

SOME STABLE STEMS

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1.1. Rational homotopy. For a (topological or simplicial) group G , BG is the classifying space of the (topological or simplicial) category with one object $*$ and morphisms G .

For G discrete, $BG = K(G, 1)$. For G discrete abelian, $B^n G = K(G, n)$. Examples: $B\mathbb{Z} \simeq S^1$, $B^2\mathbb{Z} \simeq \mathbb{C}P^\infty$.

Serre: $H_*(B^n\mathbb{Z}; \mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_n)$ for n odd and $\mathbb{Q}[x_n] = \Gamma_{\mathbb{Q}}[x_n]$ for $n \geq 2$ even.

Proof: Use the bar spectral sequence $\text{Tor}_*^{H_*(G; \mathbb{Q})}(\mathbb{Q}, \mathbb{Q}) \implies H_*(BG; \mathbb{Q})$. Specializes to $\text{Tor}_*^{\mathbb{Q}[x_n]}(\mathbb{Q}, \mathbb{Q}) = \Lambda_{\mathbb{Q}}(\sigma x_n)$ for n even, and to $\text{Tor}_*^{\Lambda_{\mathbb{Q}}(x_n)}(\mathbb{Q}, \mathbb{Q}) = \Gamma_{\mathbb{Q}}(\sigma x_n) = \mathbb{Q}[\sigma x_n]$ for n odd. Let $x_{n+1} = \sigma x_n$.

Let $y_n: S^n \rightarrow B^n\mathbb{Z}$ represent a generator in $H^n(S^n) = \mathbb{Z}$. If n is odd, $H_*(y_n; \mathbb{Q})$ is an isomorphism. Implies that $\pi_*(y_n) \otimes \mathbb{Q}$ is an isomorphism. So $\pi_*(S^n) \otimes \mathbb{Q} = \mathbb{Q}$ for $*$ = n and 0 otherwise.

If n is even, consider fiber sequence $B^{n-1}\mathbb{Z} \rightarrow S^{\langle n \rangle} \rightarrow S^n \rightarrow B^n\mathbb{Z}$. Serre spectral sequence

$$H_*(S^n; H_*(B^{n-1}\mathbb{Z}; \mathbb{Q})) \implies H_*(S^{\langle n \rangle}; \mathbb{Q})$$

has $E^2 = \Lambda_{\mathbb{Q}}(x_n, x_{n-1})$ and $d^n(x_n) = x_{n-1}$ so $E^\infty = \mathbb{Q}\{1, x_{n-1}x_n\}$. Hence $\pi_*(S^{\langle n \rangle}) \otimes \mathbb{Q} = \Lambda_{\mathbb{Q}}(x_{2n-1})$ by the odd case. Thus $\pi_*(S^n) \otimes \mathbb{Q} \cong \mathbb{Q}$ for $*$ = n and $*$ = $2n - 1$, and 0 otherwise.

Similar arguments show that $\pi_*(S^n)$ is finitely generated abelian for each n and $*$.

1.2. Topological K -theory. Let $O = \text{colim}_n O(n)$, $U = \text{colim}_n U(n)$, $\text{Sp} = \text{colim}_n \text{Sp}(n)$, $\text{SO} = \text{colim}_n \text{SO}(n)$, $\text{Spin} = \text{colim}_n \text{Spin}(n)$. $BO = \text{colim}_n \text{Gr}_n(\mathbb{R}^{n+k})$ has $H^*(BO; \mathbb{F}_2) = \mathbb{F}_2[w_i \mid i \geq 1]$, with $w_i \in H^i$ the i -th Stiefel–Whitney characteristic class. $BU = \text{colim}_n \text{Gr}_n^{\mathbb{C}}(\mathbb{C}^{n+k})$ has $H^*(BU) = \mathbb{Z}[c_i \mid i \geq 1]$, with $c_i \in H^{2i}$ the i -th Chern class.

Bott: $\Omega BU \simeq U$, $\Omega U \simeq \mathbb{Z} \times BU$. Hence $\Omega^2(\mathbb{Z} \times BU) \simeq \mathbb{Z} \times BU$. Deduce $\pi_*(U) = (0, \mathbb{Z}, 0, \mathbb{Z}, \dots)$ repeating 2-periodically. Also $\Omega BO \simeq O$, $\Omega O \simeq O/U$, $\Omega(O/U) \simeq U/\text{Sp}$, $\Omega(U/\text{Sp}) \simeq \mathbb{Z} \times B\text{Sp}$, $\Omega(\mathbb{Z} \times B\text{Sp}) \simeq \text{Sp}$, $\Omega \text{Sp} \simeq \text{Sp}/U$, $\Omega(\text{Sp}/U) \simeq U/O$ and $\Omega(U/O) \simeq \mathbb{Z} \times BO$. Hence $\Omega^8(\mathbb{Z} \times BO) \simeq \mathbb{Z} \times BO$. Fiber sequence $B(\mathbb{Z} \times BO) \rightarrow \mathbb{Z} \times BO \rightarrow \mathbb{Z} \times BU$. Deduce $\pi_*(O) = (\mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \dots)$ repeating 8-periodically.

G. Whitehead: Monoid map $j: O \rightarrow F = \Omega_{\pm 1}^\infty \mathbb{S} = \text{GL}_1(\mathbb{S}) \subset \text{colim}_n \Omega^n S^n$ by one-point compactification. Induces J -homomorphism $J: \pi_*(O) \rightarrow \pi_*(F) = \pi_*(\mathbb{S})$ for $*$ > 0. Images of $\mathbb{Z}/2 = \pi_1(O) \rightarrow \pi_1(\mathbb{S})$, $\mathbb{Z} = \pi_3(O) \rightarrow \pi_3(\mathbb{S})$, $\mathbb{Z} = \pi_7(O) \rightarrow \pi_7(\mathbb{S})$ and generated by the stable classes η , ν and σ of the complex, quaternionic and octonionic Hopf fibrations $S^3 \rightarrow S^2$, $S^7 \rightarrow S^4$ and $S^{15} \rightarrow S^8$.

$\mathbb{Z} \times BO_p^\wedge$ represents p -complete real K -theory. Admits stable Adams operations $\psi^k: \mathbb{Z} \times BO_p^\wedge \rightarrow \mathbb{Z} \times BO_p^\wedge$ for $k \in \mathbb{Z}_p^\times$. Since $\psi^3(L) = L^{\otimes 3} \cong L$ for real line bundles, $\psi^3 = 1$ on $\pi_*(BO)_2^\wedge$ for $* \leq 3$. Also, ψ^k multiplies by k^n in degree $2n$. Hereafter we implicitly complete at a fixed prime p , usually 2.

1.3. The image-of- J spectrum. Adams conjecture: The 2-complete map $Bj \circ (\psi^3 - 1): BO \rightarrow BSF$ is null-homotopic. Proved by Quillen and Sullivan (by introducing étale homotopy theory), later by Becker–Gottlieb using transfer maps. Gives space-level dashed maps.

$$\begin{array}{ccccc}
 J_\oplus & \longrightarrow & BO & \xrightarrow{\psi^3 - 1} & BSpin \\
 \downarrow \alpha & & \downarrow & & \downarrow = \\
 SF & \longrightarrow & SF/Spin & \longrightarrow & BSpin \xrightarrow{Bj} BSF \\
 \downarrow e & & \downarrow & & \downarrow \rho \\
 J_\otimes & \longrightarrow & BO_\otimes & \xrightarrow{\psi^3/1} & BSpin_\otimes
 \end{array}$$

‘Cannibalistic’ characteristic class calculation shows that the rectangle from BO to $BSpin_\otimes$ is cartesian, so $e \circ \alpha$ is an equivalence. Hence $e_*: \pi_*(\mathbb{S}) \cong \pi_*(SF) \rightarrow \pi_*(J_\otimes)$ is split surjective for $* > 0$. References: Adams’ $J(X)$ -papers. May’s E_∞ -book.

Connective spectra ko , ku with underlying infinite loop spaces $\mathbb{Z} \times BO$ and $\mathbb{Z} \times BU$. Let $bspin = \tau_{\geq 4}ko$ with underlying infinite loop space $BSpin \simeq B^4(\mathbb{Z} \times Sp)$. Image-of- J spectrum $j = j_2^\wedge$ defined additively as fiber of $\psi^3 - 1: ko \rightarrow bspin$. Gets $\mathbb{E}_\infty$ -ring structure as pullback

$$\begin{array}{ccccc}
 \mathbb{S} & & & \xrightarrow{d} & ko \\
 \searrow e & & & & \downarrow \\
 & j & \longrightarrow & ko^{h\psi^3} & \longrightarrow ko \\
 & \downarrow & & \downarrow & \\
 & \tau_{\leq 1}\mathbb{S} & \longrightarrow & \tau_{\leq 1}(ko^{h\psi^3}) &
 \end{array}$$

Recover $J_\oplus = \Omega_0^\infty j$ and $J_\otimes = \Omega_1^\infty j$. Adams conjecture implies that the e -invariant $\pi_*(e): \pi_*(\mathbb{S}) \rightarrow \pi_*(j)$ is split surjective.

Proposition: We have $\pi_0(j) = (\mathbb{Z}, \mathbb{Z}/2, \dots)$ where in degrees $* \geq 2$,

$$\pi_*(j) = \begin{cases} \mathbb{Z}_2/16n & \text{for } * = 8n - 1, \\ \mathbb{Z}/2 & \text{for } * \equiv 0 \pmod{8}, \\ (\mathbb{Z}/2)^2 & \text{for } * \equiv 1 \pmod{8}, \\ \mathbb{Z}/2 & \text{for } * \equiv 2 \pmod{8}, \\ \mathbb{Z}/8 & \text{for } * \equiv 3 \pmod{8}, \\ 0 & \text{for } * \equiv 4, 5, 6 \pmod{8}. \end{cases}$$

It follows that ν has order 8 and σ has order 16 in $\pi_*(\mathbb{S})_2^\wedge$. (The integral orders are 24 and 240.) The d -invariant $\pi_*(d): \pi_*(\mathbb{S}) \rightarrow \pi_*(ko)$ is split surjective in degrees $* \equiv 1, 2 \pmod{8}$, i.e., hits all the torsion in the target. For $n \geq 0$ and $r \in \{1, 2\}$ there are classes $\mu_{8n+r} \in \pi_{8n+r}(\mathbb{S})$ detected by (the e -invariant and) the d -invariant

$$\pi_*(j) \longrightarrow \pi_*(ko) \xrightarrow{\psi^3-1} \pi_*(bspin)$$

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow{3^6-1} & \mathbb{Z}_2 & & \\
 & \searrow & & & \searrow & & \\
 \mathbb{Z}/16 & \longrightarrow & 0 & \longrightarrow & 0 & & \\
 & \searrow & & & \searrow & & \\
 \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & & \\
 & \searrow & & & \searrow & & \\
 (\mathbb{Z}/2)^2 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & & \\
 & \searrow & & & \searrow & & \\
 \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow{3^4-1} & \mathbb{Z}_2 & & \\
 & \searrow & & & \searrow & & \\
 \mathbb{Z}/16 & \longrightarrow & 0 & \longrightarrow & 0 & & \\
 & \searrow & & & \searrow & & \\
 0 & \longrightarrow & 0 & \longrightarrow & 0 & & \\
 & \searrow & & & \searrow & & \\
 0 & \longrightarrow & 0 & \longrightarrow & 0 & & \\
 & \searrow & & & \searrow & & \\
 0 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow{3^2-1} & \mathbb{Z}_2 & & \\
 & \searrow & & & \searrow & & \\
 \mathbb{Z}/8 & \longrightarrow & 0 & \longrightarrow & 0 & & \\
 & \searrow & & & \searrow & & \\
 \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 & & \\
 & \searrow & & & \searrow & & \\
 \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 & & \\
 & \searrow & & & \searrow & & \\
 \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0 & &
 \end{array}$$

in $\pi_{8n+r}(\mathbf{k}o) = \mathbb{Z}/2$. Here $\mu_1 = \eta$, $\mu_2 = \eta^2$ and in general we may arrange that $\mu_{8n+2} = \eta\mu_{8n+1}$. The μ_{8n+1} and μ_{8n+2} form v_1^4 -periodic families, first considered by Barratt and by Adams.

2. OCTOBER 30TH, 2026

((ETC: Hopf invariant. Hopf-Steenrod invariant. Steenrod algebra. Adams spectral sequence. Classes h_i . Ext over $A(1)$ and $A(2)$..))