

TOPOLOGICAL CYCLIC HOMOLOGY OF THE SECOND TRUNCATED BROWN–PETERSON SPECTRUM

JOHN ROGNES

JOINT WITH ANGELINI-KNOLL, AUSONI, CULVER AND HÖNING

ABSTRACT. We calculate the mod (p, v_1, v_2) homotopy $V(2)_*TC(BP\langle 2 \rangle)$ of the topological cyclic homology of the truncated Brown–Peterson spectrum $BP\langle 2 \rangle$, at all primes $p \geq 7$, and show that it is a finitely generated and free $\mathbb{F}_p[v_3]$ -module on $12p + 4$ generators in explicit degrees within the range $-1 \leq * \leq 2p^3 + 2p^2 + 2p - 3$. Our computation is the first that exhibits chromatic redshift from pure v_2 -periodicity to pure v_3 -periodicity in a precise quantitative manner.

1. STATEMENT OF RESULTS

This concerns joint work with Gabriel Angelini-Knoll, Christian Ausoni, Dominic Leon Culver and Eva Höning.

Definition 1.1. Let p be a prime. The/a second truncated Brown–Peterson spectrum $BP\langle 2 \rangle$ is a complex oriented p -local ring spectrum such that

$$\mathbb{Z}_{(p)}[v_1, v_2] \subset \pi_*BP \rightarrow \pi_*MU_{(p)} \rightarrow \pi_*BP\langle 2 \rangle$$

is an isomorphism. By Basterra–Mandell and Hahn–Wilson, we may assume $BP\langle 2 \rangle$ is an $\mathbb{E}_3$ BP -algebra spectrum.

Suppose $p \geq 7$. The Smith–Toda spectrum $V(2) = S/p/v_1/v_2$ then exists as a ring spectrum and admits a v_3 self-map. Let $T(3) = v_3^{-1}V(2)$ be its mapping telescope.

Theorem 1.2 (AK–A–C–H–R). *$V(2)_*TC(BP\langle 2 \rangle)$ is a free $\mathbb{F}_p[v_3]$ -module on $12p + 4$ explicit generators, in degrees $-1 \leq * \leq 2p^3 + 2p^2 + 2p - 3$.*

Corollary 1.3. *$T(3)_*TC(BP\langle 2 \rangle) \cong T(3)_*K(BP\langle 2 \rangle) \cong T(3)_*K(BP\langle 2 \rangle_p)$ is a free $\mathbb{F}_p[v_3^{\pm 1}]$ -module on $12p + 4$ explicit generators.*

This exhibits redshift from height 2 (à la elliptic cohomology) to height 3 in a precise, quantitative, sense.

Assuming that Bhatt–Morrow–Scholze quasi-syntomic descent behaves the same way for $BP\langle 2 \rangle$ as for chromatically quasi-syntomic $\mathbb{E}_\infty$ ring spectra, in the sense of Hahn–Raksit–Wilson, it also calculates the syntomic cohomology of $BP\langle 2 \rangle$, and suggests that the stalk of TC , at a “maximal” perfectoid or semi-perfectoid quasi-regular (SPQR) cover of $BP\langle 2 \rangle$, is a connective cover of the Lubin–Tate theory E_3 .

2. TRACE METHODS

Bökstedt–Hsiang–Madsen, Hesselholt–Madsen, Bökstedt–Bruner–Lunøe–Nielsen–R.:

$$\begin{array}{ccccccc}
 & & & & THH(B)^{tS^1} & & \\
 & & & & \nearrow R^h & \times & \downarrow G \\
 & TC(B) & \xrightarrow{\pi} & THH(B)^{hS^1} & \xrightarrow{\hat{\Gamma}_1^{hS^1}} & (THH(B)^{tC_p})^{hS^1} & \\
 & \nearrow \text{trc} & & \downarrow F & & \downarrow F & \\
 S \xrightarrow{\iota} K(B) & \xrightarrow{\text{tr}} & THH(B) & \xrightarrow{\hat{\Gamma}_1} & THH(B)^{tC_p} & &
 \end{array}$$

Nikolaus–Scholze:

$$\begin{array}{ccccccc}
 & & & & TP(B) & & \\
 & & & & \nearrow \text{can} & \times & \downarrow \simeq \\
 & TC(B) & \longrightarrow & TC^-(B) & \xrightarrow{\varphi^{h\mathbb{T}}} & (THH(B)^{tC_p})^{h\mathbb{T}} & \\
 & \nearrow \text{trc} & & \downarrow & & \downarrow & \\
 S \xrightarrow{\iota} K(B) & \xrightarrow{\text{tr}} & THH(B) & \xrightarrow{\varphi} & THH(B)^{tC_p} & &
 \end{array}$$

Bökstedt–Madsen, R., Hesselholt–Madsen, Ausoni–R. calculational strategy:

- (1) Recall $V(2)_*THH(BP\langle 2 \rangle) = \Lambda(\lambda_1, \lambda_2, \lambda_3) \otimes \mathbb{F}_p[\mu]$.
- (2) Show that $\varphi = \varphi_p$ is localization away from $\mu = \mu_3$.
- (3) Deduce that π is a homotopy equalizer (in high degrees).
- (4) Calculate homotopy and Tate fixed point spectral sequences

$$\begin{aligned}
 E_{*,*}^2 &= \mathbb{F}_p[t] \otimes V(2)_*THH(BP\langle 2 \rangle) \Longrightarrow V(2)_*TC^-(BP\langle 2 \rangle) \\
 \mu^{-1}E_{*,*}^2 &= \mathbb{F}_p[t] \otimes \mu^{-1}V(2)_*THH(BP\langle 2 \rangle) \Longrightarrow V(2)_*(THH(BP\langle 2 \rangle)^{tC_p})^{h\mathbb{T}} \\
 \hat{E}_{*,*}^2 &= \mathbb{F}_p[t^{\pm 1}] \otimes V(2)_*THH(BP\langle 2 \rangle) \Longrightarrow V(2)_*TP(BP\langle 2 \rangle)
 \end{aligned}$$

by induction over $C_p \subset \cdots \subset C_{p^n} \subset \cdots \subset \mathbb{T}$.

- (5) Identify

$$G: V(2)_*TP(BP\langle 2 \rangle) \xrightarrow{\cong} V(2)_*(THH(BP\langle 2 \rangle)^{tC_p})^{h\mathbb{T}}$$

in these terms.

- (6) Find equalizer and coequalizer of $G \circ R^h$ and $\hat{\Gamma}_1^{hS^1}$, i.e., of can and $\varphi^{h\mathbb{T}}$, and recognize their extension $V(2)_*TC(BP\langle 2 \rangle)$.

This is fully carried out in [AK-A-C-H-R]. Along the way we determine the $\mathbb{F}_p[v_3]$ -modules $V(2)_*TP(BP\langle 2 \rangle)$ and $V(2)_*TC^-(BP\langle 2 \rangle)$, which play the roles of the prismatic cohomology of $BP\langle 2 \rangle$ and a step in its Nygaard filtration, respectively. These contain a large amount of v_3 -power torsion, but the syntomic cohomology “eigenspace” is v_3 -torsion free, as stated above.

A key technical point is to show that classes $\lambda_1, \lambda_2, \lambda_3 \in V(2)_*TC^-(BP\langle 2 \rangle)$ (with $F(\lambda_i) = \lambda_i$) are actually equalized by $G \circ R^h$ and $\hat{\Gamma}_1^{hS^1}$, not just up to a

unit or modulo filtrations in the Tate spectral sequence. To achieve this, we first construct classes

$$\lambda_1^K, \lambda_2^K, \lambda_3^K \in V(2)_*K(BP\langle 2 \rangle)$$

with $\text{tr}(\lambda_i^K) = \lambda_i$, and let $\lambda_i = \pi \text{trc}(\lambda_i^K)$.

3. STEP (1)

Note that $V(2) \wedge BP\langle 2 \rangle \simeq H = H\mathbb{F}_p$. We obtain

$$\begin{aligned} H_*V(2) &= \Lambda(\tau_0, \tau_1, \tau_2) \\ H_*BP\langle 2 \rangle &= \Lambda(\bar{\tau}_i \mid i \geq 3) \otimes \mathbb{F}_p[\bar{\xi}_i \mid i \geq 1] \end{aligned}$$

inside $A_* = H_*H$ (the dual Steenrod algebra). Hence

$$\begin{aligned} H_*THH(BP\langle 2 \rangle) &\cong H_*BP\langle 2 \rangle \otimes \Lambda(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes \mathbb{F}_p[\sigma\bar{\tau}_3] \\ V(2)_*THH(BP\langle 2 \rangle) &\cong \Lambda(\lambda_1, \lambda_2, \lambda_3) \otimes \mathbb{F}_p[\mu_3] \end{aligned}$$

where the σ -operator comes from the circle action on THH . The Hurewicz map $h_2: V(2) \rightarrow H$ sends λ_i to $\sigma\bar{\xi}_i$ and $\mu = \mu_3$ to $\sigma\bar{\tau}_3$.

4. POWER OPERATIONS

For $\mathbb{E}_2$ ring spectra R such as S , or $K(B)$, $TC(B)$ and $THH(B)$ for $\mathbb{E}_3$ ring spectra B , we use braided extended powers and maps

$$\begin{aligned} \bar{\eta}_0: \Sigma^{2pk-1}DV(0) &\longrightarrow Br_p S^{2k-1} \\ \bar{\eta}_1: \Sigma^{2pk-1}DV(1) &\longrightarrow Br_p \Sigma^{2k-1}DV(0) \end{aligned}$$

to define homotopy power operations

$$\begin{aligned} P^k: \pi_{2k-1}(R) &\longrightarrow V(0)_{2pk-1}(R) \\ P^k: V(0)_{2k-1}(R) &\longrightarrow V(1)_{2pk-1}(R) \end{aligned}$$

that are compatible with Cohen's homology operations $\xi_1 = Q^k: H_{2k-1}(R) \rightarrow H_{2pk-1}(R)$, and satisfy a Cartan formula (under additional commutativity hypotheses).

Bökstedt exhibited a class $\lambda \in K_{2p-1}(\mathbb{Z})$ with $\text{tr}(\lambda)$ generating $THH_{2p-1}(\mathbb{Z}) = \mathbb{Z}/p$. The map $K(BP\langle 2 \rangle) \rightarrow K(\mathbb{Z}_{(p)})$ is $2p-1$ -connected, so we can lift λ to

$$\lambda_1^K \in K_{2p-1}(BP\langle 2 \rangle).$$

We then define

$$\begin{aligned} \lambda_2^K &= P^p(\lambda_1^K) \in V(0)_{2p^2-1}K(BP\langle 2 \rangle) \\ \lambda_3^K &= P^{p^2}(\lambda_2^K) \in V(1)_{2p^3-1}K(BP\langle 2 \rangle) \end{aligned}$$

and view these as classes in $V(2)_*K(BP\langle 2 \rangle)$.

Let $\alpha_1 = j_0(v_1)$ generate $\pi_{2p-3}(S) = \mathbb{Z}/p$, and define

$$\begin{aligned} \beta_1^\circ &= P^{p-1}(\alpha_1) \in V(0)_{2p^2-2p-1} \\ \gamma_1^\circ &= P^{p^2-p}(\beta_1^\circ) \in V(1)_{2p^3-2p^2-1}. \end{aligned}$$

Their images in $V(2)_*$ agree with those of the Greek letter elements $\beta'_1 = j_1(v_2)$ and $\gamma'_1 = j_2(v_3)$.

5. STEP (2)

In the $\mathbb{T}$ -homotopy fixed points spectral sequence

$$\begin{aligned} E_{*,*}^2 &= \mathbb{F}_p[t] \otimes \Lambda(\lambda_1, \lambda_2, \lambda_3) \otimes \mathbb{F}_p[\mu] \\ &\implies V(2)_* TC^-(BP\langle 2 \rangle) \end{aligned}$$

with $t \in E_{-2,0}^2$, the classes $\lambda_i = \pi \operatorname{trc}(\lambda_i^K)$ are detected by $\lambda_i \in E_{0,2p^i-1}^\infty$. It is easy to show that $\pi \operatorname{trc} \iota(\alpha_1)$ and $\pi \operatorname{trc} \iota(v_3)$ are detected by $t\lambda_1$ and $t\mu$, respectively. The Cartan formula, and a delicate check of indeterminacies, then let us show that $\pi \operatorname{trc} \iota(\beta'_1)$ and $\pi \operatorname{trc} \iota(\gamma''_1)$ are detected by $t^p\lambda_2$ and $t^{p^2}\lambda_3$, respectively.

$$\begin{aligned} \alpha_1 &\longmapsto \{t\lambda_1\} \\ \beta'_1 &\longmapsto \{t^p\lambda_2\} \\ \gamma''_1 &\longmapsto \{t^{p^2}\lambda_3\} \\ v_3 &\longmapsto \{t\mu\} \end{aligned}$$

Consider the C_p -Tate spectral sequence

$$\begin{aligned} \hat{E}_{*,*}^2(C_p) &= \Lambda(u_1) \otimes \mathbb{F}_p[t] \otimes \Lambda(\lambda_1, \lambda_2, \lambda_3) \otimes \mathbb{F}_p[\mu] \\ &\implies V(2)_* THH(BP\langle 2 \rangle)^{tC_p} \end{aligned}$$

with $u_1 \in \hat{E}_{-1,0}^2$. The composite

$$F \text{ can } \pi \operatorname{trc} \iota: S \longrightarrow THH(B)^{tC_p}$$

factors through B , so $t\lambda_1$, $t^p\lambda_2$, $t^{p^2}\lambda_3$ and $t\mu$ must be boundaries in this spectral sequence. The only possibilities are

$$\begin{aligned} d^{2p}(t^{1-p}) &= t\lambda_1 \\ d^{2p^2}(t^{p-p^2}) &= t^p\lambda_2 \\ d^{2p^3}(t^{p^2-p^3}) &= t^{p^2}\lambda_3 \\ d^{2p^3+1}(u_1 t^{-p^3}) &= t\mu \end{aligned}$$

up to units in $\mathbb{F}_p$, which we suppress. This leaves

$$\hat{E}_{*,*}^\infty(C_p) = \mathbb{F}_p[t^{\pm p^3}] \otimes \Lambda(\lambda_1, \lambda_2, \lambda_3).$$

Hence

$$\varphi: V(2)_* THH(BP\langle 2 \rangle) \longrightarrow V(2)_* THH(BP\langle 2 \rangle)^{tC_p} \cong \Lambda(\lambda_1, \lambda_2, \lambda_3) \otimes \mathbb{F}_p[\mu^{\pm 1}]$$

inverts μ , with μ^{-1} detected by t^{p^3} in the target.

6. STEP (3)

This used to rely on Tsalidis or Bökstedt–Bruner–Lunøe–Nielsen–R., but is now handled by Nikolaus–Scholze.

7. STEP (4)

Let $V(3) = V(2)/v_3$ be the homotopy cofiber.

A posteriori, there is little room for differentials in the v_3 -Bockstein spectral sequence

$$V(3)_*TC(BP\langle 2 \rangle)[v_3] \Longrightarrow V(2)_*TC(BP\langle 2 \rangle).$$

For expositional purposes we might therefore instead calculate directly in $V(3)$ -homotopy, using modified, or “delayed”, homotopy and Tate fixed point spectral sequences. These are defined by viewing multiplication by v_3 as a filtration-shifting self-map, and passing to homotopy cofibers. This is inspired by the approach of Hahn–Raksit–Wilson, who pass to the associated graded for their even filtration, i.e., to syntomic cohomology.

The delayed $\mathbb{T}$ -Tate spectral sequence for $V(3)$ -homotopy

$$\begin{aligned} {}_{\text{del}}\hat{E}_{*,*}^2 &= \hat{E}_{*,*}^2/(t\mu) = \mathbb{F}_p[t^{\pm 1}] \otimes \Lambda(\lambda_1, \lambda_2, \lambda_3) \\ &\Longrightarrow V(3)_*TP(BP\langle 2 \rangle) \end{aligned}$$

exhibits the same even length differentials

$$\begin{aligned} d^{2p}(t^{1-p}) &= t\lambda_1 \\ d^{2p^2}(t^{p-p^2}) &= t^p\lambda_2 \\ d^{2p^3}(t^{p^2-p^3}) &= t^{p^2}\lambda_3 \end{aligned}$$

as the C_p -Tate spectral sequence, and then collapses to

$${}_{\text{del}}\hat{E}_{*,*}^\infty = \mathbb{F}_p[t^{\pm p^3}] \otimes \Lambda(\lambda_1, \lambda_2, \lambda_3).$$

Similarly, the delayed $\mathbb{T}$ -homotopy spectral sequence

$$\begin{aligned} {}_{\text{del}}E_{*,*}^2 &= E_{*,*}^2/(t\mu) = \frac{\mathbb{F}_p[t] \otimes \Lambda(\lambda_1, \lambda_2, \lambda_3) \otimes \mathbb{F}_p[\mu]}{(t\mu)} \\ &\Longrightarrow V(3)_*TC^-(BP\langle 2 \rangle) \end{aligned}$$

has differentials

$$\begin{aligned} d^{2p}(t) &= t^{1+p}\lambda_1 \\ d^{2p^2}(t^p) &= t^{p+p^2}\lambda_2 \\ d^{2p^3}(t^{p^2}) &= t^{p^2+p^3}\lambda_3, \end{aligned}$$

leaving

$$\begin{aligned} {}_{\text{del}}E_{*,*}^\infty &= \mathbb{F}_p[t^{p^3}]\{t^{p^3}\} \otimes \Lambda(\lambda_1, \lambda_2, \lambda_3) \\ &\quad \oplus \Lambda(\lambda_2, \lambda_3)\{t^d\lambda_1 \mid 0 < d < p\} \\ &\quad \oplus \Lambda(\lambda_1, \lambda_3)\{t^{dp}\lambda_2 \mid 0 < d < p\} \\ &\quad \oplus \Lambda(\lambda_1, \lambda_2)\{t^{dp^2}\lambda_3 \mid 0 < d < p\} \\ &\quad \oplus \Lambda(\lambda_1, \lambda_2, \lambda_3) \otimes \mathbb{F}_p[\mu]. \end{aligned}$$

Finally, the delayed μ -inverted $\mathbb{T}$ -homotopy spectral sequence

$$\begin{aligned} \mu_{\text{del}}^{-1}E_{*,*}^2 &= \mu^{-1}E_{*,*}^2/(t\mu) = \Lambda(\lambda_1, \lambda_2, \lambda_3) \otimes \mathbb{F}_p[\mu^{\pm 1}] \\ &\Longrightarrow V(3)_*(THH(BP\langle 2 \rangle)^{tC_p})^{h\mathbb{T}} \end{aligned}$$

collapses immediately, so that

$$\mu_{\text{del}}^{-1}E_{*,*}^{\infty} = \Lambda(\lambda_1, \lambda_2, \lambda_3) \otimes \mathbb{F}_p[\mu^{\pm 1}].$$

8. STEP (5)

Since the λ_i come from λ_i^K they are equalized by $G \circ R^h = \text{can}$ and $\hat{\Gamma}_1^{hS^1} = \varphi^{h\mathbb{T}}$, so $V(3)_*G$ maps λ_i to λ_i and $t^{\pm p^3}$ to $\mu^{\mp 1}$, the latter up to some unit in $\mathbb{F}_p$.

9. STEP (6)

- The summand

$$\mathbb{F}_p[t^{p^3}]\{t^{p^3}\} \otimes \Lambda(\lambda_1, \lambda_2, \lambda_3)$$

maps isomorphically under can and trivially under $\varphi^{h\mathbb{T}}$, hence does not contribute to their equalizer or coequalizer.

- The summands

$$\begin{aligned} & \Lambda(\lambda_2, \lambda_3)\{t^d \lambda_1 \mid 0 < d < p\} \\ & \oplus \Lambda(\lambda_1, \lambda_3)\{t^{dp} \lambda_2 \mid 0 < d < p\} \\ & \oplus \Lambda(\lambda_1, \lambda_2)\{t^{dp^2} \lambda_3 \mid 0 < d < p\} \end{aligned}$$

map trivially under can and under $\varphi^{h\mathbb{T}}$, hence contribute to their equalizer.

- As in Step (5), the summand

$$\Lambda(\lambda_1, \lambda_2, \lambda_3)$$

is equalized by can and $\varphi^{h\mathbb{T}}$ (not just up to a unit in $\mathbb{F}_p$), hence contributes both to their equalizer and coequalizer.

- The summand

$$\Lambda(\lambda_1, \lambda_2, \lambda_3) \otimes \mathbb{F}_p[\mu]\{\mu\}$$

maps trivially under can and isomorphically under $\varphi^{h\mathbb{T}}$, hence does not contribute to their equalizer or coequalizer.

It follows that there is an extension

$$\begin{aligned} 0 \rightarrow \Sigma^{-1}\Lambda(\lambda_1, \lambda_2, \lambda_3) \xrightarrow{\partial} V(3)_*TC(BP\langle 2 \rangle) \xrightarrow{\pi} \Lambda(\lambda_1, \lambda_2, \lambda_3) \oplus \\ \oplus \Lambda(\lambda_2, \lambda_3)\{t^d \lambda_1\} \oplus \Lambda(\lambda_1, \lambda_3)\{t^{dp} \lambda_2\} \oplus \Lambda(\lambda_1, \lambda_2)\{t^{dp^2} \lambda_3\} \rightarrow 0, \end{aligned}$$

where $0 < d < p$ in each case. The total dimension is $8 + 8 + 4(p-1) + 4(p-1) + 4(p-1) = 12p + 4$, the bottom generator is $\partial(1)$ in degree -1 , and the top generator is $\lambda_1 \lambda_2 \lambda_3$ in degree $2p^3 + 2p^2 + 2p - 3$.

With this approach, additional information is needed to prove that the v_3 -Bockstein spectral sequence collapses, but only the primary Bockstein, of degree $2p^3 - 1$, needs to be considered.

The actual argument in [AK-A-C-H-R] calculates in $V(2)$ -homotopy, where the $\mathbb{T}$ -homotopy fixed point and Tate spectral sequences each have infinitely many families of differentials. The associated filtrations suffice to ensure that in $V(2)$ -homotopy the equalizer and coequalizer are both v_3 -torsion free.