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A SPECTRUM LEVEL RANK FILTRATION IN ALGEBRAIC K-THEORY

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INTRODUCTION

THIS PAPER introduces a new filtration of the algebraic K-theory spectrum $\mathbf{K}R$ of a ring R, and investigates the subquotients of this filtration. $\mathbf{K}R$ is constructed from the category $\mathscr{P}(R)$ of finitely generated projective R-modules, and its homotopy groups are the algebraic K-groups of R as defined by Quillen [14]. There is also a free K-theory $\mathbf{K}^f R$, constructed from the weakly cofinal subcategory $\mathscr{F}(R)$ of $\mathscr{P}(R)$ consisting of finitely generated free R-modules. Inclusion induces a covering map $\mathbf{K}^f R \to \mathbf{K}R$, which in turn induces an isomorphism on π_i for $i > 0$ [7, 17]. In particular the higher free K-groups $\pi_i \mathbf{K}^f R$ for $i > 0$ agree with Quillen's K-groups.

We construct a sequence of 'unstable' algebraic K-theory spectra $\{F_k \mathbf{K}R\}_k$ filtering $\mathbf{K}^f R$. We will assume that R has the invariant dimension property [12] so that it makes sense to talk about the rank of a finitely generated free R-module. Then for a fixed rank $k \geq 0$, $F_k \mathbf{K}R$ is constructed as a subspectrum of $\mathbf{K}^f R$, built from free R-modules of rank less than or equal to k. As k increases, we obtain an increasing *rank filtration* $\{F_k \mathbf{K}R\}_k$ of spectra converging to $\mathbf{K}^f R$. It turns out that each subquotient spectrum $F_k \mathbf{K}R / F_{k-1} \mathbf{K}R$ is a homotopy orbit spectrum $\mathbf{D}(R^k)/hGL_k R$ for some spectrum-with-$GL_k R$-action $\mathbf{D}(R^k)$. Furthermore we prove that $\mathbf{D}(R^k)$ is stably equivalent to the suspension spectrum on a finite dimensional $GL_k R$-complex $D(R^k)$, which we call the *stable building* of R^k. Here is a description of the stable building, related to Volodin and Wagoner's constructions of K-theory [20, 21]:

Definition 14.5′. Let $\Sigma^{-1} D^V(R^k)$ be a simplicial set with q-simplices the $(q+1)$-tuples $\{M_0, \ldots, M_q\}$ of free, proper, nontrivial submodules $M_i \subset R^k$, satisfying the following condition: There exists an R-basis $\mathscr{B}$ for R^k for which each submodule M_i has a subset of $\mathscr{B}$ as an R-basis. The *stable building* $D(R^k)$ is the suspension $\Sigma(\Sigma^{-1} D^V(R^k))$.

Understanding the $GL_k R$-homotopy type of $D(R^k)$ is thus sufficient to describe the subquotient spectra $F_k \mathbf{K}R / F_{k-1} \mathbf{K}R$. (Here, and throughout, we are referring to the weak notion of $GL_k R$-homotopy type, where a $GL_k R$-map which is a nonequivariant homotopy equivalence is treated as an equivalence.) To this end, we explicitly describe a $\mathbb{Z}GL_k R$-complex $E_*(k)$ of length $(2k-2)$ whose homology computes $H_* D(R^k) = H_* \mathbf{D}(R^k)$. There is then a spectral sequence converging to $H_*(F_k \mathbf{K}R / F_{k-1} \mathbf{K}R)$ with E^1-term $E^1_{s,t} = H_t(GL_k R; E_s(k))$. Using it we see that when R is finite the unstable K-groups $\pi_i F_k \mathbf{K}R$ are $|GL_k R|$-torsion for $i > 0$. From computations, we are led to conjecture that $D(R^k)$ is $(2k-3)$-connected for R local or Euclidean, so that its homology is concentrated in degree $(2k-2)$. If so, $F_k \mathbf{K}R \to \mathbf{K}^f R$ is $(2k-1)$-connected, and the free R-modules of rank up through k determine the K-groups up through $K_{2k-2} R$, except $K_0 R$. In the final sections of the paper we give some evidence for this conjecture.

We now outline the constructions involved in our analysis of the GL_kR-homotopy type of $D(R^k)$. We study $\mathbf{D}(R^k)$ through the stable GL_kR-type of its nth space $D^n(R^k)$. For $n = 1$, the complex $D^1(R^k)$ is the double suspension of the Tits building [21] on R^k, and this is the motivation for calling $D^n(R^k)$ the *n-dimensional building*, and $D(R^k)$ the stable building. (Note however that these higher dimensional buildings are not buildings in the abstract sense of Jacques Tits [18].) From an explicit description of this n-dimensional building, we find a filtration of $D^n(R^k)$ indexed by the isomorphism classes $[\omega]$ of partial orderings (posets) on the set $\{1, \ldots, k\}$, which we call the *poset filtration* $\{F_{[\omega]}D^n(R^k)\}_{[\omega]}$. This is a filtration because the indexing set is again partially ordered by decreasing *strength* of the partial orderings involved.

We build a covering of $D^n(R^k)$ by *apartments*, which are certain subcomplexes homeomorphic to S^{nk}. We are able to completely compute the behavior of the poset filtration restricted to an apartment A, $\{F_\omega A\}_\omega$. First we find that each $F_\omega A$ is stably contractible, except when the poset ω is *indiscrete* (setting $1 = 2 = \cdots = k$), in which case $F_\omega A \cong S^n$. Next let $F_{<\omega}A \subset F_\omega A$ denote the subcomplex covered by the $F_{\omega'}A$ where ω' is a strictly stronger ordering than ω, so that $F_{\omega'}A \subset F_\omega A$ precedes $F_\omega A$ in the poset filtration. Then each $F_{<\omega}A$ is stably contractible, except when each connected component of ω is a linear ordering. In the latter case, if ω has l components, $F_{<\omega}A$ has the stable homotopy type of a wedge of $(l-1)!$ spheres all of dimension $\text{size}(\omega) - 1$, where $\text{size}(\omega)$ is roughly the number of relations lacking for ω to be indiscrete. To summarize, $F_\omega A/F_{<\omega}A$ is stably contractible if ω has some nonlinear component, otherwise its stable homology is concentrated in degree $\text{size}(\omega)$, and in that degree it is free abelian of rank $(l-1)!$.

Lastly we express each subquotient of the poset filtration of $D^n(R^k)$ as a balanced product of subquotient spaces $F_\omega A/F_{<\omega}A$ and a set of suitable configurations of free submodules of R^k. GL_kR acts on the whole by permuting these configurations, through the GL_kR-action on R^k. This gives a complete description of the stable GL_kR-type of the subquotients in the poset filtration.

We can now describe the chain complex $E_*(k)$ as the stabilization of the spectral sequence converging to $H_*D^n(R^k)$, associated to the poset filtration of $D^n(R^k)$. First we introduce some notation. Let $\mathscr{L}_s(k)$ denote the set of isomorphism classes of componentwise linear posets of size s on $\{1, \ldots, k\}$. Each poset ω determines a parabolic subgroup $P_\omega \subset GL_kR$, and a group of self-equivalences $(\Sigma_k)_\omega$. Let $|\omega|$ be the number of components of ω, and let δ_k denote the *discrete* poset with no relations other than the identities. Lastly, write W_l for a certain Σ_l-representation of rank $(l-1)!$ over $\mathbb{Z}$, namely the $\mathbb{Z}$-dual of the free abelian subgroup of the free Lie algebra on l generators $x_1, \ldots, x_l$ generated by iterated Lie brackets involving each x_i exactly once. Then:

THEOREM 12.1′. *The $\mathbb{Z}GL_kR$-module $H_*\mathbf{D}(R^k)$ equals the homology of the complex of length $(2k-2)$ below*

$$0 \to \mathbb{Z}GL_kR/P_{\delta_k} \underset{\Sigma_k}{\otimes} W_k \xrightarrow{d_{2k-2}} \cdots$$

$$(E_*(k)) \qquad \xrightarrow{d_{s+1}} \bigoplus_{[\omega]\in\mathscr{L}_s(k)} \mathbb{Z}GL_kR/P_\omega \underset{(\Sigma_k)_\omega}{\otimes} W_{|\omega|} \xrightarrow{d_s} \cdots \xrightarrow{d_1} \mathbb{Z} \to 0.$$

Our result does not explicitly describe the boundary maps d_s, but they are easy to write down for small k.

In some cases the rank filtration of a K-theory specializes to classical constructions. The first stage $F_1\mathbf{K}R$ is always the suspension spectrum of the classifying space of the unit group

$R^* = GL_1 R$, i.e. $F_1 \mathbf{K}R \cong \Sigma^\infty BGL_1 R_+$. (The added base point shows that the filtration fails to split for rather trivial reasons.) Also the K-theory of $\mathscr{E}$, the category of based finite sets, maps to $\mathbf{K}R$ for each ring R, by taking a finite set to the free R-module generated by its non-basepoint elements. $\mathbf{K}\mathscr{E}$ has a rank filtration analogous to that for $\mathbf{K}R$, and the above map preserves these filtrations. In particular the rank filtration gives a quick proof of the Barratt–Priddy–Quillen theorem, as $\mathbf{S} \cong F_1 \mathbf{K}\mathscr{E} \subsetneq \mathbf{K}\mathscr{E}$ is a stable equivalence.

The conjecture mentioned earlier holds for $k = 2$ when R is a Euclidean domain or a local ring, so we have a short exact sequence:

$$0 \to H_2 D(R^2) \to \mathbb{Z}GL_2R/T_2 \underset{\Sigma_2}{\otimes} \mathbb{Z}[-1] \xrightarrow{d_2} \tilde{\mathbb{Z}}GL_2R/P_1 \to 0.$$

Here $T_k \cong (R^*)^k$ is the diagonal torus, $\mathbb{Z}[-1]$ the sign representation, $P_1 \subset GL_2 R$ the upper triangular matrices, and if X is a set, $\tilde{\mathbb{Z}}X$ denotes the kernel of $\mathbb{Z}X \xrightarrow{\varepsilon} \mathbb{Z}$. The differential d_2 is given by $d_2(gT_2 \otimes 1) = g(1 - \xi)P_1$ with $\xi = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. For $k = 3$, the complex $E_*(3)$ appears as

$$0 \to \mathbb{Z}GL_3R/T_3 \underset{\Sigma_3}{\otimes} W_3 \xrightarrow{d_4} \mathbb{Z}GL_3R/P_{1,12,3} \xrightarrow{d_3} \mathbb{Z}GL_3R/P_{1,12} \oplus \mathbb{Z}GL_3R/P_{12,3}$$

$$\xrightarrow{d_2} \mathbb{Z}GL_3R/P_1 \oplus \mathbb{Z}GL_3R/P_{12} \xrightarrow{d_1} \mathbb{Z} \to 0,$$

with $W_3 \cong \tilde{\mathbb{Z}}^3 \otimes \mathbb{Z}[-1]$, where $\mathbb{Z}^3$ is the permutation representation of Σ_3, and $\tilde{\mathbb{Z}}^3$ its augmentation ideal. See Definitions 8.5 and 14.1 for the parabolic subgroups, and Section 15 for the boundary maps. This complex is known to be exact at E_0, at E_1 for R local or Euclidean, at E_2 for R a field, and at E_3 for $R = \mathbb{F}_2$.

These two complexes can be used for direct group homological computations towards determining the spectrum homology of respectively the second and third stages in the rank filtration of $\mathbf{K}R$ for concrete rings R.

THE RANK FILTRATION OF K-THEORY

1. The simplicial categories describing K-theory spectra

We study the algebraic K-theory of a ring R using Waldhausen's [22] construction of the K-theory spectrum $\mathbf{K}R$. This is constructed from the category with cofibrations and weak equivalences $\mathscr{P}(R)$ of finitely generated projective R-modules, with split injections as the cofibrations and isomorphisms as the weak equivalences. We also consider the free K-theory of R, $\mathbf{K}^f R$, constructed from the subcategory with cofibrations and weak equivalences $\mathscr{F}(R)$ of finitely generated free R-modules. In this case, cofibrations are split injections with free quotient modules, while the weak equivalences are still isomorphisms. Reformulated, a morphism $f: N \to M$ in $\mathscr{F}(R)$ is a cofibration precisely if $M \cong N \oplus N'$ for two free R-modules N and N', and f is inclusion on the first summand.

For a general category with cofibrations and weak equivalences $\mathscr{C}$, the K-theory spectrum $\mathbf{K}\mathscr{C}$ has nth space $B^n\mathbf{K}\mathscr{C}$ $(n \geq 1)$, defined as the geometric realization of the n-multisimplicial category $wS_\bullet S_\bullet \ldots S_\bullet \mathscr{C}$, where $S_\bullet$ is Waldhausen's $S_\bullet$-construction. ($\mathbf{K}R = \mathbf{K}\mathscr{P}(R)$ and $\mathbf{K}^f R = \mathbf{K}\mathscr{F}(R)$.) Equivalently we may consider the diagonal simplicial category

$$[q] \mapsto wS_q S_q \ldots S_q \mathscr{C} = wS_q^n \mathscr{C},$$

and we shall take the realization of this simplicial category as our model for $B^n\mathbf{K}\mathscr{C}$.

For concreteness, and to settle notation, the remainder of this section recalls the definition of a category with cofibrations and the $S_\bullet$-construction from [22].

Definition 1.1. A category $\mathscr{C}$ is *pointed* if it has a chosen zero object $*$, i.e. an object which is both initial and terminal. A *category with cofibrations* is a pointed category $\mathscr{C}$ with a subcategory $co\mathscr{C}$, the *category of cofibrations* in $\mathscr{C}$. A morphism in $co\mathscr{C}$ is called a cofibration, denoted $A \rightarrowtail B$. $co\mathscr{C}$ is assumed to contain the isomorphisms of $\mathscr{C}$, and the morphism $* \to A$ for each object A in $\mathscr{C}$. $co\mathscr{C}$ is also assumed to be closed under arbitrary cobase change in $\mathscr{C}$, i.e. if $A \rightarrowtail B$ is a cofibration and $A \to C$ is any morphism in $\mathscr{C}$, the categorical pushout $C \cup_A B$ should exist in $\mathscr{C}$, and the canonical morphism $C \to C \cup_A B$ should be a cofibration.

A *category with cofibrations and weak equivalences* is a category with cofibrations $\mathscr{C}$, ($co\mathscr{C}$ is implicit) together with a subcategory $w\mathscr{C}$, the *category of weak equivalences* in $\mathscr{C}$. A morphism in $w\mathscr{C}$ is called a weak equivalence. $w\mathscr{C}$ is assumed to contain the isomorphisms of $\mathscr{C}$. Also, if in a commutative diagram

$$\begin{array}{ccccc} C & \leftarrow & A & \rightarrowtail & B \\ \downarrow & & \downarrow & & \downarrow \\ C' & \leftarrow & A' & \rightarrowtail & B' \end{array}$$

the vertical morphisms are in $w\mathscr{C}$, so is the canonical morphism $C \cup_A B \to C' \cup_{A'} B'$.

A functor between categories with cofibrations (and weak equivalences) is called *exact* if it preserves these structures, i.e. the zero object, the subcategory of cofibrations, and the categorical pushout diagrams (and the subcategory of weak equivalences).

Definition 1.2. For $q \in \mathbb{N}$ let $\mathbf{q}$ denote the poset $\{1 < \cdots < q\}$ and $[q]$ the poset $\{0 < \cdots < q\}$. View them as categories, and give their n-fold products $(\mathbf{q})^n$ and $([q])^n$ the product poset structures. Call (a diagram on) $([q])^n$ a *size q n-cube*. Also use $\mathbf{q}$ and $[q]$ to denote the underlying sets.

For a category $\mathscr{D}$, let the *arrow category* $\mathrm{Ar}\,\mathscr{D}$ have objects the morphisms $(i \to j)$ of $\mathscr{D}$, and morphisms from $(i \to j)$ to $(i' \to j')$ the set of commutative diagrams in $\mathscr{D}$:

$$\begin{array}{ccc} i & \to & j \\ \downarrow & & \downarrow \\ i' & \to & j' \end{array}$$

If $\mathscr{C}$ and $\mathscr{D}$ are categories, let $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ denote the category of functors from $\mathscr{C}$ to $\mathscr{D}$ and natural transformations of these.

Definition 1.3. For a category with cofibrations $\mathscr{C}$, let $S_q\mathscr{C}$ denote the category of diagrams (functors) $F: \mathrm{Ar}[q] \to \mathscr{C}$ such that for every $j \in [q]$, $F(j \to j) = *$, and for every triple $i \le j \le k$ in $[q]$, $F(i \to j) \rightarrowtail F(i \to k)$ is a cofibration, *and* the diagram

$$\begin{array}{ccc} F(i \to j) & \rightarrowtail & F(i \to k) \\ \downarrow & & \downarrow \\ * = F(j \to j) & \rightarrowtail & F(j \to k) \end{array}$$

is a pushout square. The morphisms of $S_q\mathscr{C}$ are natural transformations of such functors. Now the association $[q] \mapsto S_q\mathscr{C}$ defines a simplicial category $S_\bullet\mathscr{C}$. Its simplicial structure is given as a simplicial subcategory of the composite $[q] \mapsto \mathrm{Ar}[q] \mapsto \mathrm{Fun}(\mathrm{Ar}[q], \mathscr{C})$.

$S_\bullet\mathscr{C}$ is naturally a simplicial category with cofibrations. This means a simplicial object in the category whose objects are categories with cofibrations, and morphisms the exact functors between these. To see this, let $coS_q\mathscr{C}$ have morphisms the natural transformations $\eta: F \to G$ such that $\eta(i \to j): F(i \to j) \rightarrowtail G(i \to j)$ is a cofibration for each $(i \to j)$ in $\mathrm{Ar}[q]$, and for each triple $i \le j \le k$ the canonical map $F(i \to k) \cup_{F(i \to j)} G(i \to j) \rightarrowtail G(i \to k)$ is a cofibration.

If $\mathscr{C}$ has a category of weak equivalences $w\mathscr{C}$, $S_\bullet\mathscr{C}$ is similarly a simplicial category with cofibrations and weak equivalences: Let $wS_q\mathscr{C}$ have morphisms the $\eta: F \to G$ such that each $\eta(i \to j)$ is in $w\mathscr{C}$.

By naturality of these extended structures on $S_\bullet\mathscr{C}$, we can iterate the $S_\bullet$-construction, obtaining for each $n \ge 1$ an n-multisimplicial category with cofibrations and weak equivalences $S_\bullet S_\bullet \ldots S_\bullet\mathscr{C}$. By inspection, we can identify the 1-skeleton of $|wS_\bullet\mathscr{C}|$ with the suspension $\Sigma|w\mathscr{C}|$, yielding an inclusion $\Sigma|w\mathscr{C}| \hookrightarrow |wS_\bullet\mathscr{C}|$. Applying this to the n-multisimplicial category above, we obtain a (pre-)spectrum

$$n \mapsto |w\underbrace{S_\bullet S_\bullet \ldots S_\bullet}_{n \text{ times}}\mathscr{C}| \cong |wS^n_\bullet\mathscr{C}|$$

which turns out to be an Ω-spectrum after the first stage. Waldhausen defines this to be the *K-theory spectrum of $\mathscr{C}$*.

After this review of Waldhausen's definitions, we can now define:

Definition 1.4. The nth space $B^n\mathbf{K}\mathscr{C}$ of the K-theory spectrum of $\mathscr{C}$ is the realization of the diagonal simplicial category

$$[q] \mapsto wS^n_q\mathscr{C} \qquad \text{for } n \ge 1.$$

2. *The lattice conditions on cubical diagrams*

We often prefer to think of an object in $S_q\mathscr{C}$ in terms of the subdiagram it determines over the full subcategory spanned by the $(0 \to j)$ in $\mathrm{Ar}[q]$ with $j \in [q]$. Note that the restricted diagram determines the full diagram up to isomorphism, due to the pushout conditions on objects in $S_q\mathscr{C}$. We identify $[q]$ with this full subcategory, using the injection $r: [q] \hookrightarrow \mathrm{Ar}[q]$ given by $j \mapsto (0 \to j)$. Then for each q, r induces a restriction functor r^* taking a diagram F on $\mathrm{Ar}[q]$ to a diagram $\sigma = r^*F$ on $[q]$. Explicitly, r^*F is a chain of cofibrations in $\mathscr{C}$ of the form

$$* = \sigma(0) \rightarrowtail \sigma(1) \rightarrowtail \cdots \rightarrowtail \sigma(q).$$

The following lemmas indicate why this is a workable simplification.

LEMMA 2.1. *In Definition* 1.3 *of $S_\bullet\mathscr{C}$, $coS_\bullet\mathscr{C}$ and $wS_\bullet\mathscr{C}$, it suffices to check the hypotheses as stated when $i = 0$.*

Proof. Use identities of the form $A \cup_B B \cup_C D \cong A \cup_C D$ repeatedly to reduce to the axioms for categories with cofibrations and weak equivalences. ■

LEMMA 2.2. *For each q and $n \ge 1$,*

$$(r^*)^n: S^n_q\mathscr{C} \to (r^*S_q)^n\mathscr{C}$$

$$(r^*)^n: coS^n_q\mathscr{C} \to co(r^*S_q)^n\mathscr{C}$$

$$(r^*)^n: wS^n_q\mathscr{C} \to w(r^*S_q)^n\mathscr{C}$$

are all equivalences of categories.

Proof. We view an object F of $S_q\mathscr{C}$ as a diagram $\sigma = r^*F$ in $co\mathscr{C} \subseteq \mathscr{C}$ on $[q]$, with $\sigma(0) = *$, together with choices of subquotients. The last phrase means choices of pushouts $\sigma(k)/\sigma(j) = F(j \to k)$ for all the diagrams below

$$\begin{array}{ccc} \sigma(j) & \rightarrowtail & \sigma(k) \\ \downarrow & & \downarrow \\ * & \rightarrowtail & F(j \to k) \end{array}$$

with $j \leq k$ in $[q]$, always choosing $F(j \to j) = *$. Then a cofibration $\sigma \rightarrowtail \tau$ in $coS_q\mathscr{C}$ is viewed as a commutative diagram

$$\begin{array}{ccccccc} * = \sigma(0) & \rightarrowtail & \sigma(1) & \rightarrowtail & \dots & \rightarrowtail & \sigma(q) \\ \downarrow & & \downarrow & & & & \downarrow \\ * = \tau(0) & \rightarrowtail & \tau(1) & \rightarrowtail & \cdots & \rightarrowtail & \tau(q) \end{array}$$

where all the columns are in $co\mathscr{C}$, such that each pushout map $\tau(j) \cup_{\sigma(j)} \sigma(k) \to \tau(k)$ is a cofibration for $j \leq k$ (checking for $k = j + 1$ is sufficient). Given choices of subquotients for σ, τ, there are then canonical quotient cofibrations extending the transformation diagram above to $\mathrm{Ar}[q]$.

A weak equivalence $\sigma \to \tau$ in $wS_q\mathscr{C}$ is a commutative diagram as above, where all the column morphisms are in $w\mathscr{C}$. Again, choosing subquotients for σ and τ determines canonical quotient weak equivalences extending the diagram.

Thus, by Lemma 2.1 the forgetful functor r^* is an equivalence when applied to any of the categories $S_q\mathscr{C}$, $coS_q\mathscr{C}$ and $wS_q\mathscr{C}$, as any consistent choice of subquotients provides an inverse. This proves the lemma for $n = 1$, and the general case follows by induction. ■

Definition 2.3. Observe that an object σ of $w(r^*S_q)^n\mathscr{C}$ is "a diagram on $[q]$, $*$ at 0, in the category of diagrams on $[q]$, $*$ at 0, . . . , repeated n times, . . . in the category of diagrams on $[q]$, $*$ at 0, in $\mathscr{C}$", subject to a hierarchy of conditions requiring certain pushout morphisms to be cofibrations. Clearly this is equivalent to a diagram on $([q])^n$, $*$ off $(\mathbf{q})^n$, subject to said conditions. Call a cube in $\mathscr{C}$ on $([q])^n$, taking the value $*$ off $(\mathbf{q})^n$, satisfying them a *lattice*. We refer to these conditions as the *lattice conditions*, and will make them explicit in Lemma 2.5.

A morphism $\sigma \to \tau$ in $w(r^*S_q)^n\mathscr{C}$ is a morphism of lattices such that $\sigma(\vec{x}) \to \tau(\vec{x})$ is in $w\mathscr{C}$ for each $\vec{x} \in ([q])^n$.

To formulate the lattice conditions, we need some notation for cubes and subcubes.

Definition 2.4. Take two *sites* (*vertices*) $\vec{a}$, $\vec{b} \in ([q])^n$ with $\vec{b} \leq \vec{a}$. Write $\vec{a} = (a_1, \dots, a_n) \in ([q])^n$, and similarly for $\vec{b}$. $\vec{a}$ and $\vec{b}$ span a sub-cube of $([q])^n$, namely the full subcategory of $([q])^n$ with objects $\vec{x} \in ([q])^n$ such that each $x_i = a_i$ or b_i. If $m \leq n$ is the number of coordinates $(i_1 < i_2 < \dots < i_m)$ different in $\vec{a}$ and $\vec{b}$, this is an m-cube. Alternatively we can index the vertices of this cube by all subsets $U \subseteq \mathbf{m}$. Let $\vec{x}_U = (x_1, \dots, x_n)$ with

$$x_i = \begin{cases} a_i & \text{if } a_i = b_i, \text{ or if } i = i_j \text{ and } j \in U, \\ b_i & \text{if } i = i_j \text{ and } j \notin U. \end{cases}$$

Then $U \mapsto M_U \stackrel{\text{def}}{=} \sigma(\vec{x}_U)$ determines an m-dimensional subcube of σ in $\mathscr{C}$, denoted $\sigma_{\vec{a},\vec{b}}$, with $\vec{a}$ as the *top*, $\vec{b}$ as the *bottom* site.

This generalizes to cubes with longer edges. Just as we identified vertices of a size 1 n-cube with subsets $U \subseteq \mathbf{n}$, there is a bijective correspondence taking a vertex

$\vec{a} = (a_1, \ldots, a_n) \in (\mathbf{q})^n$ to the increasing chain of subsets

$$\varnothing = U_0 \subseteq U_1 \subseteq \ldots \subseteq U_q = \mathbf{n}$$

with $U_i = U_i(\vec{a}) = \{ j \in \mathbf{n} \mid a_j \leq i \}$.

LEMMA 2.5. *Let σ be a diagram in $\mathscr{C}$ on $([q])^n$, such that $\sigma(\vec{x}) = *$ for $\vec{x} \notin (\mathbf{q})^n$. σ is a lattice precisely if for each $m \leq n$ and all m-dimensional subcubes $\sigma_{\vec{a},\vec{b}}$, the iterated pushout map*

$$\operatorname*{colim}_{U \subset \mathbf{m}} M_U \rightarrowtail M_{\mathbf{m}} = \sigma(\vec{a})$$

is a cofibration in $\mathscr{C}$. The colimit, or iterated pushout, is here over the proper subsets U of $\mathbf{m}$, i.e. over the cube $\sigma_{\vec{a},\vec{b}}$ punctured by removing the top vertex. (These requirements are the lattice conditions.)

Proof. We induct on n. For $n = 1$, only $m = 1$ is interesting, when we recover the description of $wr^*S_q\mathscr{C}$ given in the proof of Lemma 2.2.

For $n \geq 2$, an object σ of $(r^*S_q)^n\mathscr{C}$ is a diagram

$$* = \sigma(0) \rightarrowtail \sigma(1) \rightarrowtail \cdots \rightarrowtail \sigma(q)$$

in $co(r^*S_q)^{n-1}\mathscr{C}$. If $a_n = b_n$, $m < n$, and the m-cube $\sigma_{\vec{a},\vec{b}}$ is a subcube of $\sigma(a_n)$ on $([q])^{n-1}$. Since $\sigma(a_n)$ is a lattice, the iterated pushout map is a cofibration by induction.

If $a_n > b_n$, the m-cube $\sigma_{\vec{a},\vec{b}}$ has a *lower* $(m-1)$-face indexed by $U \subseteq \mathbf{m}$ with $m \notin U$, and an *upper* $(m-1)$-face indexed by $U \subseteq \mathbf{m}$ with $m \in U$. By induction,

$$A \stackrel{\text{def}}{=} \operatorname*{colim}_{m \in U \subset \mathbf{m}} M_U \rightarrowtail M_{\mathbf{m}} = \sigma(\vec{a})$$

and

$$B \stackrel{\text{def}}{=} \operatorname*{colim}_{U \subset \mathbf{m} - \{m\}} M_U \rightarrowtail M_{\mathbf{m} - \{m\}}$$

are both cofibrations. Each map $M_{U - \{m\}} \rightarrowtail M_U$ for $m \in U \subset \mathbf{m}$ is a cofibration, because $\sigma(b_n) \rightarrowtail \sigma(a_n)$ is a cofibration in $(r^*S_q)^{n-1}\mathscr{C}$. By closure under cobase change, it follows that $B \rightarrowtail A$ is also a cofibration. We obtain a commutative square of cofibrations

$$\begin{array}{ccc} B & \rightarrowtail & A \\ \downarrow & & \downarrow \\ M_{\mathbf{m} - \{m\}} & \rightarrowtail & M_{\mathbf{m}} \end{array}$$

where the columns are also cofibrations. Using the second requirement for $\sigma(b_n) \rightarrowtail \sigma(a_n)$ to be a cofibration in $(r^*S_q)^{n-1}\mathscr{C}$, the canonical morphism

$$A \cup_B M_{\mathbf{m} - \{m\}} \rightarrowtail M_{\mathbf{m}}$$

must be a cofibration. But the left hand side is exactly $\operatorname{colim}_{U \subset \mathbf{m}} M_U$.

This proves necessity of the lattice conditions. Sufficiency is proved by reversing the argument. ∎

Remark 2.6. Note that $[q] \mapsto w(r^*S_q)^n\mathscr{C}$ is not a simplicial category. The face maps ∂_i for $i = 1, \ldots, q$ are well defined as restriction to the size $(q-1)$ n-subcube of $([q])^n$ where no coordinate is equal to i. Also the degeneracy maps s_j inserting identity morphisms restrict well over r^*. However, the face map ∂_0 of $S_q\mathscr{C}$ uses the choices of subquotients inherent in a diagram on $\mathrm{Ar}[q]$, which are forgotten by r^*. It is possible to define face maps ∂_0 on the level of lattices, but they will generally only satisfy the simplicial identities up to isomorphism. We shall see in Section 3 that this difficulty disappears when we limit attention to the subquotients of the rank filtration.

3. The stable rank filtration

Hypothesis 3.1. We now turn to the algebraic K-theory of a ring. Let R be an associative ring with unit. Throughout the paper we assume that R satisfies the (*strong*) *invariant dimension property* [12], i.e. if there exists a split injection of R-modules $R^n \hookrightarrow R^m$ then $n \leq m$. This implies that R^n and R^m are isomorphic if and only if $n = m$. Examples of such rings are commutative rings, or algebras over commutative rings which additively are finitely generated and free. This includes finite matrix algebras over commutative rings, and group rings of commutative rings over finite groups. Other examples are those rings which admit a homomorphism onto a skew-field.

For the remainder of the paper, we will only be considering free K-theory, and so let $\mathbf{K}R$ denote the free K-theory $\mathbf{K}\mathscr{F}(R)$ of R in place of Quillen's K-theory $\mathbf{K}\mathscr{P}(R)$. As previously noted, these two theories agree in all positive degrees. By our hypothesis, finitely generated free R-modules have a well-defined rank, and $\pi_0 \mathbf{K}^f R \cong \mathbb{Z}$.

We now filter $\mathscr{F}(R)$ by the full subcategories $F_k\mathscr{F}(R)$ of submodules of rank k or less. These are not closed under pushouts, hence do not inherit a structure of category with cofibrations from $\mathscr{F}(R)$:

$$* \simeq F_0\mathscr{F}(R) \subset F_1\mathscr{F}(R) \subset \cdots \subset F_k\mathscr{F}(R) \subset \cdots \subset \mathscr{F}(R).$$

From this we obtain a rank filtration of the nth space $B^n\mathbf{K}R$ of the (free) K-theory spectrum of R, by restricting to diagrams of submodules in $F_k\mathscr{F}(R) \subset \mathscr{F}(R)$, as the rank k varies. This is a special case of an $S_\bullet$-construction with coefficients, where the coefficients are a functor from an arbitrary category to the category with cofibrations and weak equivalences in question.

Definition 3.2. $F_kB^n\mathbf{K}R \subset B^n\mathbf{K}R$ is the subcomplex realizing the simplicial full subcategory of $[q] \mapsto wS^n_q\mathscr{F}(R)$, denoted $[q] \mapsto wS^n_qF_k\mathscr{F}(R)$, consisting of diagrams in $\mathscr{F}(R)$ which factor through $F_k\mathscr{F}(R)$. For each n this gives a filtration

$$* \simeq F_0 B^n\mathbf{K}R \subset F_1 B^n\mathbf{K}R \subset \cdots \subset F_kB^n\mathbf{K}R \subset \cdots \subset B^n\mathbf{K}R.$$

LEMMA 3.3. *The connecting maps $\Sigma B^n\mathbf{K}R \hookrightarrow B^{n+1}\mathbf{K}R$ restrict to connecting maps $\Sigma F_kB^n\mathbf{K}R \hookrightarrow F_kB^{n+1}\mathbf{K}R$ making each $\{F_kB^n\mathbf{K}R\}_n$ a prespectrum.*

Proof. The connecting map identifies a suspended q-simplex F in $\Sigma B^n\mathbf{K}R$ with the $(1, q)$-bisimplex $(0 \rightarrowtail F)$ in $wS_\bullet S^n_\bullet\mathscr{F}(R)$, which has realization $B^{n+1}\mathbf{K}R$. This identification preserves rank, hence is respected by the rank filtration. ∎

Definition 3.4. Denote the associated spectrum $F_k\mathbf{K}R$. It is the *kth unstable K-theory* of R. The *rank filtration* of $\mathbf{K}R$ is the sequence of spectra:

$$* \simeq F_0\mathbf{K}R \to F_1\mathbf{K}R \to \cdots \to F_k\mathbf{K}R \to \cdots \to \mathbf{K}R.$$

LEMMA 3.5. $\operatorname{colim}_{k\to\infty} \pi_iF_k\mathbf{K}R \xrightarrow{\cong} \pi_i\mathbf{K}R = K_iR$ *for all i, that is, the unstable K-theories approximate* $\mathbf{K}R$.

Proof. In spectrum level n, the space $|B^n\mathbf{K}R|$ is a CW-complex and has an exhaustive filtration by the subcomplexes $|F_k B^n\mathbf{K}R|$. Hence $\operatorname{colim}_{k\to\infty}\pi_i F_k B^n\mathbf{K}R \xrightarrow{\cong} \pi_i B^n\mathbf{K}R$, and the lemma follows as n passes to infinity. ∎

Remark 3.6. $\Omega^\infty\mathbf{K}\mathscr{P}(R) \simeq K_0 R \times BGL(R)^+$ [7], where X^+ denotes Quillen's plus construction on X. For finite ranks k, $BGL_k R^+$ is not generally an infinite loop space, so we cannot expect a description of this type for the spaces $\Omega^\infty F_k\mathbf{K}R$, (for neither free nor ordinary K-theory). In this sense, our rank filtration differs from the one offered by the plus construction on finite rank matrices.

Next we investigate the subquotients of the nth spaces of the rank filtration.

Definition 3.7. If X is a based G-space, $X/hG = EG_+ \wedge_G X$ is its (*reduced*) *homotopy orbit space*, also known as the Borel construction.

PROPOSITION 3.8. *$F_k B^n\mathbf{K}R/F_{k-1}B^n\mathbf{K}R \simeq D^n(R^k)/hGL_k R$, where $D^n(R^k)$ is a finite dimensional $GL_k R$-complex. The sequence of spaces $\{D^n(R^k)\}_n$ inherits the structure of a subspectrum of $F_k\mathbf{K}R/F_{k-1}\mathbf{K}R$, denoted $\mathbf{D}(R^k)$. In the category of spectra*:

$$F_k\mathbf{K}R/F_{k-1}\mathbf{K}R \simeq \mathbf{D}(R^k)/hGL_k R.$$

Proof. $F_k B^n\mathbf{K}R/F_{k-1}B^n\mathbf{K}R$ is the realization of a simplicial category which we denote $X'_\bullet$. In degree q, X'_q has objects the diagrams on $(\mathrm{Ar}[q])^n$ in $S^n_q\mathscr{F}(R)$ where the top (largest) module has rank exactly k, together with a base object $*_q$. Morphisms in X'_q are isomorphisms of such diagrams.

Let X_q be the analogous category of lattices. X_q has objects the lattices on $([q])^n$ of free R-modules with top module isomorphic to R^k, together with a base object $*_q$. Again, morphisms in X_q are isomorphisms of such diagrams. Restriction over $r:[q]\to\mathrm{Ar}[q]$ induces an equivalence of categories $r^*:X'_q\to X_q$ for each q, by a filtered version of Lemma 2.2. We claim that $[q]\mapsto X_q$ is a simplicial category, and that r induces an equivalence of simplicial categories $r^*:X'_\bullet \xrightarrow{\cong} X_\bullet$.

Note that on a nondegenerate q-simplex σ of X'_q, $\partial_0(\sigma) = *_{q-1}$. On degenerate simplices, ∂_0 is determined by this observation and the simplicial identities. By Remark 2.6, $X_\bullet$ supports all the simplicial maps except the 0th face maps. So clearly we can define ∂_0 by mapping all nondegenerate simplices of $X_\bullet$ to $*$, and extend to degenerate simplices as dictated by the simplicial identities. This gives $X_\bullet$ the structure of a simplicial category, compatible under the map r^* with the simplicial structure on $X'_\bullet$. Hence r^* is a simplicial equivalence. The simplicial identities referred to in Remark 2.6, which in general would hold only up to isomorphism, now reduce to the true identity $* = *$.

Having established the equivalence $r^*:F_k B^n\mathbf{K}R/F_{k-1}B^n\mathbf{K}R \simeq |X_\bullet|$, we turn to studying $X_\bullet$. It has two simplicial small subcategories $Y_\bullet$ and $Z_\bullet$, ($X_\bullet \supset Y_\bullet \supset Z_\bullet$) defined as follows. Both $Y_\bullet$ and $Z_\bullet$ have objects the lattices where the top module is equal to R^k (not just isomorphic) and the cofibration morphisms are genuine inclusions (not just injective maps). $Y_\bullet$ is the simplicial full subcategory of $X_\bullet$, while $Z_\bullet$ has only the identity morphisms, and is effectively a simplicial set, namely $D^n(R^k)$. (Both $Y_\bullet$ and $Z_\bullet$ also contain the base object in every degree).

Every object of $X_\bullet$ is isomorphic to one in $Y_\bullet$. Choosing such an isomorphism for every object of $X_\bullet$, taking the identity whenever possible, suffices to describe a deformation retraction from $|X_\bullet|$ down to $|Y_\bullet|$.

Furthermore, the morphisms in $Y_\bullet$ are precisely determined by the source object and the morphism's action on the top module R^k, i.e. an element of GL_kR. This means that $Y_\bullet$ is the simplicial based translation category for the GL_kR-action on $Z_\bullet$, and we easily obtain $|Y_\bullet| \cong |Z_\bullet|/hGL_kR$. Identifying the realization of $Z_\bullet$ in the category direction with its simplicial object set, which we call $D^n(R^k)$, we have now established the needed homotopy equivalence $|X_\bullet| \simeq D^n(R^k)/hGL_kR$.

For the statements about spectra, note that the inclusions $Z_\bullet \subset Y_\bullet \subset X_\bullet$ respect the connecting maps coming from $\mathbf{K}R$. ■

By untangling the description of the simplicial set $|Z_\bullet|$ in the proof above, we extract the following definition:

Definition 3.9. The *n-dimensional building* $D^n(R^k)$ is the simplicial set with q-simplices the lattices on $([q])^n$ of inclusions of free submodules of R^k, with those lattices where the top module is not R^k identified to a base simplex $*_q$.

A face map ∂_i for $i = 1, \ldots, q$ deletes the vertices in $([q])^n$ lying in the n 'hyperplanes' where at least one coordinate is equal to i. ∂_0 iteratively divides out by the modules at the vertices with at least one coordinate equal to 1. The degeneracies s_j insert identity morphisms across the n hyperplanes where at least one coordinate is equal to j.

By inspection, any simplex of dimension greater than nk is degenerate, so $D^n(R^k)$ is (at most) nk-dimensional. GL_kR acts on $D^n(R^k)$ as automorphisms of R^k by permuting the submodules.

APARTMENTS

4. *Axial submodules*

Definition 4.1. Let $\mathscr{E}$ be the category of finite sets, with injections as cofibrations and bijections as weak equivalences. Its K-theory spectrum has 0th space $\mathbb{Z} \times B\Sigma_\infty^+$, and has a rank filtration (filter by cardinality) with analogous properties to those indicated for $\mathbf{K}R$ in the previous section. ($\mathscr{E}$ should really be the category of *based* finite sets and functions, but we suppress the base point.)

We have a functor $\mathscr{E} \to \mathscr{F}(R)$ respecting the rank filtration, taking a finite set I to R^I, the free R-module generated by I. Identify $R^{\mathbf{k}}$ and R^k.

Definition 4.2. The *axial submodules* of R^k are the submodules $R^I \subseteq R^k$, for $I \subseteq \mathbf{k}$.

LEMMA 4.3. *$D^n(\mathbf{k}) \subsetneq D^n(R^k)$ is identified with the subcomplex of lattice diagrams in axial submodules of R^k.* ■

Definition 4.4. *Call* $A_{n,k} = D^n(\mathbf{k}) \subseteq D^n(R^k)$ the *standard apartment* in $D^n(R^k)$. We will omit the subscripts when n and k are clear from the context. The sequence of spaces $\{A_{n,k}\}_n$ inherits the structure of a subspectrum of $\mathbf{D}(R^k)$, denoted $\mathbf{A}_k$ or simply $\mathbf{A}$.

PROPOSITION 4.5. $A_{n,k} \cong S^{nk}$.

COROLLARY 4.6. (*Barratt–Priddy–Quillen Theorem*) $\mathbf{K}\mathscr{E} \cong \mathbf{S}$ (*the sphere spectrum*), *i.e.* $\mathbb{Z} \times B\Sigma_\infty^+ \simeq Q(S^0)$.

Proof of Corollary. $F_kB^n\mathbf{K}\mathscr{E}/F_{k-1}B^n\mathbf{K}\mathscr{E} \cong D^n(\mathbf{k})/h\Sigma_k \cong S^{nk}/h\Sigma_k$ is $(2n-1)$-connected for $k \geq 2$ and homeomorphic to S^n for $k = 1$. Thus $\mathbf{S} \cong F_1\mathbf{K}\mathscr{E} \subsetneq \mathbf{K}\mathscr{E}$ is a stable equivalence. ■

Definition 4.7. For a set X, Let $\Delta(X)$ be the complex (*simplex*) with vertices X, and faces the finite nonempty subsets of X.

Proof of Proposition 4.5. First consider the case $k = 1$. A q-simplex $\sigma \neq *_q$ is a lattice on $([q])^n$ of subsets of $\mathbf{1} = \{1\}$. We claim that the lattice conditions are equivalent to that $\sigma(\vec{x}) = \mathbf{1}$ precisely for the $\vec{x} \geq \vec{a}$ for some unique $\vec{a} \in (\mathbf{q})^n$, and $\sigma(\vec{x}) = \varnothing$ for all other $\vec{x}$.

To prove this take vertices $\vec{a}, \vec{b}$ in the same axial 2-plane of $([q])^n$. Let $\vec{c} = \min\{\vec{a}, \vec{b}\}$ and $\vec{d} = \max\{\vec{a}, \vec{b}\}$ (i.e. $c_i = \min\{a_i, b_i\}$ for each i, and similarly for $\vec{d}$). If $\sigma(\vec{a}) = \sigma(\vec{b}) = \mathbf{1}$, necessarily $\sigma(\vec{d}) = \mathbf{1}$, because $\sigma(\vec{a}) \subsetneq \sigma(\vec{d}) \subseteq \mathbf{1}$. So for the pushout map

$$\sigma(\vec{a}) \underset{\sigma(\vec{c})}{\cup} \sigma(\vec{b}) \to \sigma(\vec{d})$$

to be an injection, we must have $\sigma(\vec{c}) = \mathbf{1}$. The claim now follows by this observation and an induction on n.

Hence we can identify a q-simplex $\sigma \neq *_q$ of $D^n(\mathbf{1})$ with the minimal vertex $\vec{a} \in (\mathbf{q})^n$ where $\sigma(\vec{a}) = \mathbf{1}$. Using this we shall now recognize this simplicial set as the smash product of n copies of the simplicial circle $\Delta(\mathbf{1})/\partial\Delta(\mathbf{1})$.

The q-simplices in the product $\Delta(\mathbf{1})^n$ are all the chains $\varnothing \subseteq U_0 \subseteq U_1 \subseteq \cdots \subseteq U_q \subseteq \mathbf{n}$. To any such chain with $\varnothing = U_0$ and $U_n = \mathbf{n}$, we can bijectively associate a vertex $\vec{a} \in (\mathbf{q})^n$ as in Definition 2.4, with $U_i = U_i(\vec{a})$. The remaining chains correspond to simplices in $\partial(\Delta(\mathbf{1})^n)$, and are identified to $*$. This induces a simplicial isomorphism $(\Delta(\mathbf{1})/\partial\Delta(\mathbf{1}))^n \to D^n(\mathbf{1})$.

For general k, a q-simplex $\sigma \neq *_q$ is a lattice on $([q])^n$ of subsets of $\mathbf{k}$. Any such lattice factors as a product over $\mathbf{k}$ of q-simplices in $D^n(\mathbf{1})$, with the ith ($i \in \mathbf{k}$) factor detecting at what vertex i first appears in the subset. If the base point of $D^n(\mathbf{1})$ appears in some factor i, none of the subsets of $\mathbf{k}$ occurring can contain i, so that top subset of σ cannot be all of $\mathbf{k}$. Hence the entire fat wedge in $(D^n(\mathbf{1}))^k$ is collapsed to $*$. This proves that $D^n(\mathbf{k}) \cong D^n(\mathbf{1})^{\wedge k} \cong S^{nk}$. ■

5. *Pick sites and submodule configurations*

We can characterize the lattice diagrams on $([q])^n$ by certain distinguished vertices of $(\mathbf{q})^n$, called the *pick sites*, and the R-modules occurring there, called the *submodule configuration*. This section defines these terms.

LEMMA 5.1. *The isomorphism class of an object σ in $wS_q^n\mathscr{F}(R)$, i.e. a q-simplex of $B^n\mathbf{K}R$, is precisely determined by the ranks* $\{\mathrm{rank}(\sigma(\vec{a}))\}_{\vec{a}\in(\mathbf{q})^n}$.

Proof. Use upward induction in $(\mathbf{q})^n$ and the lattice conditions to reduce the lemma to finding isomorphisms $f: M' \to N'$ completing the diagram below, where $\mathrm{rank}(M') = \mathrm{rank}(N')$:

$$\begin{array}{ccc} M & \rightarrowtail & M' \\ \downarrow\cong & & \downarrow f \\ N & \rightarrowtail & N' \end{array}$$

The existence of such f follows from splitness of cofibrations. ■

Definition 5.2. The *rank data* of a simplex σ in $B^n\mathbf{K}R$ is the n-cube $\mathrm{rd}(\sigma) = \{\vec{a} \mapsto \mathrm{rank}\,\sigma(\vec{a})\}$.

This can be viewed as sitting in the n-fold (diagonal) bar construction $B^n\mathbb{N} \simeq K(\mathbb{Z}, n)$ on the additive monoid of the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. The simplicial map

$$\mathrm{rd}: B^n\mathbf{K}R \twoheadrightarrow B^n\mathbb{N}$$

can also be viewed, by Lemma 5.1, as applying $X \to \pi_0(X)$ (path components) levelwise to the simplicial space $([q] \mapsto |wS^n_q \mathscr{F}(R)|) = B^n\mathbf{K}R$.

What do the lattice conditions determine about the ranks that can occur as rank data of a $\sigma \in B^n\mathbf{K}R$? Fix a σ, and consider a vertex $\vec{a} \in (\mathbf{q})^n$. Let $\vec{b} = (b_1, \ldots, b_n)$ with each $b_i = a_i - 1$. $\vec{a}$ and $\vec{b}$ span an elementary subcube of σ as in Section 2. Let $U \mapsto \sigma(\vec{x}_U) = M_U$ for $U \subseteq \mathbf{n}$ be an indexing of this subcube. The lattice condition—that

$$\operatorname*{colim}_{U \subset \mathbf{n}} M_U \rightarrowtail M_{\mathbf{n}}$$

is a cofibration—implies that the alternating sum of ranks

$$\sum_{U \subseteq \mathbf{n}} (-1)^{n-|U|} \operatorname{rank}(M_U)$$

is non-negative; in fact this sum equals the rank of the quotient of the cofibration above.

Definition 5.3. Call the sum above the *rank jump* of σ at $\vec{a}$, denoted $\mathrm{rj}_\sigma(\vec{a})$.

LEMMA 5.4. *Let σ be a q-simplex of $B^n\mathbf{K}R$. For any $\vec{x} \in (\mathbf{q})^n$*

$$\sum_{\vec{a} \leq \vec{x}} \mathrm{rj}_\sigma(\vec{a}) = \operatorname{rank}(\sigma(\vec{x})).$$

Proof. Expand the definition of the rank jumps. For each $\vec{a} \leq \vec{x}$ in $([q])^n$ the term $\operatorname{rank}(\sigma(\vec{a}))$ occurs equally often with opposite signs, except when $\vec{a} = \vec{x}$ or $\vec{a} \notin (\mathbf{q})^n$. In the latter cases $\sigma(\vec{a}) = 0$. ■

Now restrict to $D^n(R^k)$. Let $\sigma \neq *$ be a q-simplex in $D^n(R^k)$.

COROLLARY 5.5.

$$\sum_{\vec{a} \in (\mathbf{q})^n} \mathrm{rj}_\sigma(\vec{a}) = k.$$

Hence $\mathrm{rj}_\sigma(\vec{a})$ is nonzero only at k vertices of $(\mathbf{q})^n$, counting multiplicities. ($\vec{a}$ has multiplicity $\mathrm{rj}_\sigma(\vec{a})$.) ■

Definition 5.6. The vertices $\vec{a} \in (\mathbf{q})^n$ where $\mathrm{rj}_\sigma(\vec{a}) > 0$ are called the *pick sites* of σ. The submodules of R^k occurring at the pick sites, $\{\sigma(\vec{a}) \mid \mathrm{rj}_\sigma(\vec{a}) > 0\}$, are called the *submodule configuration* of σ. The pick sites are viewed as an unordered set. The submodule configuration is viewed as a family indexed by the pick sites.

LEMMA 5.7. *The pick sites and submodule configuration precisely determine σ.*

Proof. If we know $\sigma(\vec{a}) \subseteq R^k$ for all $\vec{a}$ where $\mathrm{rj}_\sigma(\vec{a}) > 0$, we can reconstruct all other $\sigma(\vec{a})$ by upward induction along $([q])^n$, since for all $\vec{a}$ other than the pick sites,

$$\operatorname*{colim}_{U \subset \mathbf{n}} M_U \to M_{\mathbf{n}} = \sigma(\vec{a})$$

is the identity map. This determines all $\sigma(\vec{a})$, and the inclusion morphisms in σ are of course unique. ■

Clearly any k-tuple of vertices in $(\mathbf{q})^n$ can occur as pick sites. However, given the pick sites there are restrictions on what submodules can appear there. In particular, the rank of $\sigma(\vec{a})$ must equal the number of pick sites $\vec{p} \in (\mathbf{q})^n$ with $\vec{p} \leq \vec{a}$.

6. *Apartments covering $D^n(R^k)$*

We now give an alternative description of how to construct apartments in $D^n(R^k)$. These are subcomplexes of $D^n(R^k)$ homeomorphic to S^{nk}, obtained as GL_kR-translates of the standard apartment discussed in Section 4. For $n = 1$ this recovers the doubly-suspended apartments of the Tits building. We show that the apartments cover the n-dimensional building $D^n(R^k)$.

Fix a $g \in GL_kR$. Its columns $g_1, \ldots, g_k \in R^k$ form an R-basis for R^k. Each g_i spans a *line* $l_i = Rg_i \subseteq R^k$, i.e. a rank one free submodule of R^k with $l_i \rightarrowtail R^k$. For any k-tuple of vertices $\vec{p}_1, \ldots, \vec{p}_k$ in $(\mathbf{q})^n$, there is associated a q-simplex $\sigma(g; \vec{p}_1, \ldots, \vec{p}_k)$ in $D^n(R^k)$:

$$\vec{a} \mapsto \bigoplus_{\vec{p}_i \leq \vec{a}} l_i \subseteq R^k$$

for $\vec{a} \in ([q])^n$, with the obvious inclusions.

Definition 6.1. Let $A(g) \subseteq D^n(R^k)$ be the union of the simplices $\sigma(g; \vec{p}_1, \ldots, \vec{p}_k)$ as the $\{\vec{p}_i\}$ vary, together with the base point $*$. These subcomplexes $A(g)$, with $g \in GL_kR$, are the *apartments* of $D^n(R^k)$.

LEMMA 6.2. *The q-simplex $\sigma(g; \vec{p}_1, \ldots, \vec{p}_k)$ has pick sites $\{\vec{p}_1, \ldots, \vec{p}_k\}$, and its submodule configuration has the module*

$$\bigoplus_{\vec{p}_j \leq \vec{p}_i} l_j \subseteq R^k.$$

at $\vec{p}_i$, $i \in \mathbf{k}$. ■

LEMMA 6.3. *Let $e \in GL_kR$ be the identity. $A(e)$ equals the standard apartment A, and $A(g) = g \cdot A \subseteq D^n(R^k)$.* ■

This lemma is clear, and indicates how to check that $\sigma(g; \vec{p}_1, \ldots, \vec{p}_k)$ is a lattice from the analysis in Section 4 for $A = D^n(\mathbf{k})$.

PROPOSITION 6.4. *The apartments $\{A(g)\}_{g \in GL_kR}$ cover $D^n(R^k)$.*

Proof. Fix a q-simplex $\sigma \in D^n(R^k)$. If $\sigma = *_q$, it occurs in each $A(g)$. Otherwise σ has k pick sites $\vec{p}_1, \ldots, \vec{p}_k \in (\mathbf{q})^n$. At each pick site $\vec{p}_i$ choose a basis $\mathscr{B}_i$ for the kernel of a splitting of the cofibration

$$\operatorname*{colim}_{U \subset \mathbf{n}} M_U \rightarrowtail M_{\mathbf{n}} = \sigma(\vec{p}_i)$$

associated to the elementary cube directly below $\vec{p}_i$. In other terms, choose a basis for a complementary free summand to the image of the colimit in $M_{\mathbf{n}}$. Use the same basis $\mathscr{B}_i$ for $\vec{p}_i$ and $\vec{p}_j$ if $\vec{p}_i = \vec{p}_j$ in $(\mathbf{q})^n$.

This gives a collection of k elements $\mathscr{B} = \bigcup_i \mathscr{B}_i = \{g_1, \ldots, g_k\}$ in R^k. We claim that for each $\vec{a}$, $\sigma(\vec{a})$ has R-basis the $\{g_j\}_j = \bigcup_{\vec{p}_i \leq \vec{a}} \mathscr{B}_i$ associated to the pick sites $\vec{p}_i \leq \vec{a}$. This is immediate by upward induction along $([q])^n$, as each colimit (or iterated pushout)

$\operatorname{colim}_{U\subset\mathfrak{n}} M_U$ will be spanned by the union of the $\mathscr{B}_i$ associated with pick sites strictly below $\hat{a}$. In particular, $\mathscr{B} = \{g_1, \ldots, g_k\}$ forms an R-basis for R^k. Let $g \in GL_kR$ have ith column g_i. Then $\sigma = \sigma(g; \tilde{p}_1, \ldots, \tilde{p}_k) \in A(g)$. ∎

THE POSET FILTRATION

7. *Posets*

The results we need about partially ordered sets (posets) are collected here.

Definition 7.1. A *poset* ω is a small category where for each pair of objects a and b there is at most one morphism from a to b, denoted $a \to b$. If the morphism exists, we write $a \le b$, and if further there is no morphism from b to a, we write $a < b$. A poset is depicted by listing its objects and a (minimal) set of generating morphisms, separating components by commas. $a \leftrightarrows b$ is shorthand for $a \to b$ and $b \to a$.

Fixing an underlying (object) set $\mathbf{k}$, the set of poset structures ω on $\mathbf{k}$ forms a poset again, $\mathrm{Po}(\mathbf{k})$. We set $\omega \le \omega'$ if ω is the *stronger* poset, i.e. ω' is a subcategory of ω. The *discrete* poset on $\mathbf{k}$, δ_k, has only the identity morphisms, and is maximal in $\mathrm{Po}(\mathbf{k})$. The *indiscrete* poset on $\mathbf{k}$, ι_k, has a morphism from each a to each b, and is minimal in $\mathrm{Po}(\mathbf{k})$.

The symmetric group Σ_k acts on $\mathrm{Po}(\mathbf{k})$. If $\pi \in \Sigma_k$, and ω is a poset on $\mathbf{k}$, $\pi \cdot \omega$ is the poset on $\mathbf{k}$ such that $a \to b$ in $\pi \cdot \omega$ if and only if $\pi^{-1}a \to \pi^{-1}b$ in ω. The Σ_k-orbit of ω is its isomorphism class $[\omega]$, and the isomorphism classes $[\omega]$ of posets on $\mathbf{k}$ form a poset $\mathrm{IPo}(\mathbf{k})$, analogously to $\mathrm{Po}(\mathbf{k})$. Let $(\Sigma_k)_\omega = \{\pi \in \Sigma_k \mid \pi \cdot \omega = \omega\}$ denote the isotropy subgroup.

Write $\omega_i = \{j \in \mathbf{k} \mid j \to i \text{ in } \omega\}$. The collection of sets $\{\omega_i\}_i$ determines ω.

The *components* of ω are the maximal subposets with connected underlying space (as a category). Let $|\omega|$ denote the number of components. ω is *linear* if for any a and b we have $a \le b$ or $b \le a$. The indiscrete poset is thus linear. A poset ω is *componentwise linear* if each connected component is linear. The *size* of a finite poset ω is

$$\mathrm{size}(\omega) = |\omega| + (\text{number of equivalence classes in } \omega) - 2.$$

Here a and b in ω are *equivalent* if $a \leftrightarrows b$ in ω. Let $\mathscr{L}_s(k) \subseteq \mathrm{IPo}(\mathbf{k})$ denote the set of isomorphism classes of componentwise linear posets of size s.

LEMMA 7.2. *If* $\omega \le \omega'$ *then* $\mathrm{size}(\omega) \le \mathrm{size}(\omega')$. *If* ω' *is componentwise linear, then* $\omega < \omega'$ *implies* $\mathrm{size}(\omega) < \mathrm{size}(\omega')$. ∎

Definition 7.3. Call $i \to j$ in ω *indecomposable* if $j \not\to i$ and there is no $a \in \mathbf{k}$ with $i \to a \to j$ such that $a \not\to i$ and $j \not\to a$. The *length* of a linear poset is one less than the number of non-equivalent objects in it, or equivalently its maximal number of composable indecomposable morphisms.

A morphism $i \to j$ is *extremal with respect to* ω if $i \not\to j$ in ω, i is minimal among the a such that $a \not\to j$ in ω, and j is maximal among the b such that $i \not\to b$ in ω. Let $\omega + (i \to j)$ denote the poset obtained by adjoining the morphism $i \to j$ to ω.

LEMMA 7.4. *The maximal elements in* $\{\omega' \in \mathrm{Po}(\mathbf{k}) \mid \omega' < \omega\}$ *are exactly the* $\omega' = \omega + (i \to j)$ *where* $i \to j$ *is extremal with respect to* ω.

Proof. If $\omega' < \omega$, there exists $i \to j$ in ω' but not in ω. Either $i \to j$ is extremal with respect to ω, or we can find $i' \to i$ and $j \to j'$ in ω such that $i' \to j'$ in ω', but not in ω, and $i' \not\leftrightarrows i$ or $j \not\leftrightarrows j'$. Either $i' \to j'$ is now extremal with respect to ω, or we can repeat the process. As long as the latter holds, we obtain a decreasing sequence $i, i', i'', \ldots$ and an increasing sequence $j, j', j'', \ldots$. As $\mathbf{k}$ is finite, the process must eventually halt. Hence we can find $i \to j$ in ω' but not in ω which is extremal with respect to ω. Then $\omega' \leq \omega + (i \to j) < \omega$. ■

Definition 7.5. Suppose the poset ω has underlying set $\mathbf{k}$. A subset $I \subseteq \mathbf{k}$ is called an *order ideal* if $\omega_i \subseteq I$ for each $i \in I$.

8. *The poset filtration of* $D^n(R^k)$

In this section we introduce the *poset filtration* on $D^n(R^k)$ by observing that the pick sites of a lattice inherit a partial ordering from the n-cube in which they are located. A filtration indexed over the isomorphism classes of partial orderings on $\{1, \ldots, k\}$ results, and we describe the subquotients of this filtration in terms of an analogous filtration on the standard apartment A, and a $GL_k R$-set of submodule configurations.

Consider a q-simplex σ of $D^n(R^k)$, different from $*$. Choose some numbering of its pick sites $(\vec{p}_1, \ldots, \vec{p}_k)$. This determines a function $\mathbf{k} \to (\mathbf{q})^n$, and hence induces a poset structure ω_σ on $\mathbf{k}$ by pullback; this is the strongest ordering for which the function is order preserving. Choosing a different numbering of the pick sites induces an isomorphic poset structure, as the results will differ only by the action of an element in Σ_k. Hence σ determines a unique isomorphism class $[\omega_\sigma] \in \mathrm{IPo}(\mathbf{k})$. Thus there is a filtration $\{F_{[\omega]} D^n(R^k)\}$ of $D^n(R^k)$ indexed by the poset of isomorphism classes of posets, $\mathrm{IPo}(\mathbf{k})$:

Definition 8.1. Let $F_{[\omega]} D^n(R^k)$ consist of (the base simplex $*$ and) the simplices whose pick sites satisfy the relations of $[\omega]$, i.e. $[\omega_\sigma] \leq [\omega]$. As $[\omega]$ varies, this is the *poset filtration* of $D^n(R^k)$.

The standard apartment carries the further structure of a poset filtration indexed by $\mathrm{Po}(\mathbf{k})$, i.e. the actual poset structures on $\mathbf{k}$. This is because there is a canonical numbering of the pick sites of a (non-basepoint) simplex σ in A, given by insisting that the ith pick site $\vec{p}_i$ is the initial site $\vec{x} \in (\mathbf{q})^n$ where $i \in \mathbf{k}$ occurs, i.e. $i \in \sigma(\vec{x})$. Thus we have a filtration $\{F_\omega A\}_{\omega \in \mathrm{Po}(\mathbf{k})}$ of the standard apartment, as defined below:

Definition 8.2. Let $F_\omega A$ consist of ($*$ and) the simplices $\sigma \in D^n(\mathbf{k}) \subsetneq D^n(R^k)$ for which the pick sites with the canonical numbering satisfy $\vec{p}_i \leq \vec{p}_j$ in $(\mathbf{q})^n$ whenever $i \leq j$ in ω.

Definition 8.3. Let $\{F_{[\omega]} A(g)\}$ denote the restriction of the poset filtration to an apartment $A(g)$ of $D^n(R^k)$, i.e. $F_{[\omega]} A(g) = F_{[\omega]} D^n(R^k) \cap A(g)$.

Write $F_{<[\omega]} D^n(R^k) = \bigcup_{\omega' < \omega} F_{[\omega']} D^n(R^k)$ for the subcomplex of $F_{[\omega]} D^n(R^k)$ occurring 'earlier' in the filtration, and similarly for $F_{<[\omega]} A(g)$ and $F_{<\omega} A$.

LEMMA 8.4.

$$F_{[\omega]} A = \Sigma_k \cdot F_\omega A = \bigcup_{\omega' \cong \omega} F_{\omega'} A \subseteq A$$

$$F_{[\omega]} D^n(R^k) = \bigcup_{g \in GL_k R} F_{[\omega]} A(g)$$

$$F_{[\omega]} A / F_{<[\omega]} A \cong \Sigma_{k+} \underset{(\Sigma_k)_\omega}{\wedge} (F_\omega A / F_{<\omega} A).$$ ■

Note that $F_{[\omega]} D^n(R^k)$ is $GL_k R$-invariant, and preserved by the connecting maps in the quotient spectrum $F_k \mathbf{K} R / F_{k-1} \mathbf{K} R$. Hence we obtain a spectrum-with-$GL_k R$-action

$F_{[\omega]}\mathbf{D}(R^k) = \{F_{[\omega]}D^n(R^k)\}_n$, and similarly there are spectra $F_\omega \mathbf{A}_k$, $F_{[\omega]}\mathbf{A}_k$, $F_{<\omega}\mathbf{A}_k$, $F_{<[\omega]}\mathbf{A}_k$ and $F_{<[\omega]}\mathbf{D}(R^k)$.

Next consider a simplex σ in the standard apartment A, with $\omega_\sigma = \omega$. Its submodule configuration consists of the axial submodules R^{ω_i} in R^k as i varies through $\mathbf{k}$. Hence the isotropy of the GL_kR-action on this submodule configuration, *which fixes the indexing of the axial submodules*, is the parabolic subgroup defined below. As usual the orbit of this action can be identified with GL_kR/P_ω.

Definition 8.5. For any partial ordering ω on $\mathbf{k}$ let P_ω denote the *parabolic subgroup* $\{g \in GL_kR \mid gR^{\omega_i} = R^{\omega_i} \text{ for all } i \in \mathbf{k}\}$.

We have the following description of the subquotients of the poset filtration.

PROPOSITION 8.6. *For each* ω,

$$F_{[\omega]}D^n(R^k)/F_{<[\omega]}D^n(R^k) \cong GL_kR/P_{\omega+} \underset{(\Sigma_k)_\omega}{\wedge} (F_\omega A/F_{<\omega}A)$$

GL_kR*-equivariantly.*

Proof. We use the inclusion $A \subsetneq D^n(R^k)$ to identify $Y \overset{\text{def}}{=} F_\omega A/F_{<\omega}A$ with a subcomplex of $X \overset{\text{def}}{=} F_{[\omega]}D^n(R^k)/F_{<[\omega]}D^n(R^k)$. Every simplex (except $*$) of X is a lattice on $([q])^n$ whose pick sites precisely satisfy ω when suitably numbered. The simplices in Y are those lattices where furthermore all the modules in the submodule configuration are axial.

As usual GL_kR acts on X by moving submodules of R^k around. We claim that every simplex of X is in the GL_kR-orbit of some simplex in Y.

For any simplex $\sigma \neq *$ of Y, the subgroups P_ω and $(\Sigma_k)_\omega$ map σ into Y. We claim that the set of elements of GL_kR which take σ into Y is precisely the product of subgroups $(\Sigma_k)_\omega \cdot P_\omega$. In particular $(\Sigma_k)_\omega \cdot P_\omega$ acts on Y.

Note that $(\Sigma_k)_\omega$ normalizes P_ω, so that $(\Sigma_k)_\omega$ acts on GL_kR/P_ω. Using the two claims, we can now conclude

$$X \cong GL_kR_+ \underset{(\Sigma_k)_\omega \cdot P_\omega}{\wedge} Y \cong GL_kR/P_{\omega+} \underset{(\Sigma_k)_\omega}{\wedge} Y$$

as P_ω acts trivially on Y. It remains to prove the two claims.

To prove the first claim, we use Proposition 6.4. Any simplex $\sigma \neq *$ in X lies in an apartment $A(g)$, for some $g \in GL_kR$. Then $g^{-1} \cdot \sigma$ lies in A, and the partial ordering of its pick sites is isomorphic to ω. The isomorphism corresponds to a renumbering of the pick sites, which can be effected by applying a permutation from $\Sigma_k \subseteq GL_kR$. Hence σ can be moved into Y by an element of GL_kR.

To prove the second claim, consider a simplex $\sigma \neq *$ in Y, and a $g \in GL_kR$ such that $g \cdot \sigma$ is in Y. Both σ and $g \cdot \sigma$ have the same pick sites and submodule configuration as sets, but the pick sites may be numbered differently. The renumbering corresponds to the action of an element π of Σ_k, which actually must come from $(\Sigma_k)_\omega$ as σ and $g \cdot \sigma$ both have their pick sites ordered precisely as per ω. Then $\pi^{-1} \cdot g$ fixes both the pick sites and the submodule configuration, and hence must lie in P_ω. ∎

9. Homotopy type of apartment subcomplexes

This section computes the weak equivariant homotopy type of the various subcomplexes occurring in the poset filtration of the standard apartment, up to one special case.

That case will be handled in Section 11.

Let $A = A_{n,k} \subseteq D^n(R^k)$ be the standard apartment, ω a poset on $\mathbf{k}$. Then:

PROPOSITION 9.1. *$F_\omega A_{n,k} \cong S^{nl}$ if ω has l components, all indiscrete. Otherwise it is contractible.*

COROLLARY 9.2.

$$F_\omega \mathbf{A} \cong \begin{cases} \mathbf{S} & \text{if } \omega = \iota_k, \\ * & \text{otherwise.} \end{cases}$$ ■

The proof begins with an easy lemma.

LEMMA 9.3. *If ω has components $\omega_1, \ldots, \omega_l$, ($\omega = \omega_1 \amalg \ldots \amalg \omega_l$) then*

$$F_\omega A \cong F_{\omega_1} A \wedge \cdots \wedge F_{\omega_l} A.$$ ■

Proof of Proposition 9.1. By the lemma we may assume ω is connected. If ω is indiscrete, all the pick sites of a simplex ($\neq *$) coincide in $(\mathbf{q})^n$, hence carry the full module R^k in the submodule configuration. Identify a simplex with this pick site as before, to recover

$$F_{\iota_k} A_{n,k} \cong D^n(\mathbf{1}) \cong S^n.$$

On the other hand, if ω is not indiscrete there exists $i < j$ in ω.

Recall that a q-simplex $\sigma \in A$ ($\sigma \neq *$) corresponds to its k pick sites $(\vec{p}_1, \ldots, \vec{p}_k)$ where e_i first appears in $\sigma(\vec{p}_i)$. Identify each pick site with an increasing chain of subsets

$$\varnothing = U_0(\vec{p}_i) \subseteq U_1(\vec{p}_i) \subseteq \cdots \subseteq U_q(\vec{p}_i) = \mathbf{n}$$

as in Section 2. So σ corresponds to a k-tuple of increasing chains of subsets of $\mathbf{n}$, or equivalently an increasing chain of k-tuples of subsets of $\mathbf{n}$.

To complete the proof, we introduce some notation. Let T' denote the poset of k-tuples $Z = (Z_1, \ldots, Z_k)$ of arbitrary subsets of $\mathbf{n}$, ordered by inclusion in each of the k components. T'_ω is the subposet of T' of k-tuples satisfying the reverse inclusions of ω, i.e. $Z_i \supseteq Z_j$ if $i \leq j$ in ω. Let $\hat{0} = (\varnothing, \ldots, \varnothing)$, and $\hat{n} = (\mathbf{n}, \ldots, \mathbf{n})$. Set $T = T' - \{\hat{0}, \hat{n}\}$, $T_\omega = T'_\omega - \{\hat{0}, \hat{n}\}$. Then we can identify $F_\omega A$ with the quotient of the nerve of T'_ω obtained by collapsing to $*$ all chains not beginning at $\hat{0}$ and ending at $\hat{n}$. From this it follows easily that $F_\omega A \cong \Sigma^2 |T_\omega|$.

Now fix $i < j$ in ω. Let $T_1 \subseteq T_\omega$ be the subposet consisting of $Z = (Z_1, \ldots, Z_k)$ such that $Z_a = \mathbf{n}$ if $a \in \omega_i$. Then $r: T_\omega \to T_1$ defined by

$$r(Z)_a = \begin{cases} \mathbf{n} & \text{if } a \in \omega_i \\ Z_a & \text{else,} \end{cases}$$

is a deformation retraction, since $Z \subseteq r(Z)$ for all Z.

Lastly let $T_2 \subseteq T_1$ be the subposet where $Z_a = \varnothing$ if $a \notin \omega_i$. Then $s: T_1 \to T_2$ defined by

$$s(Z)_a = \begin{cases} \mathbf{n} & \text{if } a \in \omega_i \\ \varnothing & \text{else,} \end{cases}$$

is again a deformation retraction, as $Z \supseteq s(Z)$ for all Z. Using $i < j$ we see that T_2 has precisely one element, so $|T_2| = *$, and $F_\omega A$ must be contractible. ■

Definition 9.4. Suppose a complex X is covered by subcomplexes $(X_i)_{i \in V}$. Consider the diagram of complexes indexed on the poset of nonempty finite subsets $U \subseteq V$, ordered by reverse inclusion, with the complex $X[U] \overset{\text{def}}{=} \bigcap_{i\in U} X_i$ at U. The *Mayer–Vietoris blowup* $X^\wedge$ of X is the homotopy direct limit

$$X^\wedge = \operatorname*{hocolim}_{\substack{\varnothing \neq U \subseteq V \\ \text{finite}}} X[U].$$

Explicitly

$$X^\wedge = \bigcup_{\substack{\varnothing \neq U \subseteq V \\ \text{finite}}} \left(\Delta(U) \times \bigcap_{i\in U} X_i \right) \subseteq \Delta(V) \times X.$$

There is also a reduced version, for based complexes.

LEMMA 9.5. *The projection* $\Delta(V) \times X \xrightarrow{pr_2} X$ *induces an equivalence* $X^\wedge \xrightarrow{\simeq} X$.

Proof. Use Quillen's theorem A [14]. ■

We will study $F_{<\omega}A$ in terms of the covering given in the lemma below. It is a direct consequence of Lemma 7.4.

LEMMA 9.6. *$F_{<\omega}A$ is covered by the $F_{\omega'}A$ where $\omega' = \omega + (i \to j)$ with $i \to j$ extremal with respect to ω. Also*

$$\bigcap_{s=0}^{q} F_{\omega + (i_s \to j_s)} A = F_{\omega'} A$$

whenever $\omega' = \omega + (i_0 \to j_0) + \ldots + (i_q \to j_q)$. ■

Definition 9.7. Let $E_\omega = \{\omega + (i \to j) | i \to j$ is extremal with respect to $\omega\}$ denote the indexing set for the covering in Lemma 9.6. Let the *non-indiscrete complex* $\mathrm{NI}(\omega)$ be the subcomplex of $\Delta(E_\omega)$ with faces $\sigma = \{\omega + (i_0 \to j_0), \ldots, \omega + (i_q \to j_q)\}$ such that the *span* $\omega + (i_0 \to j_0) + \cdots + (i_q \to j_q)$ is not indiscrete, i.e. not the poset ι_k. If $i \to j$ is extremal with respect to ω, we will think of $i \to j$ as the element $\omega + (i \to j)$ it determines in E_ω.

LEMMA 9.8. *There exists a $2n$-connected map $F_{<\omega}A^\wedge \to S^n * \mathrm{NI}(\omega)$ for each n, making $F_{<\omega}\mathbf{A} \simeq \Sigma^\infty \Sigma\, \mathrm{NI}(\omega)$ as spectra. Here $*$ denotes the join of spaces.*

Proof. We construct a map from $X \overset{\text{def}}{=} F_{<\omega}A^\wedge$ to $Y \overset{\text{def}}{=} \mathrm{NI}(\omega) \times D^{n+1} \cup \Delta(E_\omega) \times S^n$. Here D^{n+1} denotes an $n+1$-ball with S^n as its boundary. As $\Delta(E_\omega)$ is contractible, Y is homotopy equivalent to the join $S^n * \mathrm{NI}(\omega)$. View X and Y as spaces over $\Delta(E_\omega)$, by projection on the first factor.

Every $F_{\omega'}A$ contains $F_{\iota_k}A \cong S^n$, so by Lemma 9.6 both X and Y contain $\Delta(E_\omega) \times S^n$. We define $f: X \to Y$ to be the identity on this part. The remainder sits above $\mathrm{NI}(\omega)$, and here we extend f over one simplex at a time, by increasing dimension. This is possible as the relevant target is contractible. The resulting map is $2n$-connected by a spectral sequence argument, as above each simplex of $\mathrm{NI}(\omega)$ the 'fiber' in X is some $F_{\omega'}A$ which is $(2n-1)$-connected by Proposition 9.1, while the 'fiber' in Y is contractible. ■

PROPOSITION 9.9. *If ω is not componentwise linear,* $\mathrm{NI}(\omega) \simeq *$.

Proof. Take unrelated a, b in the same component of ω. Pick a path $(a = c_0, c_1, \ldots, c_l = b)$ of minimal length connecting a to b, such that for each

$s = 1, \ldots, l$, c_{s-1} and c_s are related by an indecomposable morphism $c_{s-1} \to c_s$ or $c_s \to c_{s-1}$. Since a and b are unrelated, the path must 'change direction' at least once. Hence we can replace a and b with vertices adjacent to such a turn, and assume there exists a $c \in \mathbf{k}$ with $c \to a$, $c \to b$ both indecomposable. There is also an entirely equivalent case with the arrows going in the opposite direction.

Let ω' be ω with all the extremal morphisms of E_ω adjoined. We now claim that $a \not\to c$ in ω'.

If $a \to c$ in ω' there must be a shortest chain $a = d_0 \to d_1 \to \cdots \to d_m = c$ with each $d_{s-1} \to d_s$ in ω or E_ω. Clearly no $d_{s-1} = d_s$, since the chain is as short as possible. So if $d_{s-1} \to d_s$ is in ω and $d_s \to d_{s+1}$ is in E_ω, by minimality of d_s the composite $d_{s-1} \to d_{s+1}$ must be in ω. Similarly if $d_{s-1} \to d_s$ is in E_ω and $d_s \to d_{s+1}$ is in ω, $d_{s-1} \to d_{s+1}$ is in ω.

Hence either $a = d_0 \to d_1 = c$ in ω, contradicting the choice of a and c, or each $d_{s-1} \to d_s$ is in E_ω. If $m = 1$, this states that $a = d_0 \to d_1 = c$ is in E_ω, but since $a \not\to b$ in ω and $c \to b$, c cannot satisfy the maximality condition in the definition of extremal morphisms with respect to ω.

The case $m \geq 2$ remains. Since each $d_{s-1} \to d_s$ is in E_ω, by upward induction on s, $c \to d_s$ is in ω for all $s = 1, \ldots, m$. Also the composite $d_{m-1} \to d_m = c \to a$ must be in ω. But then the factorisation $c \to d_{m-1} \to a$ in ω contradicts the indecomposability of $c \to a$. This proves the claim.

Hence $\omega' > \iota_k$, and so for each face of $\Delta(E_\omega)$ the span

$$\omega + (i_0 \to j_0) + \cdots + (i_q \to j_q) \geq \omega' > \iota_k$$

is not indiscrete. Therefore $\mathrm{NI}(\omega) = \Delta(E_\omega) \simeq *$. ∎

PROPOSITION 9.10. *If ω is componentwise linear, then*

$$\mathrm{NI}(\omega) \simeq \Sigma^m \mathrm{NI}(\delta_l),$$

where $l = |\omega|$ is the number of components of ω and m is the sum of the lengths of these linear components. We have the formula $\mathrm{size}(\omega) = 2l + m - 2$.

Proof. The extremal morphisms $i \to j$ in E_ω divide into two disjoint classes. On one hand there are $l(l-1)$ 'minmax' morphisms $X_\omega \subseteq E_\omega$ for which i is minimal in one component and j is maximal in some other component. On the other hand there are m 'reverse jump' morphisms $Y_\omega \subseteq E_\omega$ for which j is the immediate predecessor of i in ω, i.e. $j \to i$ in ω is indecomposable. Let X denote the simplex spanned by the former set, Y the simplex spanned by the latter set. Clearly $X * Y \cong \Delta(E_\omega)$. By collapsing the l components of ω to single objects we can identify X with $\Delta(E_{\delta_l})$.

To obtain ι_k by adjoining morphisms to ω, it is necessary and sufficient to adjoin all the 'reverse jump' morphisms and then enough of the 'minmax' morphisms to generate the indiscrete poset ι_l on the set of components. So a face $\{\omega + (i_0 \to j_0), \ldots, \omega + (i_q \to j_q)\}$ in $\Delta(E_\omega) \cong X * Y$ is in $\mathrm{NI}(\omega)$ precisely when either some morphism in Y_ω is missing, or the morphisms in X_ω are not indiscrete on the collapsed components $\mathbf{l}$:

$$\begin{aligned} \mathrm{NI}(\omega) &\cong (X * \partial Y) \cup (\mathrm{NI}(\delta_l) * Y) \\ &\cong \Sigma(\mathrm{NI}(\delta_l) * \partial Y) \\ &\cong \Sigma^m \mathrm{NI}(\delta_l). \end{aligned}$$ ∎

We have now reduced the determination of the stable homotopy type of $F_\omega A / F_{<\omega} A$ to that of $A/F_{<\delta_l} A \simeq \Sigma^2 \mathrm{NI}(\delta_l)$ for varying l, i.e. only the top subquotient of the poset filtration of A remains. This is resolved in Section 11.

THE SPECTRUM HOMOLOGY OF $\mathbf{D}(R^k)$

10. Stable buildings

To show that $\mathbf{D}(R^k)$ is equivalent to a suspension spectrum, we consider the covering of $D^n(R^k)$ by the apartments $\{A(g)\}$ as $g \in GL_k R$ varies, and let n tend to infinity. The Mayer–Vietoris blowup of this covering will provide a model for the stable building $D(R^k)$.

Definition 10.1. Let $T_k = P_{\delta_k} \subseteq GL_k R$ denote the *diagonal torus*. $T_k \cong (R^*)^k$.

LEMMA 10.2. *$GL_k R$ acts transitively on the set of apartments $\{A(g) \mid g \in GL_k R\}$. The isotropy of the standard apartment A is $T_k \Sigma_k \subseteq GL_k R$.* ■

First we need to investigate the intersections $g_0 A \cap \cdots \cap g_q A$. These are related to a poset determined by the $g_0, \ldots, g_q$.

Let $g \in GL_k R$. Choose a $\pi \in \Sigma_k$ such that $g\pi$ has nonzero entries on the diagonal, i.e. all $(g\pi)_{ii} \neq 0$. To see that this can be done, note that every set of c columns of g contains nonzero entries in at least c distinct rows, by the assumed strong invariant dimension property, and apply Hall's theorem [11] (also known as the solution to the marriage problem).

Definition 10.3. Let $\omega(g)$ denote the poset on $\mathbf{k}$ generated by the morphisms $i \to j$ where $(g\pi)_{ij} \neq 0$.

LEMMA 10.4. *$\omega(g)$ is independent of the choice of π.*

Proof. Momentarily let $\omega(g)$ be defined only for $g \in GL_k R$ with nonzero diagonal entries, as the poset generated by the $i \to j$ with $g_{ij} \neq 0$. Suppose g and $g\pi$ both have nonzero diagonal entries, with $g \in GL_k R$, $\pi \in \Sigma_k$. For all j, $0 \neq g_{jj} = (g\pi)_{j\pi^{-1}j}$, so $j \to \pi^{-1}(j)$ in $\omega(g\pi)$. Since π is of finite order, $\pi^{-1}(j) \to j$ too in $\omega(g\pi)$. So if $0 \neq g_{ij} = (g\pi)_{i\pi^{-1}j}$, then $i \to \pi^{-1}(j) \to j$ in $\omega(g\pi)$. Hence $\omega(g\pi) \leq \omega(g)$, and the same argument for $g\pi$ and $g\pi \cdot \pi^{-1} = g$ shows that $\omega(g) = \omega(g\pi)$. ■

Definition 10.5. If $g_1, \ldots, g_q \in GL_k R$, let $\omega(g_1, \ldots, g_q) = \omega(g_1) + \cdots + \omega(g_q)$ be the span of the posets on the right.

Using this definition we can state:

LEMMA 10.6. *$g_0 A \cap g_1 A \cap \cdots \cap g_q A = g_0 F_\omega A$, where $\omega = \omega(g_0^{-1} g_1, \ldots, g_0^{-1} g_q)$.*

Proof. We prove the special case $A \cap gA = F_{\omega(g)} A$, from which the general case follows by the second part of Lemma 9.6.

We can assume g has nonzero diagonal entries. A q-simplex $\sigma \neq *_q$ in $A \cap gA \subseteq A$ is an axial lattice on $([q])^n$, such that each submodule $\sigma(\vec{x}) = R^I \subseteq R^k$ occurring is also the span of a set of column 'vectors' of g, i.e. $\sigma(\vec{x}) = R^I = gR^J$ for some $J \subseteq \mathbf{k}$. Since $g_{ii} \neq 0$ for all i, $I = J$ and so $gR^I = R^I$.

First we prove $A \cap gA \subseteq F_{\omega(g)} A$. Let $\sigma \in A \cap gA$ have pick sites $(\vec{p}_i)$ with $\sigma(\vec{p}_i) = R^{I_i}$, with $i \in I_i = \{s \in \mathbf{k} \mid \vec{p}_s \leq \vec{p}_i\}$. Suppose $g_{ij} \neq 0$ so $i \to j$ in $\omega(g)$. From $\sigma \in A \cap gA$ we saw that $gR^{I_j} = R^{I_j}$, so $g \cdot e_j \in gR^{I_j} = R^{I_j}$ has nonzero ith coordinate, whence $i \in I_j$ and $\vec{p}_i \leq \vec{p}_j$. So the pick sites of σ satisfy $\omega(g)$, and $\sigma \in F_{\omega(g)} A$.

Conversely, if $\sigma \in F_{\omega(g)} A$, again write $(\vec{p}_i)$ for its pick sites, with I_i as above. We claim that $gR^{I_i} = R^{I_i}$ for all i. For if some $g_{st} \neq 0$ with $t \in I_i$, then $s \to t$ in $\omega(g)$, so $\vec{p}_s \leq \vec{p}_t$.

Thus $I_s \subseteq I_t \subseteq I_i$ and $s \in I_i$, which proves the claim. Hence $g^{-1}\sigma$ is also axial, and $\sigma \in A \cap gA$. ■

COROLLARY 10.7. *When $\omega(g_0^{-1}g_1, \ldots, g_0^{-1}g_q) = \iota_k$, $(g_0 A \cap \cdots \cap g_q A) \cong S^n$. Otherwise $(g_0 A \cap \cdots \cap g_q A)$ is at least $(2n-1)$-connected.* ■

In view of this high connectivity result, we obtain a good approximation to the Mayer–Vietoris blowup associated to this covering as follows:

Definition 10.8. Let $\Sigma^{-1}D(R^k) \subseteq \Delta(GL_k R/T_k\Sigma_k)$ denote the $GL_k R$-subcomplex with faces $\{g_0 T_k \Sigma_k, \ldots, g_q T_k \Sigma_k\}$ such that $\omega(g_0^{-1}g_1, \ldots, g_0^{-1}g_q) \neq \iota_k$. The *stable building* is the suspension $D(R^k) = \Sigma(\Sigma^{-1}D(R^k))$.

THEOREM 10.9. *$\Sigma^\infty(D(R^k)) \cong \mathbf{D}(R^k)$ $GL_k R$-equivariantly.*

Proof. Let $D^n(R^k)^\wedge$ be the Mayer–Vietoris blowup associated to the covering $\{A(g)\}$ of $D^n(R^k)$. By an argument similar to that in the proof of Lemma 9.8, and Corollary 10.7, there is a $2n$-connected $GL_k R$-map $D^n(R^k)^\wedge \to S^n * \Sigma^{-1}D(R^k)^\wedge \cong \Sigma^n D(R^k)$. ■

11. Configuration spaces

In Sections 8 and 9 we described the subquotients of the poset filtration on $D^n(R^k)$ in terms of submodule configurations and the complex $F_{<\delta_l}A_{n,l} \subset A_{n,l}$ for various l. Here we relate this space to configuration spaces.

$F_{<\delta_k}A \subset A \cong S^{nk}$ is the subcomplex of k-tuples of pick sites which satisfy some relation in $(\mathbf{q})^n$. It has a geometric description.

Definition 11.1. Let the *constellation space* $\mathscr{C}(\mathbb{R}^n, k) \subseteq (\mathbb{R}^n)^k$ be the space of k-tuples $(\vec{x}_1, \ldots, \vec{x}_k)$ in $\mathbb{R}^n$ such that no $\vec{x}_i \leq \vec{x}_j$ in the product ordering on $\mathbb{R}^n$ when $i \neq j$.

$\mathscr{C}(\mathbb{R}^n, k)$ embeds as an open submanifold of $(S^n)^{\wedge k} \cong S^{nk}$.

Definition 11.2. Let the *singular space* $\mathscr{S}(\mathbb{R}^n, k)$ be the subcomplex $S^{nk} - \mathscr{C}(\mathbb{R}^n, k)$ of $(S^n)^{\wedge k}$, consisting of $*$ and $(\vec{x}_1, \ldots, \vec{x}_k)$ in $\mathbb{R}^n$ such that some $\vec{x}_i \leq \vec{x}_j$ with $i \neq j$.

LEMMA 11.3. *Σ_k-equivariantly $F_{<\delta_k}A_{n,k} \cong \mathscr{S}(\mathbb{R}^n, k)$.*

Proof. Using the simplicial isomorphism ($I = \Delta(\mathbf{1})$, the unit interval) $A_{n,k} \cong (I^n/\partial I^n)^{\wedge k}$ we obtain homeomorphisms for each poset ω:

$$f: F_\omega A \xrightarrow{\cong} F_\omega(I^n/\partial I^n)^{\wedge k}.$$

Here $F_\omega(I^n/\partial I^n)^{\wedge k} \subseteq (I^n/\partial I^n)^{\wedge k}$ is the space of k-tuples $(y_1, \ldots, y_k)$ of points in I^n with $y_i \geq y_j$ whenever $i \to j$ in ω, identified to $*$ if some $y_i \in \partial I^n$. Gluing shows that $f: F_{<\delta_k}A \to F_{<\delta_k}(I^n/\partial I^n)^{\wedge k}$ is a homeomorphism. Clearly $F_{<\delta_k}(I^n/\partial I^n)^{\wedge k} \cong \mathscr{S}(\mathbb{R}^n, k)$ by a suitable homeomorphism. ■

Definition 11.4. Suppose W is some Euclidean space. Let $\mathscr{F}(W, k)$ denote the *configuration space* of ordered k-tuples of distinct points in W.

LEMMA 11.5. *Σ_k-equivariantly $\mathscr{C}(\mathbb{R}^n, k) \simeq \mathscr{F}(\mathbb{R}^{n-1}, k)$.*

Proof. Let $s: \mathbb{R}^n \to \mathbb{R}$ take $\vec{x} = (x_i)_i$ to $\sum_{i=1}^n x_i$. Set $W = \ker(s) \subset \mathbb{R}^n$, and project $\pi: \mathbb{R}^n \to W$ along lines $\vec{x} + \mathbb{R}\cdot(1, \ldots, 1)$. (π, s) determines an isomorphism $\mathbb{R}^n \cong W \oplus \mathbb{R}$.

We obtain a deformation retraction from $\mathscr{C}(\mathbb{R}^n, k)$ to $\mathscr{F}(W, k)$ by Σ_k-equivariantly deforming each k-tuple of unrelated points $(\vec{x}_1, \ldots, \vec{x}_k)$ along the lines $\vec{x}_i + \mathbb{R}\cdot(1, \ldots, 1)$ to W. The condition that two $\vec{a}, \vec{b} \in \mathbb{R}^n$ are unrelated amounts to $|s(\vec{a}) - s(\vec{b})| < \sqrt{n}\cdot \|\pi(\vec{a}) - \pi(\vec{b})\|$. Here $|\ |$ denotes the absolute value in $\mathbb{R}$, and $\|\ \|$ the Euclidean norm in $W \subset \mathbb{R}^n$. An explicit deformation down to $\mathscr{F}(W, k)$ is then given by:

$$(\vec{x}_i)_i, t \mapsto (\pi(\vec{x}_i), (1-t)s(\vec{x}_i))_i$$

as t goes from 0 to 1. ■

Definition 11.6. Let $V = \mathbb{R}^k$ be the Σ_k-representation given by permutation of the coordinates.

PROPOSITION 11.7. *Σ_k-stably $A/F_{<\delta_k}A$ is S^{nV}-dual to $\mathscr{F}(\mathbb{R}^{n-1}, k)$.*

Proof. Σ_k-stably $A/F_{<\delta_k}A \simeq \Sigma F_{<\delta_k}A$. Also $F_{<\delta_k}A_{n,k}$ and $\mathscr{C}(\mathbb{R}^n, k)$ are complementary Σ_k-subspaces of $A_{n,k}$, which is Σ_k-homeomorphic to S^{nV}. So the proposition follows directly from Lemma 11.5. ■

Fred Cohen [3, 4] has computed the cohomology algebra of configuration spaces:

COMPUTATION 11.8. *$H^*\mathscr{F}(\mathbb{R}^n, k) \cong A(n, k)$ as $\mathbb{Z}\Sigma_k$-algebras, where $A(n, k)$ has generators a_{uv} in degree $(n-1)$ for $k \geq u > v \geq 1$, and relations $a_{uv}^2 = 0$ and $a_{uw}a_{uv} = a_{vw}(a_{uv} - a_{uw})$ if $u > v > w$. Set $a_{vu} = (-1)^n a_{uv}$. Then $\pi \in \Sigma_k$ acts by $\pi\cdot a_{uv} = a_{\pi(u)\pi(v)}$.*

Remark 11.9. This is a left Σ_k-representation on cohomology, $\mathbb{Z}$-dual to the right representation on homology.

Hence the cohomology of $\mathscr{F}(\mathbb{R}^{n-1}, k)$ sits in degrees $0, n-2, \ldots, (k-1)(n-2)$, with top group free on $(k-1)!$ generators $a_{2j_2}a_{3j_3}\cdot\ldots\cdot a_{kj_k}$ with each $i > j_i \geq 1$. By duality, the homology of $A/F_{<\delta_k}A$ sits in degrees $n + (2k-2), n + (2k-2) + (n-2), \ldots, nk$.

Definition 11.10. Let W_k be the $\mathbb{Z}\Sigma_k$-representation given by the bottom homology group of $A/F_{<\delta_k}A$, for n large.

Comparison with Lemma 9.8 shows that this is well-defined. $W_1 = \mathbb{Z}$, the trivial representation, and $W_2 = \mathbb{Z}[-1]$, the sign representation. As an abelian group W_k is free of rank $(k-1)!$. Cohen's computation gives an explicit presentation, including the Σ_k-action, which we will find a simple description of in Section 13.

COROLLARY 11.11. *The stable homology of $\mathbf{A}/F_{<\delta_k}\mathbf{A}$ is isomorphic to W_k, and concentrated in degree $(2k-2)$.* ■

PROPOSITION 11.12. *Stably $H_*(F_\omega\mathbf{A}/F_{<\omega}\mathbf{A})$ is concentrated in degree* size(ω). *It is W_l if ω has l components, all linear, else the homology is 0.*

Proof. By Lemma 9.8, the homology of $\mathrm{NI}(\delta_k)$ is concentrated in degree $2k-4$. The proposition then follows from Propositions 9.9 and 9.10. ■

12. The complex of length $(2k-2)$

From the complete homological analysis of the subquotients of the poset filtration, we can now write down an expression for the spectrum homology of $\mathbf{D}(R^k)$. Our main results follow:

THEOREM 12.1. *The $\mathbb{Z}GL_kR$-module $H_*\mathbf{D}(R^k)$ equals the homology of the complex of length $(2k-2)$ below*

$$\begin{aligned}
&0 \to (\mathbb{Z}GL_kR/P_{\delta_k}) \underset{\Sigma_k}{\otimes} W_k \xrightarrow{d_{2k-2}} (\mathbb{Z}GL_kR/P_{1\to 2}) \underset{(\Sigma_k)_{1\to 2}}{\otimes} W_{k-1} \xrightarrow{d_{2k-3}} \cdots \\
&\xrightarrow{d_{s+1}} \bigoplus_{[\omega]\in\mathscr{L}_s(k)} (\mathbb{Z}GL_kR/P_\omega) \underset{(\Sigma_k)_\omega}{\otimes} W_{|\omega|} \xrightarrow{d_s} \cdots \\
(E^1_*(R^k))\qquad &\xrightarrow{d_1} \mathbb{Z} \to 0.
\end{aligned}$$

Thus the homology is concentrated in degrees 0 *through* $(2k-2)$.

Proof. Filter $\{D^n(R^k)\}_n$ by Φ_s, the union of the $\{F_{[\omega]}D^n(R^k)\}_n$ for which $\mathrm{size}(\omega) \le s$. Here s varies from 0 through $(2k-2)$.

By Propositions 8.6, 9.1 and 9.9, and Lemma 9.8, $\{F_{[\omega]}D^n(R^k)/F_{<[\omega]}D^n(R^k)\}_n$ is stably contractible when $[\omega]\notin\mathscr{L}_s(k)$. If $[\omega]\neq[\omega']\in\mathscr{L}_s(k)$, then by Lemma 7.2 $[\omega]$ and $[\omega']$ are unrelated, i.e. $[\omega]\not\le[\omega']$ and $[\omega']\not\le[\omega]$. Hence Φ_s/Φ_{s-1} stably splits into the summands $\{F_{[\omega]}D^n(R^k)/F_{<[\omega]}D^n(R^k)\}_n$ over the $[\omega]\in\mathscr{L}_s(k)$. So associated to this filtration, by Proposition 11.12, we obtain a one-line ($t=0$) spectral sequence with E^1-term:

$$E^1_{s,0}(R^k) = \bigoplus_{[\omega]\in\mathscr{L}_s(k)} \mathbb{Z}GL_kR/P_\omega \underset{(\Sigma_k)_\omega}{\otimes} W_{|\omega|}$$

converging to $H_*(E^1_*(R^k), d^1) \cong H_*\mathbf{D}(R^k)$ as a $\mathbb{Z}GL_kR$-module. ■

Remark 12.2. The inclusion $\mathbf{A}_k \subsetneq \mathbf{D}(R^k)$ allows a similar filtration by Lemma 8.4, and induces a map of spectral sequences $E^*_{**}(\mathbf{k}) \to E^*_{**}(R^k)$. The analogous complex $E^1_*(\mathbf{k})$ of length $(2k-2)$ is exact for $k\ge 2$, as its homology computes $H_*\mathbf{A}_k = 0$:

$$\begin{aligned}
&0 \to W_k \xrightarrow{d_{2k-2}} \mathbb{Z}\Sigma_k \underset{(\Sigma_k)_{1\to 2}}{\otimes} W_{k-1} \xrightarrow{d_{2k-3}} \cdots \\
(E^1_*(\mathbf{k}))\qquad &\xrightarrow{d_{s+1}} \bigoplus_{[\omega]\in\mathscr{L}_s(k)} \mathbb{Z}\Sigma_k \underset{(\Sigma_k)_\omega}{\otimes} W_{|\omega|} \xrightarrow{d_s} \cdots \\
&\xrightarrow{d_1} \mathbb{Z} \to 0.
\end{aligned}$$

Furthermore the differentials in $E^1_*(R^k)$ are left adjoint to those of $E^1_*(\mathbf{k})$; hence recognizing the differentials on the standard apartment is sufficient to describe the differentials in the theorem.

The exactness of the complex above is also sufficient to reprove by induction on k that W_k is free abelian and, in principle, to recover that its rank is $(k-1)!$.

We will see computational evidence for the following connectivity conjecture in Sections 14 and 15.

CONJECTURE 12.3. *Suppose R is a Euclidean domain or a local ring. Then the $\mathbb{Z}GL_kR$-complex* $E_*(k) \overset{\mathrm{def}}{=} (E^1_*(R^k), d^1)$ *is exact except at* $* = (2k-2)$.

This conjecture implies that $H_*\mathbf{D}(R^k)$ is concentrated in degree $(2k-2)$, and equal to the kernel of d^1_{2k-2}. In this case

$$H_{s+(2k-2)}(\mathbf{D}(R^k)/hGL_kR) \cong H_s(GL_kR; H_{2k-2}E_*(k)),$$

so $F_k\mathbf{K}R/F_{k-1}\mathbf{K}R$ is at least $(2k-3)$-connected, and K_iR is realized as $\pi_iF_k\mathbf{K}R$ for $k \geq i/2 + 1$.

The following three results do not depend on this conjecture:

PROPOSITION 12.4. *There exists a spectral sequence with E^1-term*

$$E^1_{s,t} = H_t(GL_kR; E_s(k)) \cong \bigoplus_{[\omega]\in\mathscr{L}_s(k)} H_t(P_\omega \cdot (\Sigma_k)_\omega; W_{|\omega|})$$

converging to $H_{s+t}(F_k\mathbf{K}R/F_{k-1}\mathbf{K}R)$.

Proof. This is a special case of a based G-complex X, with a finite G-filtration $* = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_{2k-2} = X$ such that $\tilde{H}_*(X_s/X_{s-1})$ is concentrated in degree s for each s. The G-cofibration sequences $X_{s-1} \hookrightarrow X_s \twoheadrightarrow X_s/X_{s-1}$ induce long exact sequences

$$\to \tilde{H}_i(X_{s-1}/hG) \to \tilde{H}_i(X_s/hG) \to \tilde{H}_i((X_s/X_{s-1})/hG) \xrightarrow{\partial}$$

forming an exact couple with

$$E^1_{s,t} = \tilde{H}_{s+t}((X_s/X_{s-1})/hG)$$

converging to $\tilde{H}_{s+t}(X/hG)$. Furthermore $\tilde{H}_{s+t}((X_s/X_{s-1})/hG) \cong H_t(G; \tilde{H}_s(X_s/X_{s-1}))$, as $\tilde{H}_*(X_s/X_{s-1})$ is concentrated in degree s. Hence the E^1-term can be written $E^1_{s,t} = H_t(G; \tilde{H}_s(X_s/X_{s-1}))$.

In our situation, $G = GL_kR$, $X = D^n(R^k)$ for n large, $X_s = \Phi_s$, and $\tilde{H}_s(\Phi_s/\Phi_{s-1}) = E_s(k)$. Recall that

$$E_s(k) = \bigoplus_{[\omega]\in\mathscr{L}_s(k)} \mathbb{Z}GL_kR/P_\omega \underset{(\Sigma_k)_\omega}{\otimes} W_{|\omega|}$$

and use

$$\begin{aligned} H_*(GL_kR; \mathbb{Z}GL_kR/P_\omega \underset{(\Sigma_k)_\omega}{\otimes} W_{|\omega|}) &= H_*(GL_kR; \mathbb{Z}GL_kR \underset{P_\omega\cdot(\Sigma_k)_\omega}{\otimes} W_{|\omega|}) \\ &\cong H_*(P_\omega \cdot (\Sigma_k)_\omega; W_{|\omega|}) \end{aligned}$$

where $P_\omega \cdot (\Sigma_k)_\omega \subseteq GL_kR$ is a subgroup because $(\Sigma_k)_\omega$ acts on P_ω by conjugation. ∎

COROLLARY 12.5. *Suppose R is finite. Then the higher unstable K-groups $\pi_iF_k\mathbf{K}R$, $i > 0$, are $|GL_kR|$-torsion* ($k < \infty$).

Proof. By induction, starting at $F_0\mathbf{K}R \cong *$, it suffices to prove that the spectrum homology groups $H_*(F_k\mathbf{K}R/F_{k-1}\mathbf{K}R)$ are $|GL_kR|$-torsion, except for $k = 1$, $* = 0$. For $k \geq 1$, $\pi_0F_k\mathbf{K}R \cong \mathbb{Z}$. We will suppress that summand for the remainder of the proof. Let E^*_{**} denote the spectral sequence in Proposition 12.4. There exists an analogous spectral sequence for $\mathbf{K}\mathscr{E}$, $\bar{E}^*_{**}$, converging to $H_*(F_k\mathbf{K}\mathscr{E}/F_{k-1}\mathbf{K}\mathscr{E}) = 0$, with

$$\bar{E}^1_{s,t} = \bigoplus_{[\omega]\in\mathscr{L}_s(k)} H_t((\Sigma_k)_\omega; W_{|\omega|}),$$

and $\mathbf{K}\mathscr{E} \to \mathbf{K}R$ induces a map of spectral sequences.

For $t > 0$, $E^1_{s,t}$ is $|GL_kR|$-torsion and $\bar{E}^1_{s,t}$ is $k!$-torsion. For $t = 0$ note that the terms $E^1_{s,0}$ are independent of the ring R. Furthermore

$$\bar{E}^1_{s,0} = H_0((\Sigma_k)_\omega; W_{|\omega|}) \to H_0(P_\omega\cdot(\Sigma_k)_\omega; W_{|\omega|}) = E^1_{s,0}$$

is an isomorphism for each s, as P_ω acts trivially on $W_{|\omega|}$. Hence $\bar{E}^2_{s,0} \cong E^2_{s,0}$.

But since $\bar{E}^\infty_{**} = 0$, $\bar{E}^2_{s,0}$ is $k!$-torsion. $k!$ divides $|GL_kR|$, so all of $E^2_{s,t}$ is $|GL_kR|$-torsion, and as E^2_{**} converges to $H_*(F_k\mathbf{K}R/F_{k-1}\mathbf{K}R)$, this proves the corollary. ■

COROLLARY 12.6. *For each k, $\pi_iF_k\mathbf{K}\mathbb{Z}/p^n \to \pi_iF_k\mathbf{K}\mathbb{F}_p$ is an isomorphism away from the prime p ($k \le \infty$).*

Proof. Let $\varphi: R = \mathbb{Z}/p^n \to \mathbb{F}_p$. Write $P_\omega R \subseteq GL_kR$ and $P_\omega\mathbb{F}_p \subseteq GL_k\mathbb{F}_p$ for the parabolic subgroups. The surjection

$$P_\omega R\cdot(\Sigma_k)_\omega \xrightarrow{\varphi} P_\omega\mathbb{F}_p\cdot(\Sigma_k)_\omega$$

has a kernel which is a finite p-group. Hence φ induces an isomorphism away from p of the spectral sequences in Proposition 12.4 for R and $\mathbb{F}_p$, and so $F_k\mathbf{K}R \xrightarrow{\varphi} F_k\mathbf{K}\mathbb{F}_p$ is a homology equivalence away from p. ■

Assuming the connectivity Conjecture 12.3, we obtain a surjection

$$\pi_{2i-1}F_i\mathbf{K}R \twoheadrightarrow \pi_{2i-1}F_{i+1}\mathbf{K}R \cong K_{2i-1}R,$$

and $\pi_{2i-2}F_i\mathbf{K}R \cong K_{2i-2}R$. So $K_{2i-1}R$ and $K_{2i-2}R$ must be $|GL_iR|$-torsion. Note how this fits neatly with Quillen's computation for $R = \mathbb{F}_q$ [13]. If l is a prime dividing $q^i - 1$ but not $q^{i-1} - 1$, a nontrivial l-component of $\pi_{2i-1}F_k\mathbf{K}\mathbb{F}_q$ will first appear in $F_i\mathbf{K}\mathbb{F}_q$ by Corollary 12.5, then may be hit from the bottom ($2i$th) homotopy of $F_{i+1}\mathbf{K}\mathbb{F}_q/F_i\mathbf{K}\mathbb{F}_q$, but already at $F_{i+1}\mathbf{K}\mathbb{F}_q$ the unstable K-theory realizes actual K-theory; $K_{2i-1}\mathbb{F}_q \cong \mathbb{Z}/(q^i - 1)$. Similarly, a nontrivial l-component may first appear in $\pi_{2i}F_i\mathbf{K}\mathbb{F}_q$, but then dies again in $\pi_{2i}F_{i+1}\mathbf{K}\mathbb{F}_q \cong K_{2i}\mathbb{F}_q = 0$, for $i > 0$.

COMPUTATIONS

13. $(k+1)$-ad homotopy

We will reexpress the top homology of a configuration space as the bottom $(k+1)$-ad homotopy group of a wedge of spheres. This argument is based on an approach due to Gunnar Carlsson. The proof we give uses a construction related to Thomas Goodwillie's kth derivative of an analytic functor [6]. Using this, we give a simple explicit description of the Σ_k-representation W_k in terms of free Lie algebras.

First we recall the definition of $(k+1)$-ad homotopy [19, 1]. A $(k+1)$-ad of spaces $\mathscr{Y} = (Y; Y_1, \ldots, Y_k)$ is a based space Y with k subspaces $(Y_1, \ldots, Y_k)$ such that $* \in \bigcap_{i\in\mathbf{k}} Y_i$. $\mathscr{Y}$ determines an inclusion k-cube indexed by $U \subseteq \mathbf{k}$:

$$U \mapsto \sigma(U) = \bigcap_{i\notin U} Y_i \subseteq Y$$

with $\sigma(\mathbf{k}) = Y$, and the obvious inclusion maps $\sigma(U) \hookrightarrow \sigma(U')$ for $U \subseteq U'$. Let $F: Spaces_* \to Spaces_*$ be a covariant functor from the category of based topological spaces to itself.

Definition 13.1.

$$S(\mathscr{Y}, F) = \mathrm{hofib}(F\sigma(\varnothing) \to \operatorname*{holim}_{\varnothing \neq U \subseteq \mathbf{k}} F\sigma(U)),$$

i.e. the homotopy fiber of the canonical map from $F\sigma(\varnothing) = F(\bigcap_{i\in\mathbf{k}} Y_i)$ to the homotopy inverse limit over the cube punctured at $U = \varnothing$, with F applied. Equivalently, this is the k-fold iterated homotopy fiber over a k-cube of spaces indexed by $U \subseteq \mathbf{k}$.

LEMMA 13.2. *The homotopy of $S(\mathscr{Y}, \mathrm{id})$ is the classical $(k+1)$-ad homotopy of $\mathscr{Y}$, shifted down k degrees, i.e.* $\pi_{i-k} S(\mathscr{Y}, \mathrm{id}) = \pi_i \mathscr{Y}$.

Proof. To see this, note that $S(\mathscr{Y}, \mathrm{id})$ is the k-fold iterated homotopy fiber over the cube σ, and compare with Toda's definition using a function space. ■

Next we consider a particular $(k+1)$-ad with Σ_k-action for which there is a dual formulation of $(k+1)$-ad homotopy.

Given a based space X let $X_i = X$ for all $i \in \mathbf{k}$. There is a natural $(k+1)$-ad $\mathscr{Y} = (Y; Y_1, \ldots, Y_k)$ on the k-fold wedge $Y = \bigvee_{i\in\mathbf{k}} X_i$, given by $Y_i = \bigvee_{j\neq i} X_j$.

Definition 13.3. $S_k(X, F) = S(\mathscr{Y}, F)$ with $\mathscr{Y}$ as above.

Explicitly $\sigma(U) = \bigvee_{i\in U} X_i$. There is a natural Σ_k-action on Y and $S_k(X, F)$, which permutes the wedge summands. Furthermore, each inclusion map $\sigma(U) \hookrightarrow \sigma(U')$ for $U \subseteq U'$ is naturally split by a map $\bigvee_{i\in U'} X_i \twoheadrightarrow \bigvee_{i\in U} X_i$ collapsing each X_i with $i \in U' - U$ to $*$. Hence we can form a contravariant k-cube

$$U \mapsto \tau(U) = \bigvee_{i\in U} X_i$$

with collapse maps $\tau(U') \twoheadrightarrow \tau(U)$ for $U \subseteq U'$.

Definition 13.4.

$$T_k(X, F) = \mathrm{hofib}(F\tau(\mathbf{k}) \to \operatorname*{holim}_{U \subset \mathbf{k}} F\tau(U)).$$

LEMMA 13.5. $\Omega^k T_k(X, F) \simeq S_k(X, F)$.

Proof. Note that if $i: A \hookrightarrow B$ is split by $p: B \twoheadrightarrow A$, there is an equivalence (the composite map)

$$\Omega\,\mathrm{hofib}(p) \to \Omega B \to \mathrm{hofib}(i).$$

Hence both spaces in the lemma are k-fold iterated homotopy fibers of split maps $F\sigma(U) \hookrightarrow F\sigma(U')$ for $U \subseteq U'$. ■

So $T_k(X, F)$ recovers the $(k+1)$-ad homotopy without a dimenson shift.

When X is a suspension, and $F = \mathrm{id}$, by the Hilton–Milnor theorem [8, 10, 23], $\pi_* T_k(X, \mathrm{id})$ is the summand of $\pi_* Y$ which maps to 0 by each of the collapse maps $Y \twoheadrightarrow Y_i$, $i \in \mathbf{k}$.

Definition 13.6. Let FL_k denote the free Lie algebra on k generators. Let $XL_k \subset FL_k$ denote the free abelian subgroup generated by the k-fold iterated brackets in FL_k involving each generator exactly once. It has rank $(k-1)!$. Σ_k acts on both FL_k and XL_k by permuting the generators.

From now on we will analyze the case $X = S^m$. The Hilton–Milnor theorem gives an explicit description of $\pi_* T_k(S^m, \mathrm{id})$, and by the lemma below, also of $\pi_* T_k(S^m, \Omega^n\Sigma^n)$.

LEMMA 13.7. $T_k(X, \Omega^n\Sigma^n) = \Omega^n T_k(\Sigma^n X, \mathrm{id})$. ∎

LEMMA 13.8. *$\pi_n T_k(S^m, \mathrm{id})$ is the direct summand of $\pi_n(\overbrace{S^m \vee \cdots \vee S^m}^{k \text{ times}})$ generated by composition of $\pi_n(S^q)$ with basic iterated Whitehead brackets $p \in \pi_q(S^m \vee \cdots \vee S^m)$ involving each generator $\gamma_i \in \pi_m(S_i^m)$ at least once. (Here q is the dimension of the iterated bracket p.)*

Proof. See [8, 3] for the definition of basic products. Clearly an iterated bracket maps to 0 by every collapsing map $Y \twoheadrightarrow Y_i$ precisely if it involves each generator at least once. ∎

COROLLARY 13.9. *The first nonzero group of $\pi_* T_k(S^m, \Omega^n\Sigma^n) = \pi_{*+n} T_k(S^{m+n}, \mathrm{id})$ sits in degree $b = k(m+n-1) + 1 - n = km + (k-1)(n-1)$. If $m+n$ is even, the iterated Whitehead brackets on the k generators $\gamma_1, \ldots, \gamma_k \in \pi_m Y \subsetneq \pi_m \Omega^n\Sigma^n Y$ generate a copy of FL_k in $\pi_* \Omega^n\Sigma^n Y$, and $\pi_b T_k(S^m, \Omega^n\Sigma^n) \cong XL_k$.*

Proof. Refer to [8, 3] for the anti-commutativity and Jacobi identities for the Whitehead bracket. ∎

We will now detect the image of the *bottom* $(k+1)$*-ad group* $\pi_b T_k(S^m, \Omega^n\Sigma^n)$ in $H_b\Omega^n\Sigma^n Y$ by the Hurewicz map h, using the Browder operations λ_{n-1} on $H_*(\Omega^n\Sigma^n Y; \mathbb{F}_p)$ for each prime p [2].

There is a Σ_k-equivariant commutative diagram:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\pi_* T_k(S^m, \Omega^n\Sigma^n) & \xrightarrow{h} & \ker(g) & \cong & \bigoplus_{j\geq 1} \ker(g_j) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_* \Omega^n\Sigma^n Y & \xrightarrow{h} & \tilde{H}_* \Omega^n\Sigma^n Y & \cong & \bigoplus_{j \geq 1} \tilde{H}_*(\mathscr{F}(\mathbb{R}^n, j)_+ \underset{\Sigma_j}{\wedge} Y^{\wedge j}) \\
\downarrow f & & \downarrow g & & \downarrow \oplus g_j \\
\prod_{i \in k} \pi_* \Omega^n\Sigma^n Y_i & \xrightarrow{h} & \prod_{i \in k} \tilde{H}_* \Omega^n\Sigma^n Y_i & \cong & \prod_{i \in k} \bigoplus_{j \geq 1} \tilde{H}_*(\mathscr{F}(\mathbb{R}^n, j)_+ \underset{\Sigma_j}{\wedge} Y_i^{\wedge j})
\end{array}$$

where f, g, and the g_js are induced by the collapse maps $Y \twoheadrightarrow Y_i$, and the columns are exact. The isomorphisms on the right follow from the Snaith splitting [16, 5].

LEMMA 13.10. *In degree $b = km + (k-1)(n-1)$, $\ker(g) = \ker(g_k)$ for m sufficiently large.*

Proof. If $j < k$, g_j has no kernel, since a summand $X_{i_1} \wedge \cdots \wedge X_{i_j}$ of $Y^{\wedge j}$ can only involve j of the $\{X_i\}$, and thus sits in some $Y_i^{\wedge j}$. If $j > k$, $\mathscr{F}(\mathbb{R}^n, j)_+ \wedge_{\Sigma_j} Y^{\wedge j}$ is at least $(jm-1)$-connected. For large m this exceeds b. ∎

Lemma 13.11. *$\ker(g_k) \cong \tilde{H}_*(\mathscr{F}(\mathbb{R}^n, k)_+ \wedge (X_1 \wedge \cdots \wedge X_k))$. In degree b,*

$$\ker(g_k)_b \cong H_{(k-1)(n-1)}\mathscr{F}(\mathbb{R}^n, k) \otimes \mathbb{Z}[(-1)^m]$$

as $\mathbb{Z}\Sigma_k$-modules.

Proof. $Y^{\wedge k}$ splits as a wedge sum $\bigvee_{i_1,\ldots,i_k \in \mathbf{k}} X_{i_1} \wedge \cdots \wedge X_{i_k}$. If any $i \notin \{i_1, \ldots, i_k\}$, the summand $X_{i_1} \wedge \cdots \wedge X_{i_k}$ survives to $Y_i^{\wedge k}$. Otherwise $j \mapsto i_j$ is a permutation of $\mathbf{k}$. Hence

$$\begin{aligned}\ker(g_k) &= \tilde{H}_*(\mathscr{F}(\mathbb{R}^n, k)_+ \underset{\Sigma_k}{\wedge} \bigvee_{\pi \in \Sigma_k} (X_{\pi(1)} \wedge \cdots \wedge X_{\pi(k)})) \\ &\cong \tilde{H}_*(\mathscr{F}(\mathbb{R}^n, k)_+ \wedge (X_1 \wedge \cdots \wedge X_k)).\end{aligned}$$

Here the Σ_k-action permuting wedge summands of Y translates to the diagonal action on $\mathscr{F}(\mathbb{R}^n, k)_+$ and $X_1 \wedge \cdots \wedge X_k \cong S^{mV}$. Clearly $\tilde{H}_*(S^{mV}) \cong \mathbb{Z}[(-1)^m]$ in degree km. ∎

Lastly we check that:

Lemma 13.12. *$h: \pi_* T_k(S^m, \Omega^n\Sigma^n) \to \ker(g)$ is a Σ_k-isomorphism in degree b.*

Proof. Both sides are free abelian of rank $(k-1)!$. By [3] there exists operations for each prime p

$$\lambda_{n-1}: H_i(\Omega^n\Sigma^n Y; \mathbb{F}_p) \otimes H_j(\Omega^n\Sigma^n Y; \mathbb{F}_p) \to H_{i+j+n-1}(\Omega^n\Sigma^n Y; \mathbb{F}_p),$$

the *Browder operations* [2], commuting with the Whitehead bracket under Ω^n followed by the mod p Hurewicz map h_p. The generators γ_i in $\pi_m S_i^m \subsetneq \pi_m \Omega^n\Sigma^n Y$, $i \in \mathbf{k}$, map to generators $\zeta_i = h_p(\gamma_i) \in H_m(\Omega^n\Sigma^n Y; \mathbb{F}_p)$. Furthermore, iterated basic Whitehead brackets in the γ_i map to iterated basic λ_{n-1} products in the ζ_i. Hence h_p takes the generators for $XL_k \cong \pi_b T_k(S^m, \Omega^n\Sigma^n)$ to iterated basic λ_{n-1} products in $H_b(\Omega^n\Sigma^n Y; \mathbb{F}_p)$. By Theorem 3.2 and the formulas preceding Lemma 3.8 of [3], these are linearly independent. Therefore each h_p is injective, and h must be an isomorphism. ∎

Corollary 13.13. *$W_k \cong (XL_k)^*$ as $\mathbb{Z}\Sigma_k$-modules.*

Proof.

$$\begin{aligned}W_k &= H_{n+2k-2}(A/F_{<\delta_k} A) \\ &\cong H^{(k-1)(n-2)}\mathscr{F}(\mathbb{R}^{n-1}, k) \otimes \mathbb{Z}[(-1)^n]\end{aligned}$$

by S^{nV}-duality [9]

$$\begin{aligned}&\cong (H_{(k-1)(n-2)}\mathscr{F}(\mathbb{R}^{n-1}, k))^* \otimes \mathbb{Z}[(-1)^n] \\ &\cong (XL_k)^* \otimes \mathbb{Z}[(-1)^m] \otimes \mathbb{Z}[(-1)^n]\end{aligned}$$

for $m + n$ even and m sufficiently large

$$\cong (XL_k)^*.$$ ∎

With this, we have completely determined the modules in the complex $E_*^1(R^k)$ in Theorem 12.1. Note that we have also effectively computed the coefficient spectra in the Taylor series for the identity functor viewed as an analytic functor in the sense of [6].

14. Connectivity results

In this section we prove the connectivity Conjecture 12.3 for the case $k = 2$. We also give an alternative description of the stable building, which shows that it has the type of a finite $GL_k R$-complex. Lastly we obtain partial results towards the connectivity conjecture for higher k when $R = \mathbb{Z}/p^2$.

First we extend Definition 8.5:

Definition 14.1. For $I \subseteq \mathbf{k}$, $P_I = (GL_k R)_{R^I}$, the isotropy of R^I under $GL_k R$-action, is a *parabolic subgroup*. $P_{I_0 \ldots I_q} = P_{I_0} \cap \cdots \cap P_{I_q}$. Note that $P_\omega = P_{\omega_1 \ldots \omega_k}$.

LEMMA 14.2. *If R is a Euclidean domain or a local ring, $D(R^k)$ is simply-connected for $k \geq 2$.*

Proof. We prove that $\Sigma^{-1} D(R^k)$ is connected. There is an edge connecting $hT_k\Sigma_k$ to $hgT_k\Sigma_k$ whenever $\omega(h^{-1}hg) = \omega(g)$ is not indiscrete. In particular this is the case when g is maximal parabolic (i.e. $g \in P_I$ for some proper, nonempty subset $I \subset \mathbf{k}$) or a permutation matrix. But when R is Euclidean or local, it is easy to see that the subgroups $\{P_I\}_{\emptyset \neq I \subset \mathbf{k}}$ and Σ_k generate $GL_k R$ by using (the Euclidean algorithm followed by) Gauss elimination. ■

Next we construct a Volodin- or BN-type description [20, 21] of the stable building. By Lemma 10.6, $g_0 A \cap \cdots \cap g_q A$ is highly connected precisely when $\omega = \omega(g_0^{-1} g_1, \ldots, g_0^{-1} g_q)$ is not indiscrete, i.e. when ω has some proper nontrivial order ideal $I \subset \mathbf{k}$. We can assume $I = \omega_i$ for some i.

LEMMA 14.3. *With the notation above, for every $s = 0, \ldots, q$, $g_0\, R^I$ is spanned by a subset of column 'vectors' of g_s.*

Proof. I is an order ideal under $\omega(g_0^{-1} g_s)$. Thus R^I occurs in the $([q])^n$-diagram for a simplex $\sigma \in F_{\omega(g_0^{-1} g_s)} A$, and $g_0 R^I$ occurs in $g_0 \sigma \in g_0 A \cap g_s A \subseteq g_s A$. ■

COROLLARY 14.4. *$\{g_0, \ldots, g_q\}$ spans a simplex in $\Sigma^{-1} D(R^k)$ if and only if there exists a proper, nontrivial submodule $M \subset R^k$ such that for each g_s, $M = g_s R^J$ for some $J \subset \mathbf{k}$.* ■

Hence $\Sigma^{-1} D(R^k)$ is covered by the contractible subcomplexes

$$C(M) = \Delta(\{g \in GL_k R / T_k \Sigma_k \mid M = gR^J \text{ for some } J \subset \mathbf{k}\})$$

as M varies through the proper nontrivial $M \rightarrowtail R^k$.

Furthermore, if $\{M_0, \ldots, M_q\}$ is a collection of such submodules, $C(M_0) \cap \cdots \cap C(M_q)$ is the span on the set of vertices $gT_k\Sigma_k$ for which each M_s is spanned by some columns of g. Hence each such intersection is either contractible or empty. So the Mayer–Vietoris blowup of $\Sigma^{-1} D(R^k)$, covered by the $\{C(M)\}$, is homotopy equivalent to the complex defined below:

Definition 14.5. Let $\Sigma^{-1} D^V(R^k) \subseteq \Delta(\{M \mid \text{proper nontrivial } M \rightarrowtail R^k\})$ be the subcomplex with faces $\{M_0, \ldots, M_q\}$ such that there exists a $g \in GL_k R$ with each M_i spanned by column 'vectors' of g.

We have proved

LEMMA 14.6. *$GL_k R$-equivariantly $D(R^k) \simeq \Sigma(\Sigma^{-1} D^V(R^k))$.* ■

Let us now consider $R = \mathbb{Z}/p^2$, p a prime. Write $\varphi: R \to \mathbb{F}_p$ for the residue field map. Let N_k denote the kernel of the induced homomorphism $\varphi: GL_k R \to GL_k \mathbb{F}_p$. It is an elementary abelian p-group on k^2 generators. There is a $GL_k R$-map $\varphi: D^n(R^k) \to D^n(\mathbb{F}_p{}^k)$.

Definition 14.7. Let the *small n-dimensional building* $d^n(R^k) = \varphi^{-1}(A) \subseteq D^n(R^k)$ be the preimage of the standard apartment $A \subseteq D^n(\mathbb{F}_p{}^k)$. It is an N_k-complex. Let $\Sigma^{-1} d(R^k)$ in $\Delta(N_k/(N_k \cap T_k)\Sigma_k)$ be the subcomplex with faces $\{n_0, \ldots, n_q\}$ whose associated poset $\omega(n_0^{-1}n_1, \ldots, n_0^{-1}n_q)$ is not indiscrete. As before we write $d(R^k) = \Sigma(\Sigma^{-1}d(R^k))$. Let $\Sigma^{-1}d^V(R^k) \subseteq \Sigma^{-1}D^V(R^k)$ be the complex with q-simplices $\{M_0 \ldots, M_q\}$ with every M_i spanned by some columns of one $g \in N_k$.

The analogous result to Lemma 14.6 is still true. Also:

Lemma 14.8. $d^n(R^k) = \bigcup_{g \in N_k} A(g) \subset D^n(R^k)$. ■

Remark 14.9. $D^n(R^k)$ is covered by $\{g \cdot d^n(R^k)\}$ for $g \in GL_k R/N_k \cong GL_k \mathbb{F}_p$, sitting over the covering of $D^n(\mathbb{F}_p{}^k)$ by $A(g)$ for $g \in GL_k \mathbb{F}_p$. So since we understand $\mathbf{K}\mathbb{F}_p$ and the stable covering by apartments of $D^n(\mathbb{F}_p{}^k)$, in principle we should be able to reduce the study of $D(R^k)$ to that of $d(R^k)$.

Definition 14.10. View $H_t(N_k; \mathbb{F}_p)$ for $p = 2$ as the symmetric coalgebra on the $\mathbb{F}_p$-vector space N_k with basis vectors $\{x_{i,j}\}$ corresponding to the i,jth matrix entries (resp. for p odd, the analogous exterior tensor symmetric coalgebra). A monomial $z \in H_t(N_k; \mathbb{F}_p)$ determines a poset ω_z on $\mathbf{k}$, generated by $(i \to j)$ if z involves $x_{i,j}$ for $p = 2$ (resp. the reduced homology of the i,jth factor if p is odd). z is *mixing* if and only if ω_z is indiscrete.

Proposition 14.11. *For $R = \mathbb{Z}/p^2$, $d(R^k)/hN_k$ is $(k-1)$-connected.*

Proof. The q-simplices of the simplicial N_k-set $\Sigma^{-1}d^V(R^k)$ is an N_k-set with orbit representatives $\{\{R^{I_0}, \ldots, R^{I_q}\}\}$ and associated N_k-isotropy subgroups $Q_{I_0 \ldots I_q} = P_{I_0 \ldots I_q} \cap N_k$. Suspending the skeletal filtration for $\emptyset \subset \Sigma^{-1}d^V(R^k)$ once, and applying $(\ )/hN_k$, induces a spectral sequence for $\tilde{H}_*(d(R^k)/hN_k; \mathbb{F}_p)$ with

$$E^1_{s,t} = \begin{cases} H_t(N_k; \mathbb{F}_p) & \text{for } s = 0, \\ \oplus_{I_1 \ldots I_s} H_t(Q_{I_1 \ldots I_s}; \mathbb{F}_p) & \text{for } s \geq 1, \end{cases}$$

and differentials d^1 commuting with the inclusions $H_t(Q_{I_1 \ldots I_s}; \mathbb{F}_p) \hookrightarrow H_t(N_k; \mathbb{F}_p)$. Notice that the (E^1, d^1)-term of the spectral sequence splits additively over the basis monomials of $H_t(N_k; \mathbb{F}_p)$.

Fix a basis monomial $z \in H_t(N_k; \mathbb{F}_p)$. By the splitting, the summand of E^1 corresponding to z is a horizontal chain complex. It is equal to the augmented chain complex of a complex Δ_z, shifted upward (to the right) one degree. Here Δ_z has a q-simplex for every $\{I_0, \ldots, I_q\}$ such that $z \in H_t(Q_{I_0 \ldots I_q}; \mathbb{F}_p)$, i.e. z occurs at position $(q+1, t)$ in the spectral sequence. It follows that Δ_z is the simplex spanned by its vertices. So $\Delta_z \simeq *$ unless $\Delta_z = \emptyset$, which happens precisely when z is mixing.

Clearly E^2 is the sum of the homologies of the augmented chain complexes of the Δ_z:

$$E^2_{s,t} = \begin{cases} J_t & \text{if } s = 0 \\ 0 & \text{if } s > 0 \end{cases}$$

where $J_t \subseteq H_t(N_k; \mathbb{F}_p)$ is the submodule generated by mixing monomials $z \in H_t(N_k; \mathbb{F}_p)$. Since $J_t = 0$ for $t < k$, this completes the proof. ■

15. Computations for low ranks

We gather here some notes and added details about Theorem 12.1 for $k \leq 3$.

LEMMA 15.1. *For any R, $F_0\mathbf{K}R \cong *$ and $F_1\mathbf{K}R \simeq \Sigma^\infty BGL_1R_+$.*

Proof. For each n, $D^n(R^1) = A_{n,1} \cong S^n$ by Proposition 6.4, so $F_1\mathbf{K}R$ has nth space $S^n/hGL_1R \cong \Sigma^n(BGL_1R_+)$ by Proposition 3.8. ∎

LEMMA 15.2. *For R local or Euclidean*

$$0 \to H_2\mathbf{D}(R^2) \to \mathbb{Z}GL_2R/T_2 \underset{\Sigma_2}{\otimes} \mathbb{Z}[-1] \xrightarrow{d_2} \mathbb{Z}GL_2R/P_1 \xrightarrow{d_1} \mathbb{Z} \to 0$$

is exact. Here $d_2(gT_2 \otimes 1) = g(1-\xi)P_1$, $d_1(gP_1) = 1$, with $\xi = (12) \in \Sigma_2 \subsetneq GL_2R$. In terms of submodule configurations, $gT_2 \in GL_2R/T_2$ determines a pair of basis lines (l_1, l_2). Then $d_2(l_1, l_2) = l_1 - l_2$. $d_1(l_1) = d_1(l_2) = 1$.

Proof. The exactness of the sequence is the content of Theorem 12.1 and Lemma 14.2 for $k = 2$. To evaluate the differentials we consider the sequence in Remark 12.2 and employ adjointness. We omit the details.

The note about submodule configurations is immediate from the comment preceding Definition 8.5. ∎

Let $\varphi = (12)$ and $\psi = (123)$ generate $\Sigma_3 \subsetneq GL_3R$.

LEMMA 15.3. *The complex $E_*(3)$ is*

$$0 \to \mathbb{Z}GL_3R/T_3 \underset{\Sigma_3}{\otimes} W_3 \xrightarrow{d_4} \mathbb{Z}GL_3R/P_{1,12,3} \xrightarrow{d_3} \mathbb{Z}GL_3R/P_{1,12} \oplus \mathbb{Z}GL_3R/P_{12,3}$$

$$\xrightarrow{d_2} \mathbb{Z}GL_3R/P_1 \oplus \mathbb{Z}GL_3R/P_{12} \xrightarrow{d_1} \mathbb{Z} \to 0.$$

$W_3 \cong \tilde{\mathbb{Z}}^3 \otimes \mathbb{Z}[-1] \cong (\tilde{\mathbb{Z}}^3)^* \cong (XL_3)^* \cong \mathbb{Z}\{w_1, w_2\}$. In the basis $\{w_1, w_2\}$, $\varphi = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $\psi = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$.

$$d_4(gT_3 \otimes w_1) = g(-e + \varphi + \psi^2 - \varphi\psi)P_{1,12,3}$$

$$d_4(gT_3 \otimes w_2) = g(+e - \varphi - \psi + \psi\varphi)P_{1,12,3}$$

$$d_3(gP_{1,12,3}) = (g(+e - \varphi\psi + \psi^2)P_{1,12}, gP_{12,3})$$

$$d_2(gP_{1,12}, 0) = (-gP_1, gP_{12})$$

$$d_2(0, gP_{12,3}) = (g\psi^2P_1, -gP_{12})$$

$$d_1(gP_1, 0) = d_1(0, gP_{12}) = 1.$$

Note that $T_3 = P_{\delta_3}$, $P_{1,12,3} = P_{1\to 2}$, $P_{1,12} = P_{1\to 2\to 3}$, $P_{12,3} = P_{1=2}$, $P_1 = P_{1\to 2=3}$ and $P_{12} = P_{1=2\to 3}$.

Proof. As for the previous lemma, the proof reduces to a detailed inspection of the poset filtration on $A_{n,3} \cong S^{3n}$. We find that $E^1_*(3)$ is the exact complex:

$$0 \to \mathbb{Z}\{w_1, w_2\} \hookrightarrow \mathbb{Z}\Sigma_3 \xrightarrow{\binom{1-\varphi\psi+\psi^2}{1}} \mathbb{Z}\Sigma_3 \oplus \mathbb{Z}\Sigma_3/(\varphi)$$

$$\xrightarrow{\begin{pmatrix} -1 & \psi^2 \\ 1 & -1 \end{pmatrix}} \mathbb{Z}\Sigma_3/(\varphi\psi) \oplus \mathbb{Z}\Sigma_3/(\varphi) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

and the lemma follows by adjointness. ∎

LEMMA 15.4. *$E_*(3)$ is exact at E_0, at E_1 if R is local or Euclidean, at E_2 if R is a field, and at E_3 if $R = \mathbb{F}_2$.*

Proof. The augmentation is surjective for any ring with units, which amounts to exactness at E_0.

$\ker(d_1)$ is generated by the $(e, -g) \overset{\text{def}}{=} (eP_1, -gP_{12})$ for $g \in GL_3R$, as a $\mathbb{Z}GL_3R$-module. $\operatorname{im}(d_2)$ contains $(e, -g)$ for $g \in P_1$, $g \in P_{12}$ and $g = \psi$ by Lemma 15.3. Furthermore, the $g \in GL_3R$ with $(e, -g) \in \operatorname{im}(d_2)$ form a subgroup, as $(e, -g^{-1}h) = (e, -e) - g^{-1}(e, -g) + g^{-1}(e, -h)$. For R local or Euclidean, P_1, P_{12} and ψ generate all of GL_3R, so $\operatorname{im}(d_2) = \ker(d_1)$.

We call a free submodule of R^k of rank two, included by a cofibration, a *plane*. We interpret a GL_3R-coset of $P_1, P_{12}, P_{1,12}, P_{12,3}, P_{1,12,3}$ and T_3 as a line l, a plane α, a line in a plane $(l \subset \alpha)$ (l is included into α by a cofibration), a line transverse to a plane $(l \pitchfork \alpha)$ (R^3 is the direct sum $(l + \alpha)$), a line in a plane with a second transverse line $(l_1 \subset \alpha \pitchfork l_2)$, and three basis lines (l_1, l_2, l_3) respectively. Then $d_1(l) = d_1(\alpha) = 1$, $d_2(l \subset \alpha) = -d_2(l \pitchfork \alpha) = \alpha - l$, and $d_3(l_1 \subset \alpha_1 \pitchfork l_2) = (l_1 \subset \alpha_1) - (l_1 \subset \alpha_2) + (l_2 \subset \alpha_2) + (l_2 \pitchfork \alpha_1)$ where $\alpha_2 = l_1 + l_2$ is a plane.

Let (l, α) denote $(l \subset \alpha)$ or $-(l \pitchfork \alpha)$. Then $\ker(d_2)$ consists of sums $\sum_i n_i(l_i, \alpha_i)$ with each l or α occurring algebraically zero times. By induction on $\sum_i |n_i|$ we see that these sums are generated by expressions $\square(l_1, l_2, \alpha_1, \alpha_2) = +(l_1, \alpha_1) - (l_1, \alpha_2) - (l_2, \alpha_1) + (l_2, \alpha_2)$ with $l_1 \neq l_2$ and $\alpha_1 \neq \alpha_2$. $\operatorname{im}(d_3)$ contains $\square(l_1, l_2, \alpha_1, \alpha_2)$ when $l_1 \subset \alpha \pitchfork l_2$ and $\alpha_2 = l_1 + l_2$. We claim that if R is a field, so that for any l, α either $l \subset \alpha$ or $l \pitchfork \alpha$, then these terms generate all of $\ker(d_2)$.

Let $l_3 = \alpha_1 \cap \alpha_2$. If $l_1 \neq l_3$, use

$$\square(l_1, l_2, \alpha_1, \alpha_2) = \square(l_3, l_2, \alpha_1, \alpha_2) - \square(l_3, l_1, \alpha_1, \alpha_2)$$

to reduce to the case $l_1 = \alpha_1 \cap \alpha_2$. Since $l_2 \neq l_1$, we may assume $l_2 \pitchfork \alpha_1$. If $l_2 \subset \alpha_2, \alpha_2 = l_1 + l_2$ and $\square(l_1, l_2, \alpha_1, \alpha_2) = d_3(l_1 \subset \alpha_1 \pitchfork l_2)$. Otherwise $l_2 \pitchfork \alpha_2$ and $\square(l_1, l_2, \alpha_1, \alpha_2) = d_3(l_1 \subset \alpha_1 \pitchfork l_2) - d_3(l_1 \subset \alpha_2 \pitchfork l_2)$. This proves the claim of exactness at E_2.

The check that $E_*(3)$ is exact at $* = 3$ for $R = \mathbb{F}_2$ was done using *Mathematica*™, a formula manipulator program, by listing a basis for E_4, E_3 and E_2, writing d_4 and d_3 in matrix form, and solving the necessary linear systems. $\ker(d_4) \cong H_4\mathbf{D}(\mathbb{F}_2{}^3)$ is free abelian of rank 8. Hence the connectivity conjecture holds for $k = 3$, $R = \mathbb{F}_2$. ∎

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