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The product on topological Hochschild homology of the integers with mod 4 coefficients

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Abstract

We compute the natural multiplication on mod 4 homotopy for the ring spectrum $T(\mathbb{Z}) = THH(\mathbb{Z})$, and its associated natural module action upon the mod 2 homotopy. © 1999 Elsevier Science B.V. All rights reserved.

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0. Introduction

Bökstedt's *topological Hochschild homology* $THH(A) = T(A)$ of a ring A can be thought of as the spectrum defined by forming the Hochschild complex for A, but replacing A with its Eilenberg–Mac Lane spectrum HA everywhere, and replacing all tensor products with smash products of spectra. Intuitively, this gives a simplicial spectrum $[q] \mapsto T(A)_q = HA \wedge \cdots \wedge HA$ (with $q+1$ factors), and although there are technical problems with such a statement, these were overcome by Bökstedt in [2]. Just like the Hochschild complex is a cyclic complex, $T(A)$ admits a circle action, and can be given the structure of an S^1-spectrum. Its 0-simplices $T(A)_0$ are HA, and the circle action defines an interesting map

$$\lambda : S^1_+ \wedge HA \to T(A),$$

mapping into the 1-skeleton of $T(A)$. When A is commutative, $T(A)$ is a ring spectrum, so $\pi_* T(A)$ is an algebra. The inclusion on 0-simplices $HA \to T(A)$ is a ring map, so

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$T(A)$ is a retract of $HA \wedge T(A)$ and thus equivalent to a wedge of Eilenberg–Mac Lane spectra. We will be interested in the case when A equals the integers. Then $\pi_* T(\mathbb{Z})$ is trivial in positive even degrees, so the algebra structure is trivial. Instead we will study the nontrivial algebra structure on its homotopy with finite coefficients.

Ultimately, we are interested in studying the trace map $\mathrm{tr} : K(\mathbb{Z}) \to T(\mathbb{Z})$ defined in [2] from algebraic K-theory to topological Hochschild homology, and its refinements $\mathrm{tr}_{p^n} : K(\mathbb{Z}) \to T(\mathbb{Z})^{C_{p^n}}$ defined in [4] for the various finite subgroups $C_{p^n} \subset S^1$. These refinements combine to define the *cyclotomic trace map* $\mathrm{trc} : K(\mathbb{Z}) \to TC(\mathbb{Z})$, which was used in [5] to compute the (completed) algebraic K-theory of the p-adic integers for odd p. We can extend these calculations to the case $p=2$, and an essential ingredient of this program is to algebraically structure the mod 2 homotopy of the fixed-point spectra $T(\mathbb{Z})^{C_{2^n}}$ for varying n. This note provides the foundations for these calculations, by establishing the properties summarized in Theorem 3.2 below.

1. Homotopy with finite coefficients

Recall the *Moore spectrum* S^0/n defined by the cofiber sequence

$$S^0 \xrightarrow{n} S^0 \xrightarrow{i} S^0/n \xrightarrow{j} S^1.$$

The mod n homotopy of a spectrum X is defined as $\pi_*(X;\mathbb{Z}/n) = \pi_*(X \wedge S^0/n)$. Given a prime p, we let i_k and j_k denote the maps i and j above in the case $n = p^k$. If X is a ring spectrum and S^0/n has a product map $S^0/n \wedge S^0/n \to S^0/n$, then the mod n homotopy $\pi_*(X;\mathbb{Z}/n)$ naturally inherits a product structure from the composite

$$(X \wedge S^0/n) \wedge (X \wedge S^0/n) \xrightarrow{(23)} (X \wedge X) \wedge (S^0/n \wedge S^0/n) \to X \wedge S^0/n.$$

Here (23) denotes the shuffle map. The mod p Moore spectrum admits such a product map for all odd primes p, but not for $p=2$. This is because the cofiber sequence

$$S^0/2 \wedge S^0 \xrightarrow{1 \wedge 2} S^0/2 \wedge S^0 \xrightarrow{1 \wedge i_1} S^0/2 \wedge S^0/2 \xrightarrow{1 \wedge j_1} S^0/2 \wedge S^1$$

cannot be split, and so there is no (right) unital product map $\mu_1 : S^0/2 \wedge S^0/2 \to S^0/2 \cong S^0/2 \wedge S^0$. To see this note that $Sq^1(a_0) = a_1$ in $H^*(S^0/2;\mathbb{Z}/2) \cong \mathbb{Z}/2\{a_0, a_1\}$, whence by the Cartan formula $Sq^2(a_0 \wedge a_0) = a_1 \wedge a_1 \neq 0$ in $H^*(S^0/2 \wedge S^0/2;\mathbb{Z}/2)$, and so there can be no splitting. It follows easily that twice the identity map on the Moore spectrum $S^0/2$ has the essential factorization

$$2 = i_1 \eta j_1 : S^0/2 \xrightarrow{j_1} S^1 \xrightarrow{\eta} S^0 \xrightarrow{i_1} S^0/2,$$

up to homotopy. Hence, there is no algebra structure on $\pi_*(S^0/2)$, nor a natural algebra structure on $\pi_*(X;\mathbb{Z}/2)$ for general ring spectra X. The observations go back to Barratt [1].

We are concerned with the two-primary homotopy of $T(\mathbb{Z})$ in an equivariant sense. To be precise, we wish to study the mod 2 homotopy of the C_{2^n}-fixed points of $T(\mathbb{Z})$

for all n, where $C_{2^n} \subset S^1$ is the cyclic group of order 2^n. As noted above $T(\mathbb{Z})$ is a wedge of Eilenberg–Mac Lane spectra, so in fact $1 \wedge 2 : T(\mathbb{Z}) \wedge S^0/2 \to T(\mathbb{Z}) \wedge S^0/2$ factors through $\eta : T(\mathbb{Z}) \wedge S^1 \to T(\mathbb{Z})$, which is inessential. Hence, the obstruction to finding a product on $T(\mathbb{Z}) \wedge S^0/2$ vanishes in this case. But $\eta : T(\mathbb{Z}) \wedge S^1 \to T(\mathbb{Z})$ is not equivariantly inessential, in the sense that its restriction even to the C_2-fixed sets is essential. Thus, it is not clear that $T(\mathbb{Z}) \wedge S^0/2$ can be given a product structure in any natural equivariant way.

However, $2\eta \simeq *$, so four times the identity map on $S^0/2$ is inessential. Hence, the cofiber sequence

$$S^0 \wedge S^0/2 \xrightarrow{4 \wedge 1} S^0 \wedge S^0/2 \xrightarrow{i_2 \wedge 1} S^0/4 \wedge S^0/2 \xrightarrow{j_2 \wedge 1} S^1 \wedge S^0/2$$

splits, and there is a pairing of spectra

$$m : S^0/4 \wedge S^0/2 \to S^0/2$$

inducing a (left) unital pairing of mod 4 and mod 2 homotopy. In a similar way, $S^0/4$ admits a multiplication map

$$\mu_2 : S^0/4 \wedge S^0/4 \to S^0/4.$$

Oka proved in [9] that μ_2 can be chosen to be left and right unital and associative up to homotopy, but not homotopy commutative. The commutator satisfies

$$\mu_2 - \mu_2 \circ (12) = i_2 \circ \eta^2 \circ (j_2 \wedge j_2),$$

up to homotopy, where (12) is the twist map. Hence, if X is a ring spectrum such that $\eta^2 : X \wedge S^2 \to X \wedge S^0$ is equivariantly inessential, then $\pi_*(X; \mathbb{Z}/4)$ is equivariantly commutative. This will be the case with $T(\mathbb{Z})$ since η^2 maps to zero in each $\pi_* T(\mathbb{Z})^{C_{2^n}}$, so $\pi_*(T(\mathbb{Z})^{C_{2^n}}; \mathbb{Z}/4)$ is naturally a commutative algebra for all n.

There is also a question of associativity (or transitivity) for the action m, i.e., whether the maps $m(\mu_2 \wedge 1)$ and $m(1 \wedge m)$ are homotopic as maps $S^0/4 \wedge S^0/4 \wedge S^0/2 \to S^0/2$. The associator (their difference) factors up to homotopy as

$$m(\mu_2 \wedge 1) - m(1 \wedge m) = i_1 \circ \alpha \circ (j_2 \wedge j_2 \wedge j_1)$$

for some map $\alpha : S^3 \to S^0$, and the obstruction to associativity is its mod 2 residue class in $\pi_3(S^0)/2$. For ring spectra X that are algebras over $K(\mathbb{Z})$ this obstruction naturally vanishes, since the obstruction (which comes from the stable three-stem) is divisible by two in $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$. Hence, in the case of $T(\mathbb{Z})$ the action of the mod 4 homotopy of $T(\mathbb{Z})^{C_{2^n}}$ upon the mod 2 homotopy is really associative for all n.

The *homotopy Bockstein* $\delta_k = i_k j_k : S^0/2^k \to S^1 \to S^1/2^k$ acts on mod 2^k homotopy. Oka also proved that μ_2 can be chosen so that δ_2 is a derivation, i.e., so that

$$\delta_2 \mu_2 = \mu_2(\delta_2 \wedge 1) + \mu_2(1 \wedge \delta_2),$$

up to homotopy. There are natural maps $\rho : S^0/4 \to S^0/2$ (*coefficient reduction*) and $\varepsilon : S^0/2 \to S^0/4$ (*coefficient extension*) mapping between the following horizontal cofiber sequences:

$$\begin{tikzcd} S^0 \arrow[r, "i_2"] \arrow[d, equal] & S^0/4 \arrow[r, "j_2"] \arrow[d, "\rho"] & S^1 \arrow[d, "2"] \\ S^0 \arrow[r, "i_1"] \arrow[d, "2"] & S^0/2 \arrow[r, "j_1"] \arrow[d, "\varepsilon"] & S^1 \arrow[d, equal] \\ S^0 \arrow[r, "i_2"] & S^0/4 \arrow[r, "j_2"] & S^1. \end{tikzcd} \tag{1.1}$$

Lemma 1.1. *These maps satisfy*

$$\rho\mu_2 = m(1 \wedge \rho) \quad \textit{and} \quad \varepsilon m = \mu_2(1 \wedge \varepsilon),$$

up to homotopy.

Proof. Oka constructed m in [9] so as to satisfy the first formula.

The second formula also follows from [9], by the following argument. Let $\hat{m} : S^1/2 \to S^0/4 \wedge S^0/2$ and $\hat{\mu}_2 : S^1/4 \to S^0/4 \wedge S^0/4$ be the homotopy fiber maps of m and μ_2 respectively. Then $\hat{\mu}_2\varepsilon = (1 \wedge \varepsilon)\hat{m}$ by Lemma 26 of [9], and there is an induced map of cofibers $f : S^0/2 \to S^0/4$ satisfying $fm = \mu_2(1 \wedge \varepsilon)$. We compute

$$f = fm(i \wedge 1) = \mu_2(1 \wedge \varepsilon)(i \wedge 1) = \mu_2(i \wedge 1)(1 \wedge \varepsilon) = 1 \wedge \varepsilon$$

as maps $S^0 \wedge S^0/2 = S^0/2 \to S^0/4 = S^0 \wedge S^0/4$. So $f = \varepsilon$ and the statement follows. □

Clearly, μ_2, m, ρ, ε and the i_k, j_k and δ_k are compatible with the corresponding pairings and operations in homology (with integral or finite coefficients), under the Hurewicz homomorphisms induced by $S^0 \to H\mathbb{Z}$ or $S^0/2^k \to H\mathbb{Z}/2^k$ as appropriate. In the diagram

$$\begin{tikzcd} \pi_*(T(\mathbb{Z}); \mathbb{Z}/4) \arrow[r, "h_2"] \arrow[d, shift left, "\rho"] & H_*(T(\mathbb{Z}); \mathbb{Z}/4) \arrow[d, shift left, "\rho"] \\ \pi_*(T(\mathbb{Z}); \mathbb{Z}/2) \arrow[r, "h_1"] \arrow[u, shift left, "\varepsilon"] & H_*(T(\mathbb{Z}); \mathbb{Z}/2) \arrow[u, shift left, "\varepsilon"] \end{tikzcd} \tag{1.2}$$

the two squares commute, and h_2 and the right-hand side ρ are multiplicative. We shall see below that the horizontal maps are injective.

2. Mod 2 homology of $T(\mathbb{Z})$

We recall Bökstedt's calculation of $H_*(T(\mathbb{Z});\mathbb{Z}/2)$ (spectrum homology) given in [3]. The skeleton filtration on $T(\mathbb{Z})$ induces a spectral sequence

$$E^2_{**} = HH_*(H_*(H\mathbb{Z};\mathbb{Z}/2)) \Rightarrow H_*(T(\mathbb{Z});\mathbb{Z}/2),$$

where $H_*(H\mathbb{Z};\mathbb{Z}/2) = \bar{\mathscr{A}} = \mathbb{Z}/2[\xi_1^2, \chi\xi_2, \chi\xi_3, \dots]$ is contained in the dual of the Steenrod algebra $H_*(H\mathbb{Z}/2;\mathbb{Z}/2) = \mathscr{A} = \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \dots]$. Here the ξ_i are the Milnor generators in degrees $2^i - 1$, and χ is the canonical involution on $\mathscr{A}$ induced by the twist map (see [8]). By the Künneth theorem

$$E^2_{**} \cong \bar{\mathscr{A}} \otimes E(1 \otimes \xi_1^2, 1 \otimes \chi\xi_2, 1 \otimes \chi\xi_3, \dots),$$

where $E(\;)$ denotes the exterior algebra on the listed generators. Letting $[S^1]$ denote the fundamental class in $H_1(S^1_+;\mathbb{Z}/2)$, the circle action map λ induces a homomorphism

$$\sigma : \Sigma\bar{\mathscr{A}} \to H_*(T(\mathbb{Z});\mathbb{Z}/2)$$

given by $\sigma(x) = \lambda_*([S^1] \otimes x)$. Then $\sigma(x)$ maps to $1 \otimes x$ in the spectral sequence above, and so the exterior generators above are $\sigma(\xi_1^2)$, $\sigma(\chi\xi_2)$, $\sigma(\chi\xi_3)$ and so on. In fact σ is a derivation because of the form of the second Hochschild boundary:

$$b_2(1 \otimes x \otimes y) = x\sigma(y) - \sigma(xy) + y\sigma(x).$$

As noted $T(\mathbb{Z})$ is a $H\mathbb{Z}$-algebra, so the spectral sequence consists of (free) $\bar{\mathscr{A}}$-modules. Since E^2 is multiplicatively generated by classes in filtration 1, clearly, $d^r = 0$ for all $r \geq 2$ and so the spectral sequence collapses at the E^2-term.

Let $e_3 = \sigma(\xi_1^2)$ and $e_{2^i} = \sigma(\chi\xi_i)$ for all $i \geq 2$ define classes in $H_*(T(\mathbb{Z});\mathbb{Z}/2)$, indexed by their degrees. Our classes differ from those of Bökstedt only in that we apply the involution χ to eliminate the decomposable indeterminacy present in [3].

Lemma 2.1. *In* $H_*(T(\mathbb{Z});\mathbb{Z}/2)$ (i) $e_3^2 = 0$, (ii) $e_{2^i}^2 = e_{2^{i+1}}$ *for all* $i \geq 2$, (iii) $\beta_1(e_4) = e_3$, *and* (iv) $\hat{Q}^4(e_3) = 0$.

Proof. $T(\mathbb{Z})$ is an E_∞ ring spectrum, so its homology $H_*(T(\mathbb{Z});\mathbb{Z}/2)$ admits multiplicative Dyer–Lashof operations $\hat{Q}^r$ (see [6] or [7]). Likewise, there are additive Dyer–Lashof operations Q^r on $H_*(H\mathbb{Z};\mathbb{Z}/2) = \bar{\mathscr{A}}$, which are recorded in [10, Ch. III, Theorem 2.2]. In particular, $Q^{2^i}(\chi\xi_i) = \chi\xi_{i+1}$. By Bökstedt's lemma [3, Lemma 2.9] these operations are compatible under σ, i.e., $\hat{Q}^r(\sigma(x)) = \sigma(Q^r(x))$ for all x. Hence, we can compute

$$e_3^2 = \hat{Q}^3(e_3) = \sigma(Q^3(\xi_1^2)) = 0$$

by the Cartan formula. Similarly for $i \geq 2$

$$e_{2^i}^2 = \hat{Q}^{2^i}(e_{2^i}) = \sigma(Q^{2^i}(\chi\xi_i)) = \sigma(\chi\xi_{i+1}) = e_{2^{i+1}}$$

as claimed. Next, $\chi\xi_2 = \xi_2 + \xi_1^3$, $\beta_1(\xi_1) = 1$ and $\beta_1(\xi_i) = 0$ for $i \geq 2$. Hence,

$$\beta_1(e_4) = \beta_1(\sigma(\chi\xi_2)) = \sigma(\beta_1\xi_2 + \beta_1\xi_1^3) = \sigma(\xi_1^2) = e_3.$$

Finally,

$$\hat{Q}^4(e_3) = \sigma(Q^4(\xi_1^2)) = \sigma(Q^2(\xi_1)^2) = 0,$$

using the Cartan formula and that σ is a derivation. $\square$

Thus, as algebras and free $\bar{\mathscr{A}}$-modules we have

$$H_*(T(\mathbb{Z}); \mathbb{Z}/2) \cong \bar{\mathscr{A}}[e_3, e_4]/(e_3^2 = 0).$$

Since $T(\mathbb{Z})$ is a wedge of Eilenberg–Mac Lane spectra the integral and mod 2^k Hurewicz homomorphisms

$$h : \pi_* T(\mathbb{Z}) \to H_* T(\mathbb{Z}) \quad \text{and} \quad h_k : \pi_*(T(\mathbb{Z}); \mathbb{Z}/2^k) \to H_*(T(\mathbb{Z}); \mathbb{Z}/2^k)$$

are all injective. In the mod 2 case the image consists of $\bar{\mathscr{A}}$-module generators. There is at most one of these in each degree, so under h_1 we can identify

$$\pi_*(T(\mathbb{Z}); \mathbb{Z}/2) \cong \mathbb{Z}/2[e_3, e_4]/(e_3^2 = 0).$$

The algebra structure here is that inherited by $\pi_*(T(\mathbb{Z}); \mathbb{Z}/2)$ as a subgroup of $H_*(T(\mathbb{Z}); \mathbb{Z}/2)$ which turns out to be closed under multiplication, and thus becomes a subalgebra. This algebraically defined product does not need a priori to be well related to a spectrum level product on $T(\mathbb{Z}) \wedge S^0/2$. We shall prove instead that it is well related to the natural pairings on homotopy induced by μ_2 and m.

Remark 2.2. From Lemma 2.1(iv) the higher Bockstein structure on $\pi_*(T(\mathbb{Z}); \mathbb{Z}/2)$ may be deduced. Combining these results with the corresponding odd primary calculations, Bökstedt obtained the integral answer

$$\pi_* T(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } * = 0, \\ \mathbb{Z}/n & \text{for } * = 2n-1, \\ 0 & \text{otherwise.} \end{cases}$$

3. Homotopy of $T(\mathbb{Z})$ with finite coefficients

We recall the additive structure on $\pi_*(T(\mathbb{Z}); \mathbb{Z}/4)$. The mod 2 homotopy Bockstein δ_1 is a derivation, because it is detected in homology where it corresponds to the mod 2 homology Bockstein (β_1 dual to Sq^1), which is a derivation. From $\beta_1(e_4) = e_3$ we have $\pi_3 T(\mathbb{Z})_2^\wedge = \mathbb{Z}/2$. In general, $\pi_{4k-1} T(\mathbb{Z})_2^\wedge$ is $\mathbb{Z}/2$ for k odd, and cyclic of order 4 or more for k even. So

$$\pi_*(T(\mathbb{Z});\mathbb{Z}/4)\cong\begin{cases}\mathbb{Z}/4 & \text{for } *\equiv 0,7 \bmod 8,\\ \mathbb{Z}/2 & \text{for } *\equiv 3,4 \bmod 8,\\ 0 & \text{otherwise.}\end{cases}$$

We let $f_i\in\pi_i(T(\mathbb{Z});\mathbb{Z}/4)$ denote an additive generator for each $i\equiv 0,3 \bmod 4$. There is a choice of sign in f_i for $i\equiv 0,7 \bmod 8$, which we will specify shortly. The classes f_i in odd degrees are integral. We note that $2f_3=2f_4=0$.

The homomorphisms ρ and ε linking $\pi_*(T(\mathbb{Z});\mathbb{Z}/2)$ and $\pi_*(T(\mathbb{Z});\mathbb{Z}/4)$ are easily determined using the maps of cofiber sequences (1.1):

$$\rho(f_{8k+i})=\begin{cases}e_4^{2k} & \text{for } i=0,\\ e_3e_4^{2k} & \text{for } i=3,\\ 0 & \text{for } i=4,\\ e_3e_4^{2k+1} & \text{for } i=7\end{cases}$$

and

$$\varepsilon(e_3^ae_4^{2k+b})=\begin{cases}2f_{8k} & \text{for } a=0,b=0,\\ 0 & \text{for } a=1,b=0,\\ f_{8k+4} & \text{for } a=0,b=1,\\ 2f_{8k+7} & \text{for } a=1,b=1.\end{cases}$$

The mod 2 Bockstein satisfies $\delta_2(f_4)=f_3$, $\delta_2(f_8)=\pm f_7$ and is a derivation. We agree to choose f_7 so that $\delta_2(f_8)=f_7$.

We also let $g_i\in\pi_iT(\mathbb{Z})$ denote an integral generator for each $i\equiv 3 \bmod 4$. reducing mod 4 to f_i and mod 2 to e_i.

Lemma 3.1. *The product on the* mod 4 *homotopy of* $T(\mathbb{Z})$ *satisfies the following relations*:

$$\begin{aligned}&f_3f_3=0,\quad f_3f_4=2f_7,\quad f_3f_7=0,\quad f_3f_8^k=f_{8k+3},\\ &f_4f_4=0,\quad f_4f_7=0,\quad f_4f_8^k=f_{8k+4},\\ &f_7f_7=0,\quad f_7f_8^k=\pm f_{8k+7}\quad \textit{and}\quad f_8^k=\pm f_{8k}\end{aligned}$$

for all $k\geq 0$.

Proof. The claims $f_3^2=0$, $f_3f_7=0$ and $f_7^2=0$ are clear since the products lie in trivial groups.

For f_3f_4 note that f_3 is integral and ε is linear with respect to multiplication by integral classes. So $f_3f_4=g_3f_4=g_3\varepsilon(e_4)=\varepsilon(g_3e_4)=2f_7$. Likewise, $f_4f_7=f_4g_7=\varepsilon(e_4)g_7=\varepsilon(e_4g_7)=\varepsilon(e_3e_4^2)=0$.

Because δ_2 is a derivation $\delta_2(f_4^2)=2f_3f_4=0$, and since δ_2 is injective in degree eight it follows that $f_4^2=0$.

Next $f_3f_8^k$ is nonzero because its image under ρh_2 (which is multiplicative) equals $e_3e_4^{2k}$ which is nonzero. Compare with diagram (1.2). So $f_3f_8^k$ is the only nonzero class in degree $8k+3$, which is the generator f_{8k+3}. Likewise, $f_7 f_8^k$ is nonzero because its image under ρh_2 equals $e_3 e_4^{2k+1}$, and f_8^k is nonzero because its image e_4^{2k} is also nonzero.

Finally, we conclude that $f_4 f_8^k$ is nonzero by computing $\delta_2(f_4 f_8^k) = f_3 f_8^k + k f_4 f_7 f_8^{k-1} = f_3 f_8^k$ which is nonzero. □

Hence, we may take $f_i f_8^k$ as the generator in degree $8k+i$ for all $k \geq 0$ and $i = 0,3,4,7$. Modulo the sign of f_8, this precisely pins down a set of generators for $\pi_*(T(\mathbb{Z}); \mathbb{Z}/4)$. The sign of f_8 can be fixed by insisting that $h_2(f_8) = P(e_4)$ where P is the Pontrjagin squaring operation (see [6] or [7]). We omit the details of this argument.

Theorem 3.2. (i) *The* mod 4 *homotopy algebra of* $T(\mathbb{Z})$ *is*

$$\pi_*(T(\mathbb{Z}); \mathbb{Z}/4) \cong \mathbb{Z}/4[f_3, f_4, f_7, f_8]/\sim$$

with the relations $2f_3 = 2f_4 = 0$, $f_3 f_4 = 2f_7$, *and* $f_i f_j = 0$ *for all other* $i, j < 8$.

(ii) *The pairing*

$$\pi_*(T(\mathbb{Z}); \mathbb{Z}/4) \otimes \pi_*(T(\mathbb{Z}); \mathbb{Z}/2) \to \pi_*(T(\mathbb{Z}); \mathbb{Z}/2)$$

induced by m *takes* $x \otimes y$ *to* $\rho(x)y$ *in the subalgebra product on* $\pi_*(T(\mathbb{Z}); \mathbb{Z}/2)$ *inherited from* $H_*(T(\mathbb{Z}); \mathbb{Z}/2)$. *Thus,* $m(x \otimes y) = \rho(x) \cdot y$.

(iii) *The coefficient reduction* $\rho : \pi_*(T(\mathbb{Z}); \mathbb{Z}/4) \to \pi_*(T(\mathbb{Z}); \mathbb{Z}/2)$ *is an algebra homomorphism when mapping to the subalgebra product. The coefficient extension* $\varepsilon : \pi_*(T(\mathbb{Z}); \mathbb{Z}/2) \to \pi_*(T(\mathbb{Z}); \mathbb{Z}/4)$ *is a* $\pi_*(T(\mathbb{Z}); \mathbb{Z}/4)$*-module homomorphism with respect to the actions induced by* m *and* μ_2.

Proof. Part (i) summarizes Lemma 3.1 and the conventions following it.

To prove $m(x \otimes y) = \rho(x)y$ in $\pi_*(T(\mathbb{Z}); \mathbb{Z}/2)$ it suffices to apply the injection h_1 and note that $h_1 m(x \otimes y) = \rho h_2(x) h_1(y) = h_1(\rho(x)y)$.

The remaining claims follow from Lemma 1.1. □

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