by the failure of THH and TC to admit the same localization sequences as algebraic K -theory. Motivated by a connection between relative THH and the de Rham complex with log poles [37], first appearing in work of Hesselholt–Madsen [33], one strategy to overcome this is to extend the definition of THH and TC to appropriately defined prelog or log ring spectra, and to realize the terms missing from the localization sequences by the THH and TC of these objects. In this paper, we propose new notions of prelog and log $\mathbb{E}_k$ -rings, construct THH and TC for such objects, and establish localization sequences motivated by those in algebraic K -theory. Using simplicially enriched Waldhausen categories, Blumberg–Mandell [17] have in certain cases given a different construction of relative THH and TC terms fitting in such localization sequences. While their approach connects well to algebraic K -theory, it appears to have no direct relation to the de Rham complex with log poles.

Our present approach extends and strengthens our earlier work on the subject [55–57] in several regards: While our previous localization sequences and calculations were limited to $\mathbb{E}_\infty$ -rings with prelog structures generated by single homotopy classes (such as ku with the Bott element $u \in \pi_2(ku)$), the scope of our new methods encompasses even $\mathbb{E}_k$ -rings with $k \geq 2$ and prelog structures generated by several different homotopy classes, such as the truncated Brown–Peterson spectrum $BP\langle n \rangle$ at a prime p with $p, v_1, \dots, v_n \in \pi_* BP\langle n \rangle$. More substantially, while our earlier work only addressed log THH as a cyclic spectrum, we now also construct cyclotomic structures using the graded refinement of the Nikolaus–Scholze approach that appears in work of Antieau–Mathew–Morrow–Nikolaus [6]. This allows us to define log TC for prelog $\mathbb{E}_k$ -rings, and we obtain the desired localization sequences both for log THH and log TC.

1.1 | R -based prelog $\mathbb{E}_k$ -rings and Thom spectra

A (discrete) prelog ring $(A, M, \tilde{\alpha})$ consists of a commutative ring A , a commutative monoid M , and a ring homomorphism $\tilde{\alpha} : \mathbb{Z}[M] \rightarrow A$ from the integral monoid ring. It determines a localization $A[M^{-1}] = A \otimes_{A[M]} A[M^{\text{gp}}]$, and one can interpret $(A, M, \tilde{\alpha})$ as an intermediate object between

A and $A[M^{-1}]$ that does not exist within commutative rings. (Making this factorization explicit requires discussing the log condition, which we skip in this introduction for simplicity.)

When generalizing prelog rings to the homotopy theoretic setup of structured ring spectra, we want to allow A to be an $\mathbb{E}_k$ -ring for some $2 \leq k \leq \infty$. However, different choices suggest themselves when generalizing the remaining data. Most obviously, we can consider a discrete commutative monoid or an $\mathbb{E}_k$ -space M together with a structure map $\mathbb{S}[M] \rightarrow A$ from the spherical monoid ring. This works well when A is the Eilenberg–Mac Lane spectrum of a commutative ring [55], and also generalizes to the animated situation [13, 62]. However, it turns out to be too restrictive for capturing the prelog structure generated by a positive-dimensional homotopy class in an $\mathbb{E}_k$ -ring.

In [56, 57], we overcame this problem by working with a notion of graded $\mathbb{E}_\infty$ -spaces. However, the specific point set level model used there only allowed us to model $\mathbb{E}_\infty$ -spaces graded over the $\mathbb{E}_\infty$ -space $QS^0 = \Omega^\infty \Sigma^\infty S^0$, which in turn restricted the prelog structures we were able to construct. Our new solution to this problem is to use an ∞ -categorical setup and work Picard-graded relative to a base $\mathbb{E}_\infty$ -ring R . That is, we consider $\mathbb{E}_k$ -spaces M graded over the $\mathbb{E}_\infty$ -space Pic_R , defined as the maximal subgroupoid spanned by the invertible R -modules in the symmetric monoidal ∞ -category Mod_R . There is an $\mathbb{E}_k$ Thom R -algebra functor $\text{Th}_R : \text{Alg}_{\mathbb{E}_k}(\mathbb{S}/\text{Pic}_R) \rightarrow \text{Alg}_{\mathbb{E}_k}(\text{Mod}_R)$ that generalizes the spherical monoid ring $\mathbb{S}[-] : \text{Alg}_{\mathbb{E}_k}(\mathbb{S}) \rightarrow \text{Alg}_{\mathbb{E}_k}(\text{Sp})$.

We define an R -based prelog $\mathbb{E}_k$ -ring to be a triple $(A, \xi, \bar{\alpha})$ consisting of an $\mathbb{E}_k$ -ring A , an $\mathbb{E}_k$ -map $\xi : M \rightarrow \text{Pic}_R$, and an $\mathbb{E}_k$ -map $\bar{\alpha} : \text{Th}_R(\xi) \rightarrow A$ from the underlying $\mathbb{E}_k$ -ring of $\text{Th}_R(\xi)$. (We also consider a more structured variant where A is an $\mathbb{E}_k$ R -algebra and $\bar{\alpha}$ is an $\mathbb{E}_k$ R -algebra map.) When $R = \mathbb{S}$, we drop the term $\mathbb{S}$ -based and simply speak of prelog $\mathbb{E}_k$ -rings. While the examples in the present paper are mostly $\mathbb{S}$ -based, we develop a general R -based theory because this will allow us to consider $\mathbb{E}_k$ -maps $\xi : M \rightarrow \text{Pic}_R$ that do not factor through $\text{Pic}_{\mathbb{S}} \rightarrow \text{Pic}_R$.

One useful choice for the structure map ξ of a prelog $\mathbb{E}_2$ -ring is the $\mathbb{E}_2$ -map ξ_{2d} given by the composite

$$\mathbb{Z}_{\geq 0} \xrightarrow{1 \mapsto d} \mathbb{Z} \xleftarrow{\simeq} \Omega^2(BU(1)) \xrightarrow{\Omega^2(\text{incl})} \Omega^2(BU) \xleftarrow{\simeq} \mathbb{Z} \times BU \xrightarrow{J_{\mathbb{C}}} \text{Pic}_{\mathbb{S}},$$

where $J_{\mathbb{C}}$ denotes the complex J -homomorphism. It sends $1 \in \mathbb{Z}_{\geq 0}$ to $2d \in \pi_0(\text{Pic}_{\mathbb{S}})$. Using a minimal $\mathbb{E}_2$ -cell structure on $\text{Th}_{\mathbb{S}}(\xi_{2d})$ established by Galatius–Kupers–Randal-Williams [28], an obstruction theory argument allows us to build an $\mathbb{E}_2$ -map $\bar{\alpha}(a) : \text{Th}_{\mathbb{S}}(\xi_{2d}) \rightarrow A$ hitting a prescribed homotopy class $a \in \pi_{2d}(A)$ whenever $\pi_*(A)$ is sufficiently sparse, for example, even. This defines a prelog $\mathbb{E}_2$ -ring

$$(A, \xi_{2d}, \bar{\alpha}(a)) = (A, \xi_{2d} : \mathbb{Z}_{\geq 0} \rightarrow \text{Pic}_{\mathbb{S}}, \bar{\alpha}(a) : \text{Th}_{\mathbb{S}}(\xi_{2d}) \rightarrow A). \quad (1.1)$$

For the connective complex topological K -theory spectrum ku or the Adams summand ℓ , with their respective Bott and Adams elements, we thus obtain the existence of prelog $\mathbb{E}_2$ -rings

$$(ku, \langle u \rangle) := (ku, \xi_2, \bar{\alpha}(u)) \quad \text{and} \quad (\ell, \langle v_1 \rangle) := (\ell, \xi_{2p-2}, \bar{\alpha}(v_1))$$

with prelog structures generated by these classes. The obstruction theory also applies to Thom spectra of $\mathbb{E}_2$ -maps $(\mathbb{Z}_{\geq 0})^r \rightarrow \text{Pic}_{\mathbb{S}}$ and allows us, for example, to define prelog $\mathbb{E}_2$ -rings $(\ell, \langle p, v_1 \rangle)$ and $(\text{BP}\langle n \rangle, \langle p, v_1, \dots, v_n \rangle)$.

1.2 | Log THH

The topological Hochschild homology $\mathrm{THH}(A)$ of an $\mathbb{E}_k$ -ring A can be defined as the realization of the cyclic bar construction $B^{\mathrm{cy}} : [q] \mapsto A^{\otimes^{1+q}}$. It comes equipped with a circle action because it is the realization of a cyclic object, and has an $\mathbb{E}_{k-1}$ -algebra structure because B^{cy} is symmetric monoidal. Moreover, there are $\mathbb{T}$ -equivariant cyclotomic structure maps $\varphi_p : \mathrm{THH}(A) \rightarrow \mathrm{THH}(A)^{tC_p}$ to the C_p -Tate construction on $\mathrm{THH}(A)$ that equip $\mathrm{THH}(A)$ with the structure of a cyclotomic spectrum in the sense of [49], and these data can be used to define the topological cyclic homology $\mathrm{TC}(A)$ of A .

The key idea for extending THH to R -based prelog $\mathbb{E}_k$ -rings comes from a variant of the cyclic bar construction known as the *replete bar construction*. Motivated by the algebraic structure of $\pi_* \mathrm{THH}(A|K)$ from [33], the replete bar construction was introduced in [55]. For a discrete monoid M , it is defined as the pullback of $M \rightarrow M^{\mathrm{gp}} \leftarrow B^{\mathrm{cy}}(M^{\mathrm{gp}})$, that is, of the group completion map of M and the augmentation of the cyclic bar construction of M^{gp} induced by the iterated multiplication maps $(M^{\mathrm{gp}})^{\times^{1+q}} \rightarrow M^{\mathrm{gp}}$. However, generalizing the last step to the present $\mathbb{E}_k$ -setup is delicate because there is no natural augmentation $B^{\mathrm{cy}}(M) \rightarrow M$ for a general $\mathbb{E}_k$ -space M . We resolve this issue by viewing the group completion M^{gp} of an $\mathbb{E}_k$ -space M with $k \geq 2$ as an object augmented over its commutative monoid of path components $\pi_0(M^{\mathrm{gp}}) \cong (\pi_0(M))^{\mathrm{gp}}$, so that there is an augmentation $B^{\mathrm{cy}}(M^{\mathrm{gp}}) \rightarrow B^{\mathrm{cy}}(\pi_0(M^{\mathrm{gp}})) \rightarrow \pi_0(M^{\mathrm{gp}})$ and we can base change along $\pi_0(M) \rightarrow \pi_0(M^{\mathrm{gp}})$ to (re-)define $B^{\mathrm{rep}}(M)$.

To implement this basic idea in the above setup of R -based prelog $\mathbb{E}_k$ -rings, we view the extension of an $\mathbb{E}_k$ -map $\xi : M \rightarrow \mathrm{Pic}_R$ to the group completion $\xi^{\mathrm{gp}} : M^{\mathrm{gp}} \rightarrow \mathrm{Pic}_R$ as an object ξ_*^{gp} graded over $\pi_0(M^{\mathrm{gp}})$, or over a more general commutative monoid. It gives rise to a $\pi_0(M^{\mathrm{gp}})$ -graded $\mathbb{E}_k$ Thom R -algebra $\mathrm{Th}_R(\xi_*^{\mathrm{gp}})$ and we form the THH of (the underlying $\mathbb{E}_k$ -ring of) $\mathrm{Th}_R(\xi_*^{\mathrm{gp}})$ in $\pi_0(M^{\mathrm{gp}})$ -graded spectra. Restricting it along the group completion map $\pi_0(M) \rightarrow \pi_0(M^{\mathrm{gp}})$ gives an $\mathbb{E}_{k-1}$ -ring $\mathrm{THH}(\mathrm{Th}_R(\xi), \xi)$. It comes with a canonical $\mathbb{E}_{k-1}$ *repletion map* $\rho : \mathrm{THH}(\mathrm{Th}_R(\xi)) \rightarrow \mathrm{THH}(\mathrm{Th}_R(\xi), \xi)$ and we define $\log \mathrm{THH}$ of an R -based prelog $\mathbb{E}_k$ -ring $(A, \xi, \bar{\alpha})$ to be the relative tensor product

$$\mathrm{THH}(A, \xi, \bar{\alpha}) := \mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathrm{Th}_R(\xi))} \mathrm{THH}(\mathrm{Th}_R(\xi), \xi).$$

While we show that $\mathrm{THH}(\mathrm{Th}_R(\xi), \xi)$ can also be obtained as the Thom spectrum over a suitable replete bar construction, the above approach has the advantage that we can use [6] to obtain the cyclotomic structure on $\mathrm{THH}(A, \xi, \bar{\alpha})$ that we need to define $\mathrm{TC}(A, \xi, \bar{\alpha})$.

When $k = 2$ and $R = \mathbb{S}$ or $R = \mathbb{S}_{(p)}$, our version of $\log \mathrm{THH}$ for $(A, \xi_{2d}, \bar{\alpha}(a))$, including the construction of its cyclotomic structure, agrees with the variant of $\log \mathrm{THH}$ studied by Ausoni–Bayındır–Moulinos [9]. Moreover, we show that in examples including $(ku, \langle u \rangle)$ and $(\ell, \langle v_1 \rangle)$, it is equivalent to the version studied in [56, 57]. Hence the calculations and $\log \mathrm{THH}$ -étaleness results from [57] are also valid in the present model.

It will be interesting to study the even filtration on $\mathrm{THH}(A, \xi, \bar{\alpha})$ and the resulting $\log$ prismatic and syntomic cohomology using the motivic filtration [30, 51] (compare also [14]). We intend to pursue this in future work.

1.3 | Repletion–residue sequences

In the situation of (1.1), there is an $\mathbb{E}_2$ -collapse map $\mathrm{Th}_{\mathbb{S}}(\xi_{2d}) \rightarrow \mathbb{S}$ that is a left inverse to the unit map $\mathbb{S} \rightarrow \mathrm{Th}_{\mathbb{S}}(\xi_{2d})$. Using it, we can define the residue $\mathbb{E}_1$ A -algebra $A//\bar{\alpha}(a) := A \otimes_{\mathrm{Th}_{\mathbb{S}}(\xi_{2d})} \mathbb{S}$.

The next theorem is a simplified version of our main structural result about log THH and log TC:

Theorem 1.4. *For a prelog $\mathbb{E}_2$ -ring as in (1.1), there are repletion–residue cofiber sequences*

$$\mathrm{THH}(A) \xrightarrow{\rho} \mathrm{THH}(A, \xi_{2d}, \bar{\alpha}(a)) \xrightarrow{\mathrm{res}} \Sigma \mathrm{THH}(A//\bar{\alpha}(a))$$

of cyclotomic $\mathrm{THH}(A)$ -modules and

$$\mathrm{TC}(A)_p \xrightarrow{\rho} \mathrm{TC}(A, \xi_{2d}, \bar{\alpha}(a))_p \xrightarrow{\mathrm{res}} \Sigma \mathrm{TC}(A//\bar{\alpha}(a))_p$$

of $\mathrm{TC}(A)_p$ -modules.

In the case of $(\ell, \langle v_1 \rangle)$, this specializes to a cofiber sequence

$$\mathrm{THH}(\ell) \rightarrow \mathrm{THH}(\ell, \langle v_1 \rangle) \rightarrow \Sigma \mathrm{THH}(H\mathbb{Z}_{(p)})$$

of cyclotomic $\mathrm{THH}(\ell)$ -modules and a similar sequence in log TC. This matches the analogous localization sequence in algebraic K -theory, with the periodic theory L instead of $(\ell, \langle v_1 \rangle)$, which was established by Blumberg–Mandell [16]. A comparison between our $\mathrm{THH}(\ell, \langle v_1 \rangle)$ and Blumberg–Mandell’s relative term $\mathrm{THH}(\ell|L)$ from [17], and therefore also a trace map from $K(L)$ to $\mathrm{THH}(\ell, \langle v_1 \rangle)$ or $\mathrm{TC}(\ell, \langle v_1 \rangle)$, appears in recent work by Lundemo [42].

We give an example of the computational accessibility of our construction: For $d > 0$, the Thom spectrum $\mathbb{S}[x] = \mathrm{Th}_{\mathbb{S}}(\xi_{2d})$ is an $\mathbb{E}_2$ -model for the non-negative $2d$ -periodic sphere spectrum. It is part of a canonical prelog $\mathbb{E}_2$ -ring $(\mathbb{S}[x], \xi_{2d}, \mathrm{id})$, for which we can explicitly identify the terms in the log THH and log TC repletion–residue sequences (see Proposition 11.3 and Theorem 11.7). In the TC-case, the sequence is p -equivalent to

$$\mathrm{TC}(\mathbb{S})_p \vee \bigvee_{i>0} \Sigma((\mathbb{S}^{2d})^{\otimes i})_{hC_i} \longrightarrow \mathrm{TC}(\mathbb{S})[S^1]_p \vee \bigvee_{i>0} \Sigma((\mathbb{S}^{2d})^{\otimes i})_{hC_i} \longrightarrow \Sigma \mathrm{TC}(\mathbb{S})_p.$$

1.4 | The fraction field of topological K -theory

Theorem 1.4 generalizes to prelog $\mathbb{E}_k$ -rings $(A, \xi : \langle x_1, \dots, x_r \rangle \rightarrow \mathrm{Pic}_{\mathbb{S}}, \bar{\alpha})$ where $\langle x_1, \dots, x_r \rangle$ is the free commutative monoid on r generators. In this case, we obtain an r -dimensional cube of cofiber sequences with initial term $\mathrm{THH}(A)$ and final term $\Sigma^r \mathrm{THH}(A//\bar{\alpha})$. The general case is spelled out in Theorem 10.6.

In the special case $(\ell, \langle p, v_1 \rangle)$ of the Adams summand with the $\mathbb{E}_2$ -prelog structure generated by $p \in \pi_0(\ell)$ and $v_1 \in \pi_{2p-2}(\ell)$, we get a diagram of horizontal and vertical cofiber sequences of cyclotomic $\mathrm{THH}(\ell)$ -modules

$$\begin{array}{ccccc} \mathrm{THH}(\ell) & \xrightarrow{\rho} & \mathrm{THH}(\ell, \langle p \rangle) & \xrightarrow{\mathrm{res}} & \Sigma \mathrm{THH}(\ell/p) \\ \rho \downarrow & & \rho \downarrow & & \Sigma \rho \downarrow \\ \mathrm{THH}(\ell, \langle v_1 \rangle) & \xrightarrow{\rho} & \mathrm{THH}(\ell, \langle p, v_1 \rangle) & \xrightarrow{\mathrm{res}} & \Sigma \mathrm{THH}(\ell/p, \langle v_1 \rangle) \\ \mathrm{res} \downarrow & & \mathrm{res} \downarrow & & \Sigma \mathrm{res} \downarrow \\ \Sigma \mathrm{THH}(\mathbb{Z}_{(p)}) & \xrightarrow{\Sigma \rho} & \Sigma \mathrm{THH}(\mathbb{Z}_{(p)}, \langle p \rangle) & \xrightarrow{\Sigma \mathrm{res}} & \Sigma^2 \mathrm{THH}(\mathbb{Z}/p), \end{array} \quad (1.2)$$

where the lower right-hand square anti-commutes and the three other squares commute. For this diagram, we extend our definition of $\log \mathrm{THH}$ to cover the case of the $\mathbb{E}_1$ -ring ℓ/p by setting $\mathrm{THH}(\ell/p, \langle v_1 \rangle) = \mathrm{THH}(\ell/p) \otimes_{\mathrm{THH}(\ell)} \mathrm{THH}(\ell, \langle v_1 \rangle)$. Such a diagram of cofiber sequences was considered by Ausoni and the first author in the unpublished version [10] of [11]. There, the $\log \mathrm{THH}$ terms constructed here were defined in terms of iterated homotopy cofibers. Based on the calculations made there, the authors hypothesized the existence of a *fraction field of topological K-theory* whose THH , TC and algebraic K -theory should participate in localization sequences of the type we construct. In view of this, the present results confirm that the prelog $\mathbb{E}_2$ -ring $(\ell, \langle p, v_1 \rangle)$ should be viewed as a model for this fraction field of (the Adams summand of) topological K -theory. We intend to demonstrate the connection between the calculations in [10, 11] and our present constructions in more detail in future work.

1.5 | Organization

Section 2 is about R -algebra Thom spectra. Section 3 discusses $\mathbb{E}_k$ -group completions and relative variants thereof. Section 4 contains the basic definitions and constructions of R -based prelog $\mathbb{E}_k$ -rings. Section 5 features the construction of $\log \mathrm{THH}$ and in Section 6 we introduce $\log \mathrm{TC}$. Section 7 is about (log) THH of Thom spectra. In Section 8, we introduce the log condition and the logification construction. Sections 9 and 10 contain the construction of the repletion–residue sequences, including a proof of Theorem 1.4 and the construction of the diagram (1.2). The two appendices discuss the relations between the present approach and our previous results in [56, 57].

Notation and conventions

Except from in the appendices, we work ∞ -categorically and use [38, 45, 47] as primary references. We write $\mathcal{S}$ for the ∞ -category of spaces and Sp for the ∞ -category of spectra, often viewed as the symmetric monoidal ∞ -categories $(\mathcal{S}, \times, *)$ and $(\mathrm{Sp}, \otimes, \mathbb{S})$.

2 | THOM SPECTRA

Let R be an $\mathbb{E}_\infty$ -ring. The goal of this section is to review the symmetric monoidal Thom R -module functor Th_R and to study its right adjoint.

2.1 | A construction principle for Thom spectrum functors

We follow the approach to Th_R in [36, §3] (see also [2, 3, 8]) and work in a slightly more general setup that will become useful in Appendix A.

Let P be an $\mathbb{E}_\infty$ -space. The slice ∞ -category $S_{/P}$ inherits a symmetric monoidal structure [47, Remark 2.2.2.5] and there is a canonical symmetric monoidal functor $y_P : P \rightarrow S_{/P}$ sending an object p to the corresponding map $\Delta^0 \rightarrow P$ (compare [36, Notation 3.1.2]). The functor ∞ -category $\mathrm{Fun}(P^{\mathrm{op}}, \mathcal{S})$ has a symmetric monoidal convolution product, and under the symmetric monoidal

equivalence

$$\mathrm{Fun}(P^{\mathrm{op}}, S) \simeq S_{/P} \quad (2.1)$$

(see [36, Notation 3.1.2; 52, Corollary 5.9]), y_P corresponds to the Yoneda embedding of P into its cocompletion $\mathrm{Fun}(P^{\mathrm{op}}, S)$.

Let $(D, \otimes)$ be a presentably symmetric monoidal ∞ -category, that is, D admits all small colimits and $\otimes$ preserves small colimits separately in each variable. The next proposition is a minor generalization of [36, Proposition 3.1.3].

Proposition 2.2. *Composition with $y_P : P \rightarrow S_{/P}$ induces an equivalence of ∞ -categories $\mathrm{LFun}^{\otimes}(S_{/P}, D) \rightarrow \mathrm{Fun}^{\otimes}(P, D)$ from the ∞ -category of colimit-preserving symmetric monoidal functors $S_{/P} \rightarrow D$ to the ∞ -category of symmetric monoidal functors $P \rightarrow D$. $\square$*

Proof. The argument for [36, Proposition 3.1.3] applies also in this situation (see [47, Proposition 4.8.1.10 and Corollary 4.8.1.12] for more details). $\square$

Corollary 2.3 [36, Corollary 3.1.4]. *Let $\iota : P \rightarrow D$ be a symmetric monoidal functor. Then there is an essentially unique colimit-preserving symmetric monoidal functor $F_{\iota} : S_{/P} \rightarrow D$ such that $F_{\iota} \circ y_P \simeq \iota$. $\square$*

Remark 2.4. In the motivating example where ι is the inclusion $\mathrm{Pic}_R \rightarrow \mathrm{Mod}_R$, this F_{ι} is the Thom R -module functor Th_R ; see Definition 2.9.

Under the equivalence (2.1), the functor F_{ι} can by construction be identified with the left Kan extension of $\iota : P \rightarrow S$ along the Yoneda-embedding y_P . Since y_P is fully faithful, we can use [45, Definition 4.3.2.2] to describe the value of this left Kan extension on a functor $Z : P^{\mathrm{op}} \rightarrow S$: It is the colimit over the full subcategory of $\mathrm{Fun}(P^{\mathrm{op}}, S)_{/Z}$ spanned by the objects $y_P(p) \rightarrow Z$ with p an object of P . Using again (2.1), Z corresponds to a map $\xi : X \rightarrow P$ and y_P to the map sending an object of P to the map $\Delta^0 \rightarrow P$ it determines. Under this identification, the ∞ -category over which we take the colimit corresponds to X , and the functor we are taking the colimit over is the composite of the projection to P with ι .

Corollary 2.5. *For an object $\xi : X \rightarrow P$ of $S_{/P}$, the object $F_{\iota}(\xi)$ is naturally equivalent to $\mathrm{colim}_X \iota \xi$. $\square$*

The next construction is motivated by [8, Definition 3.3]. Given an object A of D , we define $P_{\downarrow A}$ by the following pullback of ∞ -categories:

$$\begin{array}{ccc} P_{\downarrow A} & \longrightarrow & D_{/A} \\ \downarrow & & \downarrow \\ P & \xrightarrow{\iota} & D. \end{array} \quad (2.2)$$

Here $D_{/A}$ is the slice of D over A . Since $D_{/A} \rightarrow D$ is a right fibration [45, dual of Proposition 2.1.2.1] and P is assumed to be an ∞ -groupoid, [45, Lemma 2.1.3.3] implies that $P_{\downarrow A}$ is an ∞ -groupoid. In particular, we can view $P_{\downarrow A} \rightarrow P$ as an object of $S_{/P}$. This defines a functor

$$G_{\iota} : D \rightarrow S_{/P}, \quad A \mapsto (P_{\downarrow A} \rightarrow P).$$

For an object A in $\mathcal{D}$, we note that the functor F_l sends $G_l(A) = (P_{\downarrow A} \rightarrow P)$ to the colimit of the composite $P_{\downarrow A} \rightarrow P \rightarrow \mathcal{D}$. Since this map factors through the canonical map $\mathcal{D}_{/A} \rightarrow \mathcal{D}$, and the latter map preserves colimits [45, Proposition 1.2.13.8], we can form the colimit in $\mathcal{D}_{/A}$ and then project to $\mathcal{D}$. This defines a map from the colimit of $P_{\downarrow A} \rightarrow \mathcal{D}$ to A , which we denote by $\varepsilon_A : F_l(G_l(A)) \rightarrow A$. It is natural by the naturality of taking slices, pullbacks, and colimits.

Proposition 2.6. *The functor F_l is left adjoint to G_l with counit $\varepsilon : F_l \circ G_l \rightarrow \text{id}_{\mathcal{D}}$.*

Proof. We apply [38, Proposition 5.1.13], which is the dual of [48, Tag 02FX]. For this we need to check that the composite

$$\text{Map}_{S/P}(\xi, G_l(A)) \xrightarrow{F_l} \text{Map}_{\mathcal{D}}(F_l(\xi), F_l(G_l(A))) \xrightarrow{(\varepsilon_A)_*} \text{Map}_{\mathcal{D}}(F_l(\xi), A) \quad (2.3)$$

is an equivalence for all $\xi : X \rightarrow P$ in S_P and A in $\mathcal{D}$. Writing the ∞ -groupoid X as the colimit over X of a terminal diagram reduces the claim to the case where X is a point.

To verify this case, recall that $\mathcal{D}_{/A}$ is cocomplete and that $\text{pr} : \mathcal{D}_{/A} \rightarrow \mathcal{D}$ preserves colimits. Writing C' for the colimit of the map $P_{\downarrow A} \rightarrow \mathcal{D}_{/A}$ from (2.2), it follows that $C = \text{pr}(C') = F_l(G_l(A))$ is a colimit of the composite $P_{\downarrow A} \rightarrow P \rightarrow \mathcal{D}$. The choice of a colimit cone for C' provides a canonical map $P_{\downarrow A} \rightarrow (\mathcal{D}_{/A})_{/C'}$ that sends the objects of $P_{\downarrow A}$ to the values of the natural transformation from $P_{\downarrow A} \rightarrow \mathcal{D}_{/A}$ to the constant $P_{\downarrow A}$ -diagram with value C' . We now consider the composite

$$P_{\downarrow A} \rightarrow (\mathcal{D}_{/A})_{/C'} \rightarrow \mathcal{D}_{/C} \rightarrow \mathcal{D}_{/A} \quad (2.4)$$

induced by the map just discussed, the projection $\text{pr} : \mathcal{D}_{/A} \rightarrow \mathcal{D}$, and $\varepsilon_A : C \rightarrow A$. Informally, the first map sends an object $(\xi(*), \iota\xi(*) \rightarrow A)$ of $P_{\downarrow A}$ to

$$\begin{array}{ccc} \iota\xi(*) & \longrightarrow & C \\ \downarrow & & \downarrow \\ A & \xrightarrow{=} & A, \end{array}$$

the second map sends this object to $\iota\xi(*) \rightarrow C$, and the last map sends the latter to $\iota\xi(*) \rightarrow A$. More formally, the natural transformation associated with the colimit cone for C' provides an equivalence between the composite in (2.4) and the base change of ι in (2.2). Next we consider the following diagram

$$\begin{array}{ccccc} \Delta^0 & \xrightarrow{\xi} & P & \longleftarrow & P_{\downarrow A} \\ \parallel & & \downarrow \iota & & \downarrow \\ \Delta^0 & \xrightarrow{\iota\xi} & \mathcal{D} & \longleftarrow & \mathcal{D}_{/C} \\ \parallel & & \parallel & & \downarrow \\ \Delta^0 & \xrightarrow{\iota\xi} & \mathcal{D} & \longleftarrow & \mathcal{D}_{/A} \end{array} \quad (2.5)$$

in which the upper right-hand vertical map is the composite of the first two maps in (2.4). The natural equivalences $\text{Map}_S(\Delta^0, P) \simeq P$ and $\text{Map}_S(\Delta^0, P_{\downarrow A}) \simeq P_{\downarrow A}$ and [38, Proposition 3.3.18] imply that the horizontal homotopy pullback of the first row is a model for the mapping space $\text{Map}_{S/P}(\xi, P_{\downarrow A} \rightarrow P)$. By [38, Corollary 2.5.34], the horizontal homotopy pullbacks of the second

and third rows are models for the mapping spaces $\mathrm{Map}_D(\iota\xi(*), C)$ and $\mathrm{Map}_D(\iota\xi(*), A)$, respectively. By construction, when passing to pullbacks the vertical maps in (2.5) induce the maps in the composite (2.3). The composite of the two right-hand vertical squares in (2.5) is equivalent to the homotopy pullback (2.2) defining $P_{\downarrow A}$. Hence the induced map of horizontal homotopy fibers is an equivalence, showing the claim. $\square$

Corollary 2.7. *The functor $G_i : D \rightarrow S/P$ is lax symmetric monoidal, that is, it is a map of ∞ -operads between the underlying ∞ -operads of the symmetric monoidal categories D and S/P .*

Proof. This is a consequence of [47, Corollary 7.3.2.7]. $\square$

2.2 | Thom R -modules

Recall that R denotes an $\mathbb{E}_\infty$ -ring. Let Mod_R be the symmetric monoidal ∞ -category of R -modules and Pic_R the maximal subgroupoid of Mod_R spanned by the invertible R -modules, which inherits the structure of a grouplike $\mathbb{E}_\infty$ -space.

Definition 2.9. The Thom R -module functor is the colimit-preserving symmetric monoidal left adjoint in the adjunction

$$\mathrm{Th}_R : S/\mathrm{Pic}_R \rightleftarrows \mathrm{Mod}_R : \Omega^R$$

that arises from applying Corollary 2.3 and Proposition 2.6 to the inclusion $\iota : \mathrm{Pic}_R \rightarrow \mathrm{Mod}_R$. In particular, $\mathrm{Th}_R(\xi : M \rightarrow \mathrm{Pic}_R) \simeq \mathrm{colim}(\iota\xi : M \rightarrow \mathrm{Mod}_R)$. We write Ω^R for the right adjoint, which is lax symmetric monoidal by Corollary 2.7. Informally, $\Omega^R(A)$ is the space over Pic_R of R -module maps $P \rightarrow A$, with P ranging through Pic_R .

Let $1 \leq k \leq \infty$ count a number of coherently commuting multiplications. As a consequence of [47, Remark 7.3.2.13], we get an induced adjunction

$$\mathrm{Th}_R : \mathrm{Alg}_{\mathbb{E}_k}(S/\mathrm{Pic}_R) \rightleftarrows \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Mod}_R) : \Omega^R \quad (2.6)$$

on the categories of $\mathbb{E}_k$ -algebras (or, more generally, for algebras over an ∞ -operad). By abuse of notation, we often also write $\Omega^R(A)$ for the total space of $\Omega^R(A) = ((\mathrm{Pic}_R)_{\downarrow A} \rightarrow \mathrm{Pic}_R)$ and leave the $\mathbb{E}_k$ -structure map implicit.

We will often be interested in the composites

$$u\mathrm{Th}_R : S/\mathrm{Pic}_R \rightarrow \mathrm{Sp} \quad \text{and} \quad u\mathrm{Th}_R : \mathrm{Alg}_{\mathbb{E}_k}(S/\mathrm{Pic}_R) \rightarrow \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Sp}) \quad (2.7)$$

of Th_R with the lax symmetric monoidal functor $u = (\mathbb{S} \rightarrow R)^* : \mathrm{Mod}_R \rightarrow \mathrm{Sp}$. More explicitly, $u\mathrm{Th}_R(\xi : M \rightarrow \mathrm{Pic}_R) \simeq \mathrm{colim}(u\xi : M \rightarrow \mathrm{Sp})$.

Remark 2.10. The adjunction (2.6) induces an equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Mod}_R)}(\mathrm{Th}_R(\xi), A) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_k}(S/\mathrm{Pic}_R)}(\xi, \Omega^R(A)),$$

which recovers the universal property of Thom spectra from [8, Theorem 3.5].

We will now explain why Ω^R generalizes the functor sending a ring to its underlying multiplicative monoid. For this we specialize to the case $R = \mathbb{S}$ and $k = 1$, consider an $\mathbb{E}_1$ -ring A , and define a $\mathbb{Z}$ -graded monoid $\pi_{0,*}(\Omega^{\mathbb{S}}(A))$ by letting $\pi_{0,n}(\Omega^{\mathbb{S}}(A))$ be the set of path components of the homotopy fiber of the $\text{Pic}_{\mathbb{S}}$ -augmented $\mathbb{E}_1$ -space $\Omega^{\mathbb{S}}(A)$ over $n \in \pi_0(\text{Pic}_{\mathbb{S}})$. Then $\pi_{0,*}(\Omega^{\mathbb{S}}(A))$ inherits an associative and unital graded multiplication from A . Moreover, on each $\pi_{0,n}(\Omega^{\mathbb{S}}(A))$ we get a $\{\pm 1\} = \pi_1(\text{Pic}_{\mathbb{S}})$ -action as in the long exact sequence of homotopy groups.

With this multiplication and sign action, $\pi_{0,*}(\Omega^{\mathbb{S}}(A))$ is a *graded signed monoid* in the sense of [60, Definition 4.15], meaning that sign action and multiplication are suitably compatible. The $\mathbb{Z}$ -graded ring $\pi_*(R)$ also forms a graded signed monoid by forgetting all of the additive structure but the sign action. The following statement will be proved at the end of Appendix A:

Proposition 2.11. *Let A be an $\mathbb{E}_1$ -ring. Then $\pi_{0,*}(\Omega^{\mathbb{S}}(A))$ and $\pi_*(A)$ are isomorphic as graded signed monoids.*

This proposition motivates our approach to (pre-)log structures in the next sections, although only the comparison in Lemma 8.12 relies on it.

Remark 2.12. While there are model categorical versions of the right adjoint to the Thom spectrum functor when working with $\mathbb{Z} \times BO$ instead of $\text{Pic}_{\mathbb{S}}$ or Pic_R (see Appendix A or [63]), we are not aware of a reference where this adjunction is established for general R . In line with this, we are not aware of an ∞ -categorical proof of Proposition 2.11, but have not pursued such an ∞ -categorical proof ourselves.

2.3 | Weight-graded Thom spectra

We will often consider functors with values in slice categories and begin with the following general observation:

Lemma 2.14. *Let K be a simplicial set, C an ∞ -category, and X an object of C . Then a lift of a functor $F : K \rightarrow C$ to $C_{/X}$ is the same data as a natural transformation $F \rightarrow \text{const}_K X = (K \rightarrow \Delta^0)^* X$ from F to the constant K -diagram with value X .*

In fact, $\text{Fun}(K, C_{/X}) \rightarrow \text{Fun}(K, C)$ and $\text{Fun}(K, C)_{/\text{const}_K X} \rightarrow \text{Fun}(K, C)$ are equivalent as right fibrations over $\text{Fun}(K, C)$, but we will not use this stronger statement.

Proof of Lemma 2.14. Using [38, Proposition 2.5.27], we may replace $C_{/X}$ by the fat slice $C^{/X}$. By [38, Lemma 4.4.7 and Proposition 4.3.14], the fiber of the right fibration $\text{Fun}(K, C^{/X}) \rightarrow \text{Fun}(K, C)$ over F is $\text{map}_{\text{Fun}(K, C)}(F, \text{const}_K X)$. $\square$

We now return to Thom spectra. In our applications, they will often come with an additional *weight* grading. To explain this structure, let $(I, +, 0)$ be a discrete commutative monoid, such as $\{0\}$, $\mathbb{Z}_{\geq 0}$ or $\mathbb{Z}$, viewed as a symmetric monoidal ∞ -category. Moreover, let C be a symmetric monoidal ∞ -category with all small colimits. Then an I -graded object in C is a functor $I \rightarrow C$, and we view the functor ∞ -category $\text{Fun}(I, C)$ as a symmetric monoidal ∞ -category equipped with the Day convolution product [47, §2.2.6].

When $C = S$, we note that any map of spaces $\tau : M \rightarrow I$ is automatically a co-Cartesian fibration, classified as in [45, §3.2] by the functor $M_* : I \rightarrow S$ mapping $i \in I$ to $M_i = \tau^{-1}(i)$. We refer to M_i as the *weight i graded piece* of M . The colimit of M_* recovers $M = \coprod_{i \in I} M_i$. We have the following characterization of multiplicative structure on the weight grading:

Lemma 2.15. *Let I be a discrete commutative monoid and let $\tau : M \rightarrow I$ be a map of spaces. Then $\tau : M \rightarrow I$ is an $\mathbb{E}_k$ -map of spaces if and only if the functor M_* is lax $\mathbb{E}_k$ -monoidal, which holds if and only if M_* is an $\mathbb{E}_k$ -algebra in $\text{Fun}(I, S)$. $\square$*

Suppose also given an $\mathbb{E}_k$ -map $\xi : M \rightarrow \text{Pic}_R$, and let $\xi_i = \xi|_{M_i} : M_i \rightarrow \text{Pic}_R$ for each $i \in I$. Then mapping $i \in I$ to ξ_i defines a lax $\mathbb{E}_k$ -monoidal functor $\xi_* : I \rightarrow S/\text{Pic}_R$, hence also an $\mathbb{E}_k$ -algebra in $\text{Fun}(I, S/\text{Pic}_R)$. To emphasize the domains of the ξ_i , we often present ξ_* as a map $M_* \rightarrow c^* \text{Pic}_R$, where c^* is the restriction along $c : I \rightarrow \{0\}$ (and we implicitly apply Lemma 2.14). The colimit of ξ_* in S/Pic_R over I recovers ξ in S/Pic_R . The functor Th_R of Definition 2.9 induces a symmetric monoidal functor $\text{Fun}(I, S/\text{Pic}_R) \rightarrow \text{Fun}(I, \text{Mod}_R)$ taking ξ_* to the $\mathbb{E}_k$ -algebra $\text{Th}_R(\xi_*)$ in $\text{Fun}(I, \text{Mod}_R)$, mapping $i \in I$ to $\text{Th}_R(\xi_i)$ in Mod_R . Its colimit over I recovers $\text{Th}_R(\xi) = \bigvee_{i \in I} \text{Th}_R(\xi_i)$. Since the monoid pairing $\cdot$ on M takes $M_i \times M_j$ to M_{i+j} for all $i, j \in I$, the underlying R -algebra pairing on $\text{Th}_R(\xi)$ takes $\text{Th}_R(\xi_i) \otimes_R \text{Th}_R(\xi_j)$ to $\text{Th}_R(\xi_{i+j})$. Likewise, for the functor $u\text{Th}_R$ of (2.7), $u\text{Th}_R(\xi_*)$ is an $\mathbb{E}_k$ -algebra in $\text{Fun}(I, \text{Sp})$, with underlying $\mathbb{E}_1$ -ring pairing taking $u\text{Th}_R(\xi_i) \otimes u\text{Th}_R(\xi_j)$ to $u\text{Th}_R(\xi_{i+j})$.

The following definition is our main source for weight gradings:

Definition 2.16. Let M be an $\mathbb{E}_k$ -space such that $\pi_0(M)$ is commutative. The 0th Postnikov truncation $\tau : M \rightarrow \tau_{\leq 0} M \xrightarrow{\cong} \pi_0(M)$ is an $\mathbb{E}_k$ -map and defines the *canonical $\pi_0(M)$ -grading* of M .

Example 2.17. For an $\mathbb{E}_k$ -ring A , Proposition 2.11 implies that $\pi_0(\Omega^{\mathbb{S}}(A)) \cong \pi_*(A)/\{\pm 1\}$. If $k \geq 2$, then $\pi_*(A)$ is graded commutative, $\pi_*(A)/\{\pm 1\}$ is commutative, and $\Omega^{\mathbb{S}}(A)$ is canonically graded by $\pi_*(A)/\{\pm 1\}$.

2.4 | Collapse maps

Let $f : J \rightarrow I$ be a homomorphism of discrete commutative monoids. Then restriction and left Kan extension along f induce an adjunction

$$f_! : \text{Fun}(J, \text{Mod}_R) \rightleftarrows \text{Fun}(I, \text{Mod}_R) : f^* \quad (2.8)$$

with a lax symmetric monoidal right adjoint f^* and a symmetric monoidal left adjoint $f_!$. If we assume that $f : J \rightarrow I$ is the inclusion of a submonoid, then the left and right Kan extensions

$$f_! \simeq f_* : \text{Fun}(J, \text{Mod}_R) \longrightarrow \text{Fun}(I, \text{Mod}_R)$$

along f both map $Y_* : J \rightarrow \text{Mod}_R$ to $f_! Y_* \simeq f_* Y_*$ given by sending $i \in I$ to Y_i if $i \in J$ and to 0 for $i \in I \setminus J$. These naturally equivalent functors are both fully faithful and symmetric monoidal.

The following condition on a submonoid $J \subseteq I$ will be useful for controlling the multiplicative properties of collapsing the part of $\mathrm{Th}_R(\xi_*)$ complementary to J .

Definition 2.19 [50, §1.4]. A *face* of a discrete commutative monoid I is a submonoid J such that $i + j \in J$ for $i, j \in I$ only if $i \in J$ and $j \in J$.

Equivalently, $J \subseteq I$ is a face if the complement $I \setminus J$ is a prime ideal in I (again, compare [50, §1.4]).

If J is a face of I , then f^* is symmetric monoidal because we have

$$(f^*(X_* \otimes_R Y_*))_n = \bigvee_{\substack{i, j \in I \\ i+j=n}} X_i \otimes_R Y_j = \bigvee_{\substack{i, j \in J \\ i+j=n}} X_i \otimes_R Y_j = (f^*(X_*) \otimes_R f^*(Y_*))_n$$

for $n \in J$, and f^* preserves the unit. Moreover, the localization functor $f_* f^*$ is compatible with the symmetric monoidal structure on $\mathrm{Fun}(I, \mathrm{Mod}_R)$, so by [47, Proposition 2.2.1.9] it is a symmetric monoidal functor receiving a symmetric monoidal natural transformation from the identity. Applied to the $\mathbb{E}_k$ -algebra $\mathrm{Th}_R(\xi_*)$ we get a natural collapse $\mathbb{E}_k$ -map of I -graded $\mathbb{E}_k$ R -algebras

$$\mathrm{coll}_*^{I \setminus J} : \mathrm{Th}_R(\xi_*) \longrightarrow f_* f^* \mathrm{Th}_R(\xi_*)$$

sending i to $\mathrm{Th}_R(\xi_i)$ for $i \in J$ and to 0 for $i \in I \setminus J$. Passing to total objects, that is, colimits over I , we obtain the $\mathbb{E}_k$ R -algebra collapse map

$$\mathrm{coll}^{I \setminus J} : \mathrm{Th}_R(\xi) = \bigvee_{i \in I} \mathrm{Th}_R(\xi_i) \longrightarrow \bigvee_{i \in J} \mathrm{Th}_R(\xi_i) = \mathrm{Th}_R(\xi|_{M_J}) \quad (2.9)$$

collapsing the summands with $i \in I \setminus J$ to 0. Here $M_J = \tau^{-1}(J)$. We note that this construction is functorial for face inclusions, that is, for faces $K \subseteq J$ and $J \subseteq I$ we have $\mathrm{coll}^{I \setminus K} = \mathrm{coll}^{J \setminus K} \circ \mathrm{coll}^{I \setminus J}$.

These collapse maps will become relevant for the construction of the quotient objects appearing in our localization sequences (see Sections 4.11, 9 and 10). For these applications, it is useful to observe that $\mathrm{coll}^{I \setminus J}$ has structure not captured by the above construction. To see this, we consider the following more general construction also relevant in the next section.

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a symmetric monoidal left adjoint functor between presentably symmetric monoidal ∞ -categories, with right adjoint G , and let A be an $\mathbb{E}_k$ -algebra in $\mathcal{C}$. Then $F(A)$ is an $\mathbb{E}_k$ -algebra in $\mathcal{D}$, the categories LMod_A (of left A -modules) and $\mathrm{LMod}_{F(A)}$ inherit $\mathbb{E}_{k-1}$ -monoidal structures, and F induces the left adjoint in an adjunction

$$\mathrm{Alg}_{\mathbb{E}_{k-1}}(\mathrm{LMod}_A(\mathcal{C})) \rightleftarrows \mathrm{Alg}_{\mathbb{E}_{k-1}}(\mathrm{LMod}_{F(A)}(\mathcal{D})). \quad (2.10)$$

Since F is symmetric monoidal, its right adjoint G is lax symmetric monoidal. So the right adjoint in (2.10) can be identified with the composite of the functor

$$\mathrm{Alg}_{\mathbb{E}_{k-1}}(\mathrm{LMod}_{F(A)}(\mathcal{D})) \rightarrow \mathrm{Alg}_{\mathbb{E}_{k-1}}(\mathrm{LMod}_{GF(A)}(\mathcal{C}))$$

induced by G and the restriction along the adjunction unit $A \rightarrow G(F(A))$.

In the above situation where $J \subseteq I$ is a face, we can apply this to $F = f^*$ and $A = \mathrm{Th}_R(\xi_*)$. Evaluating the adjunction unit on the $\mathbb{E}_{k-1} \mathrm{Th}_R(\xi_*)$ -algebra $\mathrm{Th}_R(\xi_*)$ and passing to total objects, we obtain the following result:

Lemma 2.20. *The underlying $\mathbb{E}_{k-1}$ -map of the collapse map (2.9) extends to an $\mathbb{E}_{k-1} \mathrm{Th}_R(\xi)$ -algebra structure on $\mathrm{Th}_R(\xi|M_J)$. $\square$*

3 | GROUP COMPLETION AND REPLETION

As before, let R be an $\mathbb{E}_\infty$ -ring, $k \geq 1$ a number of coherently commuting multiplications, and I a discrete commutative monoid.

3.1 | Group completion

For any $\mathbb{E}_k$ -space M , the $\mathbb{E}_k$ -space $M^{\mathrm{gp}} = \Omega BM$ provides a group completion of M that comes with a natural group completion $\mathbb{E}_k$ -map $\gamma : M \rightarrow M^{\mathrm{gp}}$ (see [39, Proposition 2.8] for a modern treatment).

Lemma 3.2. *A map $M \rightarrow M'$ in $\mathrm{Alg}_{\mathbb{E}_k}(S)$ is equivalent to the group completion $M \rightarrow M^{\mathrm{gp}}$ if and only if M' is grouplike and $BM \rightarrow BM'$ is an equivalence.*

Proof. This follows because [39, Proposition 2.8] implies that $M' \rightarrow (M')^{\mathrm{gp}}$ is an equivalence if M' is grouplike. $\square$

To lift $(-)^{\mathrm{gp}}$ to our relative and graded situations, it will be convenient to view it as a localization. For this, we write $\mathbb{E}_k(*)$ for the free $\mathbb{E}_k$ -space on the one-point space in (unbased) spaces and consider the set of morphisms

$$W = \{\gamma_{\mathbb{E}_k(*)} : \mathbb{E}_k(*) \rightarrow \mathbb{E}_k(*)^{\mathrm{gp}}, \mathrm{id} : \mathbb{E}_k(*)^{\mathrm{gp}} \rightarrow \mathbb{E}_k(*)^{\mathrm{gp}}\}$$

of $\mathrm{Alg}_{\mathbb{E}_k}(S)$. By arguing with the path components, we see that $\mathrm{Alg}_{\mathbb{E}_k}(S)^{\mathrm{gp}}$ is the collection of *weakly W -local objects* in the sense of [48, Tag 04LH]. Moreover, since the pushout of

$$\mathbb{E}_k(*)^{\mathrm{gp}} \leftarrow \mathbb{E}_k(*) \rightarrow \mathbb{E}_k(*)^{\mathrm{gp}} \text{ in } \mathrm{Alg}_{\mathbb{E}_k}(S) \text{ is } \mathbb{E}_k(*)^{\mathrm{gp}},$$

the set W is closed under taking *relative codiagonals* [48, Tag 04K7], that is, the relative codiagonal of every morphism in W belongs to W . Hence [48, Tag 04LY] implies that $\mathrm{Alg}_{\mathbb{E}_k}(S)^{\mathrm{gp}}$ also coincides with the collection of W -local objects in $\mathrm{Alg}_{\mathbb{E}_k}(S)$.

Lemma 3.3.

- (i) *The subcategory $\mathrm{Alg}_{\mathbb{E}_k}(S)^{\mathrm{gp}}$ of grouplike objects is a reflective localization of $\mathrm{Alg}_{\mathbb{E}_k}(S)$.*
- (ii) *The $\mathrm{Alg}_{\mathbb{E}_k}(S)^{\mathrm{gp}}$ -reflection of an $\mathbb{E}_k$ -space M can be obtained by applying the ∞ -categorical small object argument [48, Tag 04MA] to W and M .*
- (iii) *The $\mathrm{Alg}_{\mathbb{E}_k}(S)^{\mathrm{gp}}$ -reflection of M is equivalent to the group completion of M .*

Proof. Part (i) follows from [48, Tag 04MD] and part (ii) follows from [48, Tag 04MD]. The last part follows from (ii), Lemma 3.2, and the observation that B preserves colimits. $\square$

Let G be a grouplike $\mathbb{E}_k$ -space and let W/G be the set of morphisms

$$\{(M \rightarrow G) \rightarrow (M' \rightarrow G) \mid M \rightarrow M' \in W\}$$

in $\text{Alg}_{\mathbb{E}_k}(S)_{/G}$.

Lemma 3.4. *The set W/G is closed under taking relative codiagonals, and an object $M \rightarrow G$ in $\text{Alg}_{\mathbb{E}_k}(S)_{/G}$ is (weakly) W/G -local if and only if M is grouplike.*

Proof. The first assertion follows from the definition of pushouts in $\text{Alg}_{\mathbb{E}_k}(S)_{/G}$ and the second from the definitions of W/G and of weakly local objects. $\square$

Let I^{gp} be the algebraic group completion of I . In the rest of this subsection, we will use the shorthand notation

$$\text{Alg}^{(k)} = \text{Alg}_{\mathbb{E}_k}(\text{Fun}(I^{\text{gp}}, S_{/\text{Pic}_R}))$$

for the category of I^{gp} -graded $\mathbb{E}_k$ -algebras in $S_{/\text{Pic}_R}$. We say that an object $\zeta_* : N_* \rightarrow c^*\text{Pic}_R$ in $\text{Alg}^{(k)}$ is *grouplike* if the underlying total $\mathbb{E}_k$ -space N is.

Corollary 3.5. *The full subcategory $\text{Alg}^{(k),\text{gp}} \subseteq \text{Alg}^{(k)}$ of grouplike objects is a reflective localization. It is spanned by the (weakly) W' -local objects for a set W' of morphisms that is closed under taking relative codiagonals.*

Proof. Identifying $\text{Alg}^{(k)}$ with $(\text{Alg}_{\mathbb{E}_k}(S)_{/\text{Pic}_R})_{/I^{\text{gp}}}$ as in Lemma 2.15, this follows from the previous lemma, taking G to be $I^{\text{gp}} \times \text{Pic}_R$. $\square$

In analogy with Lemma 3.3, this shows that the reflection provides a group completion that can be constructed using the small object argument with respect to W' .

Our next aim is to construct a relative group completion. For this we assume $k \geq 2$. As in adjunction (2.10), we consider $\mathbb{E}_{k-1}$ -algebras in left modules over an object $\zeta_* : N_* \rightarrow c^*\text{Pic}_R$ in $\text{Alg}^{(k)}$ and denote this ∞ -category by

$$\text{Alg}_{\zeta_*}^{(k-1)} = \text{Alg}_{\mathbb{E}_{k-1}}(\text{LMod}_{\zeta_*}(\text{Fun}(I^{\text{gp}}, S_{/\text{Pic}_R}))).$$

Writing ι_* for the initial object of $\text{Alg}^{(k)}$, the ∞ -category $\text{Alg}_{\iota_*}^{(k-1)}$ is equivalent to $\text{Alg}^{(k-1)}$ and the unit map $i : \iota_* \rightarrow \zeta_*$ induces an adjunction

$$i_! : \text{Alg}^{(k-1)} \rightleftarrows \text{Alg}_{\zeta_*}^{(k-1)} : i^* . \quad (3.1)$$

We say that an $\mathbb{E}_{k-1}$ ζ_* -algebra ζ'_* is *grouplike* if the underlying $\mathbb{E}_{k-1}$ algebra $i^*(\zeta'_*)$ is grouplike in $\text{Alg}^{(k-1)}$.

Corollary 3.6.

- (i) The set $i_!(W')$ arising from the set W' of Corollary 3.5 is closed under taking relative codiagonals.
- (ii) The full subcategory $\text{Alg}_{\zeta_*}^{(k-1), \text{gp}} \subseteq \text{Alg}_{\zeta_*}^{(k-1)}$ of grouplike objects is a reflective localization. It is spanned by the (weakly) $i_!(W')$ -local objects and the group completion can be constructed using the small object argument for $i_!(W')$.

Proof. Part (i) follows because $i_!$ preserves pushouts. Part (ii) is again analogous to Lemma 3.3 and uses the adjunction $(i_!, i^*)$. $\square$

This corollary provides another way to group-complete an object ζ_* of $\text{Alg}^{(k)}$: Viewing ζ_* as an object of $\text{Alg}_{\zeta_*}^{(k-1)}$, we can apply the group completion functor from the last corollary to get an $\mathbb{E}_{k-1}$ ζ_* -algebra ζ_*^{gp} . This construction neither determines nor is determined by the group completion in $\text{Alg}^{(k)}$ (compare [47, Warning 7.1.3.9]). However, we have the following compatibility:

Proposition 3.7. *The unit map $\zeta_* \rightarrow \zeta_*^{\text{gp}}$ of the $\mathbb{E}_{k-1}$ ζ_* -algebra structure on ζ_*^{gp} is a group completion of ζ_* in $\text{Alg}^{(k-1)}$. In particular, it is equivalent to the $\mathbb{E}_{k-1}$ -map underlying the group completion $\zeta_* \rightarrow \zeta_*^{\text{gp}}$ of ζ_* as an $\mathbb{E}_k$ -algebra.*

Proof. Lemma 3.2 implies that for the $\mathbb{E}_k$ -group completion $\zeta_* \rightarrow \zeta_*^{\text{gp}}$, the underlying $\mathbb{E}_{k-1}$ -map is an $\mathbb{E}_{k-1}$ -group completion.

For the remaining first claim, we need to show that the unit map $\zeta_* \rightarrow \zeta_*^{\text{gp}}$ is a group completion in $\text{Alg}^{(k-1)}$. We may assume that ζ_*^{gp} is obtained from ζ_* by the small object argument with respect to $i_!(W')$. Since i^* preserves filtered (or, more generally, sifted) colimits, it is enough to show that when f in $\text{Alg}_{\zeta_*}^{(k-1)}$ is a pushout of a morphism in $i_!(W')$, then $i^*(f)$ is the pushout of a morphism in W' . This holds because i^* preserves pushouts (compare Remark 4.13) and any map in $i^*i_!(W')$ is the pushout of a map in W' . $\square$

3.2 | Repletion

We now set up the relative group completion that will be a crucial ingredient for our constructions of log THH.

Construction 3.9. Suppose given $\mathbb{E}_k$ -maps $\xi : M \rightarrow \text{Pic}_R$ and $\tau : M \rightarrow I$. Writing $\gamma : I \rightarrow I^{\text{gp}}$ for the algebraic group completion of I , we can form $\xi_*^{\text{gp}} = (\gamma_! \xi_*)^{\text{gp}}$ in $\text{Alg}_{\mathbb{E}_k}(\text{Fun}(I^{\text{gp}}, \mathcal{S}_{/\text{Pic}_R}))$ and view it as an $\mathbb{E}_{k-1}$ $\gamma_! \xi_*$ -algebra or an $\mathbb{E}_k$ -algebra under $\gamma_! \xi_*$ (compare Proposition 3.7). Encoding the gradings by maps to I and I^{gp} , these maps can be summarized by the commutative diagram

$$\begin{array}{ccc}
 I & \xleftarrow{\tau} & M \xrightarrow{\xi} \text{Pic}_R \\
 \gamma \downarrow & & \downarrow \gamma \\
 I^{\text{gp}} & \xleftarrow{\tau^{\text{gp}}} & M^{\text{gp}} \xrightarrow{\xi^{\text{gp}}} \text{Pic}_R
 \end{array} \quad (3.2)$$

By the symmetric monoidality of Th_R , the group completion map $\gamma : M \rightarrow M^{\text{gp}}$ induces a map $\text{Th}_R(\gamma) : \gamma_! \text{Th}_R(\xi_*) \rightarrow \text{Th}_R(\xi_*^{\text{gp}})$ of $\mathbb{E}_k$ R -algebras. By Proposition 3.7, its underlying $\mathbb{E}_{k-1}$ -map extends to an $\mathbb{E}_{k-1}$ $\gamma_! \text{Th}_R(\xi_*)$ -algebra structure on $\text{Th}_R(\xi_*^{\text{gp}})$. In particular, $\text{Th}_R(\xi_*^{\text{gp}})$ is an I^{gp} -graded $\mathbb{E}_k$ R -algebra, that is, an $\mathbb{E}_k$ -algebra in $\text{Fun}(I^{\text{gp}}, \text{Mod}_R)$.

Definition 3.10. In this situation, the *repletion map* is the natural map

$$\rho_* = \mathrm{Th}_R(\gamma_*) : \mathrm{Th}_R(\xi_*) \longrightarrow \gamma^* \mathrm{Th}_R(\xi_*^{\mathrm{gp}}) = \mathrm{Th}_R(\xi_*^{\mathrm{gp}}) \circ \gamma \quad (3.3)$$

of I -graded $\mathbb{E}_k$ R -algebras, given in weight $i \in I$ by the map of Thom R -modules $\rho_i = \mathrm{Th}_R(\gamma_i) : \mathrm{Th}_R(\xi_i) \rightarrow \mathrm{Th}_R(\xi_{\gamma(i)}^{\mathrm{gp}})$ induced by the map $\gamma_i : M_i \rightarrow M_{\gamma(i)}^{\mathrm{gp}}$ over Pic_R .

Arguing with the adjunction (2.10) in the case of the symmetric monoidal left adjoint $\gamma_!$ implies the following statement:

Corollary 3.11. *The underlying $\mathbb{E}_{k-1}$ -map of the repletion map (3.3) extends to an $\mathbb{E}_{k-1}$ $\mathrm{Th}_R(\xi_*)$ -algebra structure on $\gamma^* \mathrm{Th}_R(\xi_*^{\mathrm{gp}})$. $\square$*

Definition 3.12. Let $k \geq 2$. If $\tau : M \rightarrow I$ is an $\mathbb{E}_k$ -map map to a discrete commutative monoid I , then M is *I -replete* if the canonical map $M \rightarrow I \times_{I^{\mathrm{gp}}} M^{\mathrm{gp}}$ is an equivalence. An $\mathbb{E}_k$ -space M is π_0 -*replete* if it is I -replete for the canonical $\pi_0(M)$ -grading, that is, if $M \rightarrow \pi_0(M) \times_{\pi_0(M)^{\mathrm{gp}}} M^{\mathrm{gp}}$ is an equivalence.

It is clear from the definitions that ρ_* is an equivalence if M is I -replete.

Remark 3.13. For discrete and commutative M , the condition that M is I -replete corresponds to $\tau : M \rightarrow I$ being *exact* in the sense of [50, Definition 2.1.15], and repletion corresponds to *exactification* in the terminology of [50, §4.2]. The term ‘replete’ was chosen in [55] to avoid further overloading the word ‘exact’, and ‘repletion’ seems to be easier to say than ‘exactification’.

4 | PICARD-GRADED PRELOGARITHMIC STRUCTURES

As before, let R in $\mathrm{CAlg}(\mathrm{Sp})$ be a fixed base $\mathbb{E}_\infty$ -ring, such as $\mathbb{S}$, MU or $H\mathbb{Z}$, let Mod_R be the stable presentably symmetric monoidal ∞ -category of R -modules, and let $u : \mathrm{Mod}_R \rightarrow \mathrm{Sp}$ be the lax symmetric monoidal forgetful functor. Moreover, let $1 \leq k \leq \infty$ count a number of coherently commuting multiplications.

The following (Picard-graded) notions generalize the prelog and log rings of logarithmic geometry (see, for example, [50]) and the versions of prelog and log ring spectra studied by the authors in earlier work (see, for example, [55, 56]). Appendix B contains a comparison with the latter objects.

Definition 4.1. An R -based prelog $\mathbb{E}_k$ -ring $(A, \xi, \tilde{\alpha})$ consists of an $\mathbb{E}_k$ -ring A , an $\mathbb{E}_k$ -map $\xi : M \rightarrow \mathrm{Pic}_R$, and an $\mathbb{E}_k$ -ring map $\tilde{\alpha} : u\mathrm{Th}_R(\xi) \rightarrow A$.

A prelog $\mathbb{E}_k$ R -algebra $(A, \xi, \tilde{\alpha})$ consists of an $\mathbb{E}_k$ R -algebra A , an $\mathbb{E}_k$ -map $\xi : M \rightarrow \mathrm{Pic}_R$, and an $\mathbb{E}_k$ R -algebra map $\tilde{\alpha} : \mathrm{Th}_R(\xi) \rightarrow A$.

Prelog $\mathbb{E}_k$ R -algebras are the objects of the ∞ -category given by the pullback of

$$\mathrm{Alg}_{\mathbb{E}_k}(S/\mathrm{Pic}_R) \xrightarrow{\mathrm{Th}_R} \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Mod}_R) \xleftarrow{\mathrm{ev}_0} \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Mod}_R)^{\Delta^1},$$

and analogously for R -based prelog $\mathbb{E}_k$ -rings. A map of prelog $\mathbb{E}_k$ R -algebras $(f, f^b) : (A, \xi, \tilde{\alpha}) \rightarrow (B, \eta, \tilde{\beta})$ consists of maps $f : A \rightarrow B$ in $\text{Alg}_{\mathbb{E}_k}(\text{Mod}_R)$ and $f^b : M \rightarrow N$ in $\text{Alg}_{\mathbb{E}_k}(S_{/\text{Pic}_R})$ making the obvious square commutative, and analogously for R -based prelog $\mathbb{E}_k$ -rings.

Each prelog $\mathbb{E}_k$ R -algebra has an underlying R -based prelog $\mathbb{E}_k$ -ring. If $R = \mathbb{S}$, these two notions coincide, and we speak of a *prelog $\mathbb{E}_k$ -ring*. When the context is clear, we refer to $(\xi, \tilde{\alpha})$ as a *prelog structure* on A .

Example 4.2. The *canonical* prelog $\mathbb{E}_k$ R -algebra associated to an $\mathbb{E}_k$ -map $\xi : M \rightarrow \text{Pic}_R$ is $(\text{Th}_R(\xi), \xi, \text{id})$, where $\text{id} : \text{Th}_R(\xi) \rightarrow \text{Th}_R(\xi)$ is the identity map.

4.1 | $\mathbb{E}_2$ prelog structures

Our next goal is to build $\mathbb{E}_2$ prelog structures on an $\mathbb{E}_2$ -ring determined by a given set of homotopy classes.

Construction 4.4. Let $r \geq 1$, let $d_1, \dots, d_r \in \mathbb{Z}_{\geq 0}$, and let $\mathbb{Z}_{\geq 0}^r = (\mathbb{Z}_{\geq 0})^{\times r}$. We consider the composite

$$\xi_{2d_1, \dots, 2d_r} : \mathbb{Z}_{\geq 0}^r \xrightarrow{(d_1, \dots, d_r)} \mathbb{Z}^r \xrightarrow{(J_{\mathbb{C}} \circ \Omega^2(\text{incl}))^r} \text{Pic}_{\mathbb{S}}^r \rightarrow \text{Pic}_{\mathbb{S}} \quad (4.1)$$

of the following maps: The first is the entry-wise multiplication by d_i and the last is the iterated monoidal product of the $\mathbb{E}_{\infty}$ -space $\text{Pic}_{\mathbb{S}}$. The second is the r -fold product of the composite

$$\mathbb{Z} \xleftarrow{\sim} \Omega^2(BU(1)) \xrightarrow{\Omega^2(\text{incl})} \Omega^2(BU) \xleftarrow{\sim} \mathbb{Z} \times BU \xrightarrow{J_{\mathbb{C}}} \text{Pic}_{\mathbb{S}}, \quad (4.2)$$

where $\Omega^2(\text{incl})$ is induced by the inclusion $\text{incl} : BU(1) \rightarrow BU$ and $J_{\mathbb{C}}$ is the complex J -homomorphism, given as the composite of the realification $\mathbb{Z} \times BU \rightarrow \mathbb{Z} \times BO$ with the real J -homomorphism $J_{\mathbb{R}} : \mathbb{Z} \times BO \rightarrow \text{Pic}_{\mathbb{S}}$ (see, for example, Appendix A). Since $\Omega^2(\text{incl})$ is $\mathbb{E}_2$ and $J_{\mathbb{C}}$ is $\mathbb{E}_{\infty}$, the composite (4.2) is $\mathbb{E}_2$. The composite $\xi_{2d_1, \dots, 2d_r}$ in (4.1) is an $\mathbb{E}_2$ -map because the first and last map are $\mathbb{E}_{\infty}$ and the middle one is $\mathbb{E}_2$. The composite takes $(i_1, \dots, i_r) \in \mathbb{Z}_{\geq 0}^r$ to a point in the path component of $2(d_1 i_1 + \dots + d_r i_r) \in \mathbb{Z} \cong \pi_0(\text{Pic}_{\mathbb{S}})$, whence the name $\xi_{2d_1, \dots, 2d_r}$. We write

$$\mathbb{S}[x_1, \dots, x_r] = \text{Th}_{\mathbb{S}}(\xi_{2d_1, \dots, 2d_r})$$

for the associated Thom $\mathbb{E}_2$ -ring. The underlying $\mathbb{E}_1$ -ring of $\mathbb{S}[x_1, \dots, x_r]$ is by construction the smash product $\mathbb{S}[x_1] \otimes \dots \otimes \mathbb{S}[x_r]$ of the r free $\mathbb{E}_1$ -rings $\mathbb{S}[x_1], \dots, \mathbb{S}[x_r]$ on generators in degrees $|x_i| = 2d_i$. The identity of $\mathbb{Z}_{\geq 0}^r$ provides a canonical weight grading in the sense of Section 2.13.

Remark 4.5. This method cannot be improved to give an $\mathbb{E}_3$ -structure: Since $\widetilde{KU}^*(K(\mathbb{Z}, 3)) = 0$ by [1, Theorem III], we get $\widetilde{KO}^*(K(\mathbb{Z}, 3)) = 0$. So any map $K(\mathbb{Z}, 3) \rightarrow B^3(\mathbb{Z} \times BO)$ is null-homotopic. Therefore, any $\mathbb{E}_3$ -map $\mathbb{Z} \rightarrow \mathbb{Z} \times BO$ is null-homotopic, and we cannot realize $\mathbb{S}[x]$ with $|x| > 0$ as a Thom spectrum of an $\mathbb{E}_3$ -map $\mathbb{Z}_{\geq 0} \rightarrow \text{Pic}_{\mathbb{S}}$ that factors through $J_{\mathbb{R}} : \mathbb{Z} \times BO \rightarrow \text{Pic}_{\mathbb{S}}$. There are also obstructions to other potential constructions of such $\mathbb{E}_3$ -structures: when $|x| = 2$ the $\mathbb{E}_2$ -

structure on $\mathbb{S}[x]$ (or its 2-localization) cannot be lifted to an $\mathbb{E}_3$ -structure (see [27, Remark 3.11]). In Proposition 11.1 we show an analogous odd primary statement for $p \nmid |x|$.

However, the further composites of $\mathbb{Z} \rightarrow \text{Pic}_{\mathbb{S}}$ with the first or both the $\mathbb{E}_{\infty}$ -maps

$$\text{Pic}_{\mathbb{S}} \longrightarrow \text{Pic}_{MU} \longrightarrow \text{Pic}_{HZ}$$

become $\mathbb{E}_{\infty}$ -maps, cf. [26, Lemma 2.1.3; 35, p. 85]. The composites

$$\mathbb{Z} \rightarrow \text{Pic}_{MU} \rightarrow \mathbb{Z} \quad \text{and} \quad \mathbb{Z} \rightarrow \text{Pic}_{HZ} \rightarrow \mathbb{Z}$$

with the 0th Postnikov truncations $\text{Pic}_{MU} \rightarrow \mathbb{Z}$ and $\text{Pic}_{HZ} \rightarrow \mathbb{Z}$ send 1 to 2, so there are factorizations $\text{Pic}_{MU}^{\text{ev}} \simeq \mathbb{Z} \times BGL_1(MU)$ and $\text{Pic}_{HZ}^{\text{ev}} \simeq \mathbb{Z} \times BGL_1(\mathbb{Z})$ as $\mathbb{E}_{\infty}$ -spaces, where Pic^{ev} denotes the fiber over $2\mathbb{Z} \subset \mathbb{Z}$. Hence it is possible to consider MU -based prelog $\mathbb{E}_{\infty}$ -rings with $M = \mathbb{Z}_{\geq 0}$ and $\mathcal{U}\text{Th}_{MU}(\xi_{2d}) \simeq \bigvee_{i \geq 0} \Sigma^{2di} MU$. When $d = 1$, this defines the non-negative even periodic complex bordism spectrum $MUP_{\geq 0}$.

The inclusions of the generators $(0, \dots, 0, 1, 0, \dots, 0)$ of $\mathbb{Z}_{\geq 0}^r$ induce a map

$$\mathbb{S}^{2d_1} \vee \dots \vee \mathbb{S}^{2d_r} \rightarrow \mathbb{S}[x_1, \dots, x_r] \quad (4.3)$$

of spectra that is adjoint to a map of $\mathbb{E}_2$ -rings

$$\mathbb{E}_2(\mathbb{S}^{2d_1} \vee \dots \vee \mathbb{S}^{2d_r}) \rightarrow \mathbb{S}[x_1, \dots, x_r] \quad (4.4)$$

from the free $\mathbb{E}_2$ -ring on the source of (4.3).

Proposition 4.6. *The map (4.4) can be obtained by attaching $\mathbb{E}_2$ -cells in positive even degrees, with equally many d -cells as the rank of the homogeneous degree $d + 2$ part of the cokernel of $\text{incl} : \mathbb{Z}\{\sigma^2 x_1, \dots, \sigma^2 x_r\} \rightarrow \Gamma_{\mathbb{Z}}(\sigma^2 x_1, \dots, \sigma^2 x_r)$, where $|\sigma^2 x_i| = 2d_i + 2$.*

Here $\Gamma_{\mathbb{Z}}(\sigma^2 x_1, \dots, \sigma^2 x_r) = \mathbb{Z}\{\gamma_{k_1} \sigma^2 x_1 \cdots \gamma_{k_r} \sigma^2 x_r \mid k_1, \dots, k_r \geq 0\}$ denotes the divided power algebra over $\mathbb{Z}$ on the indicated generators.

Proof. We will use [28] and the claim that for $d \geq 0$, the map

$$H_d^{\mathbb{E}_2}(\mathbb{E}_2(\mathbb{S}^{2d_1} \vee \dots \vee \mathbb{S}^{2d_r}); \mathbb{Z}) \rightarrow H_d^{\mathbb{E}_2}(\mathbb{S}[x_1, \dots, x_r]; \mathbb{Z}) \quad (4.5)$$

in the $\mathbb{E}_2$ -homology of augmented $\mathbb{E}_2$ -rings induced by (4.4) is isomorphic to the homogeneous degree $d + 2$ part of $\text{incl} : \mathbb{Z}\{\sigma^2 x_1, \dots, \sigma^2 x_r\} \rightarrow \Gamma_{\mathbb{Z}}(\sigma^2 x_1, \dots, \sigma^2 x_r)$. Here $\mathbb{E}_2$ -homology is understood in the sense of [28, Definition 10.7].

We first explain how to deduce the proposition from this claim and [28, Theorem 11.21]. In the notation of that theorem, we take $\mathcal{S}$ to be the category of spectra, $\mathcal{G}$ to be the discrete commutative monoid $\mathbb{Z}_{\geq 0}^r$, and $\mathcal{O}$ a non-unitary Σ -cofibrant $\mathbb{E}_2$ -operad (in the sense of [28, Remark 12.3]). The (canonical) rank functor $\mathbb{Z}_{\geq 0}^r \rightarrow \mathbb{Z}_{\geq 0}$, $(i_1, \dots, i_r) \mapsto i_1 + \dots + i_r$, witnesses that $\mathcal{G}$ is Artinian in the sense of [28, Definition 11.10]. In this situation, the map (4.4) is a map of $\mathcal{G}$ -graded objects, with the degree $(i_1, \dots, i_r)$ -part of $\mathbb{S}[x_1, \dots, x_r]$ a $2(d_1 i_1 + \dots + d_r i_r)$ -fold suspension of $\mathbb{S}$. Passing to the fibers of the augmentation to $\mathbb{S}$ (viewed as a $\mathcal{G}$ -graded object concentrated in degree $(0, \dots, 0)$), we obtain

a map $f: \mathbf{R} \rightarrow \mathbf{S}$ of $\mathcal{O}$ -algebras in $\mathbf{S}^G$. Our choice of grading ensures that $\mathbf{R}$ and $\mathbf{S}$ are reduced in the sense of [28, Definition 11.11]. Finally, we choose the abstract connectivity in the sense of [28, §11.1] to be the map $c: G \rightarrow [-\infty, \infty]$ with $c(g) = 1$ if $g = (0, \dots, 0)$ and $c(g) = 0$ otherwise. (An abstract connectivity leading to sharper estimates is not relevant for us since we will count cells in each dimension.)

Now our claim and the long exact sequence [28, (10.1)] imply that

$$\bigoplus_{g \in G} H_{g,d}^{\mathcal{O}}(\mathbf{S}, \mathbf{R}; \mathbb{Z}) \cong (\text{coker}(\text{incl}: \mathbb{Z}\{\sigma^2 x_1, \dots, \sigma^2 x_r\} \rightarrow \Gamma_{\mathbb{Z}}(\sigma^2 x_1, \dots, \sigma^2 x_r)))_{d+2}.$$

Hence [28, Theorem 11.21] applies, and it remains to determine the number of d -cells in the $\mathbb{E}_2$ -cell structure resulting from the theorem. We cannot apply the statement after that theorem directly because [28, Axiom 11.19] requires us to work with $\mathbb{Z}$ -coefficients. However, the previous statement shows that each $H_{g,d}^{\mathcal{O}}(\mathbf{S}, \mathbf{R}; \mathbb{Z})$ is finitely generated and free. With this, an inductive argument with the long exact sequences of the triples $\mathbf{Z}_{\epsilon-1} \rightarrow \mathbf{Z}_{\epsilon} \rightarrow \mathbf{S}$ appearing in the proof of the theorem shows that the sets of (g, ϵ) -cells $I_{g,\epsilon}$ constructed there can be chosen so that the Hurewicz map and the inclusions induce isomorphisms

$$\mathbb{Z}[I_{g,\epsilon}] \xrightarrow{\cong} H_{g,\epsilon}^{\mathcal{O}}(\mathbf{S}, \mathbf{Z}_{\epsilon-1}; \mathbb{Z}) \xleftarrow{\cong} H_{g,\epsilon}^{\mathcal{O}}(\mathbf{S}, \mathbf{R}; \mathbb{Z}).$$

This implies the desired statement about the number of $\mathbb{E}_2$ -cells.

To verify our claim about (4.5), we use the bar spectral sequence [28, Theorem 14.1]. First, we observe that $H_*(\mathbb{S}[x_1, \dots, x_r]; \mathbb{Z}) \cong \mathbb{Z}[x_1, \dots, x_r]$. Next we apply the spectral sequence to compute $H_*(B^{\mathbb{E}_1}(\mathbb{S}[x_1, \dots, x_r]); \mathbb{Z})$. The E^2 -page is the exterior algebra $\Lambda_{\mathbb{Z}}(\sigma x_1, \dots, \sigma x_r)$ with $|\sigma x_i| = (1, 2d_i)$. Then each σx_i is an infinite cycle for filtration reasons, so there are no differentials from the E^2 -page on, and no multiplicative extensions because $(\sigma x_i)^2$ and $(\sigma x_i)(\sigma x_j) + (\sigma x_j)(\sigma x_i)$ cannot be detected in filtrations 0 or 1. This implies that

$$H_*(B^{\mathbb{E}_1}(\mathbb{S}[x_1, \dots, x_r]); \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(\sigma x_1, \dots, \sigma x_r)$$

with $|\sigma x_i| = 2d_i + 1$. We apply the spectral sequence a second time to obtain an additive isomorphism

$$H_*(B^{\mathbb{E}_2}(\mathbb{S}[x_1, \dots, x_r]); \mathbb{Z}) \cong \Gamma_{\mathbb{Z}}(\sigma^2 x_1, \dots, \sigma^2 x_r).$$

This time the spectral sequence collapses at the E^2 -page because $\sigma^2 x_i$ has even total degree $2d_i + 2$. Using the isomorphism

$$H_*^{\mathbb{E}_2}(\mathbb{S}[x_1, \dots, x_r]; \mathbb{Z}) \cong \tilde{H}_{*+2}(B^{\mathbb{E}_2}(\mathbb{S}[x_1, \dots, x_r]); \mathbb{Z})$$

from [28, Theorem 13.7], the claimed computation of $H_d^{\mathbb{E}_2}(\mathbb{S}[x_1, \dots, x_r]; \mathbb{Z})$ follows. For the source of (4.5), [28, Theorems 13.7 and 13.8] give that

$$H_d^{\mathbb{E}_2}(\mathbb{E}_2(\mathbb{S}^{2d_1} \vee \dots \vee \mathbb{S}^{2d_r}); \mathbb{Z}) \cong \tilde{H}_d(\mathbb{S}^{2d_1} \vee \dots \vee \mathbb{S}^{2d_r}; \mathbb{Z})$$

for $d \geq 0$, with the generator corresponding to $\mathbb{S}^{2d_i}$ matching the $\mathbb{E}_2$ -cell corresponding to x_i . Showing that the map in (4.5) is the claimed one reduces to the $r = 1$ case by naturality, where it follows from [28, Corollary 11.14] and the construction of (4.4). $\square$

Remark 4.7. Analogous or related $\mathbb{E}_2$ -cell structures with $r = 1$ appear in [9, Proposition 3.11; 46, §5.4], while [31, Proposition 4.2.1] contains an $\mathbb{F}_p$ -version.

Corollary 4.8. *Let A be an $\mathbb{E}_2$ -ring with $\pi_*(A)$ concentrated in even degrees, and let $a_1, \dots, a_r$ be homotopy classes of degrees $|a_i| = 2d_i$ with $d_i \geq 0$. Then there exists an $\mathbb{E}_2$ -map $\tilde{\alpha} = \tilde{\alpha}(a_1, \dots, a_r) : \mathbb{S}[x_1, \dots, x_r] \rightarrow A$ such that $\pi_{2d_i}(\tilde{\alpha})$ sends the generator corresponding to x_i to a_i .*

In general, the map $\tilde{\alpha}$ will not be unique.

Proof. The chosen generators $a_1, \dots, a_r$ determine an $\mathbb{E}_2$ -map $\mathbb{E}_2(\mathbb{S}^{2d_1} \vee \dots \vee \mathbb{S}^{2d_r}) \rightarrow A$. The obstructions to extend it over the $\mathbb{E}_2$ -cells used to obtain $\mathbb{S}[x_1, \dots, x_r]$ vanish by assumption, because we need only to attach even-dimensional cells. $\square$

Remark 4.9. In the corollary, it is sufficient to ask that $\pi_{n-1}(A) \cong 0$ for all n for which the relative cell complex (4.4) has a positive number of relative n -cells. For example, when $r = 1$, having $\pi_{j(2d_1+2)+2d_1-1}(A) \cong 0$ for all $j \geq 1$ is enough. When $r > 1$, the above weaker statement is sufficient for our examples and easier to work with.

Definition 4.10. When applied to the connective complex K -theory spectrum ku and its p -local counterpart, with their Bott elements, this construction provides prelog $\mathbb{E}_2$ -rings

$$(ku, \langle u \rangle) := (ku, \xi_2, \tilde{\alpha}(u) : \mathbb{S}[x] \rightarrow ku) \quad \text{and} \quad (4.6)$$

$$(ku_{(p)}, \langle u \rangle) := (ku_{(p)}, \xi_2, \tilde{\alpha}(u) : \mathbb{S}[x] \rightarrow ku_{(p)}) \quad (4.7)$$

with $|x| = 2$. In the case of the Adams summand, we similarly obtain

$$(\ell, \langle v_1 \rangle) := (\ell, \xi_{2p-2}, \tilde{\alpha}(v_1) : \mathbb{S}[x] \rightarrow \ell) \quad (4.8)$$

with $|x| = 2p - 2$. These prelog $\mathbb{E}_2$ -rings admit logifications that are independent of the choices made in their constructions, see Lemma 8.12 and Example 8.13.

Taking p instead of the Bott elements (with $|x| = 0$) provides

$$(ku, \langle p \rangle), \quad (ku_{(p)}, \langle p \rangle), \quad \text{and} \quad (\ell, \langle p \rangle). \quad (4.9)$$

The truncated Brown–Peterson spectrum $BP\langle n \rangle$ with $\pi_*BP\langle n \rangle \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n]$, $|v_i| = 2p^i - 2$, admits an $\mathbb{E}_3$ -structure by [31, Theorem A]. Hence Corollary 4.8 applies to the elements $p, v_1, \dots, v_n$ and provides a prelog $\mathbb{E}_2$ -ring

$$(BP\langle n \rangle, \langle p, v_1, \dots, v_n \rangle) := (BP\langle n \rangle, \xi_{0,2p-2,\dots,2p^n-2}, \tilde{\alpha}(p, v_1, \dots, v_n) : \mathbb{S}[x_0, \dots, x_n] \rightarrow BP\langle n \rangle) \quad (4.10)$$

with $|x_i| = 2p^i - 2$, which depends both on the chosen $\mathbb{E}_3$ -structure and the chosen prelog structure map. This is especially interesting in the case $n = 1$, where $BP\langle 1 \rangle = \ell$ and this prelog $\mathbb{E}_2$ -ring specializes to

$$(\ell, \langle p, v_1 \rangle) = (\ell, \xi_{0,2p-2}, \tilde{\alpha}(p, v_1) : \mathbb{S}[x_0, x_1] \rightarrow \ell). \quad (4.11)$$

Analogously, Corollary 4.8 provides $(ku, \langle p, u \rangle)$ and $(ku_{(p)}, \langle p, u \rangle)$.

4.2 | Trivial locus and collapse

We introduce two constructions associating ring spectra to R -based prelog rings and prelog R -algebras.

Definition 4.12. Let $(A, \xi : M \rightarrow \text{Pic}_R, \tilde{\alpha} : u\text{Th}_R(\xi) \rightarrow A)$ be an R -based prelog $\mathbb{E}_k$ -ring. By Proposition 3.7 and [47, Corollary 7.1.3.4], $u\text{Th}_R(\xi^{\text{gp}})$ is an $\mathbb{E}_{k-1} u\text{Th}_R(\xi)$ -algebra. The *trivial locus* of $(A, \xi, \tilde{\alpha})$ is the $\mathbb{E}_{k-1}$ A -algebra

$$A[M^{-1}] := A \otimes_{u\text{Th}_R(\xi)} u\text{Th}_R(\xi^{\text{gp}})$$

obtained as the base change of $u\text{Th}_R(\xi^{\text{gp}})$ along $\tilde{\alpha}$.

Remark 4.13. If $(A, \xi, \tilde{\alpha})$ is a prelog $\mathbb{E}_k$ R -algebra, then the relative tensor product in Definition 4.12 can be replaced by the relative tensor product

$$A \otimes_{\text{Th}_R(\xi)} \text{Th}_R(\xi^{\text{gp}})$$

in R -modules using the $\mathbb{E}_k$ R -algebra maps $\tilde{\alpha}$ and $\text{Th}_R(\gamma) : \text{Th}_R(\xi) \rightarrow \text{Th}_R(\xi^{\text{gp}})$. In this situation, [47, Theorem 4.5.2.1(2)] identifies the tensor product in Mod_R with the relative tensor product over R , which is introduced as a two-sided bar construction in [47, §4.4.2] and shown to satisfy associativity in [47, §4.4.3]. It follows that the underlying spectrum of the construction in Mod_R is equivalent to the construction in Sp , so that the former provides an R -module enrichment of the latter. In particular, we do not need to make R explicit in the notation $A[M^{-1}]$.

We also obtain an $\mathbb{E}_{k-1}$ R -algebra structure on $A[M^{-1}]$ by restricting its $\mathbb{E}_{k-1}$ A -algebra structure along $R \rightarrow A$. While this does not require $\tilde{\alpha}$ to be an $\mathbb{E}_k$ R -algebra map, we ask this property of $\tilde{\alpha}$ in the definition of a prelog $\mathbb{E}_k$ R -algebra to ensure the compatibility of this $\mathbb{E}_{k-1}$ R -algebra structure with those on $\text{Th}_R(\xi)$ and $\text{Th}_R(\xi^{\text{gp}})$. A similar discussion applies to Definition 4.16.

By construction, the prelog structure map $\tilde{\alpha} : u\text{Th}_R(\xi) \rightarrow A$ induces a map $u\text{Th}_R(\xi^{\text{gp}}) \rightarrow A[M^{-1}]$, where the map from the Thom R -module of ξ over M has been extended over M^{gp} , so that M now acts invertibly. This is intended to justify the notation $A[M^{-1}]$.

Example 4.14. Taking $r = 1$ in Construction 4.4, we get the canonical prelog $\mathbb{E}_2$ -ring $(\mathbb{S}[x], \langle x \rangle) = (\mathbb{S}[x], \xi_{2d} : \langle x \rangle \cong \mathbb{Z}_{\geq 0} \rightarrow \text{Pic}_{\mathbb{S}}, \text{id})$. In this situation, the trivial locus $A[M^{-1}] = \text{Th}_{\mathbb{S}}(\xi_{2d}^{\text{gp}})$ is the $2d$ -periodic sphere spectrum $\bigvee_{i \in \mathbb{Z}} \mathbb{S}^{2di}$. By Proposition 3.7, we can view it as an $\mathbb{E}_1 \mathbb{S}[x]$ -algebra or as an $\mathbb{E}_2$ -ring under $\mathbb{S}[x]$, with equivalent underlying $\mathbb{E}_1$ -rings.

Example 4.15. For $(\ell, \langle v_1 \rangle)$ as in (4.8), $\ell[\langle v_1 \rangle^{-1}] = \ell \otimes_{\text{Th}_{\mathbb{S}}(\xi_{2p-2})} \text{Th}_{\mathbb{S}}(\xi_{2p-2}^{\text{gp}})$ is the localization $L = E(1)$ of ℓ away from v_1 , which has a preferred $\mathbb{E}_{\infty}$ ℓ -algebra structure. Analogously, we get KU and $KU_{(p)}$ from $(ku, \langle u \rangle)$ and $(ku_{(p)}, \langle u \rangle)$.

Definition 4.16. Let $(A, \xi : M \rightarrow \text{Pic}_R, \tilde{\alpha} : u\text{Th}_R(\xi) \rightarrow A)$ be an R -based prelog $\mathbb{E}_k$ -ring, let $\tau : M \rightarrow I$ be an $\mathbb{E}_k$ -map to a discrete commutative monoid, and let J be a face of I . The *collapse* of $I \setminus J$ in $(A, \xi, \tilde{\alpha})$ is the $\mathbb{E}_{k-1}$ A -algebra given by the base change

$$A/(M_{I \setminus J}) := A \otimes_{u\mathrm{Th}_R(\xi)} u\mathrm{Th}_R(\xi|M_J)$$

of the $\mathbb{E}_{k-1} u\mathrm{Th}_R(\xi)$ -algebra $u\mathrm{Th}_R(\xi|M_J)$ from Lemma 2.20 along the $\mathbb{E}_k$ -map $\tilde{\alpha}$.

When $M = I = \mathbb{Z}_{\geq 0}^r$, $\xi = \xi_{2d_1, \dots, 2d_r}$, and $J = \{0\}$, then we denote $A/(M_{I \setminus J})$ by $A//\tilde{\alpha}$, or by $A//\langle x_1, \dots, x_r \rangle$ if $\tilde{\alpha}$ is understood from the context.

Remark 4.17. As explained in Remark 4.13, we can use the relative tensor product

$$A \otimes_{\mathrm{Th}_R(\xi)} \mathrm{Th}_R(\xi|M_J)$$

in R -modules instead when $(A, \xi, \tilde{\alpha} : \mathrm{Th}_R(\xi) \rightarrow A)$ is a prelog $\mathbb{E}_k$ R -algebra.

By construction, the collapse map $\mathrm{coll}^{I \setminus J} : \mathrm{Th}_R(\xi) \rightarrow \mathrm{Th}_R(\xi|M_J)$ induces a map $A \rightarrow A/(M_{I \setminus J})$, where $M_{I \setminus J} = \tau^{-1}(I \setminus J)$ now acts trivially on the target. This is intended to justify the notation $A/(M_{I \setminus J})$.

Example 4.18. For $(\mathbb{S}[x], \langle x \rangle)$ as in Example 4.14, we get $\mathbb{S}[x]//\langle x \rangle = \mathbb{S}$ as $\mathbb{E}_2$ -algebras (since no base change is necessary).

Example 4.19. For $(\ell, \langle v_1 \rangle)$, we get $\ell//\langle v_1 \rangle = H\mathbb{Z}_{(p)}$ from applying $\ell \otimes_{\mathbb{S}[x]} (-)$ to the cofiber sequence $\Sigma^{2p-2}\mathbb{S}[x] \rightarrow \mathbb{S}[x] \rightarrow \mathbb{S}$ with $|x| = 2p - 2$ as in (4.8). Analogously, for $(ku, \langle u \rangle)$ and $(ku_{(p)}, \langle u \rangle)$ we get $ku//\langle u \rangle = H\mathbb{Z}$ and $ku_{(p)}//\langle u \rangle = H\mathbb{Z}_{(p)}$. The same argument for the examples from (4.9) gives the mod p connective Adams summand $\ell//\langle p \rangle = \ell/p = k(1)$ and the mod p connective complex K -theory spectrum $ku//\langle p \rangle = ku_{(p)}//\langle p \rangle = ku/p$.

Example 4.20. For $(\ell, \langle p, v_1 \rangle)$, we get $\ell//\langle p, v_1 \rangle \simeq \ell//\langle p \rangle \otimes_{\ell} \ell//\langle v_1 \rangle \simeq H\mathbb{F}_p$.

Example 4.21. For $(\mathrm{BP}\langle n \rangle, \langle p, v_1, \dots, v_n \rangle)$ as in (4.10), collapsing over the complement of the face $\langle v_n \rangle$ gives the n th connective Morava K -theory spectrum $k(n)$ at the prime p : This holds additively by arguing as in Example 4.19, and as $\mathbb{E}_1$ -rings because the $\mathbb{E}_1$ -structure on $k(n)$ is unique [4]. Restricting the $\mathbb{E}_2$ -prelog structure on $\mathrm{BP}\langle n \rangle$ to $\langle v_n \rangle$ and composing with the $\mathbb{E}_1$ -map $\mathrm{BP}\langle n \rangle \rightarrow k(n)$ from the collapse defines a prelog $\mathbb{E}_1$ -ring $(k(n), \langle v_n \rangle)$. Its trivial locus is an $\mathbb{E}_0$ -ring that is equivalent as a spectrum to the n th periodic Morava K -theory spectrum $K(n)$. We can also get $K(n)$ as an $\mathbb{E}_1$ -ring from $(\mathrm{BP}\langle n \rangle, \langle p, v_1, \dots, v_n \rangle)$ by forming the relative $\otimes$ -product

$$\mathrm{BP}\langle n \rangle \otimes_{\mathbb{S}[x_0, \dots, x_n]} \mathbb{S}[x_n^{\pm 1}].$$

This amounts to performing the collapse and trivial locus in one step. The identification of the $\mathbb{E}_1$ -structure again follows from its uniqueness [4].

5 | LOGARITHMIC TOPOLOGICAL HOCHSCHILD HOMOLOGY

Our next goal is to define log THH of R -based prelog $\mathbb{E}_k$ -rings and prelog $\mathbb{E}_k$ R -algebras with $k \geq 2$, including its cyclotomic structure. We begin by reviewing the cyclic bar construction and then turn to the construction of weight-graded THH from [6, App. A].

5.1 | The cyclic bar construction

Recall that the forgetful functor from symmetric monoidal ∞ -categories to ∞ -operads has a left adjoint $C^\otimes \mapsto \text{Env}(C)^\otimes$, given by the symmetric monoidal envelope from [47, §2.2.4]. The underlying ∞ -category $\text{Env}(C)$ of $\text{Env}(C)^\otimes$ agrees with the subcategory $C_{\text{act}}^\otimes \subset C^\otimes$ spanned by all objects and the active morphisms between them [47, Definition 2.1.2.3 and Remark 2.2.4.3]. The adjunction counit $\otimes : C_{\text{act}}^\otimes \rightarrow C$ is then informally given by $(X_1, \dots, X_n) \mapsto X_1 \otimes \dots \otimes X_n$ in the symmetric monoidal structure on C , cf. [47, Proposition 2.2.4.9; 49, Proposition III.3.2].

As in [47, Definition 4.1.1.3], we write Assoc for the associative ∞ -operad. Let C be a presentably symmetric monoidal ∞ -category. Then an associative algebra $M \in \text{Alg}(C)$ is given by an ∞ -operad map $M^\otimes : \text{Assoc}^\otimes \rightarrow C^\otimes$ (compare [47, Definition 4.1.1.6]) and restricts to a functor $M_{\text{act}}^\otimes : \text{Assoc}_{\text{act}}^\otimes \rightarrow C_{\text{act}}^\otimes$ on the active morphisms. Let Δ denote the usual simplex category, let Λ denote Connes' cyclic category, and let Λ_∞ be the paracyclic category, as in [49, p. 380]. As explained in [49, Proposition B.5] in the ∞ -categorical context, the geometric realization $|X_\bullet|$ of a cyclic object $X_\bullet : N(\Lambda^{\text{op}}) \rightarrow C$ takes values in the category $C^{B\mathbb{T}}$ of objects in C with $\mathbb{T}$ -action. It is the colimit of the restriction of $X_\bullet$ to $N(\Lambda_\infty^{\text{op}})$, and the underlying object in C is equivalent to the colimit of the restriction $X_\bullet$ to $N(\Delta^{\text{op}})$. Both these colimits are sifted. Recall from [49, Proposition B.1] that there is a functor $V^\circ : N(\Lambda^{\text{op}}) \rightarrow \text{Assoc}_{\text{act}}^\otimes$ to the active part, whose restriction to $N(\Delta^{\text{op}})$ refines the functor $[q] \mapsto (\Delta^1 / \partial \Delta^1)_q \cong \{0, 1, \dots, q\}$ represented by the simplicial circle $\Delta^1 / \partial \Delta^1$.

Definition 5.2. Let C be a presentably symmetric monoidal ∞ -category. The *cyclic bar construction* is the composite

$$B^{\text{cy}} : \text{Alg}(C) \xrightarrow{M \mapsto M_{\text{act}}^\otimes} \text{Fun}(\text{Assoc}_{\text{act}}^\otimes, C_{\text{act}}^\otimes) \xrightarrow{(V^\circ)^*} \text{Fun}(N(\Lambda^{\text{op}}), C_{\text{act}}^\otimes) \xrightarrow{(\otimes)_*} \text{Fun}(N(\Lambda^{\text{op}}), C) \xrightarrow{|\cdot|} C^{B\mathbb{T}} \quad (5.1)$$

of the functors discussed above.

For $C = \text{Sp}$, the first part of the next proposition is discussed in [49, §IV.2].

Proposition 5.3.

- (i) The functor (5.1) is symmetric monoidal.
- (ii) If $F : C \rightarrow D$ is a lax symmetric monoidal functor, then there is a natural comparison map $B^{\text{cy}} \circ F \rightarrow F \circ B^{\text{cy}}$ of lax symmetric monoidal functors $\text{Alg}(C) \rightarrow D^{B\mathbb{T}}$. If $G : D \rightarrow \mathcal{E}$ is another lax symmetric monoidal functor, then the composite

$$B^{\text{cy}} \circ G \circ F \rightarrow G \circ B^{\text{cy}} \circ F \rightarrow G \circ F \circ B^{\text{cy}}$$

induced by the comparison maps for F and G is equivalent to the comparison map for $G \circ F$.

- (iii) If $F : C \rightarrow D$ is a symmetric monoidal functor commuting with sifted colimits, then the comparison map for F is an equivalence.

Proof. For (i), we use that the symmetric monoidal structure on $\text{Alg}(C)$ is defined in terms of the pointwise symmetric monoidal structure [47, Remark 2.1.3.4] on the functor category

$\text{Fun}(\text{Assoc}_{\text{act}}^{\otimes}, C_{\text{act}}^{\otimes})$. Therefore, $M \mapsto M_{\text{act}}^{\otimes}$ is symmetric monoidal. The pointwise symmetric monoidal structure is compatible with precomposition by maps of simplicial sets, and with postcomposition by symmetric monoidal functors. Being a sifted colimit, geometric realization is symmetric monoidal. Hence it follows that B^{cy} is symmetric monoidal.

For (ii), we observe that the first two functors in (5.1) commute with lax symmetric monoidal functors. To analyze the third functor, we note that the two composites in the square

$$\begin{array}{ccc} C_{\text{act}}^{\otimes} & \xrightarrow{F_{\text{act}}^{\otimes}} & D_{\text{act}}^{\otimes} \\ \otimes_C \downarrow & & \downarrow \otimes_D \\ C & \xrightarrow{F} & D \end{array} \quad (5.2)$$

are not necessarily equivalent. However, using the characterization of the right adjoint of the adjunction $\text{incl} : \text{Fun}_{\otimes}(C_{\text{act}}^{\otimes}, D) \rightleftarrows \text{Fun}_{\text{lax}}(C_{\text{act}}^{\otimes}, D) : R$ given in [49, Lemma III.3.3], we get an equivalence $\otimes_D \circ F_{\text{act}}^{\otimes} \xrightarrow{\sim} R(F \circ \otimes_C)$ of symmetric monoidal functors because the precomposition of both functors with the lax symmetric monoidal functor $C \rightarrow C_{\text{act}}^{\otimes}$ is F .

The adjoint of this equivalence is the desired comparison map $\otimes_D \circ F_{\text{act}}^{\otimes} \rightarrow F \circ \otimes_C$ of lax symmetric monoidal functors. Applying $\text{Fun}(N(\Lambda^{\text{op}}), -)$ this gives the comparison map for the third functor in (5.1). For the last one, we use that the canonical comparison map $|F(-)| \rightarrow F(|-|)$ is a map of lax symmetric monoidal functors because F is lax symmetric monoidal. The non-trivial part in the statement about $G \circ F$ is to prove that the composed map

$$\otimes_{\mathcal{E}} \circ G_{\text{act}}^{\otimes} \circ F_{\text{act}}^{\otimes} \rightarrow G \circ \otimes_D \circ F_{\text{act}}^{\otimes} \rightarrow G \circ F \circ \otimes_C$$

is equivalent to the comparison map for $G \circ F$. By precomposing with $C \rightarrow C_{\text{act}}^{\otimes}$, the adjoint of this composite can be checked to be equivalent to the equivalence $\otimes_{\mathcal{E}} \circ G_{\text{act}}^{\otimes} \circ F_{\text{act}}^{\otimes} \xrightarrow{\sim} R(G \circ F \circ \otimes_C)$, from which the result follows.

For (iii), [49, Proposition III.3.2] implies that (5.2) commutes up to equivalence when F is symmetric monoidal. Moreover, realization, being a sifted colimit, commutes with F by assumption. $\square$

By Dunn additivity [47, Theorem 5.1.2.2], we can view $\mathbb{E}_k$ -algebras in C as $\mathbb{E}_{k-1}$ -algebras in $\text{Alg}(C)$. Hence the proposition has the following consequence:

Corollary 5.4.

(i) *The cyclic bar construction induces a functor*

$$B^{\text{cy}} : \text{Alg}_{\mathbb{E}_k}(C) \rightarrow \text{Alg}_{\mathbb{E}_{k-1}}(C^{B^{\mathbb{T}}}). \quad (5.3)$$

- (ii) *If $F : C \rightarrow D$ is a lax symmetric monoidal functor and A is any $\mathbb{E}_k$ -algebra in C , then there is a natural map $B^{\text{cy}}(F(A)) \rightarrow F(B^{\text{cy}}(A))$ in $\text{Alg}_{\mathbb{E}_{k-1}}(D^{B^{\mathbb{T}}})$. This is compatible with compositions of lax symmetric monoidal functors.*
- (iii) *If $F : C \rightarrow D$ is a symmetric monoidal functor commuting with sifted colimits, then the map in (ii) is an equivalence.*

We will also need a relative version of this corollary, which again follows from the symmetric monoidal structure of B^{cy} :

Corollary 5.5. *Let R be an $\mathbb{E}_\infty$ -algebra in $\mathcal{C}$, let A be an $\mathbb{E}_k$ R -algebra in $\mathcal{C}$, and let uA denote the underlying $\mathbb{E}_k$ -algebra of A in $\mathcal{C}$.*

- (i) *The $\mathbb{E}_{k-1}$ -algebra with $\mathbb{T}$ -action $B^{\text{cy}}(uA)$ resulting from Corollary 5.4(i) is naturally the underlying object of an $\mathbb{E}_{k-1}$ $B^{\text{cy}}(R)$ -algebra with $\mathbb{T}$ -action.*
- (ii) *If $F : \mathcal{C} \rightarrow \mathcal{D}$ is a lax symmetric monoidal functor, then there is a natural map $B^{\text{cy}}(F(uA)) \rightarrow F(B^{\text{cy}}(uA))$ of $\mathbb{E}_{k-1}$ -algebras with $\mathbb{T}$ -action over the $\mathbb{E}_\infty$ -algebra with $\mathbb{T}$ -action $B^{\text{cy}}(F(R))$. This is compatible with compositions of lax symmetric monoidal functors.*
- (iii) *If $F : \mathcal{C} \rightarrow \mathcal{D}$ is a symmetric monoidal functor commuting with sifted colimits, then the map in (ii) is an equivalence of $\mathbb{E}_{k-1}$ -algebras with $\mathbb{T}$ -action over the $\mathbb{E}_\infty$ -algebra with $\mathbb{T}$ -action $B^{\text{cy}}(F(R)) \simeq F(B^{\text{cy}}(R))$.*

5.2 | Weight-graded THH

Let I be a discrete commutative monoid. An I -graded spectrum is an object X_* of $\text{Fun}(I, \text{Sp})$, with total object $X = (c : I \rightarrow \{0\})_! X_* = \bigvee_{i \in I} X_i$. We view $\text{Fun}(I, \text{Sp})$ as a symmetric monoidal ∞ -category equipped with the Day convolution product. An I -graded $\mathbb{E}_k$ -ring is an $\mathbb{E}_k$ -algebra A_* in $\text{Fun}(I, \text{Sp})$, and has an $\mathbb{E}_k$ -ring $A = c_! A_*$ as total object.

Definition 5.7. The I -graded THH

$$\text{THH} : \text{Alg}_{\mathbb{E}_k}(\text{Fun}(I, \text{Sp})) \rightarrow \text{Alg}_{\mathbb{E}_{k-1}}(\text{Fun}(I, \text{Sp})^{B\mathbb{T}}) \quad (5.4)$$

is the cyclic bar construction (5.3) for $\mathbb{E}_k$ -algebras in $\mathcal{C} = \text{Fun}(I, \text{Sp})$. It sends the underlying I -graded $\mathbb{E}_1$ -ring $A_* \in \text{Alg}(\text{Fun}(I, \text{Sp}))$ of an $\mathbb{E}_k$ -algebra to the I -graded spectrum with $\mathbb{T}$ -action $\text{THH}(A_*) = \{i \mapsto \text{THH}(A; i)\}$.

We now turn to cyclotomic structures. By [6, Definition A.1 and Lemma A.7], left Kan extension along $\ell_p : I \rightarrow I, i \mapsto pi$ defines a symmetric monoidal functor $L_p = (\ell_p)_! : \text{Fun}(I, \text{Sp}) \rightarrow \text{Fun}(I, \text{Sp})$, and (for a class of symmetric monoidal ∞ -categories including our discrete commutative monoid I) there is a canonical L_p -twisted Tate diagonal

$$\Delta_p : L_p(X_*) \longrightarrow (X_* \otimes \cdots \otimes X_*)^{tC_p}$$

on $\text{Fun}(I, \text{Sp})$, with p copies of X_* in the target. Here $(-)^{tC_p}$ denotes the C_p -Tate construction (see, for example, [49, Chapter I]). An I -graded twisted cyclotomic spectrum is an I -graded spectrum X_* with a $\mathbb{T}$ -action and a $\mathbb{T}$ -equivariant map $\varphi_p : L_p(X_*) \rightarrow X_*^{tC_p}$ for every prime p . Here $X_*^{tC_p} = \{i \mapsto (X_i)^{tC_p}\}$ has the residual $\mathbb{T} \cong \mathbb{T}/C_p$ -action for each $i \in I$. Analogously to [49, Definition II.1.6], we obtain a stable ∞ -category $\text{TwCycFun}(I, \text{Sp})$ of I -graded twisted cyclotomic spectra. Applying [49, Construction IV.2.1] provides a presentably symmetric monoidal structure on $\text{TwCycFun}(I, \text{Sp})$.

Proposition 5.8 [6, Corollary A.9]. *The I -graded THH canonically lifts to a symmetric monoidal functor*

$$\mathrm{THH} : \mathrm{Alg}(\mathrm{Fun}(I, \mathrm{Sp})) \rightarrow \mathrm{TwCycFun}(I, \mathrm{Sp}).$$

So, for an I -graded $\mathbb{E}_1$ -ring A_* there are L_p -twisted cyclotomic structure maps

$$\varphi_p : L_p \mathrm{THH}(A_*) \longrightarrow \mathrm{THH}(A_*)^{tC_p},$$

for every prime p . They can equivalently be described by their right adjoints $\varphi_{p,*} : \mathrm{THH}(A_*) \rightarrow \ell_p^* \mathrm{THH}(A_*)^{tC_p}$, having $\mathbb{T}$ -equivariant components

$$\varphi_{p,i} : \mathrm{THH}(A; i) \longrightarrow \mathrm{THH}(A; pi)^{tC_p},$$

for all $i \in I$.

If $I = \{0\}$, then an I -graded L_p -twisted cyclotomic spectrum is the same as a cyclotomic spectrum as in [49, Definition II.1.1], and $\mathrm{TwCycFun}(I, \mathrm{Sp})$ is the category CycSp of [49, Definition II.1.6].

Lemma 5.9. *Let $f : J \rightarrow I$ be a homomorphism of discrete commutative monoids.*

- (i) *Taking the left Kan extension along f induces a symmetric monoidal functor $f_! : \mathrm{TwCycFun}(J, \mathrm{Sp}) \rightarrow \mathrm{TwCycFun}(I, \mathrm{Sp})$. If B_* is a J -graded $\mathbb{E}_k$ -ring, then there is a natural equivalence $f_! \mathrm{THH}(B_*) \simeq \mathrm{THH}(f_! B_*)$ of $\mathbb{E}_{k-1}$ -algebras in $\mathrm{TwCycFun}(I, \mathrm{Sp})$.*
- (ii) *If f is a face inclusion, then restriction along f induces a symmetric monoidal functor $f^* : \mathrm{TwCycFun}(I, \mathrm{Sp}) \rightarrow \mathrm{TwCycFun}(J, \mathrm{Sp})$. For an I -graded $\mathbb{E}_k$ -ring A_* there is a natural equivalence $f^* \mathrm{THH}(A_*) \simeq \mathrm{THH}(f^* A_*)$ of $\mathbb{E}_{k-1}$ -algebras in $\mathrm{TwCycFun}(J, \mathrm{Sp})$.*

Proof. For (i), the induced functor is that of [6, Lemma A.14]. The $\mathbb{E}_{k-1}$ -algebra part of the equivalence follows from Corollary 5.4(iii) applied to the left adjoint in (2.8) and the twisted cyclotomic structure from [6, Corollary A.15].

Part (ii) uses Corollary 5.4(iii) and a slight generalization of [6, Ex. A.17], noting that f^* is symmetric monoidal when J is a face of I (compare the discussion after Definition 2.19). $\square$

Corollary 5.10. *Let A_* be an I -graded $\mathbb{E}_k$ -ring, with total $\mathbb{E}_k$ -ring $A = c_! A_*$. Then the total spectrum $c_! \mathrm{THH}(A_*) \simeq \mathrm{THH}(A)$ is an $\mathbb{E}_{k-1}$ -algebra in cyclotomic spectra. If $\{0\} \subseteq I$ is a face, then $\mathrm{THH}(A)_0 \simeq \mathrm{THH}(A_0)$ is an $\mathbb{E}_{k-1}$ -algebra in cyclotomic spectra.*

Proof. This follows from applying Lemma 5.9 to $c : I \rightarrow \{0\}$ and $\{0\} \subseteq I$. $\square$

Definition 5.11. If R is an $\mathbb{E}_\infty$ -ring, then $\mathrm{THH}(R)$ is an $\mathbb{E}_\infty$ -algebra in CycSp . We can also view R as an I -graded object concentrated in weight 0, that is, as the I -graded $\mathbb{E}_\infty$ -ring $(\{0\} \rightarrow I)_!(R)$, and $\mathrm{THH}(R)$ as an $\mathbb{E}_\infty$ -algebra in $\mathrm{TwCycFun}(I, \mathrm{Sp})$. We write $\mathrm{CycMod}_{\mathrm{THH}(R)}$ and $\mathrm{TwCycFun}(I, \mathrm{Mod}_{\mathrm{THH}(R)})$ for the symmetric monoidal categories of $\mathrm{THH}(R)$ -modules in CycSp and in $\mathrm{TwCycFun}(I, \mathrm{Sp})$.

The symmetric monoidal structure on THH implies the following statement:

Corollary 5.12. *Let A_* be an I -graded $\mathbb{E}_k$ R -algebra with underlying I -graded $\mathbb{E}_k$ -ring uA_* . Then $\mathrm{THH}(uA_*)$ is naturally an $\mathbb{E}_{k-1}$ -algebra in the symmetric monoidal category $\mathrm{TwCycFun}(I, \mathrm{Mod}_{\mathrm{THH}(R)})$.*

The underlying $\mathbb{E}_{k-1}$ $\mathrm{THH}(R)$ -algebra with $\mathbb{T}$ -action of $\mathrm{THH}(uA_*)$ coincides by construction with the one resulting from Corollary 5.5(i).

Remark 5.13. In the situation of the previous corollary, we will generally omit ‘ u ’ from the notation, as THH and its logarithmic and/or cyclic variants will always be applied to $\mathbb{E}_k$ -rings, not to $\mathbb{E}_k$ R -algebras for $R \neq \mathbb{S}$ (compare Remark 7.4).

Next we discuss an operation that will become useful in Section 10. Let I_1 and I_2 be discrete commutative monoids. Then the external product

$$\begin{aligned} \mathrm{TwCycFun}(I_1, \mathrm{Mod}_{\mathrm{THH}(R)}) \times \mathrm{TwCycFun}(I_2, \mathrm{Mod}_{\mathrm{THH}(R)}) \\ \longrightarrow \mathrm{TwCycFun}(I_1 \times I_2, \mathrm{Mod}_{\mathrm{THH}(R)}) \quad (5.5) \end{aligned}$$

is defined by first applying Lemma 5.9(i) to the inclusions of I_1, I_2 into $I_1 \times I_2$ and then forming the symmetric monoidal product in $\mathrm{TwCycFun}(I_1 \times I_2, \mathrm{Mod}_{\mathrm{THH}(R)})$ of the resulting objects. It sends objects X_*^1 and X_*^2 to

$$X_*^1 \otimes_{\mathrm{THH}(R)} X_*^2 := (\mathrm{incl} : I_1 \rightarrow I_1 \times I_2)_!(X_*^1) \otimes_{\mathrm{THH}(R)} (\mathrm{incl} : I_2 \rightarrow I_1 \times I_2)_!(X_*^2).$$

If A_*^1 is an I_1 -graded $\mathbb{E}_k$ R -algebra and A_*^2 is an I_2 -graded $\mathbb{E}_k$ R -algebra, then we can analogously use the left adjoints from (2.8) and the monoidal structure on $\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Fun}(I_1 \times I_2, \mathrm{Mod}_R))$ to form an external product $A_*^1 \otimes_R A_*^2$.

Lemma 5.14. *In this situation, the universal property of $\otimes_{\mathrm{THH}(R)}$ induces an equivalence*

$$\mathrm{THH}(A_*^1) \otimes_{\mathrm{THH}(R)} \mathrm{THH}(A_*^2) \xrightarrow{\sim} \mathrm{THH}(A_*^1 \otimes_R A_*^2)$$

of $\mathbb{E}_{k-1}$ -algebras in $I_1 \times I_2$ -graded twisted cyclotomic $\mathrm{THH}(R)$ -modules.

5.3 | Log THH of canonical prelog $\mathbb{E}_k$ -algebras

Suppose given an $\mathbb{E}_\infty$ -ring R , a discrete commutative monoid I , and an $\mathbb{E}_k$ -algebra $\xi_* : M_* \rightarrow c^*\mathrm{Pic}_R$ in the symmetric monoidal ∞ -category $\mathrm{Fun}(I, \mathcal{S}_{/\mathrm{Pic}_R})$. As in Section 2.13, let $\tau : M \rightarrow I$ be the $\mathbb{E}_k$ -map encoding the weight grading on the total object $\xi : M \rightarrow \mathrm{Pic}_R$ of ξ_* . The I -graded $\mathbb{E}_k$ R -algebra $\mathrm{Th}_R(\xi_*)$ from Section 2.13 has as total object the Thom $\mathbb{E}_k$ R -algebra $\mathrm{Th}_R(\xi) = c_! \mathrm{Th}_R(\xi_*)$. Its underlying I -graded $\mathbb{E}_k$ -ring $u\mathrm{Th}_R(\xi)$ has I -graded $\mathrm{THH}(\mathrm{Th}_R(\xi_*))$, as in Definition 5.7, which is an $\mathbb{E}_{k-1}$ -algebra in $\mathrm{TwCycFun}(I, \mathrm{Mod}_{\mathrm{THH}(R)})$. Similarly, the I^{gp} -graded $\mathbb{E}_k$ R -algebra $\mathrm{Th}_R(\xi_*^{\mathrm{gp}})$ has as total object $\mathrm{Th}_R(\xi^{\mathrm{gp}})$, and its underlying I^{gp} -graded $\mathbb{E}_k$ -ring has I^{gp} -graded $\mathrm{THH}(\mathrm{Th}_R(\xi_*^{\mathrm{gp}}))$, which is an $\mathbb{E}_{k-1}$ -algebra in $\mathrm{TwCycFun}(I^{\mathrm{gp}}, \mathrm{Mod}_{\mathrm{THH}(R)})$.

Remark 5.16. In examples, we will mostly consider the canonical $\pi_0(M)$ -grading of Definition 2.16. However, we allow for a general I -grading in our constructions because this helps studying

induced maps on $\log \mathrm{THH}$ (see Example 5.24 and Corollary B.6) and enables us to vary the notion of repletion (compare Sections 3.8 and 7.16).

Definition 5.17. The I -graded logarithmic THH of the canonical prelog $\mathbb{E}_k$ R -algebra $(\mathrm{Th}_R(\xi), \xi, \mathrm{id})$ is the pullback

$$\mathrm{THH}(\mathrm{Th}_R(\xi_*), \xi) := \gamma^* \mathrm{THH}(\mathrm{Th}_R(\xi_*^{\mathrm{gp}})) = \mathrm{THH}(\mathrm{Th}_R(\xi_*^{\mathrm{gp}})) \circ \gamma$$

of the I^{gp} -graded THH of $u\mathrm{Th}_R(\xi_*^{\mathrm{gp}})$ along $\gamma: I \rightarrow I^{\mathrm{gp}}$. It is an $\mathbb{E}_{k-1}$ -algebra in $\mathrm{TwCycFun}(I, \mathrm{Mod}_{\mathrm{THH}(R)})$, with weight i component

$$\mathrm{THH}(\mathrm{Th}_R(\xi_*), \xi; i) = \mathrm{THH}(\mathrm{Th}_R(\xi_*^{\mathrm{gp}}); \gamma(i))$$

for each $i \in I$. The L_p -twisted cyclotomic structure map has $\mathbb{T}$ -equivariant component

$$\varphi_{p; \gamma(i)}: \mathrm{THH}(\mathrm{Th}_R(\xi_*^{\mathrm{gp}}); \gamma(i)) \longrightarrow \mathrm{THH}(\mathrm{Th}_R(\xi_*^{\mathrm{gp}}); p\gamma(i))^{tC_p}$$

in weight i , keeping in mind that $p\gamma(i) = \gamma(pi)$.

The (ungraded) logarithmic THH of $(\mathrm{Th}_R(\xi), \xi, \mathrm{id})$ is

$$\mathrm{THH}(\mathrm{Th}_R(\xi), \xi) := (c: I \rightarrow \{0\})_! \mathrm{THH}(\mathrm{Th}_R(\xi_*), \xi),$$

which is an $\mathbb{E}_{k-1}$ -algebra in $\mathrm{CycMod}_{\mathrm{THH}(R)}$.

Using Corollary 5.4(ii), the I -graded repletion map (3.3) induces an I -graded repletion map

$$\rho_*: \mathrm{THH}(\mathrm{Th}_R(\xi_*)) \rightarrow \gamma^* \mathrm{THH}(\mathrm{Th}_R(\xi_*^{\mathrm{gp}})) = \mathrm{THH}(\mathrm{Th}_R(\xi_*), \xi), \quad (5.6)$$

which is an $\mathbb{E}_{k-1}$ -algebra map in $\mathrm{TwCycFun}(I, \mathrm{Mod}_{\mathrm{THH}(R)})$. Its weight i component is

$$\rho_i = \mathrm{THH}(\mathrm{Th}_R(\gamma); i): \mathrm{THH}(\mathrm{Th}_R(\xi_*); i) \longrightarrow \mathrm{THH}(\mathrm{Th}_R(\xi_*^{\mathrm{gp}}); \gamma(i)). \quad (5.7)$$

The corresponding map of total objects

$$\rho: \mathrm{THH}(\mathrm{Th}_R(\xi)) \longrightarrow \mathrm{THH}(\mathrm{Th}_R(\xi), \xi)$$

is the (ungraded) repletion map, which is an $\mathbb{E}_{k-1}$ -map in $\mathrm{CycMod}_{\mathrm{THH}(R)}$.

The repletion map (5.6) is related to an $\mathbb{E}_{k-2}$ $\mathrm{THH}(\mathrm{Th}_R(\xi_*))$ -algebra structure:

Corollary 5.18. The underlying $\mathbb{E}_{k-2}$ -map of the repletion map (5.6) extends to an $\mathbb{E}_{k-2}$ $\mathrm{THH}(\mathrm{Th}_R(\xi_*))$ -algebra structure on $\mathrm{THH}(\mathrm{Th}_R(\xi_*), \xi)$ in $\mathrm{TwCycFun}(I, \mathrm{Sp})$.

Proof. We can view ξ_*^{gp} as an $\mathbb{E}_{k-1}$ $\gamma_! \xi_*$ -algebra (see Proposition 3.7), so that $u\mathrm{Th}_R(\xi_*^{\mathrm{gp}})$ becomes an $\mathbb{E}_{k-1}$ $u\mathrm{Th}_R(\gamma_! \xi_*)$ -algebra by the symmetric monoidality of Th_R and [47, Corollary 7.1.3.4]. Using $u\mathrm{Th}_R(\gamma_! \xi_*) \simeq \gamma_! u\mathrm{Th}_R(\xi_*)$, we can argue with adjunction (2.10) as in Corollary 3.11. $\square$

While not reflected in the notation, $\mathrm{THH}(\mathrm{Th}_R(\xi), \xi)$ depends on the I -grading of ξ_* and, in general, changing the I -grading changes $\mathrm{THH}(\mathrm{Th}_R(\xi), \xi)$. We will always specify the grading in concrete examples.

To study changes of the grading, let $\kappa : I \rightarrow K$ be a homomorphism of discrete commutative monoids and consider the following commutative diagram extending (3.2):

$$\begin{array}{ccccccc} K & \xleftarrow{\kappa} & I & \xleftarrow{\tau} & M & \xrightarrow{\xi} & \text{Pic}_R \\ \gamma_K \downarrow & & \gamma_I \downarrow & & \downarrow \gamma_M & \nearrow \xi^{\text{gp}} & \\ K^{\text{gp}} & \xleftarrow{\kappa^{\text{gp}}} & I^{\text{gp}} & \xleftarrow{\tau^{\text{gp}}} & M^{\text{gp}} & & \end{array} \quad (5.8)$$

Here τ specifies the grading of $\xi_* \in \text{Fun}(I, S_{/\text{Pic}_R})$, and the composite $\kappa\tau$ specifies the grading of $\kappa_! \xi_* \in \text{Fun}(K, S_{/\text{Pic}_R})$. Analogously, τ^{gp} and $\kappa^{\text{gp}}\tau^{\text{gp}}$ encode the gradings of $\xi_*^{\text{gp}} \in \text{Fun}(I^{\text{gp}}, S_{/\text{Pic}_R})$ and $\kappa_!^{\text{gp}} \xi_*^{\text{gp}} \simeq (\kappa_! \xi_*)^{\text{gp}} \in \text{Fun}(K^{\text{gp}}, S_{/\text{Pic}_R})$, respectively.

Lemma 5.19. *In this situation, κ induces a map of total objects*

$$(c_I : I \rightarrow \{0\})_! \text{THH}(\text{Th}_R(\xi_*), \xi) \rightarrow (c_K : K \rightarrow \{0\})_! \text{THH}(\text{Th}_R(\kappa_! \xi_*), \xi)$$

of $\mathbb{E}_{k-1}$ -algebras in $\text{CycMod}_{\text{THH}(R)}$, which is natural in ξ_* . If κ is an exact homomorphism of discrete commutative monoids, then this map is an equivalence.

In other words, the ungraded log THH of canonical prelog $\mathbb{E}_k$ R -algebras does not change if we modify the grading by extending along an exact homomorphism.

Proof of Lemma 5.19. Using that Th_R commutes with the colimit defining $\kappa_!^{\text{gp}}$ and Lemma 5.9(i), we get equivalences

$$\begin{aligned} (c_K : K \rightarrow \{0\})_! \text{THH}(\text{Th}_R(\kappa_! \xi_*), \xi) &\simeq (c_K : K \rightarrow \{0\})_! \gamma_K^* \text{THH}(\text{Th}_R(\kappa_!^{\text{gp}} \xi_*^{\text{gp}})) \\ &\simeq (c_K : K \rightarrow \{0\})_! \gamma_K^* \kappa_!^{\text{gp}} \text{THH}(\text{Th}_R(\xi_*^{\text{gp}})) \\ &\simeq \coprod_{(k, i') \in K \times_{K^{\text{gp}}} I^{\text{gp}}} \text{THH}(\text{Th}_R(\xi_*^{\text{gp}}); i'). \end{aligned}$$

The counit of the adjunction $(\nu_!, \nu^*)$ associated with the canonical map $\nu : I \rightarrow K \times_{K^{\text{gp}}} I^{\text{gp}}$ induces a map from

$$(c_I : I \rightarrow \{0\})_! \text{THH}(\text{Th}_R(\xi_*), \xi) = \coprod_{i \in I} \text{THH}(\text{Th}_R(\xi_*^{\text{gp}}); \gamma_I(i))$$

into the last term, which is an equivalence when κ is exact (compare Remark 3.13 and [50, Definition 2.1.15]). $\square$

Remark 5.20. When M is an $\mathbb{E}_k$ -space and $A = \mathbb{S}[M]$ is the associated spherical monoid ring, then the cyclotomic structure map φ_p of $\text{THH}(A)$ is determined by a $\mathbb{T}$ -equivariant structure map $B^{\text{cy}}(M) \rightarrow (B^{\text{cy}}(M))^{hC_p}$ on the space level [49, Lemma IV.3.1]. If $\xi : M \rightarrow \text{Pic}_{\mathbb{S}}$ is the constant map to the basepoint, then $\text{Th}_{\mathbb{S}}(\xi) \simeq \mathbb{S}[M]$. In view of this, one may wonder if for general $\xi : M \rightarrow \text{Pic}_{\mathbb{S}}$ the cyclotomic structure map of $\text{THH}(\text{Th}_{\mathbb{S}}(\xi))$ or of $\text{THH}(\text{Th}_{\mathbb{S}}(\xi), \xi)$ arises from a similar structure on the cyclic bar construction. More generally, one can ask if these cyclotomic spectra admit a *Frobenius lift* in the sense of [49, §IV.3], that is, a factorization of the p -cyclotomic structure map through the canonical map $(-)^{hC_p} \rightarrow (-)^{tC_p}$.

It turns out that this is not the case: The existence of a Frobenius lift for $\text{THH}(\mathbb{S}[M])$ is related to the instability property of the (co-)homology of $\mathbb{S}[M]$ as a module over the Steenrod alge-

bra. Using [44, (3.9) and Proposition 5.12], one can show that for an $\mathbb{E}_1$ -ring A with $H_*(A; \mathbb{F}_p)$ of finite type, the existence of $\alpha \in H_q(A; \mathbb{F}_p)$ and $(\epsilon, j) \neq (0, 0)$ with $(\beta^\epsilon P^j)_*(\alpha) = 1$ prevent that $\varphi_p : \mathrm{THH}(A) \rightarrow \mathrm{THH}(A)^{tC_p}$ admits a Frobenius lift. This applies, for example, to the Thom spectra $A = MU$ and $A = MUP_{\geq 0}$, as well as many other $\mathbb{E}_1$ -rings.

This also shows that the construction of log TC from [14, §3.2] for prelog structures involving only (spherical monoid rings of) discrete monoids does not generalize to our setup of prelog ring spectra.

5.4 | General log THH

To extend Definition 5.17 to non-canonical prelog structures we assume that $2 \leq k \leq \infty$ for the rest of this section. This ensures that $\mathrm{THH}(\mathrm{Th}_R(\xi))$ is at least an $\mathbb{E}_1$ -ring, so that we can form a relative tensor product over it. We also recall that $\mathrm{THH}(\mathrm{Th}_R(\xi), \xi)$ depends on an implicit I -grading $\tau : M \rightarrow I$ of the domain of $\xi : M \rightarrow \mathrm{Pic}_R$.

Definition 5.22. The *logarithmic THH* of an R -based prelog $\mathbb{E}_k$ -ring $(A, \xi, \bar{\alpha})$ is the $\mathbb{E}_{k-2}$ $\mathrm{THH}(A)$ -algebra in CycSp obtained as the base change

$$\mathrm{THH}(A, \xi, \bar{\alpha}) := \mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathrm{Th}_R(\xi))} \mathrm{THH}(\mathrm{Th}_R(\xi), \xi),$$

along $\mathrm{THH}(\bar{\alpha})$ of the $\mathbb{E}_{k-2}$ $\mathrm{THH}(\mathrm{Th}_R(\xi))$ -algebra $\mathrm{THH}(\mathrm{Th}_R(\xi), \xi)$ resulting from Corollary 5.18. We write

$$\rho : \mathrm{THH}(A) \rightarrow \mathrm{THH}(A, \xi, \bar{\alpha}) \quad (5.9)$$

for the unit of this $\mathbb{E}_{k-2}$ $\mathrm{THH}(A)$ -algebra structure and also refer to it as a *repletion map*. In particular, it is both an $\mathbb{E}_{k-2}$ -algebra map in CycSp and a $\mathrm{THH}(A)$ -module map in CycSp .

When $(A, \xi, \bar{\alpha})$ is a prelog $\mathbb{E}_k$ R -algebra, then this relative tensor product can be formed in $\mathrm{CycMod}_{\mathrm{THH}(R)}$ using the $\mathbb{E}_{k-1}$ -maps $\mathrm{THH}(\bar{\alpha}) : \mathrm{THH}(\mathrm{Th}_R(\xi)) \rightarrow \mathrm{THH}(A)$ and $\rho : \mathrm{THH}(\mathrm{Th}_R(\xi)) \rightarrow \mathrm{THH}(\mathrm{Th}_R(\xi), \xi)$ in that category (compare Remark 4.13). In this case, (5.9) is both an $\mathbb{E}_{k-2}$ -algebra map in $\mathrm{CycMod}_{\mathrm{THH}(R)}$ and a $\mathrm{THH}(A)$ -module map in $\mathrm{CycMod}_{\mathrm{THH}(R)}$.

We obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{THH}(\mathrm{Th}_R(\xi)) & \xrightarrow{\mathrm{THH}(\bar{\alpha})} & \mathrm{THH}(A) \\ \rho \downarrow & & \downarrow \rho \\ \mathrm{THH}(\mathrm{Th}_R(\xi), \xi) & \longrightarrow & \mathrm{THH}(A, \xi, \bar{\alpha}) \end{array}$$

in CycSp or $\mathrm{CycMod}_{\mathrm{THH}(R)}$, according to the case. We stress that also $\mathrm{THH}(A, \xi, \bar{\alpha})$ depends on the implicit I -grading $\tau : M \rightarrow I$ of the domain of $\xi : M \rightarrow \mathrm{Pic}_R$. This results in additional conditions for obtaining induced maps:

Construction 5.23. Let $(f, f^b) : (A, \xi : M \rightarrow \mathrm{Pic}_R, \bar{\alpha}) \rightarrow (B, \eta : N \rightarrow \mathrm{Pic}_R, \bar{\beta})$ be a map of R -based prelog $\mathbb{E}_k$ -rings or of prelog $\mathbb{E}_k$ R -algebras. Suppose ξ is I -graded by $\tau_M : M \rightarrow I$ and η is K -graded by $\tau_N : N \rightarrow K$. Let $\kappa : I \rightarrow K$ be a commutative monoid homomorphism and let $f_*^b : \kappa_!(\xi_*) \rightarrow \eta_*$ be an $\mathbb{E}_k$ -map in $\mathrm{Fun}(K, S_{/\mathrm{Pic}_R})$ whose underlying map of total objects in $\mathrm{Alg}_{\mathbb{E}_k}(S_{/\mathrm{Pic}_R})$ is $f^b : M \rightarrow N$. Then (f_*^b, κ) induces a map $\mathrm{THH}(\mathrm{Th}_R(\xi), \xi) \rightarrow \mathrm{THH}(\mathrm{Th}_R(\eta), \eta)$ by

Lemma 5.19, and the compatibility of this map with $\mathrm{THH}(\mathrm{Th}_R(\xi)) \rightarrow \mathrm{THH}(\mathrm{Th}_R(\eta))$ implies that (f, f^b, f_*^b, κ) induces a map

$$\mathrm{THH}(A, \xi, \tilde{\alpha}) \rightarrow \mathrm{THH}(B, \eta, \tilde{\beta})$$

of $\mathbb{E}_{k-2}$ -algebras in CycSp (respectively, $\mathrm{CycMod}_{\mathrm{THH}(R)}$) that is compatible with $\mathrm{THH}(A) \rightarrow \mathrm{THH}(B)$ via the repletion maps ρ for the domain and codomain.

Example 5.24. Given $(f, f^b) : (A, \xi : M \rightarrow \mathrm{Pic}_R, \tilde{\alpha}) \rightarrow (B, \eta : N \rightarrow \mathrm{Pic}_R, \tilde{\beta})$, we can equip M with the canonical $\pi_0(M)$ -grading of Definition 2.16, N with the canonical $\pi_0(N)$ -grading, and set $\kappa = \pi_0(f^b) : I = \pi_0(M) \rightarrow \pi_0(N) = K$. Then f_*^b arises from the path component decomposition and we get an induced map on $\log \mathrm{THH}$ formed with respect to the canonical π_0 -gradings.

We record another consequence of Lemma 5.19.

Corollary 5.25. *Let $(A, \xi : M \rightarrow \mathrm{Pic}_R, \tilde{\alpha})$ be an R -based prelog $\mathbb{E}_k$ -ring or a prelog $\mathbb{E}_k$ R -algebra. Changing the grading $\tau : M \rightarrow I$ by composing it with an exact homomorphism of discrete commutative monoids does not change $\log \mathrm{THH}$.*

Remark 5.26. In the case where $R = \mathbb{S}$ or $R = \mathbb{S}_{(p)}$ and $k = 2$, our present definition of $\log \mathrm{THH}$ coincides with the one used in [9, Definition 6.6] (with the graded $\mathbb{E}_2$ -ring $\mathbb{S}[\sigma_k]$ used there corresponding to our $\mathrm{Th}_{\mathbb{S}}(\xi_{2d})$).

6 | LOGARITHMIC TOPOLOGICAL CYCLIC HOMOLOGY

In this section we use the previous constructions to define logarithmic TC and its relatives. We begin by recalling the basic definitions in the language of [49]. However, since we will only be interested in the p -complete TC of bounded below p -cyclotomic spectra, we deviate from their conventions by imposing $\mathbb{T}$ -actions on p -cyclotomic spectra (compare [49, Remark II.1.3]).

Definition 6.1 [49, Proposition II.1.9(i)]. Let X be a spectrum with $\mathbb{T}$ -action. The *negative topological cyclic homology* $\mathrm{TC}^-(X) := X^{h\mathbb{T}}$ and the *periodic topological cyclic homology* $\mathrm{TP}(X) := X^{t\mathbb{T}}$ are the spectra given by the $\mathbb{T}$ -homotopy fixed points and the $\mathbb{T}$ -Tate fixed points, respectively. For any $\mathbb{E}_1$ -ring A we set $\mathrm{TC}^-(A) := \mathrm{TC}^-(\mathrm{THH}(A))$ and $\mathrm{TP}(A) := \mathrm{TP}(\mathrm{THH}(A))$.

Let (X, φ_p) be a (p) -cyclotomic spectrum. The p -complete *topological cyclic homology*

$$\mathrm{TC}(X)_p := \mathrm{eq}(G \circ \mathrm{can}, \varphi_p^{h\mathbb{T}})$$

is the equalizer of the maps in the diagram

$$\begin{array}{ccc} \mathrm{TC}^-(X)_p = X_p^{h\mathbb{T}} & \xrightarrow{\mathrm{can}} & X_p^{t\mathbb{T}} = \mathrm{TP}(X)_p \\ & \searrow \varphi_p^{h\mathbb{T}} & \downarrow G \\ & & (X^{tC_p})_p^{h\mathbb{T}}, \end{array}$$

where $G : X^{t\mathbb{T}} \rightarrow (X^{tC_p})^{h\mathbb{T}}$ is the comparison map. For any $\mathbb{E}_1$ -ring A we set $\mathrm{TC}(A)_p = \mathrm{TC}(\mathrm{THH}(A))_p$. When the underlying spectrum of $\mathrm{THH}(A)$ is bounded below, G is an equivalence, and, moreover, this definition of TC is equivalent to the original definition due to Bökstedt–Hsiang–Madsen [20] (see [49, Theorem II.3.8]). For a (p) -cyclotomic spectrum (X, φ_p) , we write

$$\beta = F \circ \pi : \mathrm{TC}(X)_p \xrightarrow{\pi} \mathrm{TC}^-(X)_p = X_p^{h\mathbb{T}} \xrightarrow{F} X_p \quad (6.1)$$

for the composite forgetful map.

The functors TC^- , TP and $\mathrm{TC}(-)_p$ are lax symmetric monoidal, cf. [7, Definition 4.4], so if X is an $\mathbb{E}_{k-1}$ $\mathrm{THH}(R)$ -algebra with $\mathbb{T}$ -action, then $\mathrm{TC}^-(X)$ is an $\mathbb{E}_{k-1}$ $\mathrm{TC}^-(R)$ -algebra and $\mathrm{TP}(X)$ is an $\mathbb{E}_{k-1}$ $\mathrm{TP}(R)$ -algebra, while if X is a cyclotomic $\mathbb{E}_{k-1}$ $\mathrm{THH}(R)$ -algebra, then $\mathrm{TC}(X)_p$ is an $\mathbb{E}_{k-1}$ $\mathrm{TC}(R)_p$ -algebra. These observations apply, in particular, for $X = \mathrm{THH}(A)$ when A is an $\mathbb{E}_k$ R -algebra.

Definition 6.2. Let $(A, \xi, \bar{\alpha})$ be an R -based prelog $\mathbb{E}_k$ -ring. Its *logarithmic topological negative homology*

$$\mathrm{TC}^-(A, \xi, \bar{\alpha}) := \mathrm{THH}(A, \xi, \bar{\alpha})^{h\mathbb{T}}$$

is an $\mathbb{E}_{k-2}$ $\mathrm{TC}^-(A)$ -algebra, and its *logarithmic topological periodic homology*

$$\mathrm{TP}(A, \xi, \bar{\alpha}) := \mathrm{THH}(A, \xi, \bar{\alpha})^{t\mathbb{T}}$$

is an $\mathbb{E}_{k-2}$ $\mathrm{TP}(A)$ -algebra. Its p -complete *logarithmic topological cyclic homology* is the equalizer in the following diagram, and is an $\mathbb{E}_{k-2}$ $\mathrm{TC}(A)_p$ -algebra:

$$\begin{array}{ccccc} \mathrm{TC}(A, \xi, \bar{\alpha})_p & \xrightarrow{\pi} & \mathrm{THH}(A, \xi, \bar{\alpha})_p^{h\mathbb{T}} & \xrightarrow{\mathrm{can}} & \mathrm{THH}(A, \xi, \bar{\alpha})_p^{t\mathbb{T}} \\ & & & \searrow \varphi_p^{h\mathbb{T}} & \downarrow G \\ & & & & (\mathrm{THH}(A, \xi, \bar{\alpha})^{tC_p})_p^{h\mathbb{T}}. \end{array}$$

The repletion maps

$$\begin{aligned} \rho : \mathrm{TC}^-(A) &\longrightarrow \mathrm{TC}^-(A, \xi, \bar{\alpha}), & \rho : \mathrm{TP}(A) &\longrightarrow \mathrm{TP}(A, \xi, \bar{\alpha}), & \text{and} \\ \rho : \mathrm{TC}(A)_p &\longrightarrow \mathrm{TC}(A, \xi, \bar{\alpha})_p \end{aligned}$$

are induced by the map (5.9), and are in particular $\mathbb{E}_{k-2}$ -ring maps and left module maps in the respective module categories.

If $\bar{\alpha} : \mathrm{Th}_R(\xi) \rightarrow A$ is an $\mathbb{E}_k$ R -algebra map, so that $(A, \xi, \bar{\alpha})$ is a prelog $\mathbb{E}_k$ R -algebra, then the same constructions yield $\mathbb{E}_{k-2}$ -algebras and -maps over $\mathrm{TC}^-(R)$, $\mathrm{TP}(R)$ and $\mathrm{TC}(R)_p$, respectively.

Example 6.3. We can in particular apply the previous definition to the canonical prelog $\mathbb{E}_k$ R -algebra $(\mathrm{Th}_R(\xi), \xi, \mathrm{id})$ to obtain $\mathbb{E}_{k-1}$ -algebras $\mathrm{TC}^-(\mathrm{Th}_R(\xi), \xi)$, $\mathrm{TP}(\mathrm{Th}_R(\xi), \xi)$, and $\mathrm{TC}(\mathrm{Th}_R(\xi), \xi)_p$ over $\mathrm{TC}^-(R)$, $\mathrm{TP}(R)$ and $\mathrm{TC}(R)_p$, respectively. This uses the $\mathbb{E}_{k-1}$ -structure on $\mathrm{THH}(\mathrm{Th}_R(\xi), \xi)$ from Definition 5.17, rather than the $\mathbb{E}_{k-2}$ -structure from its $\mathrm{THH}(\mathrm{Th}_R(\xi))$ -algebra structure (compare Corollary 5.18).

6.1 | Graded log TC

We now introduce parts of TC that can be built from graded THH, and consider when the whole of TC can be recovered from these pieces.

Definition 6.5. Given a discrete commutative monoid I let $I/\sim_p$ be the set of equivalence classes $[i]_p$ with respect to the equivalence relation on the underlying set of I generated by $i \sim_p pi$ for all $i \in I$.

For example, the elements of $\mathbb{Z}_{\geq 0}/\sim_p$ are $[0]_p$ together with the $[i]_p$ for $i \in \mathbb{Z}_{>0}$ with $p \nmid i$.

Definition 6.6 [6, Remark 5.39]. Let X_* be an I -graded spectrum with $\mathbb{T}$ -action. The I -graded topological negative homology $\mathrm{TC}^-(X_*) := (X_*)^{h\mathbb{T}}$ and the I -graded topological periodic homology $\mathrm{TP}(X_*) := (X_*)^{t\mathbb{T}}$ are the I -graded spectra given by the $\mathbb{T}$ -homotopy fixed points $(X_i)^{h\mathbb{T}}$ and $\mathbb{T}$ -Tate fixed points $(X_i)^{t\mathbb{T}}$, respectively, in each weight $i \in I$. For any I -graded $\mathbb{E}_1$ -ring A_* we set $\mathrm{TC}^-(A_*) := \mathrm{TC}^-(\mathrm{THH}(A_*))$ and $\mathrm{TP}(A_*) := \mathrm{TP}(\mathrm{THH}(A_*))$.

The I -graded functors TC^- and TP are lax symmetric monoidal, so if X_* is an I -graded $\mathbb{E}_{k-1}$ $\mathrm{THH}(R)$ -algebra with $\mathbb{T}$ -action, then $\mathrm{TC}^-(X_*)$ is an I -graded $\mathbb{E}_{k-1}$ $\mathrm{TC}^-(R)$ -algebra and $\mathrm{TP}(X_*)$ is an I -graded $\mathbb{E}_{k-1}$ $\mathrm{TP}(R)$ -algebra. This applies, in particular, for $X_* = \mathrm{THH}(A_*)$ when A_* is an I -graded $\mathbb{E}_k$ R -algebra.

Remark 6.7. Let $c : I \rightarrow \{0\}$ and $X = c_! X_*$. Beware that the canonical maps

$$\bigvee_{i \in I} \mathrm{TC}^-(X_i) = c_! \mathrm{TC}^-(X_*) \longrightarrow \mathrm{TC}^-(c_! X_*) = \mathrm{TC}^-(X)$$

$$\bigvee_{i \in I} \mathrm{TP}(X_i) = c_! \mathrm{TP}(X_*) \longrightarrow \mathrm{TP}(c_! X_*) = \mathrm{TP}(X)$$

are generally not equivalences, since the colimit defining total objects does not generally commute with the limits defining TC^- and TP .

Definition 6.8. Let (X_*, φ_p) be an I -graded L_p -twisted cyclotomic spectrum. For each $\sim_p$ -equivalence class $[i]_p \in I/\sim_p$, let

$$X_{[i]_p} = \bigvee_{j \in [i]_p} X_j.$$

The composite map

$$\varphi_p : X_{[i]_p} = \bigvee_{j \in [i]_p} X_j \xrightarrow{\bigvee_{j \in [i]_p} \varphi_{p,j}} \bigvee_{j \in [i]_p} (X_j^{tC_p}) \longrightarrow (\bigvee_{j \in [i]_p} X_j)^{tC_p} = X_{[i]_p}^{tC_p}$$

makes $X_{[i]_p}$ a cyclotomic subspectrum of $X = c_! X_*$. The p -completed $I/\sim_p$ -graded topological cyclic homology is the functor

$$[i]_p \longmapsto \mathrm{TC}(X_{[i]_p})_p.$$

Definition 6.9. A spectrum Y is m -connective if $\pi_*(Y) = 0$ for all $* < m$. Let X_* be an I -graded spectrum. We say that X_* is *properly connective* if for each $m \in \mathbb{Z}$ the set

$$\{i \in I \mid X_i \text{ is } m\text{-connective}\}$$

has finite complement in I .

Example 6.10. For R connective and $\xi : \langle x \rangle \rightarrow \text{Pic}_R$, the $\mathbb{Z}_{\geq 0}$ -graded spectra $X_* = \text{THH}(\text{Th}_R(\xi_*))$ and $\text{THH}(\text{Th}_R(\xi_*), \xi)$ will be properly connective if $\xi(x)$ is at least 1-connective, since then $\xi(x^i) = \xi(x)^{\otimes_{R^i}}$ will be at least i -connective. They will typically not be properly connective if $\xi(x)$ is 0-connective, such as for $\xi = 0$ appearing in the examples $(ku, \langle p \rangle)$, $(ku_{(p)}, \langle p \rangle)$, and $(\ell, \langle p \rangle)$ from (4.9).

Similarly, for $\xi : \langle x, y \rangle \rightarrow \text{Pic}_R$ the corresponding $(\mathbb{Z}_{\geq 0})^2$ -graded spectra will be properly connective if $\xi(x)$ and $\xi(y)$ are both 1-connective, but usually not if one of them is just 0-connective as in the examples $(\text{BP}\langle n \rangle, \langle p, v_1, \dots, v_n \rangle)$ from (4.10) or $(\ell, \langle p, v_1 \rangle)$ from (4.11).

Lemma 6.11. Let X_* be an I -graded spectrum with $\mathbb{T}$ -action. Suppose that X_* is properly connective.

(i) The canonical map

$$X = \bigvee_{i \in I} X_i \xrightarrow{\sim} \prod_{i \in I} X_i$$

is an equivalence.

(ii) The middle and right-hand canonical maps

$$\bigvee_{i \in I} (X_i^{tC_p}) \longrightarrow (\bigvee_{i \in I} X_i)^{tC_p} \xrightarrow{\sim} (\prod_{i \in I} X_i)^{tC_p} \xrightarrow{\sim} \prod_{i \in I} (X_i^{tC_p})$$

are equivalences.

(iii) The middle and right-hand canonical maps

$$\bigvee_{i \in I} (X_i^{t\mathbb{T}}) \longrightarrow (\bigvee_{i \in I} X_i)^{t\mathbb{T}} \xrightarrow{\sim} (\prod_{i \in I} X_i)^{t\mathbb{T}} \xrightarrow{\sim} \prod_{i \in I} (X_i^{t\mathbb{T}})$$

are equivalences.

Proof. The first displayed map induces the inclusion $\bigoplus_{i \in I} \pi_*(X_i) \rightarrow \prod_{i \in I} \pi_*(X_i)$ in homotopy, and we are assuming that in each fixed degree $*$ there are only finitely many non-zero factors in this product. This implies that the map in (i) is an equivalence. It follows that the middle maps in the second and third displays are also equivalences.

To show that the right-hand map in (ii) is an equivalence, the map of norm-restriction cofiber sequences

$$\begin{array}{ccccc} (\prod_{i \in I} X_i)_{hC_p} & \xrightarrow{N^h} & (\prod_{i \in I} X_i)^{hC_p} & \xrightarrow{R^h} & (\prod_{i \in I} X_i)^{tC_p} \\ \downarrow & & \downarrow \simeq & & \downarrow \\ \prod_{i \in I} ((X_i)_{hC_p}) & \xrightarrow{N^h} & \prod_{i \in I} (X_i^{hC_p}) & \xrightarrow{R^h} & \prod_{i \in I} (X_i^{tC_p}) \end{array}$$

shows that it suffices to prove that the left-hand vertical map is an equivalence. Forming homotopy orbits preserves connectivity, so for each fixed degree $*$ only finitely many of the factors X_i

contribute to π_* of the two left-hand spectra. The claim then follows, because $(\bigvee_{i \in I} X_i)_{hC_p} \rightarrow \bigvee_{i \in I} ((X_i)_{hC_p})$ is an equivalence.

The proof that the right-hand map in (iii) is an equivalence is similar, using the norm-restriction cofiber sequence

$$\Sigma(X_i)_{h\mathbb{T}} \xrightarrow{N^h} X_i^{h\mathbb{T}} \xrightarrow{R^h} X_i^{t\mathbb{T}} \quad (6.2)$$

and its analogue for $\prod_{i \in I} X_i$. $\square$

Proposition 6.12. *Let (X_*, φ_p) be an I -graded L_p -twisted cyclotomic spectrum. The p -cyclotomic structure map for $X = c_i X_*$ factors as*

$$X = \bigvee_{i \in I} X_i \xrightarrow{\bigvee_{i \in I} \varphi_{p,i}} \bigvee_{i \in I} (X_i^{tC_p}) \longrightarrow (\bigvee_{i \in I} X_i)^{tC_p}.$$

If X_* is properly connective, then this composite is equivalent to the product map

$$\prod_{i \in I} X_i \xrightarrow{\prod_{i \in I} \varphi_{p,i}} \prod_{i \in I} (X_i^{tC_p}),$$

and there is a natural equivalence

$$\mathrm{TC}(X)_p \simeq \prod_{[i]_p \in I/\sim_p} \mathrm{TC}(X_{[i]_p})_p.$$

Proof. The first claims are immediate from Lemma 6.11. Moreover, there are canonical equivalences

$$\begin{aligned} X^{h\mathbb{T}} &= (\bigvee_{i \in I} X_i)^{h\mathbb{T}} \xrightarrow{\simeq} (\prod_{i \in I} X_i)^{h\mathbb{T}} \simeq \prod_{i \in I} (X_i^{h\mathbb{T}}) \\ X^{t\mathbb{T}} &= (\bigvee_{i \in I} X_i)^{t\mathbb{T}} \xrightarrow{\simeq} (\prod_{i \in I} X_i)^{t\mathbb{T}} \simeq \prod_{i \in I} (X_i^{t\mathbb{T}}) \\ (X^{tC_p})^{h\mathbb{T}} &= ((\bigvee_{i \in I} X_i)^{tC_p})^{h\mathbb{T}} \xrightarrow{\simeq} ((\prod_{i \in I} X_i)^{tC_p})^{h\mathbb{T}} \simeq \prod_{i \in I} ((X_i^{tC_p})^{h\mathbb{T}}). \end{aligned}$$

Under these identifications, the maps $\mathrm{can} : X^{h\mathbb{T}} \rightarrow X^{t\mathbb{T}}$ and $G : X^{t\mathbb{T}} \rightarrow (X^{tC_p})^{h\mathbb{T}}$ correspond to the product over $i \in I$ of the maps

$$\mathrm{can}_i : X_i^{h\mathbb{T}} \longrightarrow X_i^{t\mathbb{T}} \quad \text{and} \quad G_i : X_i^{t\mathbb{T}} \longrightarrow (X_i^{tC_p})^{h\mathbb{T}},$$

while $\varphi_p^{h\mathbb{T}} : X^{h\mathbb{T}} \rightarrow (X^{tC_p})^{h\mathbb{T}}$ corresponds to the product over $i \in I$ of the maps

$$\varphi_{p,i}^{h\mathbb{T}} : X_i^{h\mathbb{T}} \longrightarrow (X_{pi}^{tC_p})^{h\mathbb{T}}.$$

Hence $\mathrm{TC}(X)_p$ is the product over the equivalence classes $[i]_p \in I/\sim_p$ of the equalizer of the two maps

$$\prod_{j \in [i]_p} (X_j)_p^{h\mathbb{T}} \xrightarrow[\prod_j \varphi_{p,j}^{h\mathbb{T}}]{\prod_j G_j \circ \mathrm{can}_j} \prod_{j \in [i]_p} (X_j^{tC_p})_p^{h\mathbb{T}}. \quad (6.3)$$

Since X_* restricted to $[i]_p \subset I$ remains properly connective, this factor in the product is also the equalizer $\mathrm{TC}(X_{[i]_p})_p$ of the two maps

$$(X_{[i]_p})_p^{h\mathbb{T}} = \left(\bigvee_{j \in [i]_p} X_j \right)_p^{h\mathbb{T}} \xrightarrow[\varphi_p^{h\mathbb{T}}]{G \circ \mathrm{can}} \left(\left(\bigvee_{j \in [i]_p} X_j \right)^{tC_p} \right)_p^{h\mathbb{T}} = (X_{[i]_p})_p^{h\mathbb{T}}.$$

□

7 | TOPOLOGICAL HOCHSCHILD HOMOLOGY OF THOM RINGS AND ALGEBRAS

The goal of this section is to show that the THH spectra of (weight-graded) Thom rings and algebras are equivalent to (weight-graded) Thom spectra of suitable cyclic bar constructions, and that log THH spectra of canonical prelog algebras are equivalent to Thom spectra of suitable replete bar constructions. These will be equivalences of $\mathbb{E}_{k-1}$ -rings or $\mathbb{E}_{k-1}$ THH(R)-algebras with $\mathbb{T}$ -action, not generally fully capturing the cyclotomic structures constructed in Section 5.

7.1 | THH of Thom rings

We first handle the case $R = \mathbb{S}$. Let I be a discrete commutative monoid. As in Section 2.13, we consider the symmetric monoidal category $\mathrm{Fun}(I, S_{/\mathrm{Pic}_{\mathbb{S}}})$ with the Day convolution product structure. Then Corollary 5.4(i) provides a functor

$$B^{\mathrm{cy}} : \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Fun}(I, S_{/\mathrm{Pic}_{\mathbb{S}}})) \rightarrow \mathrm{Alg}_{\mathbb{E}_{k-1}}(\mathrm{Fun}(I, S_{/\mathrm{Pic}_{\mathbb{S}}})^{B\mathbb{T}}). \quad (7.1)$$

Using Lemma 2.14, we note that it sends an I -graded $\mathbb{E}_k$ -algebra $\xi_* : M_* \rightarrow c^*\mathrm{Pic}_{\mathbb{S}}$ in $S_{/\mathrm{Pic}_{\mathbb{S}}}$ to the I -graded $\mathbb{E}_{k-1}$ -algebra with $\mathbb{T}$ -action $B^{\mathrm{cy}}(M_*) = \{i \mapsto B^{\mathrm{cy}}(M_*; i)\}$ with augmentation $\xi_*^{\mathrm{cy}} : B^{\mathrm{cy}}(M_*) \rightarrow c^*\mathrm{Pic}_{\mathbb{S}}$, where $c^*\mathrm{Pic}_{\mathbb{S}}$ is the constant I -diagram on $\mathrm{Pic}_{\mathbb{S}}$ with trivial $\mathbb{T}$ -action. The map ξ_*^{cy} is the composite of $B^{\mathrm{cy}}(\xi_*) : B^{\mathrm{cy}}(M_*) \rightarrow B^{\mathrm{cy}}(c^*\mathrm{Pic}_{\mathbb{S}})$ and the map $B^{\mathrm{cy}}(c^*\mathrm{Pic}_{\mathbb{S}}) \rightarrow c^*\mathrm{Pic}_{\mathbb{S}}$ induced by the iterated multiplications of the $\mathbb{E}_{\infty}$ -object $c^*\mathrm{Pic}_{\mathbb{S}}$. When $I = \{0\}$, this specializes to an absolute cyclic bar construction sending an $\mathbb{E}_k$ -map $\xi : M \rightarrow \mathrm{Pic}_{\mathbb{S}}$ to an $\mathbb{E}_{k-1}$ -algebra $\xi^{\mathrm{cy}} : B^{\mathrm{cy}}(M) \rightarrow \mathrm{Pic}_{\mathbb{S}}$ in $(S_{/\mathrm{Pic}_{\mathbb{S}}})^{B\mathbb{T}}$.

We record some compatibility statements, of which part (7.3) is closely related to [15, Theorem 2.1]:

Proposition 7.2. *Let $c : I \rightarrow \{0\}$ be the collapse map and let $\xi_* : M_* \rightarrow c^*\mathrm{Pic}_{\mathbb{S}}$ be an $\mathbb{E}_k$ -algebra in $\mathrm{Fun}(I, S_{/\mathrm{Pic}_{\mathbb{S}}})$ with total object $\xi : M \rightarrow \mathrm{Pic}_{\mathbb{S}}$.*

(i) *There is a natural equivalence of I -graded $\mathbb{E}_{k-1}$ -rings with $\mathbb{T}$ -action*

$$\mathrm{THH}(\mathrm{Th}_{\mathbb{S}}(\xi_*)) \simeq \mathrm{Th}_{\mathbb{S}}(\xi_*^{\mathrm{cy}} : B^{\mathrm{cy}}(M_*) \rightarrow c^*\mathrm{Pic}_{\mathbb{S}}) \quad (7.2)$$

and an analogous equivalence of total objects

$$\mathrm{THH}(\mathrm{Th}_{\mathbb{S}}(\xi)) \simeq \mathrm{Th}_{\mathbb{S}}(\xi^{\mathrm{cy}} : B^{\mathrm{cy}}(M) \rightarrow \mathrm{Pic}_{\mathbb{S}}). \quad (7.3)$$

(ii) *There is a natural equivalence of $\mathbb{E}_{k-1}$ -algebras with $\mathbb{T}$ -action in $S_{/\text{Pic}_{\mathbb{S}}}$ between*

$$c_1(\xi_*^{\text{cy}} : B^{\text{cy}}(M_*) \rightarrow c^*\text{Pic}_{\mathbb{S}}) \quad \text{and} \quad \xi^{\text{cy}} : B^{\text{cy}}(M) \rightarrow \text{Pic}_{\mathbb{S}}.$$

(iii) *Passing to total objects, the equivalence (7.2) recovers (7.3).*

Proof. All statements follow from Corollary 5.4(iii) applied to suitable parts of the commutative square

$$\begin{array}{ccc} \text{Fun}(I, S_{/\text{Pic}_{\mathbb{S}}}) & \xrightarrow{\text{Th}_{\mathbb{S}}} & \text{Fun}(I, \text{Sp}) \\ c_1 \downarrow & & \downarrow c_1 \\ S_{/\text{Pic}_{\mathbb{S}}} & \xrightarrow{\text{Th}_{\mathbb{S}}} & \text{Sp} \end{array}$$

of colimit-preserving symmetric monoidal functors. □

7.2 | THH of Thom algebras

To analyze THH in the case of Thom R -algebras for a general base $\mathbb{E}_{\infty}$ -ring R , we must take into account that the forgetful functor $u : \text{Mod}_R \rightarrow \text{Sp}$ is usually only lax symmetric monoidal. This implies that the statement of Proposition 7.2(i) does in general not hold with R in place of $\mathbb{S}$. To alleviate this problem, we pass to a more general setting where the $\mathbb{E}_{\infty}$ -ring R is allowed to vary.

Remark 7.4. If we were concerned with the *relative* $\text{THH}^R(A) = |[q] \mapsto A^{\otimes_R 1+q}|$ of an R -algebra A , then we could work in Mod_R throughout. However, $\text{THH}^R(A)$ will only admit a cyclotomic structure under additional hypotheses on R , such as the existence of a $\mathbb{T}$ -map $R \rightarrow R^{tC_p}$ factoring $\epsilon^{tC_p} \varphi_p : \text{THH}(R) \rightarrow \text{THH}(R)^{tC_p} \rightarrow R^{tC_p}$ over $\epsilon : \text{THH}(R) \rightarrow R$, that is, that R is a *cyclotomic base* (compare [6, Example A.3; 30, Definition 3.2.1], [18], and work in progress by Devalapurkar, Hahn, Raksit and Yuan in the $R = MU$ case).

Lemma 7.5. *The functor $\text{Pic} : \text{CAlg}(\text{Sp}) \rightarrow \mathcal{S}$ is lax symmetric monoidal and preserves filtered colimits.*

Informally, for $\mathbb{E}_{\infty}$ -rings $R^1, R^2 \in \text{CAlg}(\text{Sp})$ the lax structure map

$$\hat{\otimes} : \text{Pic}_{R^1} \times \text{Pic}_{R^2} \longrightarrow \text{Pic}_{R^1 \otimes R^2}$$

takes an invertible R^1 -module P^1 and an invertible R^2 -module P^2 to the invertible $R^1 \otimes R^2$ -module $P^1 \otimes P^2$.

Proof of Lemma 7.5. This pairing is obtained from the composite

$$\text{Mod}_{R^1} \times \text{Mod}_{R^2} \longrightarrow \text{Mod}_{R^1} \otimes \text{Mod}_{R^2} \simeq \text{Mod}_{R^1 \otimes R^2}$$

from [47, Proposition 4.8.1.17 and Theorem 4.8.5.16] by passage to the maximal grouplike (with respect to $\hat{\otimes}$) ∞ -groupoids inside these symmetric monoidal ∞ -categories.

The functor $R \mapsto \text{Mod}_R$ preserves K -indexed colimits for weakly contractible simplicial sets K by [47, Corollary 4.8.5.13], and this includes all filtered K by [45, Lemma 5.3.1.20]. Moreover, $\text{Mod}_R \mapsto \text{Pic}_R$ preserves filtered colimits, since passage to the groupoid core $\text{Mod}_R^\simeq$, and to the full subcategory of invertible objects, both have this property. $\square$

Definition 7.6. Let I be a discrete commutative monoid. We define

$$\text{Fun}(I, S)_{/c^*\text{Pic}} = \text{CAlg}(\text{Sp}) \times_{\text{Fun}(\{1\}, \text{Fun}(I, S))} \text{Fun}(\Delta^1, \text{Fun}(I, S))$$

to be ∞ -category with objects pairs $(R, \xi_* : M_* \rightarrow c^*\text{Pic}_R)$ where c^* is the restriction along $c : I \rightarrow \{0\}$. To equip it with a symmetric monoidal structure, we define $(\text{Fun}(I, S)_{/c^*\text{Pic}})^\otimes$ as the pullback in ∞ -operads of

$$\text{CAlg}(\text{Sp})^\otimes \xrightarrow{(c^*\text{Pic})^\otimes} \text{Fun}(I, S)^\otimes \xleftarrow{\text{ev}_1^\otimes} \text{Fun}(\Delta^1, \text{Fun}(I, S)^\otimes),$$

where $\text{ev}_1^\otimes$ is a co-Cartesian fibration by [45, Corollary 2.4.7.12]. The composite of its pullback $(\text{Fun}(I, S)_{/c^*\text{Pic}})^\otimes \rightarrow \text{CAlg}(\text{Sp})^\otimes$ with the co-Cartesian fibration $\text{CAlg}(\text{Sp})^\otimes \rightarrow N(\mathcal{F}\text{in}_*)$ that defines the symmetric monoidal structure of $\text{CAlg}(\text{Sp})$ is the co-Cartesian fibration that exhibits the symmetric monoidal structure of $\text{Fun}(I, S)_{/c^*\text{Pic}}$.

Informally, the symmetric monoidal pairing of $\text{Fun}(I, S)_{/c^*\text{Pic}}$ takes two objects $(R^1, \xi_*^1 : M_*^1 \rightarrow c^*\text{Pic}_{R^1})$ and $(R^2, \xi_*^2 : M_*^2 \rightarrow c^*\text{Pic}_{R^2})$ to

$$(R^1 \otimes R^2, \bar{\otimes}(\xi_*^1 \times \xi_*^2) : M_*^1 \times M_*^2 \rightarrow c^*\text{Pic}_{R^1 \otimes R^2}),$$

where $\bar{\otimes}$ is the lax structure map for the functor $c^*\text{Pic}$ (that results as the composite of the lax structure map $\bar{\otimes}$ of Pic with that of c^*).

Lemma 7.7. *The ∞ -category $\text{Fun}(I, S)_{/c^*\text{Pic}}$ is presentably symmetric monoidal.*

Proof. We consider the functors $F, G : \text{Fun}(I, S) \times \text{CAlg}(\text{Sp}) \rightarrow \text{Fun}(I, S)$ given by the projection to the first factor and the projection to the second factor followed by $c^*\text{Pic}$, respectively. Then by construction, $\text{Fun}(I, S)_{/c^*\text{Pic}}$ is equivalent to the lax equalizer $\text{LEq}(F, G)$ in the sense of [49, Definition II.1.4]. Since F preserves colimits and G preserves filtered colimits by Lemma 7.5, [49, Proposition II.1.5(iv)] and its proof show that $\text{Fun}(I, S)_{/c^*\text{Pic}}$ is presentable and that the projection $\text{LEq}(F, G) \rightarrow \text{Fun}(I, S) \times \text{CAlg}(\text{Sp})$ preserves and detects colimits. Using that $\text{CAlg}(\text{Sp})$ and $\text{Fun}(I, S)$ are presentably symmetric monoidal, it follows from the construction of the symmetric monoidal product of $\text{Fun}(I, S)_{/c^*\text{Pic}}$ that it preserves colimits separately in each variable. $\square$

We define the *underlying Thom spectrum* functor to be

$$u\text{Th} : \text{Fun}(I, S)_{/c^*\text{Pic}} \rightarrow \text{Fun}(I, \text{Sp}), \quad (R, \xi_* : M_* \rightarrow c^*\text{Pic}_R) \mapsto u\text{Th}_R(\xi_*). \quad (7.4)$$

When $I = \{0\}$, this specializes to a functor $u\text{Th} : S_{/\text{Pic}} \rightarrow \text{Sp}$.

Lemma 7.8. *The functor (7.4) is symmetric monoidal and preserves sifted colimits.*

Proof. Recall that $u\mathrm{Th}_R(\xi_*) \simeq \{i \mapsto \mathrm{colim}(u\xi_i : M_i \rightarrow \mathrm{Sp})\}$. We have

$$\begin{aligned}
 & u\mathrm{Th}_{R_1}(\xi_*^1) \otimes u\mathrm{Th}_{R_2}(\xi_*^2) \\
 &= \{i \mapsto \coprod_{i_1+i_2=i} \mathrm{colim}(u\xi_{i_1}^1 : M_{i_1}^1 \rightarrow \mathrm{Sp}) \otimes \mathrm{colim}(u\xi_{i_2}^2 : M_{i_2}^2 \rightarrow \mathrm{Sp})\} \\
 &\simeq \{i \mapsto \mathrm{colim}(\coprod_{i_1+i_2=i} M_{i_1}^1 \times M_{i_2}^2 \xrightarrow{u\xi_{i_1}^1 \times u\xi_{i_2}^2} \mathrm{Sp} \times \mathrm{Sp} \xrightarrow{\otimes} \mathrm{Sp})\} \\
 &= \mathrm{colim}(u\bar{\otimes}(\xi_*^1 \times \xi_*^2) : M_*^1 \times M_*^2 \rightarrow \mathrm{Sp}) \\
 &= u\mathrm{Th}_{R_1 \otimes R_2}(\bar{\otimes}(\xi_*^1 \times \xi_*^2)),
 \end{aligned}$$

since the smash product of spectra preserves colimits in each variable.

For the rest of this proof, we will follow the terminology of [38, 45] and say that a functor between ∞ -categories is *cofinal* if restriction along it preserves colimits. As a first preparatory step, let $\mathcal{D}$ be an ∞ -category and let $\mathrm{Ar}(\mathcal{D}) = \mathrm{Fun}(\Delta^1, \mathcal{D})$ denote the category of arrows $d \rightarrow e$ in $\mathcal{D}$. We claim that the ∞ -functor $\mathcal{D} \rightarrow \mathrm{Ar}(\mathcal{D})$ sending d to id_d is cofinal. By the ∞ -categorical version of Quillen's Theorem A (see, for example, [38, Theorem 4.4.20]), it is sufficient to show that for any object $\alpha : d \rightarrow e$ in $\mathrm{Ar}(\mathcal{D})$, the slice $\mathcal{D}_{/\alpha}$ is weakly contractible. This slice is weakly equivalent to the slice $\mathcal{D}_{e/}$, which is weakly contractible because it has an initial object.

Let $\mathcal{D}$ be a sifted ∞ -category. As a second preparatory step, we claim that for any object d of $\mathcal{D}$, the forgetful functor $U : \mathcal{D}_{d/} \rightarrow \mathcal{D}$ is cofinal. Arguing as above, it is sufficient to show that for any object e of $\mathcal{D}$ the slice $(\mathcal{D}_{d/})_{e/}$ is weakly contractible. Identifying the latter with the slice of the diagonal functor $\mathcal{D} \rightarrow \mathcal{D} \times \mathcal{D}$ under (d, e) , the conclusion follows from the definition of a sifted ∞ -category.

Let $d \mapsto (R_d, (\xi_*)_d : (M_*)_d \rightarrow c^*\mathrm{Pic}_{R_d})$ be a $\mathcal{D}$ -diagram in $\mathrm{Fun}(I, S)_{/c^*\mathrm{Pic}}$, with colimit $(R, \xi_* : M_* \rightarrow c^*\mathrm{Pic}_R)$. We can extend the functor $d \mapsto u\mathrm{Th}_{R_d}((\xi_*)_d)$ over $\mathrm{Ar}(\mathcal{D})$ by sending $\alpha : d \rightarrow e$ to $u\mathrm{Th}_{R_e}((c^*\mathrm{Pic}_\alpha) \circ (\xi_*)_d)$, where $\mathrm{Pic}_\alpha : \mathrm{Pic}_{R_d} \rightarrow \mathrm{Pic}_{R_e}$ is given by base change along $\alpha : R_d \rightarrow R_e$. We then obtain equivalences

$$\begin{aligned}
 \mathrm{colim}_{d \in \mathcal{D}} u\mathrm{Th}_{R_d}((\xi_*)_d) &\simeq \mathrm{colim}_{\alpha : d \rightarrow e \in \mathrm{Ar}(\mathcal{D})} u\mathrm{Th}_{R_e}((c^*\mathrm{Pic}_\alpha) \circ (\xi_*)_d) \\
 &\simeq \mathrm{colim}_{d \in \mathcal{D}} \mathrm{colim}_{\alpha : d \rightarrow e \in \mathcal{D}_{d/}} u\mathrm{Th}_{R_e}((c^*\mathrm{Pic}_\alpha) \circ (\xi_*)_d) \\
 &\simeq \mathrm{colim}_{d \in \mathcal{D}} u\mathrm{Th}_R((c^*\mathrm{Pic}_{R_d \rightarrow R}) \circ (\xi_*)_d) \\
 &\simeq u\mathrm{Th}_R(\xi_*).
 \end{aligned}$$

The first equivalence arises from the first preparatory step and the last one from $u\mathrm{Th}_R$ commuting with colimits. The second one arises as in [34, Proposition 6.2], viewing the arrow category $\mathrm{Ar}(\mathcal{D})$ as the Grothendieck construction on the undercategories $\mathcal{D}_{d/}$ for $d \in \mathcal{D}$. For the third equivalence, the map we need to show to be an equivalence is induced by the maps $u(R_e \otimes_{R_d} \mathrm{Th}_{R_d}((\xi_*)_d)) \rightarrow u(R \otimes_{R_d} \mathrm{Th}_{R_d}((\xi_*)_d))$ via the diagram of R -modules underlying the restriction of a colimit cone for the equivalence of $\mathbb{E}_\infty$ -rings $\mathrm{colim}_{d \rightarrow e \in \mathcal{D}_{d/}} R_e \rightarrow R$ resulting from the second preparatory step. By writing $(M_*)_d$ as a colimit of its points, this reduces to the case where $\mathrm{Th}_{R_d}((\xi_*)_d)$ is $\otimes$ -invertible over R_d . So we need to show that $\mathrm{colim}_{d \rightarrow e \in \mathcal{D}_{d/}} uR_e \rightarrow uR$ is an equivalence. Because the colimit system arises by restriction along the projection $\mathcal{D}_{d/} \rightarrow \mathcal{D}$, we can apply the second preparatory step once more so that the claim reduces to showing that

$\operatorname{colim}_{e \in D} uR_e \rightarrow uR$ is an equivalence. This holds because sifted colimits of $\mathbb{E}_\infty$ -rings are sifted colimits of the underlying spectra [47, Corollary 3.2.3.2]. $\square$

The passage to total objects $c_! : \operatorname{Fun}(I, S) \rightarrow S$ induces a functor

$$c_! : \operatorname{Fun}(I, S)_{/c^* \operatorname{Pic}} \rightarrow S_{/\operatorname{Pic}} \quad (7.5)$$

sending $(R, \xi_* : M_* \rightarrow c^* \operatorname{Pic}_R)$ to the adjoint $\xi : M = c_!(M_*) \rightarrow \operatorname{Pic}_R$ of ξ_* .

Lemma 7.9. *The functor (7.5) is symmetric monoidal and preserves colimits.*

Proof. The functor preserves colimits because it has a right adjoint induced by c^* . It is symmetric monoidal because the underlying total object of the Day convolution product in $\operatorname{Fun}(I, S)$ is equivalent to the product of the underlying total objects. $\square$

We also need the category $(\operatorname{Fun}(I, S)_{/c^* \operatorname{Pic}})^{B\mathbb{T}}$ of $\mathbb{T}$ -objects in $\operatorname{Fun}(I, S)_{/c^* \operatorname{Pic}}$. Since $\operatorname{Fun}(\mathbb{T}, -)$ commutes with pullbacks, Definition 7.6 implies that such a $\mathbb{T}$ -object consists of an $\mathbb{E}_\infty$ -ring with $\mathbb{T}$ -action and $\mathbb{T}$ -map in $\operatorname{Fun}(I, S)$ to $c^* \operatorname{Pic}$ of this $\mathbb{E}_\infty$ -ring with $\mathbb{T}$ -action. We extend $u\operatorname{Th}$ to this category of $\mathbb{T}$ -objects by simply appealing to functoriality.

Applying Corollary 5.4(i) provides a cyclic bar construction

$$B^{\operatorname{cy}} : \operatorname{Alg}_{\mathbb{E}_k}(\operatorname{Fun}(I, S)_{/c^* \operatorname{Pic}}) \rightarrow \operatorname{Alg}_{\mathbb{E}_{k-1}}((\operatorname{Fun}(I, S)_{/c^* \operatorname{Pic}})^{B\mathbb{T}})$$

which can be described as follows:

Lemma 7.10. *For an object $(R, \xi_* : M_* \rightarrow c^* \operatorname{Pic}_R)$ of $\operatorname{Alg}_{\mathbb{E}_k}(\operatorname{Fun}(I, S)_{/c^* \operatorname{Pic}})$, the cyclic bar construction $B^{\operatorname{cy}}(R, \xi_*)$ is, as an object of $\operatorname{Alg}_{\mathbb{E}_{k-1}}((\operatorname{Fun}(I, S)_{/c^* \operatorname{Pic}})^{B\mathbb{T}})$, equivalent to*

$$(\operatorname{THH}(R), \xi_*^{\operatorname{cy}} := \bar{\epsilon} \circ B^{\operatorname{cy}}(\xi_*) : B^{\operatorname{cy}}(M_*) \longrightarrow B^{\operatorname{cy}}(c^* \operatorname{Pic}_R) \longrightarrow c^* \operatorname{Pic}_{\operatorname{THH}(R)}),$$

where $\bar{\epsilon} : B^{\operatorname{cy}}(c^* \operatorname{Pic}_R) \rightarrow c^* \operatorname{Pic}_{\operatorname{THH}(R)}$ is the $\mathbb{T}$ -equivariant $\mathbb{E}_\infty$ -map obtained by applying Corollary 5.4(ii) to the lax symmetric monoidal structure of $c^* \operatorname{Pic}$.

Proof. The forgetful functors from $\operatorname{Fun}(I, S)_{/c^* \operatorname{Pic}}$ to $\operatorname{CAlg}(\operatorname{Sp})$ and $\operatorname{Fun}(I, S)$ are symmetric monoidal and create the colimit in $\operatorname{Fun}(I, S)_{/c^* \operatorname{Pic}}$ defining geometric realization, using the lax symmetric monoidal structures on c^* and Pic . This explains the appearance of $\operatorname{THH}(R)$, $B^{\operatorname{cy}}(M_*)$, and $\xi_*^{\operatorname{cy}}$, respectively. $\square$

An $\mathbb{E}_k$ -algebra $\xi_* : M_* \rightarrow c^* \operatorname{Pic}_R$ in $\operatorname{Fun}(I, S_{/\operatorname{Pic}_R})$ can be viewed as an $\mathbb{E}_k$ -algebra in $\operatorname{Fun}(I, S)_{/c^* \operatorname{Pic}}$ by letting the $\mathbb{E}_k$ -algebra structure in the $\operatorname{CAlg}(\operatorname{Sp})$ -component be the restricted $\mathbb{E}_\infty$ -algebra structure. More generally, an $\mathbb{E}_k$ -algebra over an $\mathbb{E}_\infty$ -algebra in $\operatorname{Fun}(I, S_{/\operatorname{Pic}_R})$ gives rise to an $\mathbb{E}_k$ -algebra over an $\mathbb{E}_\infty$ -algebra in $\operatorname{Fun}(I, S)_{/c^* \operatorname{Pic}}$.

Proposition 7.11. *Let $\xi_* : M_* \rightarrow c^* \operatorname{Pic}_R$ be an $\mathbb{E}_k$ -algebra in $\operatorname{Fun}(I, S_{/\operatorname{Pic}_R})$, viewed as an $\mathbb{E}_k$ -algebra in $\operatorname{Fun}(I, S)_{/c^* \operatorname{Pic}}$ as explained above, and let $\xi : M \rightarrow \operatorname{Pic}_R$ be its total object.*

(i) *There is a natural equivalence of I -graded $\mathbb{E}_{k-1}$ $\operatorname{THH}(R)$ -algebras with $\mathbb{T}$ -action*

$$\operatorname{THH}(\operatorname{Th}_R(\xi_*)) \simeq \operatorname{Th}_{\operatorname{THH}(R)}(\xi_*^{\operatorname{cy}} : B^{\operatorname{cy}}(M_*) \rightarrow c^* \operatorname{Pic}_{\operatorname{THH}(R)}) \quad (7.6)$$

and an analogous equivalence of total $\mathbb{E}_{k-1}$ THH(R)-algebras with $\mathbb{T}$ -action

$$\mathrm{THH}(\mathrm{Th}_R(\xi)) \simeq \mathrm{Th}_{\mathrm{THH}(R)}(\xi^{\mathrm{cy}} : B^{\mathrm{cy}}(M) \rightarrow \mathrm{Pic}_{\mathrm{THH}(R)}). \quad (7.7)$$

(ii) There is a natural equivalence of $\mathbb{E}_{k-1}$ -algebras in $(S/\mathrm{Pic}_{\mathrm{THH}(R)})^{B\mathbb{T}}$ between

$$c_!(\xi_*^{\mathrm{cy}}) \quad \text{and} \quad \xi^{\mathrm{cy}} : B^{\mathrm{cy}}(M) \rightarrow \mathrm{Pic}_{\mathrm{THH}(R)}.$$

(iii) Passing to total objects, the equivalence (7.6) recovers (7.7).

Proof. Let $\iota_R : \Delta^0 \rightarrow \mathrm{Pic}_R$ be the unit of S/Pic_R . Then ι_R can be viewed as an $\mathbb{E}_\infty$ -algebra in S/Pic_R (resp., in $\mathrm{Fun}(I, S/\mathrm{Pic}_R)$) and ξ (resp., ξ_*) can be viewed as an $\mathbb{E}_k$ ι_R -algebra. Viewing ξ and ξ_* as ι_R -algebras in S/Pic_R and $\mathrm{Fun}(I, S)_{/c^*\mathrm{Pic}}$ in the way explained before the proposition, the claims now follow from applying Corollary 5.5(iii) to suitable parts of the commutative square

$$\begin{array}{ccc} \mathrm{Fun}(I, S)_{/c^*\mathrm{Pic}} & \xrightarrow{u\mathrm{Th}} & \mathrm{Fun}(I, \mathrm{Sp}) \\ c_! \downarrow & & \downarrow c_! \\ S/\mathrm{Pic} & \xrightarrow{u\mathrm{Th}} & \mathrm{Sp} \end{array}$$

of symmetric monoidal functors that preserve sifted colimits. $\square$

Remark 7.12. If $R = \mathbb{S}$, then the equivalences in Proposition 7.2 agree with the equivalences in Proposition 7.11, keeping in mind that $\mathrm{THH}(\mathbb{S}) = \mathbb{S}$. This boils down to the observation that as a symmetric monoidal functor, $\mathrm{Th}_{\mathbb{S}} : S/\mathrm{Pic}_{\mathbb{S}} \rightarrow \mathrm{Sp}$ factors as the composite of $S/\mathrm{Pic}_{\mathbb{S}} \rightarrow S/\mathrm{Pic}$, mapping ξ to $(\mathbb{S}, \xi)$, followed by $u\mathrm{Th}$.

7.3 | Log THH of Thom algebras

As in Proposition 7.11, we consider an $\mathbb{E}_k$ -map $\xi_* : M_* \rightarrow c^*\mathrm{Pic}_R$. By the discussion in Construction 3.9, the group completion M^{gp} of M gives rise to a map $\xi_*^{\mathrm{gp}} : M_*^{\mathrm{gp}} \rightarrow c^*\mathrm{Pic}_R$.

Definition 7.14. The I -replete bar construction of ξ_* is the $\mathbb{T}$ -equivariant $\mathbb{E}_{k-1}$ -map

$$\xi_*^{\mathrm{rep}} : B^{I\text{-rep}}(M_*) \rightarrow c^*\mathrm{Pic}_{\mathrm{THH}(R)}$$

obtained by first applying B^{cy} to $(R, \xi_*^{\mathrm{gp}} : M_*^{\mathrm{gp}} \rightarrow c^*\mathrm{Pic}_R)$ and then applying the restriction $\mathrm{Alg}_{\mathbb{E}_{k-1}}((\mathrm{Fun}(I^{\mathrm{gp}}, S)_{/c^*\mathrm{Pic}})^{B\mathbb{T}}) \rightarrow \mathrm{Alg}_{\mathbb{E}_{k-1}}((\mathrm{Fun}(I, S)_{/c^*\mathrm{Pic}})^{B\mathbb{T}})$ along $\gamma : I \rightarrow I^{\mathrm{gp}}$. Passing to total objects defines the ungraded map $\xi^{\mathrm{rep}} : B^{I\text{-rep}}(M) \rightarrow \mathrm{Pic}_{\mathrm{THH}(R)}$.

Using Corollary 5.4(ii), the group completion map $M_* \rightarrow \gamma^*(M_*^{\mathrm{gp}})$ induces a natural I -graded repletion map $\rho_* : B^{\mathrm{cy}}(M_*) \rightarrow B^{I\text{-rep}}(M_*)$, which is a $\mathbb{T}$ -equivariant $\mathbb{E}_{k-1}$ -map over $c^*\mathrm{Pic}_{\mathrm{THH}(R)}$. Passage to total objects defines the (ungraded) repletion map $\rho : B^{\mathrm{cy}}(M) \rightarrow B^{I\text{-rep}}(M)$.

Encoding the gradings of M_* and M_*^{gp} by $\mathbb{E}_k$ -maps $\tau : M \rightarrow I$ and $\tau^{\mathrm{gp}} : M^{\mathrm{gp}} \rightarrow I^{\mathrm{gp}}$ as in (3.2), the construction of $B^{I\text{-rep}}$ can be rephrased by requiring the lower left-hand and the lower middle

square in the following diagram to be pullbacks:

$$\begin{array}{ccccccc}
 I & \xleftarrow{\epsilon} & B^{\text{cy}}(I) & \xleftarrow{B^{\text{cy}}(\tau)} & B^{\text{cy}}(M) & \xrightarrow{B^{\text{cy}}(\xi)} & B^{\text{cy}}(\text{Pic}_R) \xrightarrow{\hat{\epsilon}} \text{Pic}_{\text{THH}(R)} \\
 \downarrow = & & \downarrow \rho & & \downarrow \rho & \nearrow B^{\text{rep}}(\xi) & \downarrow \simeq B^{\text{cy}}(\gamma) \\
 I & \xleftarrow{\hat{\epsilon}} & B^{\text{rep}}(I) & \xleftarrow{B^{\text{rep}}(\tau)} & B^{I\text{-rep}}(M) & & \\
 \downarrow \gamma & & \downarrow \tilde{\gamma} & & \downarrow \tilde{\gamma} & & \\
 I^{\text{gp}} & \xleftarrow{\epsilon} & B^{\text{cy}}(I^{\text{gp}}) & \xleftarrow{B^{\text{cy}}(\tau^{\text{gp}})} & B^{\text{cy}}(M^{\text{gp}}) & \xrightarrow{B^{\text{cy}}(\xi^{\text{gp}})} & B^{\text{cy}}(\text{Pic}_R^{\text{gp}}).
 \end{array} \quad (7.8)$$

Here the two maps labeled ϵ result from I being commutative and $\hat{\epsilon}$ arises from applying Corollary 5.4(ii) to the lax symmetric monoidal structures of Pic . The augmentation map $\xi^{\text{rep}} : B^{I\text{-rep}}(M) \rightarrow \text{Pic}_{\text{THH}(R)}$ from Definition 7.14 is the composite of $B^{\text{rep}}(\xi)$ and $\hat{\epsilon}$. This description of $B^{I\text{-rep}}(M)$ highlights its (implicit) dependence on the choice of the I -grading $\tau : M \rightarrow I$. In contrast, $B^{\text{cy}}(M)$ is also defined for an ungraded $\mathbb{E}_k$ -space.

Proposition 7.15. *In the situation of Proposition 7.11, there is a natural equivalence of I -graded $\mathbb{E}_{k-1}$ $\text{THH}(R)$ -algebras with $\mathbb{T}$ -action*

$$\text{THH}(\text{Th}_R(\xi_*), \xi) \simeq \text{Th}_{\text{THH}(R)}(\xi_*^{\text{rep}} : B^{I\text{-rep}}(M_*) \rightarrow c^* \text{Pic}_{\text{THH}(R)}).$$

Passing to total objects, this provides an equivalence

$$\text{THH}(\text{Th}_R(\xi), \xi) \simeq \text{Th}_{\text{THH}(R)}(\xi^{\text{rep}} : B^{I\text{-rep}}(M) \rightarrow \text{Pic}_{\text{THH}(R)})$$

of $\mathbb{E}_{k-1}$ $\text{THH}(R)$ -algebras with $\mathbb{T}$ -action.

These equivalences are compatible with those of Proposition 7.11 via the repletion maps $\rho : \text{THH}(\text{Th}_R(\xi)) \rightarrow \text{THH}(\text{Th}_R(\xi), \xi)$, $\rho : B^{\text{cy}}(M) \rightarrow B^{I\text{-rep}}(M)$ and their I -graded variants.

Proof. Arguing as in the proof of Proposition 7.11 with $\xi_*^{\text{gp}} : M_*^{\text{gp}} \rightarrow c^* \text{Pic}_R$ (in place of ξ_*) and forming the restriction gives the first equivalence, and since $u\text{Th}_{\text{THH}(R)}$ commutes with coproducts passing to total objects gives the second. For the compatibility with the equivalences from Proposition 7.11, we first note that the two lax symmetric monoidal functors $\gamma^* \circ u\text{Th}$ and $u\text{Th} \circ \gamma^*$ from $\text{Fun}(I^{\text{gp}}, S)_{/c^* \text{Pic}}$ to $\text{Fun}(I, \text{Sp})$ are equivalent. The claim then follows from Corollary 5.5(ii), making use of the compatibility of the comparison map with composites of lax symmetric monoidal functors. $\square$

7.4 | Comparison of replete bar constructions

For the rest of this section, we will allow I to be any $\mathbb{E}_\infty$ -space. If $M \rightarrow I$ is an $\mathbb{E}_\infty$ -map, we define $B^{I\text{-rep}}(M)$ to be the pullback $I \times_{I^{\text{gp}}} B^{\text{cy}}(M^{\text{gp}})$ as in (7.8). For $M \rightarrow I$ the identity, we denote the resulting replete bar construction by $B^{\text{rep}}(M)$. It coincides with the version in [55, Corollary 8.10].

Recall from Definition 3.12 that M is π_0 -replete if $M \rightarrow \pi_0(M) \times_{\pi_0(M^{\text{gp}})} M^{\text{gp}}$ is an equivalence.

Lemma 7.17. *Let M be a π_0 -replete $\mathbb{E}_\infty$ -space and let $M \rightarrow I = \pi_0(M)$ be the canonical $\pi_0(M)$ -grading from Definition 2.16. Then there is a natural equivalence $B^{I\text{-rep}}(M) \rightarrow B^{\text{rep}}(M)$ of I -graded $\mathbb{E}_\infty$ -spaces.*

Proof. Writing $B^{I\text{-rep}}(M)$ as an iterated pullback, this is immediate. $\square$

Corollary 7.18. *When $R = \mathbb{S}$ and M is a π_0 -replete $\mathbb{E}_\infty$ -space, then the notion of log THH from Definition 5.22 is equivalent to that of [55, Definition 8.11].*

Proof. This follows from the last corollary and Proposition 7.15. $\square$

An analogous compatibility result with the notion of log THH from [56, Definition 4.6; 57, Definition 4.1] is discussed in Appendix B. The π_0 -repleteness hypothesis is satisfied in all the cases for which specific calculations were made in those papers.

Remark 7.19. If M is an $\mathbb{E}_k$ -space, then [47, Corollary 5.1.1.7] implies that the Postnikov truncation $\tau_{\leq n} M$ has a preferred $\mathbb{E}_\infty$ -structure if $n + 1 < k$. In this situation, the truncation $\tau : M \rightarrow \tau_{\leq n} M$ provides a $\tau_{\leq n} M$ -grading on M that gives rise to a $\tau_{\leq n} M$ -replete bar construction. For $n = \infty$, this gives the M -replete bar construction. For $n = 0$ and $k \geq 2$, we get the I -replete bar construction for the canonical $I = \pi_0(M)$ -grading. We will not study the versions for $0 < n < \infty$ here.

8 | LOGARITHMIC STRUCTURES AND LOGIFICATION

The purpose of this section is to generalize the log condition and the logification process for discrete prelog rings to prelog $\mathbb{E}_k$ R -algebras. Based on this, we compare different notions of log THH (see Appendix B) and use the results of Section 7 to show that in some cases of interest, log THH is independent of the choices made in the construction of $\mathbb{E}_2$ -prelog structures.

8.1 | Log $\mathbb{E}_k$ R -algebras

If $(A, \xi, \bar{\alpha})$ is a prelog $\mathbb{E}_k$ R -algebra, then the adjunction (2.6) allows us to equivalently describe the structure map $\bar{\alpha}$ by its right adjoint $\alpha : M \rightarrow \Omega^R(A)$ in $\text{Alg}_{\mathbb{E}_k}(S_{/\text{Pic}_R})$. Since $u : \text{Mod}_R \rightarrow \text{Sp}$ is in general only lax symmetric monoidal, $u : \text{Alg}_{\mathbb{E}_k}(\text{Mod}_R) \rightarrow \text{Alg}_{\mathbb{E}_k}(\text{Sp})$ has no right adjoint, and so there is no adjoint structure map for an R -based prelog $\mathbb{E}_k$ -ring.

Example 8.2. Let $(B, \eta : N \rightarrow \text{Pic}_R, \bar{\beta} : \text{Th}_R(\eta) \rightarrow B)$ be a prelog $\mathbb{E}_k$ R -algebra and let $f : A \rightarrow B$ be an $\mathbb{E}_k$ R -algebra map. The *direct image* prelog $\mathbb{E}_k$ R -algebra $(A, f_*\eta : f_*N \rightarrow \text{Pic}_R, f_*\bar{\beta} : \text{Th}_R(f_*\eta) \rightarrow A)$ has as structure map $f_*\bar{\beta}$ the left adjoint of the map $f_*\beta$ defined by the pullback

$$\begin{array}{ccc} f_*N & \xrightarrow{f_*\beta} & \Omega^R(A) \\ \downarrow \tilde{f} & & \downarrow \Omega^R(f) \\ N & \xrightarrow{\beta} & \Omega^R(B) \end{array}$$

in $\text{Alg}_{\mathbb{E}_k}(S_{/\text{Pic}_R})$, while $f_*\eta = \eta \circ \tilde{f}$.

Definition 8.3. Let A be an $\mathbb{E}_k$ R -algebra. Its *space of units* $\mathrm{GL}_1^R(A)$ consists of the connected components of $\Omega^R(A)$ corresponding to units in $\pi_0(\Omega^R(A))$, and is itself an $\mathbb{E}_k$ -space by [2, Lemma 7.2]. The composite $\xi^{\mathrm{GL}_1} : \mathrm{GL}_1^R(A) \rightarrow \mathrm{Pic}_R$ of the inclusion $\iota : \mathrm{GL}_1^R(A) \rightarrow \Omega^R(A)$ and the structure map of $\Omega^R(A)$ provides an object of $\mathrm{Alg}_{\mathbb{E}_k}(S/\mathrm{Pic}_R)$ sometimes also denoted by $\mathrm{GL}_1^R(A)$.

Proposition 2.11 implies that $\mathrm{GL}_1^{\mathbb{S}}(A)$ (as a space over $\mathrm{Pic}_{\mathbb{S}}$) captures the units of the graded ring $\pi_*(A)$.

Definition 8.4. A *prelog $\mathbb{E}_k$ R -algebra* $(A, \xi, \tilde{\alpha})$ is a *log $\mathbb{E}_k$ R -algebra* if the map $\tilde{\alpha}$ in the following pullback square in $\mathrm{Alg}_{\mathbb{E}_k}(S/\mathrm{Pic}_R)$ is an equivalence:

$$\begin{array}{ccc} \alpha^{-1}(\mathrm{GL}_1^R(A)) & \xrightarrow{\tilde{\alpha}} & \mathrm{GL}_1^R(A) \\ \downarrow & & \downarrow \iota \\ M & \xrightarrow{\alpha} & \Omega^R(A). \end{array} \quad (8.1)$$

A prelog $\mathbb{E}_k$ -ring is a *log $\mathbb{E}_k$ -ring* if it is a log $\mathbb{E}_k$ $\mathbb{S}$ -algebra.

Example 8.5. Let A be an $\mathbb{E}_k$ R -algebra and let $\bar{\iota}$ be the adjoint of the inclusion ι . Then $(A, \xi^{\mathrm{GL}_1}, \bar{\iota})$ is the *trivial log $\mathbb{E}_k$ R -algebra*.

Example 8.6. If $f : B \rightarrow A$ is an $\mathbb{E}_k$ R -algebra map and $(A, \xi, \tilde{\alpha})$ is a log $\mathbb{E}_k$ R -algebra, then the direct image prelog $\mathbb{E}_k$ R -algebra $(B, f_*\xi, f_*\tilde{\alpha})$ of Example 8.2 is in fact a log $\mathbb{E}_k$ R -algebra. (This can be seen by identifying the pullback of $f_*\alpha$ along $\mathrm{GL}_1^R(B) \rightarrow \Omega^R(B)$ with the pullback of α along $\mathrm{GL}_1^R(B) \rightarrow \mathrm{GL}_1^R(A) \rightarrow \Omega^R(A)$.) In particular, any morphism of $\mathbb{E}_k$ R -algebras $f : B \rightarrow A$ gives rise to a log $\mathbb{E}_k$ R -algebra $(B, f_*\xi^{\mathrm{GL}_1}, f_*\bar{\iota})$, the direct image log structure associated with the trivial log structure on A .

8.2 | Logification

We can build log structures from prelog structures:

Construction 8.8. Let $(A, \xi, \tilde{\alpha})$ be a prelog $\mathbb{E}_k$ R -algebra with $k \geq 2$. We view the maps $M \leftarrow \alpha^{-1}(\mathrm{GL}_1^R(A)) \rightarrow \mathrm{GL}_1^R(A)$ in $\mathrm{Alg}_{\mathbb{E}_k}(S/\mathrm{Pic}_R)$ from the pullback square (8.1) as an $\mathbb{E}_{k-1}$ -algebra in the span category of $\mathrm{Alg}_{\mathbb{E}_1}(S/\mathrm{Pic}_R)$ with the pointwise symmetric monoidal structure [47, Remark 2.1.3.4]. Viewing M as a left $\alpha^{-1}(\mathrm{GL}_1^R(A))$ -module and $\mathrm{GL}_1^R(A)$ as a right $\alpha^{-1}(\mathrm{GL}_1^R(A))$ -module, we can form the two-sided bar construction $M^a := B(M, \alpha^{-1}(\mathrm{GL}_1^R(A)), \mathrm{GL}_1^R(A))$ in the symmetric monoidal category S/Pic_R . Since the two-sided bar construction is symmetric monoidal with respect to the pointwise symmetric monoidal structure, M^a inherits an $\mathbb{E}_{k-1}$ -algebra structure. The unit of $\alpha^{-1}(\mathrm{GL}_1^R(A))$ induces an $\mathbb{E}_{k-1}$ -map of spans of algebras, yielding $\mathbb{E}_{k-1}$ -maps $M \rightarrow M^a$ and $\mathrm{GL}_1^R(A) \rightarrow M^a$. The $\mathbb{E}_{k-1}$ -map to $\Omega^R(A) \leftarrow \Omega^R(A) \rightarrow \Omega^R(A)$ in the span category induces the structure map $\alpha^a : M^a \rightarrow \Omega^R(A)$. This factorization induces the *logification* map

$$(A, \xi, \tilde{\alpha}) \longrightarrow (A, \xi^a : M^a \rightarrow \mathrm{Pic}_R, \bar{\alpha}^a : \mathrm{Th}_R(\xi^a) \rightarrow A) =: (A, \xi^a, \bar{\alpha}^a) \quad (8.2)$$

of prelog $\mathbb{E}_{k-1}$ R -algebras whose codomain is the *associated log $\mathbb{E}_{k-1}$ R -algebra* of $(A, \xi, \tilde{\alpha})$.

Lemma 8.9. *The prelog $\mathbb{E}_{k-1}$ R -algebra $(A, \xi^a, \bar{\alpha}^a)$ resulting from Construction 8.8 is a log $\mathbb{E}_{k-1}$ R -algebra. Any map of prelog $\mathbb{E}_k$ R -algebras $(A, \xi, \bar{\alpha}) \rightarrow (B, \eta, \bar{\beta})$ to a log $\mathbb{E}_k$ R -algebra extends canonically as an $\mathbb{E}_{k-1}$ map over the logification map (8.2).*

Proof. We first show that $(A, \xi^a, \bar{\alpha}^a)$ is log and argue as in the case of discrete prelog rings or $\mathcal{J}$ -space prelog ring spectra (see, for example, [58, Lemma 3.12]). For a map $W \rightarrow \Omega^R(A)$, the canonical $\pi_0(\Omega^R(A))$ -grading of $\Omega^R(A)$ induces a $\pi_0(\Omega^R(A))$ -grading on W , which induces a coproduct decomposition $\tilde{W} \amalg \hat{W}$ in $\alpha^{-1}(\mathrm{GL}_1^R(A))$ -modules in $\mathcal{S}_{\mathrm{Pic}_R}$ of the parts of W over the units $\pi_0(\Omega^R(A))^\times$ and its complement. For $\alpha: M \rightarrow \Omega^R(A)$ we have $\tilde{M} = \alpha^{-1}(\mathrm{GL}_1^R(A))$, and there are equivalences

$$M^a = M \otimes_{\tilde{M}} \mathrm{GL}_1^R(A) \simeq (\tilde{M} \amalg \hat{M}) \otimes_{\tilde{M}} \mathrm{GL}_1^R(A) \simeq \mathrm{GL}_1^R(A) \amalg (\hat{M} \otimes_{\tilde{M}} \mathrm{GL}_1^R(A)),$$

where the latter coproduct is the decomposition for $M^a \rightarrow \Omega^R(A)$. This shows that $M^a \rightarrow \Omega^R(A)$ is log. The second part follows by naturality and the observation that $N \rightarrow N^a$ is an equivalence if $(B, \eta: N \rightarrow \mathrm{Pic}_R, \bar{\beta}: \mathrm{Th}_R(\eta) \rightarrow B)$ is log. $\square$

When $k = 2$, then $(A, \xi^a, \bar{\alpha}^a)$ is only $\mathbb{E}_1$ so that its log THH is not defined. To remedy this, we use the following tweak.

Definition 8.10. A map of prelog $\mathbb{E}_k$ R -algebras $(A, \xi, \bar{\alpha}) \rightarrow (B, \eta, \bar{\beta})$ is an $\mathbb{E}_k$ -logification if $(B, \eta, \bar{\beta})$ is log and the map induces an equivalence of prelog $\mathbb{E}_{k-1}$ R -algebras $(A, \xi^a, \bar{\alpha}^a) \rightarrow (B, \eta^a, \bar{\beta}^a)$.

Since logification does not change the underlying $\mathbb{E}_k$ R -algebra, the map $A \rightarrow B$ in the definition is an equivalence.

Construction 8.11. Suppose that $k = 2$, that $d > 0$, that $f: A \rightarrow B$ is a map of $\mathbb{E}_k$ -rings that exhibits A as a connective cover of B , and that $a \in \pi_{2d}(A)$ is a homotopy class such that f induces an isomorphism $\pi_*(A)[1/a] \rightarrow \pi_*(B)$. (In other words, A is the connective cover of the $2d$ -periodic $\mathbb{E}_k$ -ring B .) Suppose further that $\pi_{i(2d+2)+2d-1}(A) \cong 0$ for all $i \geq 1$. Then by Remark 4.9, there exists an $\mathbb{E}_2$ -map $\bar{\alpha}: \mathrm{Th}_{\mathbb{S}}(\xi_{2d}) = \mathbb{S}[x] \rightarrow A$ sending the generator x of degree $2d$ to a . This defines a prelog $\mathbb{E}_2$ -ring $(A, \xi_{2d}, \bar{\alpha})$. The adjoint of the composite $\mathbb{S}[x] \rightarrow A \rightarrow B$ factors through the inclusion $\mathrm{GL}_1^{\mathbb{S}}(B) \rightarrow \Omega^{\mathbb{S}}(B)$ and therefore induces a map $(A, \xi_{2d}, \bar{\alpha}) \rightarrow (A, f_* \xi_{2d}^{\mathrm{GL}_1}, f_* \bar{\alpha})$ to the direct image log $\mathbb{E}_k$ -ring determined by f (compare Example 8.6).

Lemma 8.12. *In this situation, $(A, \xi_{2d}, \bar{\alpha}) \rightarrow (A, f_* \xi_{2d}^{\mathrm{GL}_1}, f_* \bar{\alpha})$ is an $\mathbb{E}_2$ -logification of $(A, \xi_{2d}, \bar{\alpha})$, and any $\mathbb{E}_2$ -logification of $(A, \xi_{2d}, \bar{\alpha})$ is equivalent as a prelog $\mathbb{E}_2$ -ring to $(A, f_* \xi_{2d}^{\mathrm{GL}_1}, f_* \bar{\alpha})$.*

Proof. The first part is analogous to [58, Lemma 4.7]. For $\alpha: \mathbb{Z}_{\geq 0} \rightarrow \Omega^{\mathbb{S}}(A)$ we have $\alpha^{-1}(\mathrm{GL}_1^{\mathbb{S}}(A)) \simeq \{0\}$ since $\pi_*(A)$ has no units of positive degree. Hence $(\mathbb{Z}_{\geq 0})^a \simeq \mathbb{Z}_{\geq 0} \times \mathrm{GL}_1^{\mathbb{S}}(A)$ and it remains to show that $\mathbb{Z}_{\geq 0} \times \mathrm{GL}_1^{\mathbb{S}}(A) \rightarrow f_* \mathrm{GL}_1^{\mathbb{S}}(B)$ is an equivalence. For this it is convenient to use the $\mathbb{Z}$ -grading of $\Omega^{\mathbb{S}}(B)$ induced by the canonical $\pi_*(B)/\{\pm 1\}$ grading and the map $\pi_*(B)/\{\pm 1\} \rightarrow \mathbb{Z}$ sending a homotopy class to its degree. Proposition 2.11 implies that after restricting along $(\mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z})$, the map $\Omega^{\mathbb{S}}(A) \rightarrow \Omega^{\mathbb{S}}(B)$ becomes an equivalence. Hence the map

$\mathrm{GL}_1^{\mathbb{S}}(A) \rightarrow (\{0\} \rightarrow \mathbb{Z})^*(\mathrm{GL}_1^{\mathbb{S}}(B))$ is an equivalence, and so the induced map

$$\mathbb{Z}_{\geq 0} \times \mathrm{GL}_1^{\mathbb{S}}(A) \rightarrow f_*(\mathrm{GL}_1^{\mathbb{S}}(B)) \simeq (\mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z})^*(\mathrm{GL}_1^{\mathbb{S}}(B))$$

is an equivalence because $\mathrm{GL}_1^{\mathbb{S}}(B)$ is a grouplike $\mathbb{E}_2$ -space and all non-empty components in the $\mathbb{Z}$ -grading are therefore equivalent.

For the second statement, let $(A, \xi_{2d}, \bar{\alpha}) \rightarrow (A, \eta : N \rightarrow \mathrm{Pic}_{\mathbb{S}}, \bar{\beta})$ be any $\mathbb{E}_2$ -logification. Then the image of $\pi_0(N)$ in $\pi_0(\Omega^{\mathbb{S}}(B))$ is contained in the units $\pi_0(\Omega^{\mathbb{S}}(B))^{\times} = (\pi_*(B))^{\times} / \{\pm 1\}$ by the first part. Hence there is an induced map $(A, \eta, \bar{\beta}) \rightarrow (A, f_* \xi^{\mathrm{GL}_1}, f_* \bar{\iota})$, which is an equivalence since both are logifications of $(A, \xi_{2d}, \bar{\alpha})$. $\square$

Example 8.13. The previous lemma applies in particular to the prelog $\mathbb{E}_2$ -rings $(ku, \langle u \rangle)$, $(ku_{(p)}, \langle u \rangle)$, and $(\ell, \langle v_1 \rangle)$ of Definition 4.10 and shows that they admit $\mathbb{E}_2$ -logifications that do not depend on the choices made in the construction of the $\mathbb{E}_2$ -prelog structures. Moreover, these logifications are log $\mathbb{E}_{\infty}$ -rings since they arise as direct image prelog $\mathbb{E}_{\infty}$ -rings from the $\mathbb{E}_{\infty}$ -maps $A \rightarrow A[1/a]$ and the trivial log $\mathbb{E}_{\infty}$ -rings $(A[1/a], \xi^{\mathrm{GL}_1}, \bar{\iota})$ in the way explained in Example 8.6. This argument does not apply to the other examples introduced in Definition 4.10, since they all involve degree 0 elements a for which A is not the connective cover of $A[1/a]$. We do not know how to construct the logifications as direct image log structures in these cases.

8.3 | Logification invariance of log THH

The next result uses the induced map of Example 5.24. It generalizes [56, Theorem 4.24].

Theorem 8.15. *Let $(A, \xi, \bar{\alpha})$ be a prelog $\mathbb{E}_k$ R -algebra with an $\mathbb{E}_k$ -logification $(f, f^b) : (A, \xi, \bar{\alpha}) \rightarrow (B, \eta, \bar{\beta})$. Then (f, f^b) induces an equivalence $\mathrm{THH}(A, \xi, \bar{\alpha}) \rightarrow \mathrm{THH}(B, \eta, \bar{\beta})$ on log THH formed with respect to the canonical π_0 -grading.*

Proof. The argument for [56, Theorem 4.24] applies also in the present context and we explain only the necessary modifications. First, using Propositions 7.11 and 7.15 we can argue with B^{cy} and $B^{\pi_0\text{-rep}}$, the replete bar construction for the π_0 -repletion. As in [56, Theorem 4.24], we write $G = \mathrm{GL}_1^{\mathbb{S}}$ and $P = \alpha^{-1}(G)$, while N replaces the M^a there. We then get a commutative square

$$\begin{array}{ccc} P & \longrightarrow & G \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array} \quad (8.3)$$

in $\mathrm{Alg}_{\mathbb{E}_k}(S/\mathrm{Pic}_R)$ inducing an equivalence of $\mathbb{E}_{k-1}$ -algebras $M \otimes_P G \rightarrow N$.

As in [56, Theorem 4.24], the canonical map $B^{\mathrm{cy}}(P^{\mathrm{gp}}) \rightarrow B^{\pi_0\text{-rep}}(P) \otimes_{B^{\mathrm{cy}}(P)} B^{\mathrm{cy}}(P^{\mathrm{gp}})$ is an equivalence, its domain and codomain are grouplike, and it induces an equivalence after group completion (as one can check by applying B to the counterpart of the first square in the proof of [56, Theorem 4.24]). Replacing the $\mathbb{S}^J$ in the reference by Th_R , the same argument implies that for the repletion map $\rho : B^{\mathrm{cy}}(P) \rightarrow B^{\pi_0\text{-rep}}(P)$, the cobase change of $\mathrm{Th}_R(\rho)$ along $\mathrm{Th}_R(B^{\mathrm{cy}}(P) \rightarrow \mathrm{Pic}_R) \rightarrow \mathrm{THH}(A)$ is an equivalence.

Since B^{cy} is symmetric monoidal, (8.3) induces an equivalence of $\mathbb{E}_{k-2}$ -algebras $B^{\mathrm{cy}}(M) \otimes_{B^{\mathrm{cy}}(P)} B^{\mathrm{cy}}(G) \rightarrow B^{\mathrm{cy}}(N)$. Similarly, it induces an equivalence of discrete commutative monoids

$\pi_0(M) \times_{\pi_0(P)} \pi_0(G) \rightarrow \pi_0(N)$. Arguing as in [56, Proof of Lemma 4.26], this implies that also $B^{\pi_0\text{-rep}}(M) \otimes_{B^{\pi_0\text{-rep}}(P)} B^{\pi_0\text{-rep}}(G) \rightarrow B^{\pi_0\text{-rep}}(N)$ is an equivalence. Applying Th_R , we get the counterpart of the cubical diagram in [56, Proof of Lemma 4.26], and the conclusion follows analogously. $\square$

Corollary 8.16. *Log THH of the prelog $\mathbb{E}_2$ -rings discussed in Example 8.13 is independent of the choices made in the construction of the $\mathbb{E}_2$ -prelog structures.* $\square$

Theorem 8.15 will also be used in Appendix B to compare the present notion of log THH to the one used in [56, Definition 4.6] and [57, Definition 4.1].

9 | THE REPLETION-RESIDUE SEQUENCE IN THE ONE GENERATOR CASE

In this section, we specialize to $\mathbb{E}_k$ prelog structures $\xi : M \rightarrow \text{Pic}_R$ with $k \geq 2$ where $M = \langle x \rangle = \{x^i \mid i \geq 0\}$ is the free commutative monoid on one generator x , graded by $I = \mathbb{Z}_{\geq 0}$ via the $\mathbb{E}_\infty$ -isomorphism $\tau : M \rightarrow I, x^i \mapsto i$. As in Construction 3.9, we write $\xi^{\text{gp}} : M^{\text{gp}} = \langle x^{\pm 1} \rangle \rightarrow \text{Pic}_R$ for the preferred $\mathbb{E}_k$ -extension of ξ along the group completion map of M .

Our goal is to identify the fiber of the repletion map $\rho : \text{THH}(A) \rightarrow \text{THH}(A, \xi, \bar{\alpha})$ from (5.9) for an R -based prelog $\mathbb{E}_k$ -ring $(A, \xi : \langle x \rangle \rightarrow \text{Pic}_R, \bar{\alpha} : \text{Th}_R(\xi) \rightarrow A)$. To do so, we first consider the weight 0 component

$$\rho_0 : \text{THH}(\text{Th}_R(\xi_*); 0) \longrightarrow \text{THH}(\text{Th}_R(\xi_*), \xi; 0) = \text{THH}(\text{Th}_R(\xi_*^{\text{gp}}); 0) \quad (9.1)$$

of the repletion map for the canonical prelog structure, introduced in (5.7). As discussed in Section 5.15, this is a map of cyclotomic $\text{THH}(R)$ -modules.

Definition 9.1. The *repletion fiber* $\text{THH}^\xi(R)$ is the fiber of ρ_0 in cyclotomic $\text{THH}(R)$ -modules. We say that the pair (R, ξ_*) is *cyclotomically good* if $\text{THH}(R)$ and $\text{THH}^\xi(R)$ are equivalent as cyclotomic $\text{THH}(R)$ -modules.

We will see in Lemma 9.11 that the underlying $\text{THH}(R)$ -modules of $\text{THH}(R)$ and $\text{THH}^\xi(R)$ are equivalent without further assumptions on R , and refine this to a $\mathbb{T}$ -equivariant statement in Proposition 9.21. However, we know only how to control the full cyclotomic structure of $\text{THH}^\xi(R)$ under additional assumptions:

Proposition 9.2. *If $R = \mathbb{S}$, $R = MU$ or $R = H\mathbb{F}$ with $\mathbb{F}$ a perfect field of characteristic p or a perfectoid ring, then (R, ξ_*) is cyclotomically good.*

We prove this proposition in Section 9.22, but first explain its relevance for understanding the fiber of the repletion map of an R -based prelog $\mathbb{E}_k$ -ring $(A, \xi : \langle x \rangle \rightarrow \text{Pic}_R, \bar{\alpha} : \text{Th}_R(\xi) \rightarrow A)$. The $\mathbb{E}_k$ R -algebra collapse map $\text{coll} : \text{Th}_R(\xi) \rightarrow R$ obtained from the face inclusion $\{1\} = \{x^0\} \subseteq \langle x \rangle$ as an instance of (2.9) allows us to form the $\mathbb{E}_{k-1}$ A -algebra

$$A//\bar{\alpha} := A \otimes_{\text{Th}_R(\xi)} R \quad (9.2)$$

as in Definition 4.16. Restricting the $\mathrm{THH}(R)$ -module structure on $\mathrm{THH}^\xi(R)$ along $\mathrm{THH}(\mathrm{coll}) : \mathrm{THH}(\mathrm{Th}_R(\xi)) \rightarrow \mathrm{THH}(R)$, we can form the balanced smash product

$$\mathrm{THH}^\xi(A//\bar{\alpha}) := \mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathrm{Th}_R(\xi))} \mathrm{THH}^\xi(R)$$

in cyclotomic $\mathrm{THH}(A)$ -modules.

The next theorem, proved in Section 9.6, generalizes [56, Theorem 1.1], where we considered only certain $\mathbb{E}_\infty$ prelog structures over $R = \mathbb{S}$ (compare Appendix B) and did not record any cyclotomic structure. Together with Proposition 9.2 and Corollary 9.5, it implies Theorem 1.4 from the introduction.

Theorem 9.3. *Let $(A, \xi : \langle x \rangle \rightarrow \mathrm{Pic}_R, \bar{\alpha} : \mathrm{Th}_R(\xi) \rightarrow A)$ be an R -based prelog $\mathbb{E}_k$ -ring with $k \geq 2$. There is a natural repletion–residue cofiber sequence*

$$\mathrm{THH}^\xi(A//\bar{\alpha}) \xrightarrow{\mathrm{trf}} \mathrm{THH}(A) \xrightarrow{\rho} \mathrm{THH}(A, \xi, \bar{\alpha}) \xrightarrow{\mathrm{res}} \Sigma \mathrm{THH}^\xi(A//\bar{\alpha})$$

of cyclotomic $\mathrm{THH}(A)$ -modules. If the pair (R, ξ_*) is cyclotomically good, then $\mathrm{THH}^\xi(A//\bar{\alpha})$ is equivalent to $\mathrm{THH}(A//\bar{\alpha})$ as a cyclotomic $\mathrm{THH}(A)$ -module.

Informally, when $A = \mathrm{Th}_R(\xi)$, the map $\mathrm{res} : \mathrm{THH}(\mathrm{Th}_R(\xi), \xi) \rightarrow \Sigma \mathrm{THH}^\xi(R) \simeq \Sigma \mathrm{THH}(R)$ is given by first collapsing all weight $i > 0$ summands to the zero spectrum 0, then viewing the quotient as $\mathrm{Th}_{\mathrm{THH}(R)}(\xi_0^{\mathrm{rep}})$ formed over $B^{\mathrm{rep}}(\langle x \rangle; 0) \simeq S^1(0)$, then collapsing the copy of $\mathrm{THH}(R)$ over the 0-simplex (1) to 0, and finally identifying this quotient as a suspension of $\mathrm{THH}(R)$. We write trf to suggest transfer, but have not verified compatibility of this map under the (cyclotomic) trace map with the algebraic K -theory transfer map $K(R) \rightarrow K(\mathrm{Th}_R(\xi))$. Partial results in this direction can be derived from [42].

Remark 9.4. When (R, ξ_*) is not known to be cyclotomically good, then Proposition 9.21 still implies that $\mathrm{THH}^\xi(A//\bar{\alpha})$ is equivalent to $\mathrm{THH}(A//\bar{\alpha})$ as a $\mathrm{THH}(A)$ -module with $\mathbb{T}$ -action.

Corollary 9.5. *Let $(A, \xi : \langle x \rangle \rightarrow \mathrm{Pic}_R, \bar{\alpha})$ be an R -based prelog $\mathbb{E}_k$ -ring with $k \geq 2$ such that (R, ξ_*) is cyclotomically good. Then there is a repletion–residue cofiber sequence*

$$\mathrm{TC}(A//\bar{\alpha})_p \xrightarrow{\mathrm{trf}} \mathrm{TC}(A)_p \xrightarrow{\rho} \mathrm{TC}(A, \xi, \bar{\alpha})_p \xrightarrow{\mathrm{res}} \Sigma \mathrm{TC}(A//\bar{\alpha})_p$$

of $\mathrm{TC}(A)_p$ -modules.

9.1 | Construction of the repletion–residue sequence

Next we recall the cyclic and replete bar constructions for the free commutative monoid on one generator. Here and later, weak equivalences of spaces with $\mathbb{T}$ -action or $\mathbb{T}$ -equivariant weak equivalences refer to $\mathbb{T}$ -equivariant maps that are underlying weak equivalences, also known as naive equivalences.

Proposition 9.7 [32, Lemma 2.2.3; 55, Propositions 3.20 and 3.21]. *There are weight decompositions*

$$B^{\mathrm{cy}}(\langle x \rangle) = \coprod_{i \geq 0} B^{\mathrm{cy}}(\langle x \rangle; i) \quad \text{and} \quad B^{\mathrm{rep}}(\langle x \rangle) = \coprod_{i \geq 0} B^{\mathrm{rep}}(\langle x \rangle; i)$$

and weak equivalences of spaces with $\mathbb{T}$ -action

$$B^{\text{cy}}(\langle x \rangle; 0) = *, \quad B^{\text{cy}}(\langle x \rangle; i) \simeq S^1(i) \text{ if } i \geq 1, \text{ and } B^{\text{rep}}(\langle x \rangle; i) \simeq S^1(i) \text{ if } i \geq 0.$$

Here $S^1(i)$ denotes the unit circle with the $\mathbb{T}$ -action given by $z \cdot w = z^i w$ for $z \in \mathbb{T}$ and $w \in S^1(i) \subset \mathbb{C}$. The repletion map $\rho : B^{\text{cy}}(\langle x \rangle) \rightarrow B^{\text{rep}}(\langle x \rangle)$ is compatible with this decomposition. It induces a $\mathbb{T}$ -equivariant equivalence in positive weights and the inclusion of the basepoint in weight 0.

Corollary 9.8. *The unit map $R \rightarrow \text{Th}_R(\xi)$ induces an equivalence $\text{THH}(R) \rightarrow \text{THH}(\text{Th}_R(\xi_*); 0)$ of cyclotomic $\mathbb{E}_{k-1}$ $\text{THH}(R)$ -algebras.*

Proof. It is sufficient to show that it is an equivalence of underlying $\mathbb{E}_{k-1}$ $\text{THH}(R)$ -algebras with $\mathbb{T}$ -action. This follows from Proposition 7.11 and the equivalence $* \rightarrow B^{\text{cy}}(\langle x \rangle; 0)$ from the last proposition. $\square$

The next proposition is the key ingredient for the proof of Theorem 9.3. It generalizes [56, Proposition 6.11].

Proposition 9.9. *Let $\xi : \langle x \rangle \rightarrow \text{Pic}_R$ be an $\mathbb{E}_k$ -map with $k \geq 2$. There is a natural repletion–residue cofiber sequence*

$$\text{THH}^\xi(R) \xrightarrow{\text{trf}} \text{THH}(\text{Th}_R(\xi_*)) \xrightarrow{\rho} \text{THH}(\text{Th}_R(\xi_*), \xi) \xrightarrow{\text{res}} \Sigma \text{THH}^\xi(R)$$

of $\mathbb{Z}_{\geq 0}$ -graded twisted cyclotomic $\text{THH}(\text{Th}_R(\xi_*))$ -modules, with $\text{THH}^\xi(R)$ viewed as an object in this category by restriction along $\text{THH}(\text{coll})$.

Proof. As explained in Section 2.18, the face inclusion $f : \{0\} \rightarrow \mathbb{Z}_{\geq 0}$ induces a natural transformation $\text{id} \rightarrow f_* f^*$. Applied to the $\mathbb{Z}_{\geq 0}$ -graded repletion map (5.6), this provides the left-hand commutative square of $\mathbb{Z}_{\geq 0}$ -graded twisted cyclotomic $\mathbb{E}_{k-1}$ $\text{THH}(R)$ -algebras in the following diagram:

$$\begin{array}{ccccc} \text{THH}(\text{Th}_R(\xi_*)) & \xrightarrow{\rho} & \text{THH}(\text{Th}_R(\xi_*), \xi) & \xrightarrow{\text{res}} & \text{cof}(\rho) \\ \downarrow & & \downarrow & & \downarrow \\ \text{THH}(\text{Th}_R(\xi_*); 0) & \xrightarrow{\rho_0} & \text{THH}(\text{Th}_R(\xi_*), \xi; 0) & \xrightarrow{\text{res}_0} & \text{cof}(\rho_0). \end{array} \quad (9.3)$$

Using these $\mathbb{E}_{k-1}$ -algebra structures, we view the left-hand square as one of $\mathbb{Z}_{\geq 0}$ -graded twisted cyclotomic $\text{THH}(\text{Th}_R(\xi_*))$ -modules and obtain the right-hand square by forming horizontal cofibers in that category. By construction, the left-hand vertical map in (9.3) is equivalent to the composite of $\text{THH}(\text{coll})$ and the equivalence of Corollary 9.8. Therefore, $\text{cof}(\rho_0)$ is equivalent to $\Sigma \text{THH}^\xi(R)$ as a $\mathbb{Z}_{\geq 0}$ -graded twisted cyclotomic $\text{THH}(\text{Th}_R(\xi_*))$ -module, where $\Sigma \text{THH}^\xi(R)$ carries the module structure obtained by restriction along $\text{THH}(\text{coll})$.

With this identification, it suffices to show that the right-hand vertical map $\text{cof}(\rho) \rightarrow \text{cof}(\rho_0)$ is an equivalence. In weight 0, this holds by definition. In weights $i > 0$, the homotopy equivalences $B^{\text{cy}}(\langle x \rangle; i) \simeq B^{\text{rep}}(\langle x \rangle; i)$ from Proposition 9.7 and the compatible equivalences from Propositions 7.11 and 7.15 imply that the maps

$$\rho_i : \text{THH}(\text{Th}_R(\xi_*); i) \xrightarrow{\sim} \text{THH}(\text{Th}_R(\xi_*), \xi; i) \quad (9.4)$$

are equivalences of $\mathrm{THH}(R)$ -modules. So both $\mathrm{cof}(\rho)$ and $\mathrm{cof}(\rho_0)$ are contractible in positive weights. $\square$

Proof of Theorem 9.3. We apply $\mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathrm{Th}_R(\xi))} (-)$ to the repletion-residue cofiber sequence of Proposition 9.9. This provides the desired cofiber sequence. Now suppose that (R, ξ_*) is cyclotomically good, so that there is an equivalence $\mathrm{THH}(R) \simeq \mathrm{THH}^\xi(R)$ of cyclotomic $\mathrm{THH}(R)$ -modules. Restricting along $\mathrm{THH}(\mathrm{coll})$ and applying $\mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathrm{Th}_R(\xi))} (-)$ gives an equivalence relating

$$\mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathrm{Th}_R(\xi))} \mathrm{THH}(R) \simeq \mathrm{THH}(A \otimes_{\mathrm{Th}_R(\xi)} R) = \mathrm{THH}(A//\bar{\alpha})$$

and $\mathrm{THH}^\xi(A//\bar{\alpha}) = \mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathrm{Th}_R(\xi))} \mathrm{THH}^\xi(R)$. $\square$

9.2 | Identifying the repletion fiber as a $\mathrm{THH}(R)$ -module

We turn to identifying $\mathrm{THH}^\xi(R)$ as a $\mathrm{THH}(R)$ -module. While this result will be superseded by Proposition 9.21, the present analysis will be useful for the identification of the cyclotomic structure map of $\mathrm{THH}^\xi(R)$ and future calculations.

As before, we write $\xi^{\mathrm{gp}} : M^{\mathrm{gp}} = \langle x^{\pm 1} \rangle \rightarrow \mathrm{Pic}_R$ for the preferred $\mathbb{E}_k$ -extension of $\xi : M = \langle x \rangle \rightarrow \mathrm{Pic}_R$. We write $\xi_*^{\mathrm{gp}} : \langle x^{\pm 1} \rangle \rightarrow c^*\mathrm{Pic}_R$ for the graded version of our preferred $\mathbb{E}_k$ -extension of ξ to $\langle x^{\pm 1} \rangle$ and $\xi_*^{\mathrm{gp}, \mathrm{cy}} : B^{\mathrm{cy}}(\langle x^{\pm 1} \rangle) \rightarrow c^*\mathrm{Pic}_{\mathrm{THH}(R)}$ for the map induced by ξ_*^{gp} and the comparison map from Corollary 5.4(ii). Moreover, we write $P := \xi^{\mathrm{gp}}(x)$ and $P^* := \xi^{\mathrm{gp}}(x^{-1})$.

The coherence data for the $\mathbb{E}_k$ -map ξ^{gp} include a homotopy between two composites $M \times M \rightarrow \mathrm{Pic}_R$, which we can interpret as part of the coherence data for the map of cyclic bar constructions induced by ξ_*^{gp} , namely a choice of a homotopy in the diagram

$$\begin{array}{ccc} B_1^{\mathrm{cy}}(\langle x^{\pm 1} \rangle) & \longrightarrow & B_1^{\mathrm{cy}}(c^*\mathrm{Pic}_R) \\ d_0 \downarrow & & d_0 \downarrow \\ B_0^{\mathrm{cy}}(\langle x^{\pm 1} \rangle) & \longrightarrow & B_0^{\mathrm{cy}}(c^*\mathrm{Pic}_R). \end{array} \quad (9.5)$$

Likewise, the composite of the homotopy with the twist map determines the analogous coherence datum for the simplicial structure map d_1 . Evaluated at (x, x^{-1}) and (x^{-1}, x) , the homotopy provides two equivalences

$$\begin{aligned} \lambda_{x, x^{-1}} : P \otimes_R P^* &= \xi^{\mathrm{gp}}(x) \otimes_R \xi^{\mathrm{gp}}(x^{-1}) \xrightarrow{\simeq} \xi^{\mathrm{gp}}(x \cdot x^{-1}) = \xi^{\mathrm{gp}}(1) = R \quad \text{and} \\ \lambda_{x^{-1}, x} : P^* \otimes_R P &= \xi^{\mathrm{gp}}(x^{-1}) \otimes_R \xi^{\mathrm{gp}}(x) \xrightarrow{\simeq} \xi^{\mathrm{gp}}(x^{-1} \cdot x) = \xi^{\mathrm{gp}}(1) = R \end{aligned}$$

witnessing that P and P^* are mutually inverse R -modules.

Lemma 9.11. *The weight 0 component*

$$\rho_0 : \mathrm{THH}(\mathrm{Th}_R(\xi_*); 0) \longrightarrow \mathrm{THH}(\mathrm{Th}_R(\xi_*), \xi; 0)$$

of the repletion map is equivalent as a map of $\mathrm{THH}(R)$ -modules to the composite

$$\mathrm{THH}(R) \xrightarrow{\simeq} \mathrm{THH}(R) \wedge \{*\}_+ \longrightarrow \mathrm{THH}(R) \wedge S_+^1 \quad (9.6)$$

induced by the basepoint inclusion $\{*\} \rightarrow S^1$, and the repletion fiber $\mathrm{THH}^\xi(R)$ is equivalent to $\mathrm{THH}(R)$ as a $\mathrm{THH}(R)$ -module.

Proof. Proposition 7.15 provides an equivalence

$$\mathrm{THH}(\mathrm{Th}_R(\xi_*), \xi; 0) \simeq \mathrm{Th}_{\mathrm{THH}(R)}(\xi_0^{\mathrm{rep}}) = \mathrm{Th}_{\mathrm{THH}(R)}(\xi_0^{\mathrm{gp}, \mathrm{cy}}) \quad (9.7)$$

of $\mathrm{THH}(R)$ -modules. To analyze the right-hand term, we note that Lemma 7.10 and the use of the comparison map $|F(-)| \rightarrow F(|-|)$ in the proof of Proposition 5.3(ii) imply that for each q we have a commutative diagram in which the vertical maps are induced by the inclusions η_q of the q -simplices of the respective cyclic bar constructions into their realizations, that is, the colimits over Δ^{op} :

$$\begin{array}{ccccc} B_q^{\mathrm{cy}}(\langle x^{\pm 1} \rangle; 0) & \longrightarrow & B_q^{\mathrm{cy}}(\mathrm{Pic}_R) & \longrightarrow & \mathrm{Pic}_{\mathrm{THH}_q(R)} \\ \eta_q \downarrow & & \eta_q \downarrow & & \downarrow \mathrm{Pic}_{\eta_q} \\ B^{\mathrm{cy}}(\langle x^{\pm 1} \rangle) & \longrightarrow & B^{\mathrm{cy}}(\mathrm{Pic}_R) & \longrightarrow & \mathrm{Pic}_{\mathrm{THH}(R)} \end{array} \quad (9.8)$$

By the proof of Proposition 9.7 (see, for example, [55, Proposition 3.21]), the inclusion of

$$(x^{-1}, x) \xrightarrow[d_1]{d_0} (1)$$

into the Δ^{op} -diagram $B_*^{\mathrm{rep}}(\langle x \rangle; 0) = B_*^{\mathrm{cy}}(\langle x^{\pm 1} \rangle; 0)$ induces a homotopy equivalence $\Delta^1 / \partial \Delta^1 \rightarrow B^{\mathrm{rep}}(\langle x \rangle; 0)$ upon passage to colimits. Composing with the maps to $\mathrm{Pic}_{\mathrm{THH}(R)}$ induced by ξ_*^{gp} that are specified in (9.8), and passing to Thom $\mathrm{THH}(R)$ -modules, the coequalizer diagram above gets mapped to a diagram of the form

$$\mathrm{THH}(R) \otimes_{R \otimes R} P^* \otimes P \xrightarrow[d_1]{d_0} \mathrm{THH}(R) \otimes_R R. \quad (9.9)$$

Since $\eta_1 : R \otimes R \simeq \mathrm{THH}_1(R) \rightarrow \mathrm{THH}(R)$ factors up to homotopy as $\eta_0 \circ d_0$, the face operator d_0 induces equivalences

$$\begin{aligned} \mathrm{THH}(R) \otimes_{R \otimes R} P^* \otimes P &\simeq \mathrm{THH}(R) \otimes_R R \otimes_{R \otimes R} P^* \otimes P \\ &\simeq \mathrm{THH}(R) \otimes_R (P^* \otimes_R P) \simeq \mathrm{THH}(R) \otimes_R R = \mathrm{THH}(R) \end{aligned} \quad (9.10)$$

identifying the first term in (9.9). (This equivalence is not a canonical one because we could also argue with d_1 and $\eta_1 \simeq \eta_0 \circ d_1$.)

To identify the coequalizer (9.9), we note that while the 1-simplex (x^{-1}, x) represents a closed loop in $B_*^{\mathrm{cy}}(\langle x^{\pm 1} \rangle; 0)$, its image in $B_*^{\mathrm{cy}}(\mathrm{Pic}_R)$ is an edge from $P^* \otimes_R P$ to $P \otimes_R P^*$. By the discussion of coherence homotopies before the lemma, the closed loop in Pic_R represented by the image of (x^{-1}, x) is composed of the edge $\lambda_{x^{-1}, x}$ (in reverse), the above-mentioned edge from $P^* \otimes_R P$ to $P \otimes_R P^*$, and the edge $\lambda_{x, x^{-1}}$. Upon passage to Thom spectra, this means that the maps d_0 and d_1 in (9.9) are induced by $\lambda_{x^{-1}, x}$ and $\lambda_{x, x^{-1}} \circ \tau$. Our assumption that $k \geq 2$ therefore implies that d_0 and d_1 are homotopic.

Using (9.7), the coequalizer of (9.9) is equivalent to $\mathrm{THH}(\mathrm{Th}_R(\xi_*), \xi; 0)$. This equivalence is compatible with the corresponding identification for $(1) \rightarrow B^{\mathrm{cy}}(\langle x \rangle; 0)$, and yields the following diagram of $\mathrm{THH}(R)$ -modules, with vertical coequalizers and horizontal cofiber sequences. The upper right-hand entry is characterized by its suspension being the cofiber $\Sigma \mathrm{THH}^\xi(R)$ of ρ_0 , hence is equivalent to the repletion fiber $\mathrm{THH}^\xi(R)$:

$$\begin{array}{ccccc}
 0 & \longrightarrow & \mathrm{THH}(R) \otimes_{R \otimes R} P^* \otimes P & \xrightarrow{\simeq} & \mathrm{THH}^\xi(R) \\
 \downarrow \downarrow & & \downarrow d_0 \downarrow d_1 & & \downarrow \downarrow \\
 \mathrm{THH}(R) & \xrightarrow{\simeq} & \mathrm{THH}(R) \otimes_R R & \longrightarrow & 0 \\
 \simeq \downarrow & & \downarrow & & \downarrow \\
 \mathrm{THH}(\mathrm{Th}_R(\xi_*); 0) & \xrightarrow{\rho_0} & \mathrm{THH}(\mathrm{Th}_R(\xi_*), \xi; 0) & \xrightarrow{\mathrm{res}_0} & \Sigma \mathrm{THH}^\xi(R).
 \end{array} \quad (9.11)$$

With the homotopy $d_0 \simeq d_1$, the diagram implies the claimed identification of ρ_0 . The statement about $\mathrm{THH}^\xi(R)$ follows by forming the cofiber of (9.6), or by combining the upper right-hand horizontal equivalence in (9.11) with the combined equivalence in (9.10). $\square$

9.3 | Identifying the cyclotomic structure map on the repletion fiber

We now study the cyclotomic structure map $\varphi_p^\xi : \mathrm{THH}^\xi(R) \rightarrow \mathrm{THH}^\xi(R)^{tC_p}$ of the repletion fiber. Recall that in the formulation of [49, §§III.2–3], the cyclotomic structure map $\varphi_p : \mathrm{THH}(A) \rightarrow \mathrm{THH}(A)^{tC_p}$ for an $\mathbb{E}_1$ -ring A is constructed as a colimit of the Tate diagonal maps $\Delta_p : X \rightarrow (X^{\otimes p})^{tC_p}$, making squares

$$\begin{array}{ccc}
 A^{\otimes 1+q} & \xrightarrow{\eta_q} & \mathrm{THH}(A) \\
 \Delta_p \downarrow & & \downarrow \varphi_p \\
 (A^{\otimes p(1+q)})^{tC_p} & \xrightarrow{\eta_{pq+p-1}} & \mathrm{THH}(A)^{tC_p}
 \end{array} \quad (9.12)$$

commute for all $q \geq 0$.

We begin with the following refinement of Lemma 9.11:

Proposition 9.13. *Non-equivariantly, that is, ignoring the $\mathbb{T}$ -action, the cyclotomic structure map $\varphi_p^\xi : \mathrm{THH}^\xi(R) \rightarrow \mathrm{THH}^\xi(R)^{tC_p}$ is equivalent as a $\mathrm{THH}(R)$ -module map to the standard cyclotomic structure map $\varphi_p : \mathrm{THH}(R) \rightarrow \mathrm{THH}(R)^{tC_p}$, for each prime p .*

Proof. The C_p -action on THH is realized by a simplicial action on its p -fold edgewise subdivision, see [20, §1; 43, Lemma 3.4; 49, §III.2]. This leads to the following C_p -equivariant enrichment of (9.11):

$$\begin{array}{ccccc}
 0 & \longrightarrow & \mathrm{THH}(R) \otimes_{R \otimes 2p} (P^* \otimes P)^{\otimes p} & \xrightarrow{\simeq} & \mathrm{THH}^\xi(R) \\
 \downarrow \downarrow & & \downarrow d_0 \downarrow d_1 & & \downarrow \downarrow \\
 \mathrm{THH}(R) & \xrightarrow{\simeq} & \mathrm{THH}(R) \otimes_{R \otimes p} R^{\otimes p} & \longrightarrow & 0 \\
 \simeq \downarrow & & \downarrow & & \downarrow \\
 \mathrm{THH}(\mathrm{Th}_R(\xi_*); 0) & \xrightarrow{\rho_0} & \mathrm{THH}(\mathrm{Th}_R(\xi_*), \xi; 0) & \xrightarrow{\mathrm{res}_0} & \Sigma \mathrm{THH}^\xi(R).
 \end{array} \quad (9.13)$$

In particular, the middle column is a coequalizer diagram of $\mathrm{THH}(R)$ -modules with C_p -action, and therefore yields a coequalizer diagram after passage to C_p -Tate fixed points.

When applied with $A = R$ or $A = \mathrm{Th}_R(\xi_*^{\mathrm{gp}})$, and evaluated in weight $i = 0$, the square (9.12) leads to the following commutative cube:

$$\begin{array}{ccccc}
 R^{\otimes 1+q} & \xrightarrow{\eta_q} & \mathrm{THH}(R) & & \\
 \Delta_p \downarrow & \searrow & \downarrow \varphi_p & \searrow & \\
 & (\mathrm{Th}_R(\xi_*^{\mathrm{gp}})^{\otimes 1+q})_0 & \xrightarrow{\eta_q} & \mathrm{THH}(\mathrm{Th}_R(\xi_*^{\mathrm{gp}}); 0) & \\
 & \downarrow \Delta_p & \downarrow & \downarrow \varphi_{p;0} & \\
 (R^{\otimes p(1+q)})^{tC_p} & \xrightarrow{\Delta_p} & \mathrm{THH}(R)^{tC_p} & & \\
 & \downarrow & \downarrow & \downarrow & \\
 & (\mathrm{Th}_R(\xi_*^{\mathrm{gp}})^{\otimes p(1+q)})_0^{tC_p} & \xrightarrow{\quad} & \mathrm{THH}(\mathrm{Th}_R(\xi_*^{\mathrm{gp}}); 0)^{tC_p} &
 \end{array}$$

When restricted along $R \rightarrow \mathrm{Th}_R(\xi_*^{\mathrm{gp}})_0$ (for $q = 0$) and $P^* \otimes P \rightarrow \mathrm{Th}_R(\xi_*^{\mathrm{gp}})_0^{\otimes 2}$ (for $q = 1$), substituting $\mathrm{THH}(\mathrm{Th}_R(\xi_*), \xi; 0)$ for $\mathrm{THH}(\mathrm{Th}_R(\xi_*^{\mathrm{gp}}); 0)$, this leads to the following horizontal map of vertical coequalizer diagrams:

$$\begin{array}{ccc}
 \mathrm{THH}(R) \otimes_{R^{\otimes 2}} P^* \otimes P & \xrightarrow{\lambda_1(\varphi_p \otimes \Delta_p \Delta_p)} & (\mathrm{THH}(R) \otimes_{R^{\otimes 2p}} (P^* \otimes P)^{\otimes p})^{tC_p} \\
 d_0 \downarrow \downarrow d_1 & & d_0 \downarrow \downarrow d_1 \\
 \mathrm{THH}(R) \otimes_R R & \xrightarrow{\lambda_0(\varphi_p \otimes \Delta_p \Delta_p)} & (\mathrm{THH}(R) \otimes_{R^{\otimes p}} R^{\otimes p})^{tC_p} \\
 \downarrow & & \downarrow \\
 \mathrm{THH}(\mathrm{Th}_R(\xi_*), \xi; 0) & \xrightarrow{\varphi_{p;0}} & \mathrm{THH}(\mathrm{Th}_R(\xi_*), \xi; 0)^{tC_p} .
 \end{array}$$

Here

$$\begin{aligned}
 \lambda_0 : \mathrm{THH}(R)^{tC_p} \otimes_{(R^{\otimes p})^{tC_p}} (R^{\otimes p})^{tC_p} &\xrightarrow{\sim} (\mathrm{THH}(R) \otimes_{R^{\otimes p}} R^{\otimes p})^{tC_p} \quad \text{and} \\
 \lambda_1 : \mathrm{THH}(R)^{tC_p} \otimes_{(R^{\otimes 2p})^{tC_p}} ((P^* \otimes P)^{\otimes p})^{tC_p} &\longrightarrow (\mathrm{THH}(R) \otimes_{R^{\otimes 2p}} (P^* \otimes P)^{\otimes p})^{tC_p}
 \end{aligned}$$

are instances of the lax structure maps for C_p -Tate fixed points.

In analogy with (9.10), the subdivided face operator d_0 induces equivalences

$$\begin{aligned}
 \mathrm{THH}(R) \otimes_{R^{\otimes 2p}} (P^* \otimes P)^{\otimes p} &\simeq \mathrm{THH}(R) \otimes_{R^{\otimes p}} R^{\otimes p} \otimes_{R^{\otimes 2p}} (P^* \otimes P)^{\otimes p} \\
 &\simeq \mathrm{THH}(R) \otimes_{R^{\otimes p}} (P^* \otimes_R P)^{\otimes p} \simeq \mathrm{THH}(R) \otimes_{R^{\otimes p}} R^{\otimes p} = \mathrm{THH}(R)
 \end{aligned}$$

that respect the C_p -actions, hence also induce equivalences of C_p -Tate fixed points. Taken together, we obtain a commutative diagram

$$\begin{array}{ccccc}
 \mathrm{THH}^{\xi}(R) & \xleftarrow{\sim} & \mathrm{THH}(R) \otimes_{R^{\otimes 2}} (P^* \otimes P) & \xrightarrow{\sim} & \mathrm{THH}(R) \\
 \downarrow \varphi_p^{\xi} & & \downarrow \lambda_1(\varphi_p \otimes \Delta_p \Delta_p) & & \downarrow \varphi_p \\
 \mathrm{THH}^{\xi}(R)^{tC_p} & \xleftarrow{\sim} & (\mathrm{THH}(R) \otimes_{R^{\otimes 2p}} (P^* \otimes P)^{\otimes p})^{tC_p} & \xrightarrow{\sim} & \mathrm{THH}(R)^{tC_p}
 \end{array}$$

of $\mathrm{THH}(R)$ -modules, exhibiting the commensurability of φ_p^{ξ} with φ_p . $\square$

Lemma 9.19. *The cyclic map f takes $S^0_\circ$ to (1) , and induces an isomorphism $\bar{f}$ of $S^\infty_\circ/S^0_\circ$ with the cyclic subset of $B^\text{cy}_\circ(\langle x^{\pm 1} \rangle; 0)$ consisting of the $(x^{i_0}, \dots, x^{i_q})$ with $i_0 + \dots + i_q = 0$ for which the non-zero i_s alternate cyclically between $+1$ and -1 .*

Proof. The integer sequence $(i_0, \dots, i_q)$ drawn from $\{+1, 0, -1\}$ determines the object sequence $(g_0, \dots, g_q)$ drawn from $\{(+\frac{1}{2}), (-\frac{1}{2})\}$, except when the latter sequence is constant: The first non-zero i_s in $(i_0, \dots, i_q)$ gives the morphism $(-\frac{1}{2}) \rightarrow (+\frac{1}{2})$ in $\mathcal{G}$ if $i_s = 1$ and the morphism $(+\frac{1}{2}) \rightarrow (-\frac{1}{2})$ if $i_s = -1$. This determines the domain and codomain of all other i_t in the sequence and therefore $(i_0, \dots, i_q)$. $\square$

Lemma 9.20. *The square of cyclic sets*

$$\begin{array}{ccc} S^0_\circ & \longrightarrow & S^\infty_\circ \\ \downarrow & & \downarrow f \\ (1) & \longrightarrow & B^\text{cy}_\circ(\langle x^{\pm 1} \rangle; 0) \end{array} \quad (9.14)$$

is a (homotopy) pushout, in the sense that the induced $\mathbb{T}$ -map

$$\bar{f}: S^\infty/S^0 \longrightarrow B^\text{cy}(\langle x^{\pm 1} \rangle; 0)$$

is a weak equivalence.

Proof. The cyclic map f takes each of the 0-simplices $(g_0) = (+\frac{1}{2})$ and $(g_0) = (-\frac{1}{2})$ to (1) . Moreover, it takes the 1-simplex $(g_0, g_1) = ((-\frac{1}{2}), (+\frac{1}{2}))$ to (x^{-1}, x) , hence induces the $\mathbb{T}$ -map $\bar{f}$ taking a generating loop in $S^\infty/S^0 \simeq S^1$ to a generating loop in $B^\text{cy}(\langle x^{\pm 1} \rangle; 0) \simeq S^1(0)$, which is therefore a weak equivalence. $\square$

We now apply $-\odot \text{Th}_R(\xi^{\text{gp}})$ to (9.14), making the identifications $(1) \odot \text{Th}_R(\xi^{\text{gp}}) \simeq \text{THH}(R)$ and $S^0 \odot \text{Th}_R(\xi^{\text{gp}}) \simeq \text{THH}(R) \vee \text{THH}(R)$. When applying $-\odot \text{Th}_R(\xi^{\text{gp}})$ to the cyclic map f of Definition 9.18, Proposition 7.11(i) implies that the codomain of the resulting map f_* can be identified with $\text{THH}(\text{Th}_R(\xi^{\text{gp}}_*); 0)$. This produces the following (homotopy) pushout of $\text{THH}(R)$ -modules with $\mathbb{T}$ -action:

$$\begin{array}{ccc} \text{THH}(R) \vee \text{THH}(R) & \xrightarrow{(+\frac{1}{2})_* \vee (-\frac{1}{2})_*} & S^\infty \odot \text{Th}_R(\xi^{\text{gp}}) \\ \text{id} \downarrow & & \downarrow f_* \\ \text{THH}(R) & \xrightarrow{\sim} \text{THH}(\text{Th}_R(\xi_*); 0) \xrightarrow{\rho_0} & \text{THH}(\text{Th}_R(\xi^{\text{gp}}_*); 0), \end{array} \quad (9.15)$$

where the bottom composite is induced by the unit map of $\text{Th}_R(\xi^{\text{gp}})$ and therefore factors as the weight 0 component of the repletion map and the equivalence of Corollary 9.8.

We can now prove the following equivariant refinement of Lemma 9.11:

Proposition 9.21. *The weight 0 component ρ_0 of the repletion map is equivalent as a map of $\text{THH}(R)$ -modules with $\mathbb{T}$ -action to the composite*

$$\text{THH}(R) \xrightarrow{\sim} \text{THH}(R) \wedge \{\ast\}_+ \rightarrow \text{THH}(R) \wedge S^1(0)_+ \quad (9.16)$$

induced by the basepoint inclusion $\{*\} \rightarrow S^1$, and the repletion fiber $\mathrm{THH}^\xi(R)$ is equivalent to $\mathrm{THH}(R)$ as a $\mathrm{THH}(R)$ -module with $\mathbb{T}$ -action.

Proof. We claim that the two $\mathbb{T}$ -equivariant maps $(+\frac{1}{2})_*$, $(-\frac{1}{2})_*$ in (9.15) are homotopic through $\mathbb{T}$ -equivariant maps. Assuming this, we can extend $(+\frac{1}{2})_* \vee (-\frac{1}{2})_*$ over $\mathrm{THH}(R) \wedge [0, 1]_+$. The extension $\mathrm{THH}(R) \wedge [0, 1]_+ \rightarrow S^\infty \odot \mathrm{Th}_R(\xi^{\mathrm{gp}})$ is an equivalence because the inclusion $(+\frac{1}{2})_* \rightarrow S^\infty$ induces an equivalence when applying $-\odot \mathrm{Th}_R(\xi^{\mathrm{gp}})$. The statement about ρ_0 then follows from the homotopy pushout (9.15), and from this we get that the fiber of ρ_0 is equivalent to $\mathrm{THH}(R)$ as a $\mathrm{THH}(R)$ -module with $\mathbb{T}$ -action.

To verify the claim about $(+\frac{1}{2})_*$, $(-\frac{1}{2})_*$, it is sufficient to find a homotopy between the corresponding maps $(+\frac{1}{2})_*$, $(-\frac{1}{2})_* : \{*\} \rightarrow S^\infty$ in $(S/\mathrm{Pic}_{\mathrm{THH}(R)})^{B\mathbb{T}}$ and then apply $\mathrm{Th}_{\mathrm{THH}(R)}$. Using the augmentation of the codomain, this reduces to finding such a homotopy in spaces with $\mathbb{T}$ -action. Precomposing with the $\mathbb{T}$ -equivariant weak equivalence $E\mathbb{T} \rightarrow \{*\}$, this in turn becomes a statement about $\mathbb{T}$ -equivariant maps $E\mathbb{T} \rightarrow S^\infty$. Since the space of such maps is equivalent to the contractible space $(S^\infty)^{h\mathbb{T}}$, any two such maps are $\mathbb{T}$ -equivariantly homotopic. $\square$

9.5 | Criteria for cyclotomically good pairs

Given a $\mathrm{THH}(R)$ -module E we can generalize [21] slightly and say that a $\mathrm{THH}(R)$ -module Z is E_* -acyclic if $E \otimes_{\mathrm{THH}(R)} Z \simeq 0$, a map $X \rightarrow Y$ of $\mathrm{THH}(R)$ -modules is an E_* -equivalence if its (co-) fiber is E_* -acyclic, and Y is E_* -local if $\mathrm{Map}_{\mathrm{THH}(R)}(Z, Y) \simeq *$ for each E_* -acyclic Z . The map $X \rightarrow Y$ is an E_* -localization of X if it is an E_* -equivalence and Y is E_* -local. For $\mathrm{THH}(R)$ -modules with $\mathbb{T}$ -action, the notions E_* -acyclic and E_* -equivalence are defined in terms of the underlying non-equivariant $\mathrm{THH}(R)$ -modules.

If Z and Y are $\mathrm{THH}(R)$ -modules with $\mathbb{T}$ -action, then the $\mathbb{T}$ -equivariant mapping space from Z to Y has the homotopy type of $\mathrm{Map}_{\mathrm{THH}(R)}(Z, Y)^{h\mathbb{T}}$. This space is contractible for all E_* -acyclic Z with $\mathbb{T}$ -action (in which case we say that Y is $\mathbb{T}$ -equivariantly E_* -local), if and only if the underlying non-equivariant $\mathrm{THH}(R)$ -module of Y is E_* -local. A $\mathbb{T}$ -map $X \rightarrow Y$ of $\mathrm{THH}(R)$ -modules is thus a $\mathbb{T}$ -equivariant E_* -localization if and only if the underlying map of non-equivariant $\mathrm{THH}(R)$ -modules is an E_* -localization.

When $E = \mathrm{THH}(R) \otimes \mathbb{S}/p$, being E_* -local is the same as being p -complete, and E_* -localization is the same as p -completion.

Proof of Proposition 9.2. By Proposition 9.21, $\mathrm{THH}(R) \simeq \mathrm{THH}^\xi(R)$ as $\mathrm{THH}(R)$ -modules with $\mathbb{T}$ -action. Corollary 9.14 and the above discussion therefore imply that it is sufficient to show that $\varphi_p : \mathrm{THH}(R) \rightarrow \mathrm{THH}(R)^{tC_p}$ is p -completion for every prime p under the assumption of the proposition. This hypothesis is satisfied for $R = \mathbb{S}$ by the theorems of Lin [41] and Gunawardena [29], and for $R = MU$ by [43].

A similar argument applies if $\varphi_p : \mathrm{THH}(R) \rightarrow \mathrm{THH}(R)^{tC_p}$ is a smashing localization in the category of $\mathrm{THH}(R)$ -modules with $\mathbb{T}$ -action. This applies when $R = H\mathbb{F}$ with $\mathbb{F}$ any perfect field of characteristic p , since by [33, §5] the $\mathbb{T}$ -map $\varphi_p : \mathrm{THH}(\mathbb{F}) \rightarrow \mathrm{THH}(\mathbb{F})^{tC_p}$ then inverts any choice of $\mathbb{T}$ -equivariant lift in $\pi_2 \mathrm{TC}^-(\mathbb{F}) \cong W(\mathbb{F})$ of degree 2 Bökstedt periodicity element μ_0 in $\pi_* \mathrm{THH}(\mathbb{F}) \cong \mathbb{F}[\mu_0]$. Moreover, φ_q is trivial for each prime $q \neq p$. A similar argument applies for $\mathbb{F}$ any perfectoid ring, as in [12, Definition 4.18 and Proposition 6.2]. $\square$

10 | THE REPLETION-RESIDUE SEQUENCE IN THE MULTIPLE GENERATOR CASE

The purpose of this section is to generalize the repletion-residue sequence constructed in the previous section to R -based prelog $\mathbb{E}_k$ -rings where the monoid is a free commutative monoid of rank r . So throughout this section, we let $(\xi^i : \langle x_i \rangle \rightarrow \mathrm{Pic}_R)_{1 \leq i \leq r}$ be a family of $\mathbb{E}_k$ -maps with $k \geq 2$ and let

$$\xi = \xi^1 \cdots \xi^r : \langle x_1, \dots, x_r \rangle \rightarrow \mathrm{Pic}_R \quad (10.1)$$

be their product, viewed as a $\mathbb{Z}_{\geq 0}^r$ -graded object with the canonical grading. Prelog structures involving such a ξ arise, for example, from Corollary 4.8.

The next lemma shows that the condition that ξ is a product can always be enforced, albeit up to losing one level of commutativity.

Lemma 10.1. *Let $\zeta : \langle x_1, \dots, x_r \rangle \rightarrow \mathrm{Pic}_R$ be an $\mathbb{E}_k$ -map. Then the composite*

$$\zeta|_{\langle x_1 \rangle} \cdots \zeta|_{\langle x_r \rangle} : \langle x_1, \dots, x_r \rangle = \langle x_1 \rangle \times \cdots \times \langle x_r \rangle \rightarrow \mathrm{Pic}_R \times \cdots \times \mathrm{Pic}_R \rightarrow \mathrm{Pic}_R$$

is homotopic to ζ through $\mathbb{E}_{k-1}$ -maps.

Proof. The underlying $\mathbb{E}_1$ -structure on ζ gives equivalences

$$\zeta|_{\langle x_1 \rangle}(x_1^i) \otimes_R \cdots \otimes_R \zeta|_{\langle x_r \rangle}(x_r^i) \xrightarrow{\sim} \zeta(x_1^i \cdots x_r^i)$$

that define a homotopy from $\zeta^1 \cdots \zeta^r$ to ζ through $\mathbb{E}_0$ -maps. The $\mathbb{E}_k$ -case follows by Dunn additivity. $\square$

Returning to the situation of (10.1), we can use the external product (5.5) to decompose the (canonical log) THH of $\mathrm{Th}_R(\xi^1 \cdots \xi^r)$:

Lemma 10.2. *For a family $(\xi^i : \langle x_i \rangle \rightarrow \mathrm{Pic}_R)_{1 \leq i \leq r}$ of $\mathbb{E}_k$ -maps with $k \geq 2$, the symmetric monoidal structures of THH and Th_R induce compatible equivalences*

$$\mathrm{THH}(\mathrm{Th}_R(\xi_*^1)) \otimes_{\mathrm{THH}(R)} \cdots \otimes_{\mathrm{THH}(R)} \mathrm{THH}(\mathrm{Th}_R(\xi_*^r)) \xrightarrow{\sim} \mathrm{THH}(\mathrm{Th}_R(\xi_*^1 \cdots \xi_*^r)) \quad \text{and} \quad (10.2)$$

$$\mathrm{THH}(\mathrm{Th}_R(\xi_*^1), \xi^1) \otimes_{\mathrm{THH}(R)} \cdots \otimes_{\mathrm{THH}(R)} \mathrm{THH}(\mathrm{Th}_R(\xi_*^r), \xi^r) \xrightarrow{\sim} \mathrm{THH}(\mathrm{Th}_R(\xi_*^1 \cdots \xi_*^r), \xi^1 \cdots \xi^r) \quad (10.3)$$

of $\mathbb{Z}_{\geq 0}^r$ -graded twisted cyclotomic $\mathrm{THH}(R)$ -modules.

Proof. For the first equivalence, we apply the symmetric monoidal structure of $\mathrm{Th}_R : \mathcal{S}/\mathrm{Pic}_R \rightarrow \mathrm{Mod}_R$ to $\xi_*^1, \dots, \xi_*^r$ and use Lemma 5.14. For the second, we argue analogously with $(\xi_*^1)^{\mathrm{gp}}, \dots, (\xi_*^r)^{\mathrm{gp}}$ and restrict the resulting $\mathbb{Z}^r$ -graded object to $\mathbb{Z}_{\geq 0}^r$. As in (5.6), we get compatible repletion maps. $\square$

We will use the following notion for the bookkeeping of the cofiber sequences we are after:

Definition 10.3. Let C be a symmetric monoidal stable ∞ -category and let $[2]$ denote the poset $\{0 < 1 < 2\}$. An r -dimensional cube of cofiber sequences in C is a functor $C(-) : [2]^r \rightarrow C$ such that for any sequence $(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r)$ in $\{0, 1, 2\}^{r-1}$ the diagram

$$C(a_0, \dots, a_{i-1}, -, a_{i+1}, \dots, a_r) : [2] \rightarrow C$$

is a cofiber sequence in C .

Construction 10.4. We write

$$\mathrm{THH}(\mathrm{Th}_R(\xi_*); -) : [2]^r \rightarrow \mathrm{TwCycFun}(\mathbb{Z}_{\geq 0}^r, \mathrm{Mod}_{\mathrm{THH}(R)}) \quad (10.4)$$

for the r -fold external product of the cofiber sequences of $\mathbb{Z}_{\geq 0}$ -graded twisted cyclotomic $\mathrm{THH}(R)$ -modules

$$\mathrm{THH}(\mathrm{Th}_R(\xi_*^i)) \xrightarrow{\rho} \mathrm{THH}(\mathrm{Th}_R(\xi_*^i), \xi^i) \xrightarrow{\mathrm{res}} \Sigma \mathrm{THH}^{\xi^i}(R)$$

from Proposition 9.9. Via the $\mathbb{E}_k$ -equivalence (10.2) and the $\mathrm{THH}(\mathrm{Th}_R(\xi_*^i))$ -module structures from Proposition 9.9, we can view $\mathrm{THH}(\mathrm{Th}_R(\xi_*); -)$ as a diagram of $\mathbb{Z}_{\geq 0}^r$ -graded twisted cyclotomic $\mathrm{THH}(\mathrm{Th}_R(\xi_*))$ -modules.

To describe $\mathrm{THH}(\mathrm{Th}_R(\xi_*); -)$ in more detail, we introduce the notation

$$S_j = S_j(a_1, \dots, a_r) = \{i \mid a_i = j\} \subseteq \{1, \dots, r\} \quad (10.5)$$

for $0 \leq j \leq 2$ and an index $(a_1, \dots, a_r) \in \{0, 1, 2\}^r$. Then $\mathrm{THH}(\mathrm{Th}_R(\xi_*); a_1, \dots, a_r)$ is equivalent to

$$\bigotimes_{i \in S_0} \mathrm{THH}(\mathrm{Th}_R(\xi_*^i)) \otimes \bigotimes_{i \in S_1} \mathrm{THH}(\mathrm{Th}_R(\xi_*^i), \xi^i) \otimes \bigotimes_{i \in S_2} \Sigma \mathrm{THH}^{\xi^i}(R), \quad (10.6)$$

where all $\otimes$ -products are formed over $\mathrm{THH}(R)$.

Definition 10.5. For an R -based prelog $\mathbb{E}_k$ -ring

$$(A, \xi = \xi^1 \cdots \xi^r : \langle x_1, \dots, x_r \rangle \rightarrow \mathrm{Pic}_R, \tilde{\alpha} : \mathrm{Th}_R(\xi) \rightarrow A)$$

with ξ as in (10.1), we let

$$\mathrm{THH}(A, \xi, \tilde{\alpha}; -) : [2]^r \rightarrow \mathrm{CycMod}_{\mathrm{THH}(A)}$$

be the diagram of cyclotomic $\mathrm{THH}(A)$ -modules obtained from (the underlying ungraded object of) the external product (10.4) by base change along the $\mathbb{E}_{k-1}$ -map $\mathrm{THH}(\tilde{\alpha}) : \mathrm{THH}(\mathrm{Th}_R(\xi)) \rightarrow \mathrm{THH}(A)$.

For an $S \subseteq \{1, \dots, r\}$, we define $\tilde{\alpha}_{S, \emptyset}$ to be the composite $\mathbb{E}_k$ -map

$$\mathrm{Th}_R(\prod_{i \in S} \xi^i) \xrightarrow{\mathrm{incl}} \mathrm{Th}_R(\xi^1 \cdots \xi^r) \xrightarrow{\tilde{\alpha}} A.$$

Similar to (9.2), we write

$$A // \tilde{\alpha}(S) = A \otimes_{\mathrm{Th}_R(\prod_{i \in S} \xi^i)} R$$

for the $\mathbb{E}_{k-1}$ -algebra formed using $\tilde{\alpha}_{S,\emptyset}$ and the collapse map (where we understand $A//\tilde{\alpha}(\emptyset) = A$). For subsets $S, T \subseteq \{1, \dots, r\}$, we define $\tilde{\alpha}_{S,T}$ to be the composite

$$\mathrm{Th}_R(\prod_{i \in S} \xi^i) \rightarrow A \rightarrow A//\tilde{\alpha}(T)$$

of $\tilde{\alpha}_{S,\emptyset}$ and the canonical map.

Theorem 10.6. *The diagram $\mathrm{THH}(A, \xi, \tilde{\alpha}; -)$ is an n -dimensional cube of cofiber sequences of cyclotomic $\mathrm{THH}(A)$ -modules. For an object $(a_1, \dots, a_r)$ of $[2]^r$, the term $\mathrm{THH}(A, \xi, \tilde{\alpha}; a_1, \dots, a_r)$ is equivalent as a $\mathrm{THH}(A)$ -module with $\mathbb{T}$ -action to $\Sigma^{|S_2|} \mathrm{THH}(A//\tilde{\alpha}(S_2))$ if $S_1(a_1, \dots, a_r) = \emptyset$, and to*

$$\Sigma^{|S_2|} \mathrm{THH}(A//\tilde{\alpha}(S_2)) \otimes_{\mathrm{THH}(\mathrm{Th}_R(\prod_{i \in S_1} \xi^i))} \mathrm{THH}(\mathrm{Th}_R(\prod_{i \in S_1} \xi^i), \prod_{i \in S_1} \xi^i) \quad (10.7)$$

otherwise. If (R, ξ_*^i) is in addition cyclotomically good for all $1 \leq i \leq r$, then these are equivalences of cyclotomic $\mathrm{THH}(A)$ -modules.

Remark 10.7. When $k \geq 3$, then $A//\tilde{\alpha}(S_2)$ is an $\mathbb{E}_2$ -ring and (10.7) is equivalent to $\Sigma^{|S_2|} \mathrm{THH}(A//\tilde{\alpha}(S_2), \prod_{i \in S_1} \xi^i, \tilde{\alpha}_{S_1, S_2})$ in the sense of Definition 5.22. When $k = 2$ and $S_2 \neq \emptyset$, then $A//\tilde{\alpha}(S_2)$ is *a priori* only $\mathbb{E}_1$. We view (10.7) as an extension of the definition of log THH to this situation, and use the same notation.

If the (R, ξ_*^i) are all cyclotomically good, then Corollary 9.5 generalizes and we obtain a similar r -dimensional cube of cofiber sequences of TC-terms.

Proof of Theorem 10.6. We begin by observing that if $S, T \subseteq \{1, \dots, r\}$ are disjoint subsets with $S \cup T = \{1, \dots, r\}$, then using (symmetric) monoidality of Th_R we get an equivalence

$$A \otimes_{\mathrm{Th}_R(\xi^1 \dots \xi^r)} \mathrm{Th}_R(\prod_{i \in S} \xi^i) \simeq A \otimes_{\mathrm{Th}_R(\prod_{i \in T} \xi^i)} R. \quad (10.8)$$

We first assume $S_1 \neq \emptyset$. Applying $\mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathrm{Th}_R(\xi))} -$ to (10.6) and using the equivalences $\mathrm{THH}(R) \simeq \mathrm{THH}^{\xi^i}(R)$ from Proposition 9.21 as well as (10.2) and (10.3), the term is equivalent to

$$\Sigma^{|S_2|} \mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathrm{Th}_R(\xi))} \left(\bigotimes_{i \in S_0} \mathrm{THH}(\mathrm{Th}_R(\xi^i)) \otimes \bigotimes_{i \in S_1} \mathrm{THH}(\mathrm{Th}_R(\xi^i), \xi^i) \right).$$

Replacing $\mathrm{THH}(\mathrm{Th}_R(\xi^i), \xi^i)$ by $\mathrm{THH}(\mathrm{Th}_R(\xi^i)) \otimes_{\mathrm{THH}(\mathrm{Th}_R(\xi^i))} \mathrm{THH}(\mathrm{Th}_R(\xi^i), \xi^i)$, Lemma 10.2 and the equivalence (10.8) with the complementary subsets $S_0 \cup S_1$ and S_2 imply that the latter term is equivalent to the claimed one. An analogous argument applies when $S_1 = \emptyset$.

When in addition each (R, ξ_*^i) is cyclotomically good, we can argue with the definition of cyclotomically good instead of Proposition 9.21. $\square$

The maps in the cubical diagram provided by the theorem are induced by the repletion and residue maps from Theorem 9.3.

Example 10.8. We let $r = 2$ and assume the (R, ξ_*^i) to be cyclotomically good. Abbreviating all indices $\{i\}$ to i , the theorem provides a diagram

$$\begin{array}{ccccc}
 \mathrm{THH}(A) & \xrightarrow{\rho} & \mathrm{THH}(A, \xi^1, \bar{\alpha}_{1,\emptyset}) & \xrightarrow{\mathrm{res}} & \Sigma \mathrm{THH}(A//\bar{\alpha}(1)) \\
 \rho \downarrow & & \rho \downarrow & & \Sigma \rho \downarrow \\
 \mathrm{THH}(A, \xi^2, \bar{\alpha}_{2,\emptyset}) & \xrightarrow{\rho} & \mathrm{THH}(A, \xi^1 \cdot \xi^2, \bar{\alpha}) & \xrightarrow{\mathrm{res}} & \Sigma \mathrm{THH}(A//\bar{\alpha}(1), \xi^2, \bar{\alpha}_{2,1}) \\
 \mathrm{res} \downarrow & & \mathrm{res} \downarrow & & \Sigma \mathrm{res} \downarrow \\
 \Sigma \mathrm{THH}(A//\bar{\alpha}(2)) & \xrightarrow{\Sigma \rho} & \Sigma \mathrm{THH}(A//\bar{\alpha}(2), \xi^1, \bar{\alpha}_{1,2}) & \xrightarrow{\Sigma \mathrm{res}} & \Sigma^2 \mathrm{THH}(A//\bar{\alpha})
 \end{array}$$

of horizontal and vertical cofiber sequences of cyclotomic $\mathrm{THH}(A)$ -modules, where the lower right-hand square commutes up to a sign -1 and the three other squares commute. Here the sign comes from two different ways to identify a smash product of suspensions with an iterated suspension, and the two terms mapping to $\Sigma^2 \mathrm{THH}(A//\bar{\alpha}(1, 2))$ are extending Definition 5.22 to the case where the underlying $\mathbb{E}_k$ -ring is only $\mathbb{E}_1$ in the way explained in Remark 10.7. Passing to TC, we get an analogous diagram of horizontal and vertical cofiber sequences of $\mathrm{TC}(A)_p$ -modules

$$\begin{array}{ccccc}
 \mathrm{TC}(A)_p & \xrightarrow{\rho} & \mathrm{TC}(A, \xi^1, \bar{\alpha}_{1,\emptyset})_p & \xrightarrow{\mathrm{res}} & \Sigma \mathrm{TC}(A//\bar{\alpha}(1))_p \\
 \rho \downarrow & & \rho \downarrow & & \Sigma \rho \downarrow \\
 \mathrm{TC}(A, \xi^2, \bar{\alpha}_{2,\emptyset})_p & \xrightarrow{\rho} & \mathrm{TC}(A, \xi^1 \cdot \xi^2, \bar{\alpha})_p & \xrightarrow{\mathrm{res}} & \Sigma \mathrm{TC}(A//\bar{\alpha}(1), \xi^2, \bar{\alpha}_{2,1})_p \\
 \mathrm{res} \downarrow & & \mathrm{res} \downarrow & & \Sigma \mathrm{res} \downarrow \\
 \Sigma \mathrm{TC}(A//\bar{\alpha}(2))_p & \xrightarrow{\Sigma \rho} & \Sigma \mathrm{TC}(A//\bar{\alpha}(2), \xi^1, \bar{\alpha}_{1,2})_p & \xrightarrow{\Sigma \mathrm{res}} & \Sigma^2 \mathrm{TC}(A//\bar{\alpha}(1, 2))_p.
 \end{array}$$

Example 10.9. In the specific case where the prelog ring is $(\ell, \langle p, v_1 \rangle)$ from Definition 4.10, Example 10.8 specializes to diagram (1.2) from the introduction. The collapse of the complements of the faces $\langle p \rangle$, $\langle v_1 \rangle$, and $\langle p, v_1 \rangle$ appearing in that diagram was described in Examples 4.19 and 4.20.

11 | EVEN-PERIODIC SPHERE SPECTRA

The purpose of this section is to identify the terms in the log THH and log TC repletion–residue sequences for even-periodic sphere spectra.

Let $d \geq 0$. In what follows, we write $\xi = \xi_{2d} : \mathbb{Z}_{\geq 0} \rightarrow \mathrm{Pic}_{\mathbb{S}}$ for the $r = 1$ instance of the $\mathbb{E}_2$ -map (4.1) and $\mathbb{S}[x] = \mathrm{Th}_{\mathbb{S}}(\xi)$ for the associated $\mathbb{E}_2$ -ring. Moreover, we write $\xi^{\mathrm{gp}} : \mathbb{Z} \rightarrow \mathrm{Pic}_{\mathbb{S}}$ for the extension of ξ arising from Construction 3.9 and $\mathbb{S}[x^{\pm 1}] = \mathrm{Th}_{\mathbb{S}}(\xi)$ for the associated $\mathbb{E}_2$ -ring. Then

$$\mathbb{S}[x^{\pm 1}] \simeq \bigvee_{i \in \mathbb{Z}} \mathbb{S}^{2di} \quad \text{and} \quad \mathbb{S}[x] \simeq \bigvee_{i \geq 0} \mathbb{S}^{2di},$$

meaning that these $\mathbb{E}_2$ -rings are models for the $2d$ -periodic and non-negative $2d$ -periodic sphere spectra. They come with canonical $\mathbb{Z}_{\geq 0}$ - and $\mathbb{Z}$ -gradings, and give rise to the canonical prelog $\mathbb{E}_2$ -ring $(\mathbb{S}[x], \langle x \rangle) := (\mathbb{S}[x], \xi, \mathrm{id})$ with ξ the underlying total object of the $\mathbb{Z}_{\geq 0}$ -graded ξ_* (see Example 4.14).

Before analyzing log THH and log TC in this example, we return to the question whether $\mathbb{S}[x]$ admits an $\mathbb{E}_3$ -structure (compare Remark 4.5).

Proposition 11.1. *If p is odd and $p \nmid d$, then neither $\mathbb{S}[x]$ nor its p -localization admit an $\mathbb{E}_3$ -ring structure extending the given $\mathbb{E}_1$ -ring structure.*

Proof. We set $n = 2d$. Writing $\mathbb{E}_k X = \bigvee_{j \geq 0} C_{k,j} X$ for the free $\mathbb{E}_k$ -ring on X , we get canonical $\mathbb{E}_1$ - and $\mathbb{E}_3$ -maps $\mathbb{E}_1 S^n \rightarrow \mathbb{E}_3 S^n \rightarrow \mathbb{E}_\infty S^n$ given in operadic weight $j = 1$ by the identity map of S^n and in operadic weight $j = p$ by

$$S^{np} = C_{1,p} S^n \rightarrow C_{3,p} S^n \rightarrow C_{\infty,p} S^n = (S^n)_{h\Sigma_p}^{\otimes p}.$$

If the free $\mathbb{E}_1$ -structure on $\mathbb{E}_1 S^n$ extends to an $\mathbb{E}_3$ -structure, then $\mathbb{E}_1 S^n \rightarrow \mathbb{E}_3 S^n$ admits a retraction r .

To exclude the existence of this r , we argue with Cohen's mod p homology operations defined in [25, §IV.1]. Let $e \in H_n(S^n)$ be a generator. Then $Q^d(e) = e^p$ generates $H_{pn}(-)$ at the bottom of operadic weight p in each case. The 'top' operation ξ_2 is defined for $\mathbb{E}_3$ -algebras, and $\xi_2(e) \in H_{pn+2p-2}(C_{3,p} S^n)$. If $P_*^1 \xi_2(e) \in H_{pn}(C_{3,p} S^n)$ is non-zero, then no retraction r can exist, since P_*^1 commutes with spectrum maps and $P_*^1 = 0$ acting on $H_*(C_{1,p} S^n)$. To see that $P_*^1 \xi_2(e) \neq 0$, we can use that $\xi_2(e) = Q^{d+1}(e)$ for $\mathbb{E}_\infty$ -algebras, and instead show that $Q^{d+1}(e) \in H_{pn+2p-2}(C_{\infty,p} S^n)$ has non-zero image $P_*^1 Q^{d+1}(e) \in H_{pn}(C_{\infty,p} S^n)$. When p does not divide d , this now follows from the original Nishida relation $P_*^1 Q^{d+1} = d Q^d P_*^0$, see, for example, [25, Theorem IV.1.1(6)], since $P_*^0(e) = e$ and $Q^d(e) = e^p$. $\square$

11.1 | Log THH of even-periodic sphere spectra

We get maps

$$\mathrm{THH}(\mathbb{S}[x]) \xrightarrow{\rho} \mathrm{THH}(\mathbb{S}[x], \langle x \rangle) \xrightarrow{\tilde{\gamma}} \mathrm{THH}(\mathbb{S}[x^{\pm 1}]) \quad (11.1)$$

between (the total objects of) $\mathbb{Z}_{\geq 0^-}$, $\mathbb{Z}_{\geq 0^-}$ - and $\mathbb{Z}$ -graded L_p -twisted cyclotomic $\mathbb{E}_1$ -rings, where ρ is the repletion map and $\tilde{\gamma}$ is induced by the inclusion of the non-negatively graded part of $\mathrm{THH}(\mathbb{S}[x^{\pm 1}])$.

Proposition 11.3. *As a sequence of graded spectra with $\mathbb{T}$ -action, (11.1) is equivalent to*

$$\begin{aligned} \mathbb{S} \vee \bigvee_{i>0} \mathbb{T}_+ \wedge_{C_i} (\mathbb{S}^{2d})^{\otimes i} &\longrightarrow \mathbb{S}[S^1(0)] \vee \bigvee_{i>0} \mathbb{T}_+ \wedge_{C_i} (\mathbb{S}^{2d})^{\otimes i} \\ &\longrightarrow \bigvee_{i<0} \mathbb{T}_+ \wedge_{C_{-i}} (\mathbb{S}^{-2d})^{\otimes -i} \vee \mathbb{S}[S^1(0)] \vee \bigvee_{i>0} \mathbb{T}_+ \wedge_{C_i} (\mathbb{S}^{2d})^{\otimes i}. \end{aligned}$$

In each case, $C_{|i|} \subset \mathbb{T}$ acts by cyclically permuting the $|i|$ smash factors.

Proof. Propositions 7.2 and 7.15 assert that the three terms in (11.1) are equivalent as $\mathbb{Z}_{\geq 0^-}$, $\mathbb{Z}_{\geq 0^-}$ - and $\mathbb{Z}$ -graded $\mathbb{E}_1$ -rings with $\mathbb{T}$ -action to the Thom spectra over $B^{\mathrm{cy}}(\mathbb{Z}_{\geq 0}) \simeq (0) \sqcup \prod_{i>0} S^1(i)$, $B^{\mathrm{rep}}(\mathbb{Z}_{\geq 0}) \simeq \prod_{i \geq 0} S^1(i)$ and $B^{\mathrm{cy}}(\mathbb{Z}) \simeq \prod_{i \in \mathbb{Z}} S^1(i)$, respectively. Here we use the notation $S^1(i)$ from Proposition 9.7.

In weight $i > 0$, we start with the C_i -equivariant map $\mathbb{S}[x]^{\otimes i} \rightarrow \mathrm{THH}(\mathbb{S}[x])$ obtained as the inclusion of the 0-simplices in the i -fold edgewise subdivision [20, §1]. The restriction along $(\mathbb{S}^{2d})^{\otimes i} \rightarrow \mathbb{S}[x]^{\otimes i}$ factors through the inclusion $\mathrm{THH}(\mathbb{S}[x]; i) \rightarrow \mathrm{THH}(\mathbb{S}[x])$, thus giving a C_i -map

$(\mathbb{S}^{2d})^{\otimes i} \rightarrow \mathrm{THH}(\mathbb{S}[x]; i)$ that is adjoint to a $\mathbb{T}$ -equivariant map

$$\mathbb{T}_+ \wedge_{C_i} (\mathbb{S}^{2d})^{\otimes i} \rightarrow \mathrm{THH}(\mathbb{S}[x]; i). \quad (11.2)$$

Writing sd_i for the i -fold edgewise subdivision, we can obtain the map (11.2) under the equivalence of Proposition 7.2 by extending the C_i -equivariant inclusion $\{(1, \dots, 1)\} \rightarrow \mathrm{sd}_i(B^{\mathrm{cy}}(\mathbb{Z}_{\geq 0}; i))$ to a $\mathbb{T}$ -equivariant map $\mathbb{T} \times_{C_i} \{(1, \dots, 1)\} \rightarrow \mathrm{sd}_i(B^{\mathrm{cy}}(\mathbb{Z}_{\geq 0}; i))$ and then applying $\mathrm{Th}_{\mathbb{S}}$. Since the latter map is the weak equivalence from Proposition 9.7, it follows that (11.2) is an equivalence.

In weight $i < 0$, we invoke the group isomorphism $\mathbb{Z} \rightarrow \mathbb{Z}, 1 \mapsto -1$, which reverses the $\mathbb{Z}$ -grading. Its composite with $\xi_*^{\mathrm{gp}} : \mathbb{Z} \rightarrow \mathrm{Pic}_{\mathbb{S}}$ gives an $\mathbb{E}_2$ -map $\bar{\xi}_*^{\mathrm{gp}} : \mathbb{Z} \rightarrow \mathrm{Pic}_{\mathbb{S}}$ sending 1 to $\mathbb{S}^{-2d}$. Let $\bar{\xi}_* : \mathbb{Z}_{\geq 0} \rightarrow \mathrm{Pic}_{\mathbb{S}}$ be the restricted map. Then

$$\begin{aligned} \mathrm{THH}(\mathbb{S}[x^{\pm 1}]; i) &= \mathrm{THH}(\mathrm{Th}_{\mathbb{S}}(\xi_*^{\mathrm{gp}}); i) \simeq \mathrm{THH}(\mathrm{Th}_{\mathbb{S}}(\bar{\xi}_*^{\mathrm{gp}}); -i) \\ &\simeq \mathrm{THH}(\mathrm{Th}_{\mathbb{S}}(\bar{\xi}_*); -i) \simeq \mathbb{T}_+ \wedge_{C_{-i}} (\mathbb{S}^{-2d})^{\otimes -i} \end{aligned}$$

as a spectrum with $\mathbb{T}$ -action by the weight $i > 0$ case above.

In weight $i = 0$, the assertion follows from Proposition 9.21. $\square$

The following is a homological incarnation of Proposition 11.3, capturing an algebraic image of the $\mathbb{E}_1$ -ring structures and $\mathbb{T}$ -actions.

Proposition 11.4. *There are Bökstedt spectral sequences*

$$\begin{aligned} E_{**}^2 &= \mathrm{HH}_*(H_*(\mathbb{S}[x])) = \mathrm{HH}_*(\mathbb{Z}[x]) = \mathbb{Z}[x] \otimes \Lambda(dx) \\ &\implies H_*(\mathrm{THH}(\mathbb{S}[x])) \\ E_{**}^2 &= \mathrm{HH}_*(H_*(\mathbb{S}[x]), \langle x \rangle) = \mathrm{HH}_*(\mathbb{Z}[x], \langle x \rangle) = \mathbb{Z}[x] \otimes \Lambda(d \log x) \\ &\implies H_*(\mathrm{THH}(\mathbb{S}[x], \langle x \rangle)) \\ E_{**}^2 &= \mathrm{HH}_*(H_*(\mathbb{S}[x^{\pm 1}])) = \mathrm{HH}_*(\mathbb{Z}[x^{\pm 1}]) = \mathbb{Z}[x^{\pm 1}] \otimes \Lambda(d \log x) \\ &\implies H_*(\mathrm{THH}(\mathbb{S}[x^{\pm 1}])) \end{aligned}$$

with classes of (Hochschild, homological) bidegree $\|x\| = (0, 2d)$, $\|dx\| = (1, 2d)$, and $\|d \log x\| = (1, 0)$. These are algebra spectral sequences that collapse at the $E^2 = E^{\infty}$ -terms, with no additive or multiplicative extensions. The repletion map ρ takes dx to $x \cdot d \log x$. The $\mathbb{T}$ -actions induce the derivations satisfying $x \mapsto dx$, $dx \mapsto 0$, $x \mapsto x \cdot d \log x$ and $d \log x \mapsto 0$.

Proof. Applying homology to the skeleton filtration of the geometric realization defining $\mathrm{THH}(\mathbb{S}[x^{\pm 1}])$ gives the (E^1, d^1) -term in the third case, which we identify with the normalized Hochschild complex for $H_*(\mathbb{S}[x^{\pm 1}]) = \mathbb{Z}[x^{\pm 1}]$. Hence its E^2 -term is given by the stated Hochschild homology, with $d \log x \in \mathrm{HH}_1(\mathbb{Z}[x^{\pm 1}])$ the class of the Hochschild 1-cycle $x^{-1} \otimes x$. The $\mathbb{E}_2$ -ring structure on $\mathbb{S}[x^{\pm 1}]$ gives a map $\mathbb{S}[x^{\pm 1}] \otimes \mathbb{S}[x^{\pm 1}] \rightarrow \mathbb{S}[x^{\pm 1}]$ of $\mathbb{E}_1$ -rings, inducing a pairing of spectral sequences given at the E^1 -terms by the shuffle product and at the E^2 -terms by the usual product in the Hochschild homology of a (graded) commutative ring. This shows that we have an algebra spectral sequence.

There is no room for d^r -differentials with $r \geq 2$, so $E^2 = E^\infty$. Since the E^∞ -term is free abelian in each bidegree, there is no room for additive extensions. The only possible multiplicative extension concerns the value of $d \log x \cdot d \log x$ in $\mathbb{Z}\langle x \rangle$ when $d = 1$. But since $\mathrm{THH}(\mathbb{S}[x^{\pm 1}])$ is a $\mathbb{Z}$ -graded $\mathbb{E}_1$ -ring, $(d \log x)^2$ must lie in the weight 0 summand, while x lies in the weight 1 summand.

Restricting to weights ≥ 0 gives the second Bökstedt spectral sequence, with the logarithmic Hochschild homology $\mathrm{HH}_*(\mathbb{Z}[x], \langle x \rangle) \cong \mathbb{Z}[x] \otimes \Lambda(d \log x)$ defined as in [55, Definition 3.23].

Repeating the argument with $\mathbb{S}[x]$ in place of $\mathbb{S}[x^{\pm 1}]$ gives the first Bökstedt spectral sequence, now with $\mathrm{HH}_1(\mathbb{Z}[x]) = \mathbb{Z}[x]\{dx\}$ generated by the homology class of the 1-cycle $1 \otimes x$. The chain-level product $x \cdot (x^{-1} \otimes x) = 1 \otimes x$ then implies the relation $x \cdot d \log x = \rho(dx)$ in $\mathrm{HH}_1(\mathbb{Z}[x], \langle x \rangle)$ and $\mathrm{HH}_1(\mathbb{Z}[x^{\pm 1}])$.

The $\mathbb{T}$ -action on $\mathrm{THH}(\mathbb{S}[x^{\pm 1}])$ arising from the cyclic structure takes each stage of the skeleton filtration into the next, sending the Hochschild 0-cycle $x^i \in E_{0,2di}^1$ to the 1-cycle $1 \otimes x^i \in E_{1,2di}^1$, and induces the Connes B -operator on the E^2 -terms. In particular, this operator sends x to $x \cdot d \log x$, and squares to zero. It also acts as a derivation, for example, because the filtered pairing $\mathrm{THH}(\mathbb{S}[x^{\pm 1}]) \otimes \mathrm{THH}(\mathbb{S}[x^{\pm 1}]) \rightarrow \mathrm{THH}(\mathbb{S}[x^{\pm 1}])$ takes the diagonal $\mathbb{T}$ -action on the source to the given $\mathbb{T}$ -action on the target. It follows that the B -operator agrees with the algebraic differential d , mapping the generator $x^i \in H_{2di}(\mathrm{THH}(\mathbb{S}[x^{\pm 1}]))$ to $d(x^i) = ix^{i-1} dx = ix^i d \log x$, which is i times the generator $x^i d \log x \in H_{2di+1}(\mathrm{THH}(\mathbb{S}[x^{\pm 1}]))$, and mapping $x^i d \log x$ to 0 for all $i \in \mathbb{Z}$. $\square$

11.2 | Log TC of even-periodic sphere spectra

The next lemma is the first step toward the identification of $\mathrm{TC}(\mathbb{S}[x])_p$ and $\mathrm{TC}(\mathbb{S}[x], \langle x \rangle)_p$.

Lemma 11.6. *For $i > 0$, the cyclotomic structure map $\varphi_{p;i} : \mathrm{THH}(\mathbb{S}[x]; i)_p \rightarrow \mathrm{THH}(\mathbb{S}[x]; pi)_p^{tC_p}$ is an equivalence.*

Proof. Restricting diagram (9.12) with $A = \mathrm{Th}_{\mathbb{S}}(\xi)$ and $q = i - 1$ along the map $\mathbb{S}^{2d} \rightarrow \mathrm{Th}_{\mathbb{S}}(\xi)$, we obtain the C_i -equivariant commutative outer rectangle in the diagram

$$\begin{array}{ccccc} (\mathbb{S}^{2d})^{\otimes i} & \longrightarrow & \mathbb{T}_+ \wedge_{C_i} (\mathbb{S}^{2d})^{\otimes i} & \xrightarrow{\cong} & \mathrm{THH}(\mathrm{Th}_{\mathbb{S}}(\xi); i) \\ \Delta_p \downarrow & & \tilde{\Delta}_p \downarrow & & \downarrow \varphi_{p;i} \\ ((\mathbb{S}^{2d})^{\otimes pi})^{tC_p} & \longrightarrow & (\mathbb{T}_+ \wedge_{C_{pi}} (\mathbb{S}^{2d})^{\otimes pi})^{tC_p} & \xrightarrow{\cong} & \mathrm{THH}(\mathrm{Th}_{\mathbb{S}}(\xi); pi)^{tC_p}. \end{array}$$

The two left-hand horizontal arrows are the evident inclusions, and the two right-hand horizontal equivalences arise like the equivalence (11.2) in the proof of Proposition 11.3. The two maps Δ_p and $\tilde{\Delta}_p$ are p -completions by the affirmed Segal conjecture for C_p [29, 41]. It follows that $\varphi_{p;i}$ is a p -equivalence. $\square$

The pair $(\mathbb{S}, \xi_*)$ is cyclotomically good by Proposition 9.2, and the collapse $\mathbb{S}[x]//\langle x \rangle$ is equivalent to $\mathbb{S}$ (see Example 4.18). Hence the TC repletion-residue sequence for $(\mathbb{S}[x], \xi, \mathrm{id})$ from Corollary 9.5 takes the form

$$\mathrm{TC}(\mathbb{S})_p \xrightarrow{\mathrm{trf}} \mathrm{TC}(\mathbb{S}[x])_p \xrightarrow{\rho} \mathrm{TC}(\mathbb{S}[x], \langle x \rangle)_p \xrightarrow{\mathrm{res}} \Sigma \mathrm{TC}(\mathbb{S})_p.$$

Theorem 11.7. *Let $|x| = 2d$ with $d > 0$. There are p -equivalences*

$$\mathrm{TC}(\mathbb{S}[x])_p \simeq_p \mathrm{TC}(\mathbb{S}) \vee \bigvee_{i>0} \Sigma((\mathbb{S}^{2d})^{\otimes i})_{hC_i} \quad \text{and}$$

$$\mathrm{TC}(\mathbb{S}[x], \langle x \rangle)_p \simeq_p \mathrm{TC}(\mathbb{S})[S^1] \vee \bigvee_{i>0} \Sigma((\mathbb{S}^{2d})^{\otimes i})_{hC_i},$$

and the map trf is null-homotopic as a map of $\mathrm{TC}(\mathbb{S})_p$ -modules.

Proof. Let $X_* = \mathrm{THH}(\mathbb{S}[x])$ or $X_* = \mathrm{THH}(\mathbb{S}[x], \langle x \rangle)$, viewed as $\mathbb{Z}_{\geq 0}$ -graded L_p -twisted cyclotomic spectra. As spectra with $\mathbb{T}$ -action, $X_i \simeq \mathbb{T}_+ \wedge_{C_i} (\mathbb{S}^{2d})^{\otimes i}$ for each $i > 0$ by Proposition 11.3, while $X_0 \simeq \mathbb{S}$ or $X_0 \simeq \mathbb{S}[S^1(0)]$ according to the case. Hence X_i is $2di$ -connective for each $i \geq 0$, so that X_* is properly connective in the sense of Definition 6.9, and there is a natural equivalence

$$\mathrm{TC}(X)_p \simeq \mathrm{TC}(X_0)_p \times \prod_{p \nmid i > 0} \mathrm{TC}(X_{[i]_p})_p \quad (11.3)$$

by Proposition 6.12.

First consider the two cases with $i = 0$. When $X_0 = \mathrm{THH}(\mathbb{S}[x]; 0) = \mathrm{THH}(\mathbb{S})$ we immediately have $\mathrm{TC}(X_0)_p = \mathrm{TC}(\mathbb{S})_p$. When $X_0 = \mathrm{THH}(\mathbb{S}[x], \langle x \rangle; 0)$, the weight 0 part of Proposition 9.9 leads to a repletion–residue sequence

$$\mathrm{TC}(\mathbb{S})_p \xrightarrow{\mathrm{trf}_0} \mathrm{TC}(\mathbb{S})_p \xrightarrow{\rho_0} \mathrm{TC}(\mathbb{S}[x], \langle x \rangle; 0)_p \xrightarrow{\mathrm{res}_0} \Sigma \mathrm{TC}(\mathbb{S})_p \quad (11.4)$$

of $\mathrm{TC}(\mathbb{S})_p$ -modules. In the commutative square

$$\begin{array}{ccc} \mathrm{TC}(\mathbb{S})_p & \xrightarrow{\mathrm{trf}_0} & \mathrm{TC}(\mathbb{S})_p \\ \beta \downarrow & & \downarrow \beta \\ \mathrm{THH}(\mathbb{S})_p & \xrightarrow{\mathrm{trf}_0} & \mathrm{THH}(\mathbb{S})_p \end{array}$$

with β as in (6.1), both maps $\pi_0(\beta) : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ are surjective because $\mathbb{S} \rightarrow \mathrm{TC}(\mathbb{S})_p \rightarrow \mathrm{THH}(\mathbb{S})_p$ is a p -equivalence, and injective because π_0 of the homotopy fiber vanishes; cf. [20, Theorem 5.17; 54, Corollary 1.21]. Lemma 9.11 shows that $\pi_0(\mathrm{trf}_0)$ is the zero homomorphism for THH , hence also for TC . This implies that trf_0 in (11.4) is null-homotopic as a map of $\mathrm{TC}(\mathbb{S})_p$ -modules, so that

$$\mathrm{TC}(\mathbb{S}[x], \langle x \rangle; 0)_p \simeq \mathrm{TC}(\mathbb{S})_p \wedge S^1_+ = \mathrm{TC}(\mathbb{S})_p[S^1].$$

Next, fix an $i > 0$ with $p \nmid i$, so that $[i]_p = \{i, pi, p^2i, \dots\} = \{p^e i \mid e \geq 0\}$. Then $\mathrm{TC}(X_{[i]_p})_p$ is the equalizer of the two maps in (6.3), which equals the limit of the p -complete diagram in Figure 1. For brevity, let $j = p^e i$. The maps G_j in the diagram are equivalences by [19, Proposition 3.8; 49, Lemma II.4.1]. The maps $\varphi_{p;j}$ in the diagram are equivalences by Lemma 11.6, hence so are the $\varphi_{p;j}^{h\mathbb{T}}$. For $p \nmid i$ the C_p -action on $\mathbb{T}_+ \wedge_{C_i} (\mathbb{S}^{2d})^{\otimes i}$ is free, and its C_p -Tate fixed point spectrum is zero. The cofiber sequence (6.2) for X_j takes the form

$$\Sigma((\mathbb{S}^{2d})^{\otimes j})_{hC_j} \simeq \Sigma(\mathbb{T}_+ \wedge_{C_j} (\mathbb{S}^{2d})^{\otimes j})_{h\mathbb{T}} \xrightarrow{N^h} (\mathbb{T}_+ \wedge_{C_j} (\mathbb{S}^{2d})^{\otimes j})^{h\mathbb{T}} \xrightarrow{R^h} (\mathbb{T}_+ \wedge_{C_j} (\mathbb{S}^{2d})^{\otimes j})^{t\mathbb{T}}.$$

$$\begin{array}{c}
 ((\mathbb{T}_+ \wedge_{C_i} (\mathbb{S}^{2d})^{\otimes i})^{tC_p})_p^{h\mathbb{T}} = 0 \\
 \uparrow \simeq \\
 G_i \\
 \uparrow \simeq \\
 (\mathbb{T}_+ \wedge_{C_i} (\mathbb{S}^{2d})^{\otimes i})_p^{h\mathbb{T}} \xrightarrow{\text{can}_i} (\mathbb{T}_+ \wedge_{C_i} (\mathbb{S}^{2d})^{\otimes i})_p^{t\mathbb{T}} \\
 \searrow \simeq \\
 \varphi_{p;i}^{h\mathbb{T}} \\
 \downarrow \simeq \\
 ((\mathbb{T}_+ \wedge_{C_{p^i}} (\mathbb{S}^{2d})^{\otimes pi})^{tC_p})_p^{h\mathbb{T}} \\
 \uparrow \simeq \\
 G_{p^i} \\
 \uparrow \simeq \\
 (\mathbb{T}_+ \wedge_{C_{p^i}} (\mathbb{S}^{2d})^{\otimes pi})_p^{h\mathbb{T}} \xrightarrow{\text{can}_{p^i}} (\mathbb{T}_+ \wedge_{C_{p^i}} (\mathbb{S}^{2d})^{\otimes pi})_p^{t\mathbb{T}} \\
 \searrow \simeq \\
 \varphi_{p^i;p^i}^{h\mathbb{T}} \\
 \downarrow \simeq \\
 ((\mathbb{T}_+ \wedge_{C_{p^{2i}}} (\mathbb{S}^{2d})^{\otimes p^2i})^{tC_p})_p^{h\mathbb{T}} \\
 \uparrow \simeq \\
 G_{p^{2i}} \\
 \uparrow \simeq \\
 \dots \xrightarrow{\text{can}_{p^{2i}}} (\mathbb{T}_+ \wedge_{C_{p^{2i}}} (\mathbb{S}^{2d})^{\otimes p^2i})_p^{t\mathbb{T}}
 \end{array}$$

FIGURE 1 The limit system for $\text{TC}(X_{[i]_p})_p$

By Lemma 11.8, with $Y = (\mathbb{S}^{2d})^{\otimes j}$, the map $R^h = \text{can}_j$ admits a section. Hence the limit in question agrees with the limit of a sequence

$$\dots \longrightarrow (\mathbb{T}_+ \wedge_{C_{p^{e_i}}} (\mathbb{S}^{2d})^{\otimes p^{e_i}})_p^{h\mathbb{T}} \longrightarrow (\mathbb{T}_+ \wedge_{C_{p^{e-1i}}} (\mathbb{S}^{2d})^{\otimes p^{e-1i}})_p^{h\mathbb{T}} \longrightarrow \dots \longrightarrow 0,$$

where each map admits a section. The limit is therefore the product

$$\text{TC}(X_{[i]_p})_p \simeq \prod_{e \geq 0} (\Sigma((\mathbb{S}^{2d})^{\otimes p^e i})_{hC_{p^{e_i}}})_p$$

of the fibers of these maps.

We have now identified the factors in (11.3). Each $j > 0$ can be written as $p^e i$ for unique $e \geq 0$ and $p \nmid i > 0$, so we can rewrite this equivalence as

$$\text{TC}(X) \simeq_p \text{TC}(X_0) \times \prod_{j>0} \Sigma((\mathbb{S}^{2d})^{\otimes j})_{hC_j}.$$

Since $\Sigma((\mathbb{S}^{2d})^{\otimes j})_{hC_j}$ is $(2dj + 1)$ -connective for each $j > 0$, this product is equivalent to the asserted sum. $\square$

The following lemma extracts a consequence of the Segal–tom Dieck splitting and the proven Segal conjecture, and was used in the proof above.

Lemma 11.8. *Let $j = p^e i$ with $e \geq 0$ and $p \nmid i > 0$, and let Y be a finite based C_j -CW-space. Suppose that C_i acts trivially on $\pi_*(\Sigma^\infty Y \otimes \mathbb{S}/p)$. The p -completed canonical map*

$$R^h = \text{can}_j : (\mathbb{T}_+ \wedge_{C_j} \Sigma^\infty Y)_p^{h\mathbb{T}} \longrightarrow (\mathbb{T}_+ \wedge_{C_j} \Sigma^\infty Y)_p^{t\mathbb{T}}$$

admits a section.

Proof. Using the Wirthmüller equivalence [40, Theorem II.6.2; 67, Theorem 2.1], the cofiber sequence (6.2) for $\mathbb{T}_+ \wedge_{C_j} \Sigma^\infty Y$ can be rewritten in the form

$$\Sigma(\Sigma^\infty Y)_{hC_j} \xrightarrow{N^h} \Sigma(\Sigma^\infty Y)^{hC_j} \xrightarrow{R^h} \Sigma(\Sigma^\infty Y)^{tC_j}.$$

Hence it suffices to prove that the homotopy norm map

$$N^h : (\Sigma^\infty Y)_{hC_j} \longrightarrow (\Sigma^\infty Y)^{hC_j}$$

admits a retraction after p -completion. For this, we momentarily work with genuinely equivariant spectra. The Segal–tom Dieck splitting [40, Chapter V; 64; 65, Satz 2] for the based C_{p^e} -CW-space Y is an equivalence

$$\bigvee_{0 \leq d \leq e} (\Sigma^\infty Y^{C_{p^d}})_{hC_{p^{e-d}}} \xrightarrow{\cong} (\Sigma^\infty Y)^{C_{p^e}}.$$

Its $d = 0$ component is given by the norm map

$$N : (\Sigma^\infty Y)_{hC_{p^e}} \longrightarrow (\Sigma^\infty Y)^{C_{p^e}},$$

whose composite with the comparison map $\Gamma_e : (\Sigma^\infty Y)^{C_{p^e}} \rightarrow (\Sigma^\infty Y)^{hC_{p^e}}$ is N^h for C_{p^e} . A choice of retraction r_0 to the $d = 0$ component of the Segal–tom Dieck splitting then produces the central part of the following homotopy commutative diagram.

$$\begin{array}{ccccc}
 & & i \cdot \text{id} & & \\
 & & \cong_p & & \\
 & \swarrow & & \searrow & \\
 (\Sigma^\infty Y)_{hC_j} & \xrightarrow[\cong_p]{\text{trf}} & (\Sigma^\infty Y)_{hC_{p^e}} & \xrightarrow{\text{id}} & (\Sigma^\infty Y)^{hC_{p^e}} & \xrightarrow[\cong_p]{\text{pr}} & (\Sigma^\infty Y)^{tC_j} \\
 N^h \downarrow & & N^h \downarrow & \searrow N & \uparrow r_0 & \\
 (\Sigma^\infty Y)^{hC_j} & \xrightarrow[\cong_p]{\text{res}} & (\Sigma^\infty Y)^{hC_{p^e}} & \xleftarrow[\cong_p]{\Gamma_e} & (\Sigma^\infty Y)^{C_{p^e}} &
 \end{array}$$

Here pr , res and trf are the projection, restriction and transfer maps associated to the inclusion $C_{p^e} \subset C_j$. The first two are p -equivalences, since homotopy fixed point and homotopy orbit spectral sequence arguments using the hypothesis that C_i acts trivially on $\pi_*(\Sigma^\infty Y \otimes \mathbb{S}/p)$ show that the C_{p^e} -maps $(\Sigma^\infty Y)^{hC_i} \rightarrow \Sigma^\infty Y \rightarrow (\Sigma^\infty Y)_{hC_i}$ are p -equivalences. The composite map $\text{pr} \circ \text{trf}$ is multiplication by the index i of C_{p^e} in C_j , hence it (and trf) is also a p -equivalence. Finally, the comparison map Γ_e is a p -equivalence by the proven Segal conjecture [24, 53] for C_{p^e} , using the finiteness of Y . Taken together, this means that the composite

$$(\text{pr} \circ \text{trf})^{-1} \circ \text{pr} \circ r_0 \circ (\Gamma_e)^{-1} \circ \text{res}$$

(in the homotopy category, with all maps implicitly p -completed) provides a left inverse to the left-hand homotopy norm map. $\square$

APPENDIX A: COMPARISON WITH POINT-SET LEVEL THOM SPECTRA

In [61, §2], the second and third authors consider the topologically enriched symmetric monoidal category $\mathcal{W}$ given by Quillen's localization construction applied to the category $\mathcal{V}^{\text{iso}}$ of Euclidean vector spaces $\mathbb{R}^n$, $n \geq 0$, and linear isometric isomorphisms. The classifying space of $\mathcal{W}$ is $B\mathcal{W} \simeq \mathbb{Z} \times BO$ and there is a strong symmetric monoidal functor $\mathcal{V}^{\text{iso}} \rightarrow \mathcal{W}$. Its induced map of classifying spaces $B\mathcal{V}^{\text{iso}} \rightarrow B\mathcal{W}$ is an $\mathbb{E}_\infty$ -map exhibiting $B\mathcal{W}$ as the group completion of $B\mathcal{V}^{\text{iso}} \simeq \coprod_{n \geq 0} BO(n)$. The continuous functor

$$F_-(S^-) : \mathcal{W}^{\text{op}} \rightarrow \text{Sp}^O \quad (\text{A.1})$$

to orthogonal spectra from [61, Lemma 4.3] extends the functor $(\mathcal{V}^{\text{iso}})^{\text{op}} \rightarrow \text{Sp}^O$ sending V to $\Sigma^\infty S^V$. Passing to underlying ∞ -categories, the latter functor co-restricts to an $\mathbb{E}_\infty$ -map $\coprod_{n \geq 0} BO(n) \rightarrow \text{Pic}_\mathbb{S}$, and $F_-(S^-) : \mathcal{W}^{\text{op}} \rightarrow \text{Sp}^O$ induces an $\mathbb{E}_\infty$ J -map $J : B\mathcal{W}^{\text{op}} \rightarrow \text{Pic}_\mathbb{S}$, the extension of the former $\mathbb{E}_\infty$ -map over the group completion $\coprod_{n \geq 0} BO(n) \rightarrow \mathbb{Z} \times BO$.

Definition A.1.

- (i) We let $J_! : S_{/B\mathcal{W}^{\text{op}}} \rightarrow S_{/\text{Pic}_\mathbb{S}}$ be the symmetric monoidal functor obtained from applying Corollary 2.3 to $y_{\text{Pic}_\mathbb{S}} \circ J : B\mathcal{W}^{\text{op}} \rightarrow S_{/\text{Pic}_\mathbb{S}}$.
- (ii) We let $\text{Th}_\mathcal{W} : S_{/B\mathcal{W}^{\text{op}}} \rightarrow \text{Sp}$ be the symmetric monoidal functor obtained from applying Corollary 2.3 to $\iota \circ J : B\mathcal{W}^{\text{op}} \rightarrow \text{Sp}$ (with $\iota : \text{Pic}_\mathbb{S} \rightarrow \text{Sp}$ the inclusion).

Under the symmetric monoidal equivalences $\text{Fun}(P^{\text{op}}, S) \simeq S_{/P}$ for $P = B\mathcal{W}^{\text{op}}$ and $P = \text{Pic}_\mathbb{S}$, the functor $J_!$ can be identified with the left Kan extension along J^{op} .

Corollary A.2. *The functors $\text{Th}_\mathbb{S} \circ J_!$ and $\text{Th}_\mathcal{W}$ are equivalent as colimit-preserving symmetric monoidal functors $S_{/B\mathcal{W}^{\text{op}}} \rightarrow \text{Sp}$.*

Proof. By construction, we have $\text{Th}_\mathbb{S} \circ J_! \circ y_{B\mathcal{W}^{\text{op}}} \simeq \text{Th}_\mathbb{S} \circ y_{\text{Pic}_\mathbb{S}} \circ J \simeq \iota \circ J$ and $\text{Th}_\mathcal{W} \circ y_{B\mathcal{W}^{\text{op}}} \simeq \iota \circ J$. The claim follows from Proposition 2.2. $\square$

The 1-categorical functor category $S^\mathcal{W} = \text{Fun}(\mathcal{W}, S)$ admits a $\mathcal{W}$ -model structure making it Quillen equivalent to the model category $S/(\mathbb{Z} \times BO)$, that is, the 1-categorical comma category with the overcategory model structure (see [61, §3 and §6]). Via the compatibility of the model categorical overcategory with the slice category construction [45, Lemma 6.1.3.13], $S^\mathcal{W}$ is therefore a symmetric monoidal model category representing $\text{Fun}(B\mathcal{W}, S) \simeq S_{/B\mathcal{W}^{\text{op}}}$. By construction, the strong symmetric monoidal left Quillen functor $\mathbb{S}^\mathcal{W} : S^\mathcal{W} \rightarrow \text{Sp}^O$ of [61, §4.2] extends the strong symmetric monoidal functor $F_-(S^-) : \mathcal{W}^{\text{op}} \rightarrow \text{Sp}^O$ of (A.1). With this observation, Definition A.1 implies:

Corollary A.3. *The symmetric monoidal functor of ∞ -categories induced by $\mathbb{S}^\mathcal{W}$ is equivalent to $\text{Th}_\mathcal{W}$.* $\square$

In [61], $\mathbb{S}^\mathcal{W}$ was constructed as one model for a graded Thom spectrum functor for virtual real vector bundles. The combination of Corollaries A.2 and A.3 implies that this point-set level construction is symmetric monoidally equivalent to the ∞ -categorical one.

An analogous discussion applies to the $\mathcal{J}$ -spaces considered in [60, §4]: Passing to classifying spaces, the category $\mathcal{J}$ of [60] models the group completion QS^0 of $\prod_{n \geq 0} B\Sigma_n$. There is a strong symmetric monoidal functor $F_-(S^-) : \mathcal{J}^{\text{op}} \rightarrow \text{Sp}^{\Sigma}$ to symmetric spectra that extends to a strong symmetric monoidal left Quillen functor $\mathbb{S}^{\mathcal{J}} : \mathcal{S}^{\mathcal{J}} = \text{Fun}(\mathcal{J}, \mathcal{S}) \rightarrow \text{Sp}^{\Sigma}$; see [60, Lemma 4.22 and (4.5)]. The map of classifying spaces $B\mathcal{J}^{\text{op}} \rightarrow \text{Pic}_{\mathbb{S}}$ induced by $F_-(S^-) : \mathcal{J}^{\text{op}} \rightarrow \text{Sp}^{\Sigma}$ factors as the composite of the map $\rho : B\mathcal{J}^{\text{op}} \rightarrow B\mathcal{W}^{\text{op}}$ induced by the inclusions $\Sigma_n \rightarrow O(n)$ (or the unit $\mathbb{S} \rightarrow \text{ko}$) and the above $J : B\mathcal{W}^{\text{op}} \rightarrow \text{Pic}_{\mathbb{S}}$.

Arguing as in the case of Corollaries A.2 and A.3, we obtain:

Corollary A.4. *The symmetric monoidal functor $\text{Th}_{\mathcal{J}} : \mathcal{S}_{/B\mathcal{J}^{\text{op}}} \rightarrow \text{Sp}$ of ∞ -categories induced by $\mathbb{S}^{\mathcal{J}}$ is equivalent to the composite*

$$\mathcal{S}_{/B\mathcal{J}^{\text{op}}} \xrightarrow{(J \circ \rho)_!} \mathcal{S}_{/\text{Pic}_{\mathbb{S}}} \xrightarrow{\text{Th}_{\mathbb{S}}} \text{Sp}.$$

Applying Proposition 2.6 and Corollary 2.7, it follows like in Definition 2.9 that the functors considered above participate in adjunctions

$$(\text{Th}_{\mathcal{W}}, \Omega^{\mathcal{W}}), \quad (\text{Th}_{\mathcal{J}}, \Omega^{\mathcal{J}}), \quad (J_!, J^*) \text{ and } ((J \circ \rho)_!, (J \circ \rho)^*) \quad (\text{A.2})$$

in each of which the right adjoint is a lax symmetric monoidal functor. Moreover, using [47, Remark 7.3.2.13] as in the case of (2.6), each of these adjunctions induces an adjunction on the associated categories of $\mathbb{E}_k$ -algebras for $1 \leq k \leq \infty$.

Proof of Proposition 2.11. For the model categorical counterpart of $\Omega^{\mathcal{J}}$, the statement of Proposition 2.11 is verified in [60, Proposition 4.24]. Since $J \circ \rho : QS^0 \rightarrow \text{Pic}_{\mathbb{S}}$ is an isomorphism on π_* for $* = 0, 1$, the equivalence $(J \circ \rho)^* \circ \Omega^{\mathbb{S}} \simeq \Omega^{\mathcal{J}}$ resulting from the above discussion induces a bijection between the path components of the relevant homotopy fibers, which respects the multiplication and the $\{\pm 1\}$ -action. $\square$

APPENDIX B: COMPARISON WITH POINT-SET LEVEL LOG THH

The purpose of this appendix is to compare the notion of log THH introduced in Section 5 with that of [56, Definition 4.6; 57, Definition 4.1] when M is repetitive in the sense of [56, Definition 6.4].

One building block for the prelog ring spectra of the cited paper are commutative $\mathcal{J}$ -space monoids [60, §4]. When equipped with one of the positive $\mathcal{J}$ -model structures, the category of commutative $\mathcal{J}$ -space monoids $\mathcal{CS}^{\mathcal{J}}$ models the ∞ -category $\text{CAlg}(\mathcal{S}_{/B\mathcal{J}^{\text{op}}})$. It also admits a group completion model structure where the fibrant replacement is a model for the group completion [59, Theorem 1.6]. This is compatible with the group completions discussed in Section 2.13:

Lemma B.1. *If $\gamma : M \rightarrow M^{\text{gp}}$ is a group completion of commutative $\mathcal{J}$ -space monoids, then $(J \circ \rho)_!$ sends the map in $\text{CAlg}(\mathcal{S}_{/B\mathcal{J}^{\text{op}}})$ represented by γ to a group completion in $\text{CAlg}(\mathcal{S}_{/\text{Pic}_{\mathbb{S}}})$.*

Proof. Since the $\mathbb{E}_{\infty}$ -spaces $B\mathcal{J}^{\text{op}}$ and $\text{Pic}_{\mathbb{S}}$ are grouplike, group completions of $\mathbb{E}_{\infty}$ -algebras in the slice categories $\mathcal{S}_{/B\mathcal{J}^{\text{op}}}$ and $\mathcal{S}_{/\text{Pic}_{\mathbb{S}}}$ are detected on their underlying $\mathbb{E}_{\infty}$ -spaces (compare Lemma 3.4). $\square$

A $\mathcal{J}$ -space prelog ring spectrum (as considered in [56–58, 60]) consists of an $M \in CS^{\mathcal{J}}$, a commutative symmetric ring spectrum $A \in CSp^{\Sigma}$, and a structure map $\alpha : M \rightarrow \Omega^{\mathcal{J}}(A)$ in $CS^{\mathcal{J}}$ or, equivalently, a structure map $\tilde{\alpha} : \mathbb{S}^{\mathcal{J}}[M] \rightarrow A$ in CSp^{Σ} . The commutative $\mathcal{J}$ -space monoid M represents an object in the ∞ -category $\mathrm{CAlg}(\mathrm{Fun}(B\mathcal{J}, S)) \simeq \mathrm{CAlg}(S_{/B\mathcal{J}^{\mathrm{op}}})$. We write $\xi(M)$ for its image under $(J \circ \rho)_! : \mathrm{CAlg}(S_{/B\mathcal{J}^{\mathrm{op}}}) \rightarrow \mathrm{CAlg}(S_{/\mathrm{Pic}_S})$ and view $\xi(M)$ as a graded object with the canonical π_0 -grading of Definition 2.16. As in [56–58, 60], we write $M_{h\mathcal{J}} = \mathrm{hocolim}_{\mathcal{J}}(M)$ for the homotopy colimit and note that this $\mathbb{E}_{\infty}$ -space represents the total space of $\xi(M)$. So in this case, the canonical π_0 -grading is a $\pi_0(M_{h\mathcal{J}})$ -grading. By Corollary A.4, $\tilde{\alpha}$ induces a structure map $\tilde{\alpha} : \mathrm{Th}_S(\xi(M)) \rightarrow A$ that allows us to view $(A, \xi(M), \tilde{\alpha})$ as a prelog $\mathbb{E}_{\infty}$ -ring. An analogous discussion applies to the $\mathcal{W}$ -space prelog ring spectra considered in [61].

A commutative $\mathcal{J}$ -space monoid M is *repetitive* in the sense of [56, Definition 6.4] if M is not concentrated in $\mathcal{J}$ -space degree 0 and if the group completion map induces an equivalence $M \rightarrow (M^{\mathrm{gp}})_{\geq 0}$. If M is repetitive, $M_{h\mathcal{J}} \rightarrow \pi_0(M_{h\mathcal{J}}) \times_{\pi_0(M_{h\mathcal{J}})^{\mathrm{gp}}} M_{h\mathcal{J}}^{\mathrm{gp}}$ is an equivalence, meaning that M is π_0 -replete in the sense of Definition 3.12.

Proposition B.2. *Let M be a repetitive commutative $\mathcal{J}$ -space monoid that is flat as a $\mathcal{J}$ -space. Then $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)]$ represents $\mathrm{THH}(\mathrm{Th}_S(\xi(M)), \xi(M))$ in the ∞ -category $\mathrm{CAlg}(\mathrm{Sp})$.*

Proof. Let $B^{\mathrm{rep}}(M)$ be the replete bar construction of M in commutative $\mathcal{J}$ -space monoids. The observation before the proposition and Lemma B.1 imply that $(J \circ \rho)_! : \mathrm{CAlg}(S_{/B\mathcal{J}^{\mathrm{op}}}) \rightarrow \mathrm{CAlg}(S_{/\mathrm{Pic}_S})$ sends the object represented by $B^{\mathrm{rep}}(M)$ to $B^{\pi_0\text{-rep}}((J \circ \rho)_!(M))$. With this, the claim follows from Corollary A.4 and Proposition 7.2. $\square$

Corollary B.3. *Let (A, M, α) be a $\mathcal{J}$ -space prelog ring spectrum with M repetitive. Then $\log \mathrm{THH}$ spectrum $\mathrm{THH}(A, M, \alpha)$ considered in [56, Definition 4.6; 57, Definition 4.1] represents $\log \mathrm{THH}$ of the associated prelog $\mathbb{E}_{\infty}$ -ring as an $\mathbb{E}_{\infty}$ -ring.*

B.4 | Comparison with direct image log structures

Let A be an $\mathbb{E}_{\infty}$ -ring, let $d > 0$, and let $x \in \pi_{2d}(A)$ be a homotopy class such that the localization map $j : A \rightarrow A[1/x]$ exhibits A as the connective cover of $A[1/x]$. In this situation, [58, Construction 4.2] provides a $\mathcal{J}$ -space prelog ring spectrum $(A, D(x), \alpha^x)$ that we can view as a prelog $\mathbb{E}_{\infty}$ -ring $(A, \xi(D(x)), \tilde{\alpha}^x)$ by the above construction. As in Construction 8.11, the adjoint of the composite $\mathbb{S}^{\mathcal{J}}[D(x)] \simeq \mathrm{Th}_S(\xi(D(x))) \rightarrow A \rightarrow A[1/x]$ factors through $\iota : \mathrm{GL}_1^{\mathbb{S}}(A[1/x]) \rightarrow \Omega^{\mathbb{S}}(A[1/x])$ and therefore induces a canonical map $(A, \xi(D(x)), \tilde{\alpha}^x) \rightarrow (A, j_* \xi^{\mathrm{GL}_1}, j_* \tilde{\iota})$ to the direct image log $\mathbb{E}_{\infty}$ -ring determined by j (compare Example 8.6).

Lemma B.5. *The canonical map $(A, \xi(D(x)), \tilde{\alpha}^x) \rightarrow (A, j_* \mathrm{GL}_1^{\mathbb{S}}(A[1/x]), \alpha)$ is an $\mathbb{E}_{\infty}$ -logification in the sense of Definition 8.10.*

Proof. This is analogous to [58, Lemma 4.7]. $\square$

Suppose in addition that $\pi_{i(2d+2)+2d-1}(A) \cong 0$ for all $i \geq 1$. Then by Remark 4.9, we get a prelog $\mathbb{E}_2$ -ring $(A, \xi_{2d}, \tilde{\alpha})$ determined by x . Construction 8.11 and Lemma 8.12 show that in this situation the canonical map $(A, \xi_{2d}, \tilde{\alpha}) \rightarrow (A, j_* \xi^{\mathrm{GL}_1}, j_* \tilde{\iota})$ is an $\mathbb{E}_2$ -logification. Combining this with the last lemma and applying Theorem 8.15, we obtain

Corollary B.6. *The maps $(A, \xi_{2d}, \bar{\alpha}) \rightarrow (A, f_* \xi^{\mathrm{GL}_1}, j_* \bar{i}) \leftarrow (A, \xi(D(x)), \bar{\alpha}^x)$ induce equivalences when applying log THH.*

In the corollary, log THH of each outer term is formed with respect to the canonical π_0 -grading, which is a grading by the (additive) monoid $\mathbb{Z}_{\geq 0}$ in these two cases. The middle term is graded by $\pi_0(f_* \mathrm{GL}_1^{\mathbb{S}}(A[1/x])) \cong (\pi_0(A)^{\times} / \{\pm 1\})[x]$. Our connective cover assumptions imply that the group completion of this monoid is $(\pi_0(A)^{\times} / \{\pm 1\})[x^{\pm 1}]$. Hence the map from $\mathbb{Z}_{\geq 0}$ to $\pi_0(f_* \mathrm{GL}_1^{\mathbb{S}}(A[1/x]))$ is exact. Lemma 5.19 implies that we can change the implicit grading of the outer terms to a $(\pi_0(A)^{\times} / \{\pm 1\})[x]$ -grading without altering log THH.

Using Corollary B.3, we deduce that log THH of the $\mathbb{E}_2$ -prelog rings $(ku, \langle u \rangle)$, $(ku_{(p)}, \langle u \rangle)$, and $(\ell, \langle v_1 \rangle)$ from Definition 4.10 is equivalent to that defined and studied in [56, 57].

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