

INTRODUCTION TO SPECTRAL SEQUENCES, I

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1. SPECTRAL SEQUENCES

A fairly typical example of a (Serre indexed homological) *spectral sequence* is a sequence $(\mathcal{E}_{*,*}^r, d^r)_{r \geq 1}$ of bigraded abelian groups

$$(\mathcal{E}_{s,t}^r)_{s,t \in \mathbb{Z}}$$

and bidegree $(-r, r-1)$ differentials

$$(d_{s,t}^r : \mathcal{E}_{s,t}^r \rightarrow \mathcal{E}_{s-r,t+r-1}^r)_{s,t \in \mathbb{Z}}$$

together with isomorphisms

$$\mathcal{E}^{r+1} \cong H(\mathcal{E}^r, d^r) := \frac{\ker(d^r)}{\operatorname{im}(d^r)}$$

for all $r \geq 1$. How do these structures arise, and what can one do with them?

See Miller [Mil00] for the early history of spectral sequences, starting with Jean Leray around 1943. Important examples include the Serre spectral sequence for the (co-)homology of a fiber space [Ser51], the Adams spectral sequence for the homotopy of a spectrum [Ada58], and the Atiyah–Hirzebruch spectral sequence for a (co-)homology theory [AH61].

The algebraic formalism is presented using Massey’s exact couples in [Mas52], [Mas53], and using ‘Cartan–Eilenberg systems’ in [CE56]. A refined discussion of convergence was published (after a long life in preprint form) by Boardman [Boa99]. Multiplicative structure in the case of filtered (co-)chain complexes is handled in [Mas54], and for general Cartan–Eilenberg systems by Douady in [Dou58]. See also [HR19] for more on convergence, and [HR24, Ch. 4] for more on multiplicative structure.

2. SEQUENCES OF SPECTRA

Let

$$\cdots \rightarrow Y_{s-1} \xrightarrow{f_s} Y_s \xrightarrow{f_{s+1}} Y_{s+1} \rightarrow \cdots$$

be a sequence Y_* of spectra and spectrum maps, or of chain complexes and chain maps, or more general objects and morphisms in a stable ∞ -category with a chosen t -structure. We call $s \in \mathbb{Z}$ the *filtration index*.

For each s there is a cofiber sequence

$$Y_{s-1} \xrightarrow{f_s} Y_s \longrightarrow Cf_s \longrightarrow \Sigma Y_{s-1},$$

with $Cf_s = Y_s \cup_{f_s} CY_{s-1}$ the cofiber (or mapping cone) of f_s .

Let $\pi_n(Y)$ denote the n -th homotopy group of a spectrum Y , or the n -th homology group of a chain complex Y . Then there is a long exact sequence

$$\cdots \rightarrow \pi_n(Y_{s-1}) \xrightarrow{\alpha_s} \pi_n(Y_s) \xrightarrow{\beta_s} \pi_n(Cf_s) \xrightarrow{\gamma_s} \pi_{n-1}(Y_{s-1}) \rightarrow \cdots$$

for each s , with $\alpha_s = \pi_n(f_s)$.

If f_s is a cofibration of spectra, or the inclusion of one chain complex as a subcomplex of another, then $Cf_s \simeq Y_s/Y_{s-1}$, so that we can replace $\pi_n(Cf_s)$ with $\pi_*(Y_s/Y_{s-1}) = \pi_*(Y_s, Y_{s-1})$ in the above. In general, let us also write $\pi_n(Y_s, Y_{s-r})$ for $\pi_n(Cf_s^{(r)})$ where $f_s^{(r)} := f_s \cdots f_{s-r+1}$ denotes the r -fold composite

$$Y_{s-r} \xrightarrow{f_{s-r+1}} Y_{s-r+1} \longrightarrow \cdots \longrightarrow Y_{s-1} \xrightarrow{f_s} Y_s,$$

even if Y_{s-r} may not be a subobject of Y_s in the model at hand. There are then long exact sequences

$$\cdots \rightarrow \pi_n(Y_{s-1}, Y_{s-r}) \rightarrow \pi_n(Y_s, Y_{s-r}) \rightarrow \pi_n(Y_s, Y_{s-1}) \xrightarrow{\partial} \pi_{n-1}(Y_{s-1}, Y_{s-r}) \rightarrow \cdots$$

for all $s \in \mathbb{Z}$ and $r \geq 1$. Suppose we know $\pi_*(Cf_s) = \pi_*(Y_s, Y_{s-1})$ for all s . We might then use these long exact sequences to understand $\pi_*(Y_s, Y_{s-r})$ for all s and $r \geq 1$, by induction on r , or equivalently, to understand $\pi_*(Y_v, Y_u)$ for all $u \leq v$. This requires understanding the connecting map ∂ , and determining the correct extension of $\ker(\partial)$ by $\text{cok}(\partial)$.

3. (Co-)LIMITS

The (homotopy) colimit of the sequence can be presented as the cofiber

$$\bigoplus_s Y_s \xrightarrow{1-F} \bigoplus_s Y_s \longrightarrow \text{colim}_s Y_s = Y_\infty,$$

where the inclusion of Y_s in the source, followed by F , equals f_{s+1} followed by the inclusion of Y_{s+1} in the target. Then $\text{colim}_s \pi_n(Y_s) \cong \pi_n(Y_\infty)$.

The (homotopy) limit of the sequence can be presented as the fiber

$$Y_\infty = \lim_s Y_s \longrightarrow \prod_s Y_s \xrightarrow{1-F'} \prod_s Y_s,$$

where F' projected to Y_s is given by projecting to Y_{s-1} followed by f_s . There is a short exact sequence

$$0 \rightarrow \text{Rlim}_s \pi_{n+1}(Y_s) \longrightarrow \pi_n(Y_\infty) \longrightarrow \lim_s \pi_n(Y_s) \rightarrow 0,$$

where $\text{Rlim} = \lim^1$ denotes the right derived functor of the sequential limit.

Letting v increase to ∞ , we can pass from knowledge of the $\pi_*(Y_v, Y_u)$ to knowledge of $\pi_*(Y_\infty, Y_u)$. Letting u decrease to $-\infty$, we can then pass from $\pi_*(Y_\infty, Y_u)$ to $\pi_*(Y_\infty, Y_\infty)$, i.e., the homotopy groups of the cofiber of the canonical map

$$Y_\infty = \lim_s Y_s \longrightarrow \text{colim}_s Y_s = Y_\infty.$$

If $Y_\infty \simeq 0$ is contractible, then this is $\pi_*(Y_\infty)$. On the other hand, if $Y_\infty \simeq 0$ is contractible, then it equals $\pi_{*-1}(Y_\infty)$.

Hence we usually arrange to either be in the situation that $Y_\infty \simeq 0$, and use this approach to calculate $\pi_*(Y_\infty)$, or we arrange that $Y_\infty \simeq 0$ and use this approach to calculate $\pi_*(Y_\infty)$. Here we focus on the former setting.

4. UNROLLED EXACT COUPLES

The structure at hand can be drawn as an *unrolled exact couple*

$$\begin{array}{ccccccccccc}
 A_{-\infty} & \longrightarrow & \cdots & \longrightarrow & A_{s-2} & \xrightarrow{\alpha_{s-1}} & A_{s-1} & \xrightarrow{\alpha_s} & A_s & \xrightarrow{\alpha_{s+1}} & A_{s+1} & \longrightarrow & \cdots & \longrightarrow & A_{\infty} \\
 & & & & \nwarrow \gamma_{s-1} & & \downarrow \beta_{s-1} & \nwarrow \gamma_s & \downarrow \beta_s & \nwarrow \gamma_{s+1} & \downarrow \beta_{s+1} & & & & \\
 & & & & & & E_{s-1} & & E_s & & E_{s+1} & & & &
 \end{array}$$

with absolute terms $A_s = \pi_*(Y_s)$, relative terms $E_s = \pi_*(Y_s, Y_{s-1})$, and limit terms $A_{\infty} = \operatorname{colim}_s A_s$, $A_{-\infty} = \operatorname{lim}_s A_s$ and $RA_{-\infty} = \operatorname{Rlim}_s A_s$. Each of these is a graded abelian group, each α_s and β_s is degree-preserving, and each γ_s reduces degree by 1. Each triangle is exact, in the sense that it constitutes a long exact sequence.

For example, suppose that X is a CW complex with s -skeleton X_s for each s . In particular, $X_s = \emptyset$ for $s < 0$. If M is a spectrum representing the (generalized) homology theory M_* , we can let

$$Y_s = M \wedge (X_s)_+ \simeq M \otimes \Sigma_+^{\infty}(X_s),$$

so that $\pi_n(Y_s) = M_n(X_s)$. The inclusions $X_{s-1} \rightarrow X_s$ induce the maps $f_s: Y_{s-1} \rightarrow Y_s$, and we have

$$E_s = \pi_*(Y_s, Y_{s-1}) = M_*(X_s, X_{s-1})$$

for each s . Since X_s is obtained from X_{s-1} by attaching the s -cells of X , we can write the degree n part of E_s as

$$(E_s)_n = M_n(X_s, X_{s-1}) \cong \bigoplus_{\alpha} M_n(D^s, \partial D^s) \cong C_s^{\text{CW}}(X; \pi_{n-s}(M)),$$

i.e., as the cellular chains of X with coefficients in the coefficient groups $M_* = \pi_*(M)$ of the homology theory. In this case $Y_{-\infty} = 0$ and $Y_{\infty} = M \wedge X_+$, so the procedure foreshadowed above will aim to calculate the colimit $\pi_*(Y_{\infty}) = M_*(X)$ from this information.

5. ABUTMENTS

Consider the case when $Y_{-\infty} \simeq 0$, i.e., when $A_{-\infty} = 0$ and $RA_{-\infty} = 0$. We suppose that $E_s = \pi_*(Y_s, Y_{s-1})$ is known, and aim to understand $A_{\infty} = \pi_*(Y_{\infty})$. The strategy will follow a slightly different route than that suggested by the inductive argument outlined above.

Let $G = A_{\infty}$ denote *abutment*, which we aim to calculate. It comes with a *filtration*

$$\cdots \subset F_{s-1}G \subset F_sG \subset F_{s+1}G \subset \cdots \subset G,$$

where

$$F_sG = \operatorname{im}(i_s: A_s \longrightarrow A_{\infty}).$$

This filtration is always *exhaustive*, in the sense that $\operatorname{colim}_s F_sG \cong G$. There are *convergence* criteria also ensuring that it is *complete Hausdorff*, meaning that $\operatorname{lim}_s F_sG = 0$ and $\operatorname{Rlim}_s F_sG = 0$. This implies that

$$F_sG \cong \lim_r \frac{F_sG}{F_{s-r}G},$$

so that

$$G \cong \operatorname{colim}_s \lim_r \frac{F_sG}{F_{s-r}G}.$$

Lemma 5.1. *There are isomorphisms*

$$\frac{\ker(\gamma_s)}{\beta_s(\ker(i_s))} \xleftarrow{\cong} \frac{A_s}{\operatorname{im}(\alpha_s) + \ker(i_s)} \xrightarrow{\cong} \frac{F_s G}{F_{s-1} G}$$

for each $s \in \mathbb{Z}$. These are subquotients of E_s , A_s and G , respectively.

Proof. Noether isomorphisms. □

The inclusions

$$0 \subset \beta_s(\ker(i_s)) \subset \ker(\gamma_s) \subset E_s$$

show that E_s is a kind of upper bound for $F_s G / F_{s-1} G$. The task of a *spectral sequence* is to iteratively pare down this upper bound, by passing to the subobject $\ker(\gamma_s)$ of E_s and reducing modulo the further subobject $\beta_s(\ker(i_s))$, so as to obtain the exact value for $F_s G / F_{s-1} G$.

Having achieved this, we then face the *extension problems* given by the short exact sequences

$$0 \rightarrow \frac{F_{s-1} G}{F_{s-r} G} \rightarrow \frac{F_s G}{F_{s-r} G} \rightarrow \frac{F_s G}{F_{s-1} G} \rightarrow 0$$

for all s and $r \geq 1$. Granting this, and convergence, we have a good grip on G as the colimit of limits displayed above.

6. THE CONSTRUCTION

Consider the following parts of the unrolled exact couple.

$$\begin{array}{ccccccc} A_{s-r-1} & \xrightarrow{\alpha_{s-r}} & A_{s-r} & \xrightarrow{\alpha_{s-1}^{(r-1)}} & A_{s-1} & \xrightarrow{\alpha_s} & A_s & \xrightarrow{\alpha_{s+r-1}^{(r-1)}} & A_{s+r-1} & \xrightarrow{\alpha_{s+r}} & A_{s+r} \\ & \nwarrow \gamma_{s-r} & \downarrow \beta_{s-r} & & \nwarrow \gamma_s & \downarrow \beta_s & & \nwarrow \gamma_{s+r} & \downarrow \beta_{s+r} & & \\ & & E_{s-r} & & & E_s & & & E_{s+r} & & \end{array}$$

For $r \geq 1$ and $s \in \mathbb{Z}$ let

$$\begin{aligned} Z_s^r &:= \gamma_s^{-1}(\operatorname{im}(\alpha_{s-1}^{(r-1)})) \\ B_s^r &:= \beta_s(\ker(\alpha_{s+r-1}^{(r-1)})) \\ \mathcal{E}_s^r &:= Z_s^r / B_s^r \end{aligned}$$

be r -th order cycles, r -th order boundaries and the $\mathcal{E}^r$ -page, in filtration s , respectively, so that

$$0 = B_s^1 \subset \cdots \subset B_s^r \subset \cdots \subset \beta_s(\ker(i_s)) \subset \ker(\gamma_s) \subset \cdots \subset Z_s^r \subset \cdots \subset Z_s^1 = E_s.$$

In particular, $\mathcal{E}_s^1 = Z_s^1 / B_s^1 = E_s / 0 \cong E_s$.

Then each $\mathcal{E}^r = (\mathcal{E}_s^r)_s$ is a graded object in graded abelian groups, i.e., a bigraded abelian group. We write $\mathcal{E}_{s,t}^r = (\mathcal{E}_s^r)_{s+t}$ for the degree $s+t$ part of $\mathcal{E}_s^r$. Here s is the filtration degree, t is the complementary degree, and $s+t=n$ is the total degree.

The d^r -differential

$$d_{s,t}^r: \mathcal{E}_{s,t}^r \longrightarrow \mathcal{E}_{s-r,t+r-1}^r$$

reduces the filtration degree by r , and reduces the total degree by 1. For $x \in Z_s^r$ with class $[x] \in \mathcal{E}_s^r$, we choose a $y \in A_{s-r}$ with

$$\alpha_{s-1}^{(r-1)}(y) = \gamma_s(x),$$

and we let $d_s^r([x]) := [\beta_{s-r}(y)] \in \mathcal{E}_{s-r}^r$ be the (well-defined) class of $\beta_{s-r}(y)$.

$$\begin{array}{ccc}
 y & \xrightarrow{\alpha_{s-1}^{(r-1)}} & \gamma_s(x) \\
 \downarrow \beta_{s-r} & & \nwarrow \gamma_s \\
 \beta_{s-r}(y) & & x
 \end{array}$$

Then $\ker(d_s^r) = Z_s^{r+1}/B_s^r$ and $\text{im}(d_{s+r}^r) = B_s^{r+1}/B_s^r$, so that

$$H(\mathcal{E}^r, d^r) \cong Z^{r+1}/B^{r+1} = \mathcal{E}^{r+1}.$$

In other words, $(\mathcal{E}^r, d^r)_{r \geq 1}$ is a spectral sequence.

We often abbreviate the notation for a spectral sequence to $\mathcal{E}^r$ or $\mathcal{E}_{*,*}^r$.

7. WEAK AND STRONG CONVERGENCE

Passing to (co-)limits, let

$$\begin{aligned}
 Z_s^\infty &:= \bigcap_r Z_s^r \\
 B_s^\infty &:= \bigcup_r B_s^r \\
 \mathcal{E}_s^\infty &:= Z_s^\infty / B_s^\infty
 \end{aligned}$$

be the infinite cycles, infinite boundaries, and the $\mathcal{E}^\infty$ -page in filtration s , respectively. Then

$$B_s^\infty \subset \beta_s(\ker(i_s)) \subset \ker(\gamma_s) \subset Z_s^\infty.$$

Here $B_s^\infty = \beta_s(\ker(i_s))$ always holds.

Definition 7.1. If $\ker(\gamma_s) = Z_s^\infty$ we say that the spectral sequence is *weakly convergent*. This is equivalent to saying that the natural map

$$\frac{F_s G}{F_{s-1} G} \cong \frac{\ker(\gamma_s)}{\beta_s(\ker(i_s))} \longrightarrow \frac{Z_s^\infty}{B_s^\infty} = \mathcal{E}_s^\infty$$

is an isomorphism for each s .

If, moreover, the exhaustive filtration $(F_s G)_s$ of $G = A_\infty$ is complete Hausdorff, then we say that the spectral sequence is *strongly convergent* to G .

To summarize: if

$$\mathcal{E}_{*,*}^r \Longrightarrow G_*$$

is a strongly convergent spectral sequence, starting at $\mathcal{E}_{*,*}^1 = E_{*,*}$ (or some later page), then we need to determine the differentials $d^r: \mathcal{E}^r \rightarrow \mathcal{E}^r$ for all $r \geq 1$ to inductively calculate $\mathcal{E}^{r+1}$, and pass to $\mathcal{E}^\infty$ as $r \rightarrow \infty$. This determines

$$F_s G_{s+t} / F_{s-1} G_{s+t} \cong \mathcal{E}_{s,t}^\infty$$

for all s and t , by weak convergence. Next we need to resolve the extension problems to determine $F_s G_n / F_{s-r} G_n$ for all s, n and $r \geq 1$. By strong convergence we obtain the degree n part of G as

$$G_n = \text{colim}_s \lim_r \frac{F_s G_n}{F_{s-r} G_n}$$

for all n .

8. HALF-PLANE SPECTRAL SEQUENCES WITH EXITING DIFFERENTIALS

We view each $\mathcal{E}^r$ -page as located in the (s, t) -plane. We say that $\mathcal{E}^r$ is concentrated in some half-plane if $\mathcal{E}_{s,t}^r = 0$ for all (s, t) outside of that region. We say that $(\mathcal{E}^r, d^r)_{r \geq 1}$ is a half-plane spectral sequence with *exiting differentials* if $\mathcal{E}^1$ is concentrated in such a half-plane (hence so is each later $\mathcal{E}^r$ -page), and for each (s, t) and all sufficiently large r the target bidegree $(s - r, t + r - 1)$ of the differential originating in (s, t) lies outside that half-plane.

This is the case, for example, if $\mathcal{E}_{s,t}^1 = 0$ for all $s < 0$, in which case the (homologically indexed) spectral sequence is concentrated in the right half-plane, and $A_{-\infty} \cong A_s \cong A_{-1}$ for each $s < 0$.

Theorem 8.1 ([CE56, Prop. XV.4.1], [Boa99, Thm. 6.1]). *Suppose an unrolled exact couple $(A_s, E_s, \alpha_s, \beta_s, \gamma_s)_s$ with $A_{-\infty} = 0$ yields a half-plane spectral sequence $(\mathcal{E}^r, d^r)_{r \geq 1}$ with exiting differentials. Then the spectral sequence converges strongly to $G = A_{\infty}$.*

In other words, $F_s G / F_{s-1} G \cong \mathcal{E}_s^{\infty}$ for all s , and the filtration $(F_s G)_s$ is exhaustive and complete Hausdorff.

9. HALF-PLANE SPECTRAL SEQUENCES WITH ENTERING DIFFERENTIALS

We say that $(\mathcal{E}^r, d^r)_{r \geq 1}$ is a half-plane spectral sequence with *entering differentials* if $\mathcal{E}^1$ is concentrated in a half-plane (hence so is each later $\mathcal{E}^r$ -page), and for each (s, t) and all sufficiently large r the source bidegree $(s + r, t - r + 1)$ of the differential landing in (s, t) lies outside that half-plane.

This is the case, for example, if $\mathcal{E}_{s,t}^1 = 0$ for all $s > 0$, in which case the (homologically indexed) spectral sequence is concentrated in the left half-plane, and $A_0 \cong A_s \cong A_{\infty}$ for each $s \geq 0$.

Definition 9.1. Following [Boa99, Def. 5.10], we say that (the spectral sequence associated to) an unrolled exact couple $(A_s, E_s, \alpha_s, \beta_s, \gamma_s)_s$ is *conditionally convergent* if $A_{-\infty} := \lim_s A_s = 0$ and $RA_{-\infty} := \operatorname{Rlim}_s A_s = 0$.

When $A_s = \pi_*(Y_s)$ arises from a sequence of spectra, with (homotopy) limit $Y_{-\infty} := \lim_s Y_s$, the short exact sequence

$$0 \rightarrow RA_{-\infty} \longrightarrow \pi_*(Y_{-\infty}) \longrightarrow A_{\infty} \rightarrow 0$$

shows that the exact couple is conditionally convergent if and only if $Y_{-\infty} \simeq 0$. This can often be verified from the construction of the sequence of spectra $(Y_s, f_s)_s$.

Definition 9.2. Let $R\mathcal{E}_s^{\infty} = \operatorname{Rlim}_r Z_s^r$ for each $s \in \mathbb{Z}$, so that $R\mathcal{E}^{\infty}$ is the *derived $\mathcal{E}^{\infty}$ -page*.

For each bidegree (s, t) , if there are only finitely many non-zero differentials $d_{s,t}^r$ originating in that bidegree, then $R\mathcal{E}_{s,t}^{\infty} = 0$. This is the case, for example, if $\mathcal{E}_{s,t}^r$ is a finite group for some r . This can often be verified under suitable finiteness hypotheses, or by sparsity arguments involving the bigrading.

Theorem 9.3 ([Boa99, Thm. 7.3]). *Suppose $(A_s, E_s, \alpha_s, \beta_s, \gamma_s)_s$ yields a half-plane spectral sequence $(\mathcal{E}^r, d^r)_{r \geq 1}$ with entering differentials. Then any two of the following conditions implies the third.*

- (1) *The exact couple is conditionally convergent, i.e., $A_{-\infty} = RA_{-\infty} = 0$;*

- (2) $R\mathcal{E}^\infty = 0$;
- (3) *The spectral sequence converges strongly to $G = A_\infty$.*

Hence, for a conditionally convergent half-plane spectral sequence with $R\mathcal{E}^\infty = 0$, there are isomorphisms $F_s G / F_{s-1} G \cong \mathcal{E}_s^\infty$ for all s , and the filtration $(F_s G)_s$ is exhaustive and complete Hausdorff.

10. WHOLE-PLANE SPECTRAL SEQUENCES

For spectral sequences that are not concentrated in a half-plane, there is an additional whole-plane obstruction group W , which must vanish to ensure strong convergence. See [Boa99, §8] for the rather complicated definition

$$W := \operatorname{colim}_s \operatorname{Rlim}_r K_\infty \operatorname{im}^r A^s$$

in terms of the unrolled exact couple, and [HR19] for the simpler description $W = \ker(\kappa)$ in terms of a Cartan–Eilenberg system.

11. MULTIPLICATIVE STRUCTURE

Let $('\mathcal{E}^r, 'd^r)_r$, $(''\mathcal{E}^r, ''d^r)_r$ and $(\mathcal{E}^r, d^r)_r$ be three spectral sequences. A *pairing of spectral sequences*, from the first two to the third, is a sequence $(\mu^r)_r$ of homomorphisms

$$\mu^r : '\mathcal{E}^r \otimes ''\mathcal{E}^r \longrightarrow \mathcal{E}^r$$

that satisfy the Leibniz rule

$$\mu^r \circ ('d^r \otimes 1 + 1 \otimes ''d^r) = d^r \circ \mu^r$$

and are compatible with the implicit isomorphisms, so that

$$\mu^{r+1}([x] \otimes [y]) = [\mu^r(x \otimes y)]$$

for $x \in \ker('d^r)$ and $y \in \ker(''d^r)$. In other words, μ^r is a chain map when $'\mathcal{E}^r \otimes ''\mathcal{E}^r$ is equipped with the tensor product differential $'d^r \otimes 1 + 1 \otimes ''d^r$, and μ^r induces μ^{r+1} upon passage to homology. In particular, the initial pairing μ^1 determines all of the later μ^r , but not every μ^1 extends to a pairing $(\mu^r)_r$.

Keeping track of bigradings, μ^r has components

$$\mu_{p,u,q,v}^r : '\mathcal{E}_{p,u}^r \otimes ''\mathcal{E}_{q,v}^r \longrightarrow \mathcal{E}_{p+q,u+v}^r$$

and

$$('d^r \otimes 1 + 1 \otimes ''d^r)(x \otimes y) = 'd^r(x) \otimes y + (-1)^{|x|} x \otimes ''d^r(y)$$

where $|x| = p + u$ is the total degree of $x \in '\mathcal{E}_{p,u}^r$.

A passage to subquotients shows that a pairing $(\mu^r)_r$ also determines a pairing

$$\mu^\infty : '\mathcal{E}^\infty \otimes ''\mathcal{E}^\infty \longrightarrow \mathcal{E}^\infty$$

of $\mathcal{E}^\infty$ -pages.

Beware that $('\mathcal{E}^r \otimes ''\mathcal{E}^r, 'd^r \otimes 1 + 1 \otimes ''d^r)_{r \geq 1}$ is not generally a spectral sequence, since a flatness hypothesis is needed for the homology of the r -th page to be isomorphic to the $(r+1)$ -th page, cf. the Künneth formula. Hence we do not in general have a ‘tensor product spectral sequence’.

If the three spectral sequences are equal, and each μ^r is associative and unital, then we say that $\mathcal{E}^r$ is a multiplicative spectral sequence. If $\mathcal{E}^r$ is a multiplicative spectral sequence, $''\mathcal{E}^r = \mathcal{E}^r$, and μ^r is associative and unital, then we say that $'\mathcal{E}^r$ acts on $\mathcal{E}^r$.

To determine the d^r -differentials in a multiplicative spectral sequence, it suffices to find $d^r(x)$ for a set of ring generators for $\mathcal{E}^r$. The value of d^r on decomposable classes is then determined by the Leibniz rule. This can drastically reduce the number of possible differential patterns.

Suppose that $(\mathcal{E}^r, 'd^r)_r$, $(''\mathcal{E}^r, ''d^r)_r$ and $(\mathcal{E}^r, d^r)_r$ arise from unrolled exact couples, with abutments G' , G'' and G . These are filtered graded abelian groups. A pairing $\mu: G' \otimes G'' \rightarrow G$ is *filtration-preserving* if

$$\mu(F_p G' \otimes F_q G'') \subset F_{p+q} G$$

for all $p, q \in \mathbb{Z}$. It follows that μ induces pairings

$$\bar{\mu}_{p,q}: \frac{F_p G'}{F_{p-1} G'} \otimes \frac{F_q G''}{F_{q-1} G''} \longrightarrow \frac{F_{p+q} G}{F_{p+q-1} G}$$

at the level of filtration subquotients, also known as the associated graded objects. Suppose each spectral sequence converges (weakly or strongly) to the corresponding abutment, so that we have isomorphisms

$$\frac{F_p G'}{F_{p-1} G'} \cong {}'\mathcal{E}_p^\infty \quad , \quad \frac{F_q G''}{F_{q-1} G''} \cong ''\mathcal{E}_q^\infty \quad \text{and} \quad \frac{F_s G}{F_{s-1} G} \cong \mathcal{E}_s^\infty .$$

Then we say that the spectral sequence pairing $(\mu^r)_r$ *converges to the pairing* μ if $\bar{\mu}$ and μ^∞ are compatible under these isomorphisms. We can also say that we have *multiplicative convergence*.

If $(\mathcal{E}^r)_r$ with the pairing $(\mu^r)_r$ is a multiplicative spectral sequence, converging multiplicatively to G with the filtration-preserving pairing μ , then knowing μ^1 determines each μ^r and μ^∞ , hence also the pairing $\bar{\mu}$ on the associated graded $(F_s G / F_{s-1} G)_s$. This in turn suffices to determine the pairing μ on G , ‘up to filtration shifts’. In other words, if $x \in F_p G$ and $y \in F_q G$ have product $xy = \mu(x \otimes y) \in F_{p+q} G$, knowledge of $\bar{\mu}([x] \otimes [y])$ will determine $[xy] \in F_{p+q} G / F_{p+q-1} G$, but only determines xy modulo $F_{p+q-1} G$. There can be case-specific ways of sidestepping this indeterminacy, e.g. by checking that $F_{p+q-1} G = 0$ in the total degree of xy .

12. PAIRINGS OF SEQUENCES

Let $(A'_*, E'_*, \alpha'_*, \beta'_*, \gamma'_*)$, $(A''_*, E''_*, \alpha''_*, \beta''_*, \gamma''_*)$ and $(A_*, E_*, \alpha_*, \beta_*, \gamma_*)$ be exact couples, with associated spectral sequences $('\mathcal{E}^r, 'd^r)_r$, $(''\mathcal{E}^r, ''d^r)_r$ and $(\mathcal{E}^r, d^r)_r$, and abutments $G' = A'_\infty$, $G'' = A''_\infty$ and $G = A_\infty$. It appears to be difficult to specify data in terms of the exact couples that suffice to define a pairing $(\mu^r)_r$ of the spectral sequences, compatible with a pairing μ of the filtered abutments. The condition provided in [Mas54] is essentially tautological, in that it is as difficult to verify as what one is trying to prove.

However, when the exact couples arise from sequences of spectra, or of chain complexes, this can be achieved. Up to now the discussion has effectively only taken place in the homotopy 1-category of spectra, or in the derived category of abelian groups. However, at this point we must work in a topological model category of spectra, or the associated ∞ -category, or at least the underlying 2-category.

Recall that to a sequence of spectra Y_* we associated an unrolled exact couple with $A_s = \pi_*(Y_s)$ and $E_s = \pi_*(Y_s, Y_{s-1}) := \pi_*(Cf_s)$, and a spectral sequence $(\mathcal{E}^r)_r$ with starting page $\mathcal{E}_s^1 = E_s$ and filtered abutment $G = A_\infty = \pi_*(Y_\infty)$.

Definition 12.1 ([HR24, Def. 4.20]). Let $Y'_\star$, $Y''_\star$ and $Y_\star$ be sequences of spectra and spectrum maps. A *pairing of sequences*, from the first two to the third, is a collection ϕ of spectrum maps

$$\phi_{p,q}: Y'_p \otimes Y''_q \longrightarrow Y_{p+q}$$

such that the squares

$$\begin{array}{ccccc} Y'_{p-1} \otimes Y''_q & \xrightarrow{f'_p \otimes 1} & Y'_p \otimes Y''_q & \xleftarrow{1 \otimes f''_q} & Y'_p \otimes Y''_{q-1} \\ \downarrow \phi_{p-1,q} & & \downarrow \phi_{p,q} & & \downarrow \phi_{p,q-1} \\ Y_{p+q-1} & \xrightarrow{f_{p+q}} & Y_{p+q} & \xleftarrow{f_{p+q}} & Y_{p+q-1} \end{array}$$

all commute. If $Y'_\star = Y''_\star = Y_\star$ we say that $(Y_\star, \phi)$ is a *multiplicative sequence of spectra*.

A similar definition applies for sequences of chain complexes. Note that the squares are assumed to commute strictly, not just up to (chain) homotopy. The following was presumably ‘well known’.

Theorem 12.2 ([HR24, Thm. 4.27]). *Let $Y'_\star$, $Y''_\star$ and $Y_\star$ be sequences of spectra, with associated spectral sequences $(\mathcal{E}^r)_r$, $(\mathcal{E}^r)_r$ and $(\mathcal{E}^r)_r$ and abutments G' , G'' and G . Suppose given a pairing $\phi_{*,*}$ of sequences of spectra, from $Y'_\star$ and $Y''_\star$ to $Y_\star$.*

Then there is a pairing $(\mu^r)_r$ of spectral sequences, with

$$\mu_{p,q}^1: \mathcal{E}_p^1 \otimes \mathcal{E}_q^1 \longrightarrow \mathcal{E}_{p+q}^1$$

given by a natural composite

$$\pi_*(Y'_p, Y'_{p-1}) \otimes \pi_*(Y''_q, Y''_{q-1}) \xrightarrow{\otimes} \pi_*(Y'_p \otimes Y''_q, W) \xrightarrow{\phi_*} \pi_*(Y_{p+q}, Y_{p+q-1})$$

(for a suitable pushout W). There is also a filtration-preserving pairing $\mu: G' \otimes G'' \rightarrow G$ of abutments, given by a natural composite

$$\pi_*(Y'_\infty) \otimes \pi_*(Y''_\infty) \xrightarrow{\otimes} \pi_*(Y'_\infty \otimes Y''_\infty) \xrightarrow{\phi_*} \pi_*(Y_\infty).$$

If the spectral sequences are weakly convergent, then these pairings are compatible, so that $(\mu^r)_r$ converges to μ , in the sense that μ^∞ and $\bar{\mu}$ agree by way of the canonical isomorphisms.

Corollary 12.3. *Let $(Y_\star, \phi)$ be a multiplicative sequence of spectra, with associated spectral sequence $(\mathcal{E}^r)_r$ and abutment G . Then $(\mathcal{E}^r)_r$ is a multiplicative spectral sequence, and G is a filtered graded ring. If the spectral sequence is weakly convergent, then we also have multiplicative convergence.*

Thinking instead in terms of filtered chain complexes, we have the following formulation, due to Massey [Mas54, §7, §8], who also gives the corresponding statement for a pairing of filtered chain complexes.

By a differential graded ring we mean a chain complex (R_*, ∂) with an associative and unital pairing $\phi: R_* \otimes R_* \rightarrow R_*$, where the source has the differential $\partial \otimes 1 + 1 \otimes \partial$. It is filtered if we have an exhaustive sequence of subcomplexes

$$\subset F_{s-1}R_* \subset F_sR_* \subset \cdots \subset R_*$$

such that $\phi(F_pR_* \otimes F_qR_*) \subset F_{p+q}R_*$ for all p and q . If $F_{-1}R_* = 0$, we get a strongly convergent spectral sequence with exiting differentials.

Theorem 12.4. *Let $(R_*, \partial, \phi, (F_s R_*)_s)$ be a filtered differential graded ring. The associated spectral sequence, with $\mathcal{E}_s^1 = H_*(F_s R_*/F_{s-1} R_*, \partial)$ for all s , is multiplicative, and the abutment $G = H_*(R_*, \partial)$ is a graded ring. If the spectral sequence is weakly convergent, then we also have multiplicative convergence.*

Returning to the case of spectra, the commutativity assumption for a pairing of sequences can be relaxed to commutativity up to chosen homotopy, if these choices can be made ‘coherently’, so as to make the following 3-dimensional diagram commute up to 2-homotopy.

$$\begin{array}{ccccc}
 Y'_{p-1} \otimes Y''_{q-1} & \xrightarrow{\quad} & Y'_p \otimes Y''_{q-1} & & \\
 \downarrow \scriptstyle \phi_{p-1, q-1} & \searrow & \downarrow & \searrow & \\
 & Y'_{p-1} \otimes Y''_q & \xrightarrow{\quad} & Y'_p \otimes Y''_q & \\
 & \searrow \scriptstyle \phi_{p-1, q} & \downarrow \scriptstyle \phi_{p, q-1} & \downarrow \scriptstyle \phi_{p, q} & \\
 Y_{p+q-2} & \xrightarrow{\quad} & Y_{p+q-1} & \xrightarrow{\quad} & Y_{p+q}
 \end{array}$$

For example, this is the case if Y_s is the $(-s)$ -connective Whitehead cover of Y_∞ , and similarly for Y' and Y'' .

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