

TALK 6: BRAIDED MONOIDAL POWER OPERATIONS AND THEOREMS OF MAHOWALD AND HOPKINS–MAHOWALD

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FRONTISPIECE

We study symmetric monoidal categories of spaces $(\mathcal{S}, \times, \Delta^0)$, spectra $(\mathcal{S}p, \otimes, \mathbb{S})$ and $\mathbb{F}_p$ -modules $(\text{Mod}_{\mathbb{F}_p}, \otimes_{\mathbb{F}_p}, \mathbb{F}_p)$, and symmetric monoidal (left adjoint) functors $\mathbb{S}[-] = \Sigma^\infty(-)_+ : \mathcal{S} \rightarrow \mathcal{S}p$ and $\mathbb{F}_p \otimes : \mathcal{S}p \rightarrow \text{Mod}_{\mathbb{F}_p}$. Here $\mathbb{F}_p$ is the Eilenberg–MacLane $\mathbb{E}_\infty$ -ring with $\pi_*(\mathbb{F}_p)$ equal to the prime field $\mathbb{F}_p$ concentrated in degree 0. For each $0 \leq k \leq \infty$, an $\mathbb{E}_k$ -space is an $\mathbb{E}_k$ -algebra in $\mathcal{S}$, an $\mathbb{E}_k$ -ring is an $\mathbb{E}_k$ -algebra in $\mathcal{S}p$, and an $\mathbb{E}_k$ $\mathbb{F}_p$ -algebra is an $\mathbb{E}_k$ -algebra in $\text{Mod}_{\mathbb{F}_p}$. We also consider symmetric monoidal categories of spaces over the $\mathbb{E}_\infty$ -spaces BO , $\text{Pic}_{\mathbb{S}}$ or $\text{Pic}_{\mathbb{S}_p}$, and the symmetric monoidal (left adjoint) Thom spectrum functor Th to spectra or $\mathbb{S}_p$ -modules.

$$\begin{array}{c}
 \begin{array}{ccccc}
 \mathcal{C} & \mathcal{S} & \xrightarrow{\mathbb{S}[-]} & \mathcal{S}p & \xrightarrow{\mathbb{F}_p \otimes} & \text{Mod}_{\mathbb{F}_p} \\
 & & & \downarrow & \nearrow & \\
 & \mathcal{S}/\text{Pic}_{\mathbb{S}_p} & \xrightarrow{\text{Th}} & \text{Mod}_{\mathbb{S}_p} & &
 \end{array} \\
 \\
 \begin{array}{ccccccc}
 & & & & \Sigma \mathbb{S} & \dashrightarrow & \Sigma \mathbb{F}_p \\
 \begin{array}{c} \mathcal{C} \\ \downarrow \text{Free}_{\mathbb{E}_0} \\ \text{Alg}_{\mathbb{E}_0}(\mathcal{C}) \\ \downarrow \text{Free}_{\mathbb{E}_2/\mathbb{E}_0} \\ \text{Alg}_{\mathbb{E}_2}(\mathcal{C}) \end{array} & \mathcal{S}^1 & \dashrightarrow & \mathbb{S}[S^1] & \dashrightarrow & \mathbb{F}_p[S^1] \\
 \text{Free}_{\mathbb{E}_2} \curvearrowright & & & \mathcal{S}^1 \xrightarrow{\gamma_0} \text{Pic}_{\mathbb{S}_p} & \dashrightarrow & \mathbb{S}/p & \dashrightarrow \\
 & \Omega^2 S^3 & \dashrightarrow & \mathbb{S}[\Omega^2 S^3] & \dashrightarrow & \mathbb{F}_p[\Omega^2 S^3] \stackrel{2.5}{\simeq} \mathbb{F}_p \otimes \mathbb{F}_p \\
 & & & \Omega^2 S^3 \xrightarrow{\gamma_2} \text{Pic}_{\mathbb{S}_p} & \dashrightarrow & \text{Th}(\gamma_2) \underset{4.5}{\simeq} \mathbb{F}_p &
 \end{array}
 \end{array}$$

1. THE SECOND LOOP SPACE OF THE THREE-SPHERE

For any $\mathbb{E}_2$ -space (or $\mathbb{E}_2$ -ring) X , we have natural operations

$$Q_1: H_q(X; \mathbb{F}_2) \longrightarrow H_{2q+1}(X; \mathbb{F}_2)$$

for $q \geq 0$, and

$$Q_{p-1}: H_q(X; \mathbb{F}_p) \longrightarrow H_{pq+p-1}(X; \mathbb{F}_p)$$

for p and $q \geq 1$ odd. For any space (or spectrum) X we also have the homology Bockstein operators $\beta: H_q(X; \mathbb{F}_p) \rightarrow H_{q-1}(X; \mathbb{F}_p)$. The previous talk concluded with the following results, with parts (3) and (4) due to Kudo–Araki [KA56] and Dyer–Lashof [DL62], respectively.

Theorem 1.1.

- (1) $H_*(S^3; \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(z)$ with $|z| = 3$.
- (2) $H_*(\Omega S^3; \mathbb{F}_p) \cong \mathbb{F}_p[y]$ with $|y| = 2$.
- (3) For $p = 2$,

$$H_*(\Omega^2 S^3; \mathbb{F}_2) \cong \mathbb{F}_2[x, Q_1(x), Q_1 Q_1(x), \dots].$$

Here $|Q_1^{(k-1)}(x)| = 2^k - 1$ for $k \geq 1$.

- (4) For p odd,

$$H_*(\Omega^2 S^3; \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(x, Q_{p-1}(x), Q_{p-1} Q_{p-1}(x), \dots) \\ \otimes \mathbb{F}_p[\beta Q_{p-1}(x), \beta Q_{p-1} Q_{p-1}(x), \dots].$$

Here $|Q_{p-1}^{(k)}(x)| = 2p^k - 1$ for $k \geq 0$ and $|\beta Q_{p-1}^{(k)}(x)| = 2p^k - 2$ for $k \geq 1$.

Proof. ((ETC: Serre spectral sequence for $\Omega S^3 \rightarrow PS^3 \rightarrow S^3$ and $\Omega^2 S^3 \rightarrow P\Omega S^3 \rightarrow \Omega S^3$, plus the interaction of suspension with the homology operations and the Kudo transgression theorem.)) $\square$

Proposition 1.2. $\mathbb{S}[\Omega^2 S^3]$ is the free $\mathbb{E}_2$ -ring on $\Sigma \mathbb{S}$, and $\mathbb{F}_p[\Omega^2 S^3]$ is the free $\mathbb{E}_2$ $\mathbb{F}_p$ -algebra on $\Sigma \mathbb{F}_p$.

Proof. Consider the connected based space S^1 as an object in $\text{Alg}_{\mathbb{E}_0}(\mathcal{S}) = \mathcal{S}_{\Delta^0/}$. By the approximation theorem [May72, Thm. 2.7], $\text{Free}_{\mathbb{E}_2/\mathbb{E}_0}(S^1) \simeq \Omega^2 S^3$ as $\mathbb{E}_2$ -spaces.

Since $X \mapsto \mathbb{S}[X]$ is symmetric monoidal, $\text{Free}_{\mathbb{E}_2/\mathbb{E}_0}(\mathbb{S}[S^1]) \simeq \mathbb{S}[\Omega^2 S^3]$ as $\mathbb{E}_2$ -rings. Here $\mathbb{S}[S^1] \simeq \mathbb{S} \oplus \Sigma \mathbb{S}$ in $\text{Alg}_{\mathbb{E}_0}(\mathcal{S}p) = \mathcal{S}p_{\mathbb{S}/}$, because the cofiber sequence $\Delta_+^0 \rightarrow S_+^1 \rightarrow S^1$ admits a retraction, and therefore stably splits. Hence, $\text{Free}_{\mathbb{E}_2}(\Sigma \mathbb{S}) \simeq \mathbb{S}[\Omega^2 S^3]$ is the free $\mathbb{E}_2$ -ring on $\Sigma \mathbb{S}$.

Since $X \mapsto \mathbb{F}_p \otimes X$ is symmetric monoidal, $\text{Free}_{\mathbb{E}_2}^{\mathbb{F}_p}(\Sigma \mathbb{F}_p) = \text{Free}_{\mathbb{E}_2/\mathbb{E}_0}^{\mathbb{F}_p}(\mathbb{F}_p \oplus \Sigma \mathbb{F}_p) \simeq \mathbb{F}_p[\Omega^2 S^3]$ is the free $\mathbb{E}_2$ $\mathbb{F}_p$ -algebra on $\mathbb{F}_p \otimes \Sigma \mathbb{S} = \Sigma \mathbb{F}_p$. Here $\mathbb{F}_p[S^1] \simeq \mathbb{F}_p \oplus \Sigma \mathbb{F}_p$. $\square$

2. THE MOD p EILENBERG–MACLANE SPECTRUM

2.1. The dual Steenrod algebra. Recall that $H_*(X; \mathbb{F}_p) = \pi_*(\mathbb{F}_p \otimes X)$ and $H^*(X; \mathbb{F}_p) = \pi_{-*} \text{Hom}(X, \mathbb{F}_p)$ for all spectra X . By the Yoneda lemma, the natural transformations $H^*(X; \mathbb{F}_p) \rightarrow H^{*+r}(X; \mathbb{F}_p)$ correspond to elements in $\mathcal{A}^r := H^r(\mathbb{F}_p; \mathbb{F}_p) = \pi_{-r} \text{Hom}(\mathbb{F}_p, \mathbb{F}_p)$. This is the codegree r part of the mod p Steenrod algebra $\mathcal{A} := H^*(\mathbb{F}_p; \mathbb{F}_p)$, with (non-commutative) product defined by composition. The operations $\text{Sq}^r: H^*(X; \mathbb{F}_2) \rightarrow H^{*+r}(X; \mathbb{F}_2)$ and $\beta^e P^r: H^*(X; \mathbb{F}_p) \rightarrow$

$H^{*+\epsilon+(2p-2)r}(X; \mathbb{F}_p)$ (for p odd) introduced by Steenrod satisfy relations determined by José Ádem, and were shown by Jean–Pierre Serre (for $p = 2$) and Henri Cartan (for p odd) [Ser53], [Car54] to generate the Steenrod algebra. Here $r \geq 0$ and $\epsilon \in \{0, 1\}$, with β the cohomological Bockstein operation.

More precisely, Serre and Cartan obtained bases for the $\mathcal{A}$, consisting of ‘admissible’ finite length compositions of positive degree Steenrod operations. Here $\text{Sq}^{r_1} \circ \dots \circ \text{Sq}^{r_\ell}$ is admissible if $r_i \geq 2r_{i+1}$ for each $1 \leq i < \ell$, while $\beta^{\epsilon_1} P^{r_1} \circ \dots \circ \beta^{\epsilon_\ell} P^{r_\ell}$ is admissible if $\epsilon_i + (2p-2)r_i \geq p(\epsilon_{i+1} + (2p-2)r_{i+1})$ for each $1 \leq i < \ell$.

The dual Steenrod algebra

$$\mathcal{A}_* := H_*(\mathbb{F}_p; \mathbb{F}_p) = \pi_*(\mathbb{F}_p \otimes \mathbb{F}_p)$$

satisfies $\mathcal{A}_*^\vee := \text{Hom}(\mathcal{A}_*, \mathbb{F}_p) \cong \mathcal{A}$ and $(\mathcal{A})^\vee \cong \mathcal{A}_*$, by the universal coefficient theorem. It inherits a (graded) commutative product from the $\mathbb{E}_\infty$ -ring structure on $\mathbb{F}_p \otimes \mathbb{F}_p$, and a coproduct $\psi: \mathcal{A}_* \rightarrow \mathcal{A}_* \otimes \mathcal{A}_*$ from the composition product in $\mathcal{A}$. Dually, $\mathcal{A}$ inherits a cocommutative coproduct from the product in $\mathcal{A}_*$. The symmetry $\mathbb{F}_p \otimes \mathbb{F}_p \simeq \mathbb{F}_p \otimes \mathbb{F}_p$ induces an antipode/conjugation/involution $\chi: \mathcal{A}_* \cong \mathcal{A}_*$, with dual $\chi: \mathcal{A} \cong \mathcal{A}$. Hence both $\mathcal{A}_*$ and $\mathcal{A}$ are Hopf algebras [MM65]. For each spectrum X , the mod p cohomology is naturally a left $\mathcal{A}$ -module

$$\lambda: \mathcal{A} \otimes H^*(X; \mathbb{F}_p) \longrightarrow H^*(X; \mathbb{F}_p)$$

and the mod p homology is naturally a left $\mathcal{A}_*$ -comodule

$$\nu: H_*(X; \mathbb{F}_p) \longrightarrow \mathcal{A}_* \otimes H_*(X; \mathbb{F}_p).$$

The latter determines the former, as the composite

$$\mathcal{A}_*^\vee \otimes H_*(X; \mathbb{F}_p)^\vee \xrightarrow{\otimes} (\mathcal{A}_* \otimes H_*(X; \mathbb{F}_p))^\vee \xrightarrow{\nu^\vee} H_*(X; \mathbb{F}_p)^\vee.$$

Each cohomology operation $\text{Sq}^r: H^*(X; \mathbb{F}_2) \rightarrow H^{*+r}(X; \mathbb{F}_2)$ is the dual of a homology operation $\text{Sq}_*^r: H_{*+r}(X; \mathbb{F}_2) \rightarrow H_*(X; \mathbb{F}_2)$, and these assemble to a right $\mathcal{A}$ -module action

$$\begin{aligned} \rho: H_*(X; \mathbb{F}_2) \otimes \mathcal{A} &\longrightarrow H_*(X; \mathbb{F}_2) \\ x \otimes \text{Sq}_*^r &\longmapsto \text{Sq}_*^r(x) \end{aligned}$$

More generally, the action of the Steenrod operations can be encoded in eight ways: as a left or right action or coaction on homology or cohomology [Ada69], [Boa82].

The equivalence $B\mathbb{F}_p \simeq \Omega^\infty \Sigma \mathbb{F}_p$ has left adjoint $\Sigma^\infty B\mathbb{F}_p \rightarrow \Sigma \mathbb{F}_p$, inducing an additive homomorphism

$$(2.1) \quad \tilde{H}_{*+1}(B\mathbb{F}_p; \mathbb{F}_p) \longrightarrow \mathcal{A}_*$$

taking decomposable classes to zero. For $p = 2$, we have $B\mathbb{F}_2 \simeq \mathbb{R}P^\infty$ and $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[x]$ with $|x| = 1$. This is dual to $H_*(\mathbb{R}P^\infty; \mathbb{F}_2) = \Gamma_{\mathbb{F}_2}(\alpha_1) = \mathbb{F}_2\{\alpha_i \mid i \geq 0\}$. Here α_i is decomposable unless $i = 2^k$ with $k \geq 0$. The Milnor class $\zeta_k \in \mathcal{A}_*$, in degree $|\zeta_k| = 2^k - 1$, is the image of α_{2^k} under the homomorphism (2.1). In particular, $\zeta_0 = 1$ is the algebra unit.

Theorem 2.1 ([Mil58a]). *For $p = 2$, the canonical homomorphism*

$$\mathbb{F}_2[\zeta_k \mid k \geq 1] \xrightarrow{\cong} \mathcal{A}_*$$

is an algebra isomorphism. The coproduct ψ is the algebra homomorphism determined by

$$\psi(\zeta_k) = \sum_{j=0}^k \zeta_{k-j}^{2^j} \otimes \zeta_j,$$

where $\zeta_0 = 1$.

Proof. Jack Milnor showed that the matrix expressing how the Cartan–Serre admissible basis elements for $\mathcal{A}$ evaluate on the monomial basis for $\mathbb{F}_2[\zeta_k \mid k \geq 1]$ is upper-triangular with units on the diagonal, hence invertible. This required finding a clever way to order these two bases, and proved that the monomials form a basis for $\mathcal{A}_*$. $\square$

For p odd, we have $B\mathbb{F}_p = L^\infty$ (an infinite lens space), and $H^*(L^\infty; \mathbb{F}_p) = \Lambda_{\mathbb{F}_p}(x) \otimes \mathbb{F}_p[y]$ with $|x| = 1$ and $|y| = 2$. This is dual to $H_*(L^\infty; \mathbb{F}_p) = \Lambda_{\mathbb{F}_p}(\alpha_1) \otimes \Gamma_{\mathbb{F}_p}(\alpha_2) = \mathbb{F}_p\{\alpha_i \mid i \geq 0\}$, where α_i is decomposable unless $i = 1$ or $i = 2p^k$ with $k \geq 0$. The Milnor class $\tau_k \in \mathcal{A}_*$ in degree $|\tau_k| = 2p^k - 1$ is the image of α_{2p^k} under the homomorphism (2.1).

Similarly, the equivalence $B^2\mathbb{F}_p \simeq \Omega^\infty \Sigma^2 \mathbb{F}_p$ has right adjoint $\Sigma^\infty B^2\mathbb{F}_p \rightarrow \Sigma^2 \mathbb{F}_p$. When precomposed with $B^2\mathbb{Z} \rightarrow B^2\mathbb{F}_p$, we obtain additive homomorphisms

$$(2.2) \quad \tilde{H}_{*+2}(B^2\mathbb{Z}; \mathbb{F}_p) \longrightarrow \tilde{H}_{*+2}(B^2\mathbb{F}_p; \mathbb{F}_p) \longrightarrow \mathcal{A}_*.$$

Here $B^2\mathbb{Z} \simeq \mathbb{C}P^\infty$, and $H^*(\mathbb{C}P^\infty; \mathbb{F}_p) = \mathbb{F}_p[y]$ with $|y| = 2$. This is dual to $H_*(\mathbb{C}P^\infty; \mathbb{F}_p) = \Gamma_{\mathbb{F}_p}(\beta_1) = \mathbb{F}_p\{\beta_i \mid i \geq 0\}$. The Milnor class $\xi_k \in \mathcal{A}_*$ in degree $|\xi_k| = 2p^k - 2$ is the image of β_{p^k} under the composite homomorphism (2.2), see [Swi73]. In particular, $\xi_0 = 1$. (It is not the case that ξ_k is in the image from $H_{*+1}(B\mathbb{F}_p; \mathbb{F}_p)$. This has been incorrectly asserted, in at least one of the papers listed at the end.)

Theorem 2.2 ([Mil58a]). *For p odd, the canonical homomorphism*

$$\Lambda_{\mathbb{F}_p}(\tau_k \mid k \geq 0) \otimes \mathbb{F}_p[\xi_k \mid k \geq 1] \xrightarrow{\cong} \mathcal{A}_*$$

is an algebra isomorphism. The coproduct ψ is the algebra homomorphism determined by

$$\begin{aligned} \psi(\tau_k) &= \tau_k \otimes 1 + \sum_{j=0}^k \xi_{k-j}^{p^j} \otimes \tau_j \\ \psi(\xi_k) &= \sum_{j=0}^k \xi_{k-j}^{p^j} \otimes \xi_j, \end{aligned}$$

where $\xi_0 = 1$.

2.2. Power operations. The $\mathbb{E}_\infty$ -ring structure on $\mathbb{F}_p$ induces Araki–Kudo operations

$$Q_{s-q} = Q^s: \mathcal{A}_q = H_q(\mathbb{F}_2; \mathbb{F}_2) \longrightarrow H_{q+s}(\mathbb{F}_2; \mathbb{F}_2) = \mathcal{A}_{q+s}$$

for $s \geq q$, and Dyer–Lashof operations

$$\beta^\epsilon Q^s: \mathcal{A}_q = H_q(\mathbb{F}_p; \mathbb{F}_p) \longrightarrow H_{q+(2p-2)s-\epsilon}(\mathbb{F}_p; \mathbb{F}_p) = \mathcal{A}_{q+(2p-2)s-\epsilon}$$

for $2s \geq q$.

Formulas for similar operations in the homology of the $\mathbb{E}_\infty$ ring spaces BO and BU were obtained by 1970 by Stan Kochman [Koc73]. According to Ib Madsen

(if I remember correctly), Leif Kristensen used this in the early 1970s to obtain formulas for the Dyer–Lashof operations in $\mathcal{A}_*$, possibly only up to decomposable elements in the algebra structure. The formula for Q_1 was used by Madsen and Jim Milgram [MM74] to prove the theorem of Mark Mahowald [Mah77] that we are heading for, expressing $\mathbb{F}_2$ as a Thom spectrum over $\Omega^2 S^3$. A short paper by Stuart Priddy [Pri78] corrects (typographical) errors in the Madsen–Milgram proof. These formulas were also used by Larry Breen [Bre78] in an algebro-geometric calculation that we now view as the additive part of Marcel Bökstedt’s determination (ca. 1985) of $\pi_* \mathrm{THH}(\mathbb{F}_p)$. The precise general formulas appeared in print in Mark Steinberger’s thesis [BMMS86, Ch. III]. Here we shall see that only a few special cases of Steinberger’s formulas are needed for the theorem of Mahowald (and of Hopkins–Mahowald at odd primes), hence also for Breen and Bökstedt’s calculations, and these special cases can be proved with far less effort than the 18 pages of inductive proofs in the cited thesis.

These formulas are best expressed in terms of the conjugate algebra generators $\bar{\zeta}_k := \chi(\zeta_k)$ (for $p = 2$), $\bar{\tau}_k := \chi(\tau_k)$ and $\bar{\xi}_k := \chi(\xi_k)$ (for p odd). In each case, $\chi(x) \equiv -x$ modulo decomposable elements. (Many authors write ξ_k for Milnor’s ζ_k at $p = 2$, and some use ζ_k to denote the conjugate class $\bar{\zeta}_k$. This has been known to create confusion.)

The coproducts ψ coincide with the left $\mathcal{A}_*$ -comodule coactions ν on $\mathcal{A}_* = H_*(\mathbb{F}_p; \mathbb{F}_p)$, and translate to the following formulas for the right $\mathcal{A}$ -module actions ρ .

Lemma 2.3. *For $p = 2$,*

$$\mathrm{Sq}_*^r(\bar{\zeta}_k) = \begin{cases} \bar{\zeta}_{k-i}^{2^i} & \text{for } r = 2^i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

For p odd, $\beta(\bar{\tau}_k) = \bar{\xi}_k$, $\beta(\bar{\xi}_k) = 0$, $P_*^r(\bar{\tau}_k) = 0$ for $r \geq 1$, and

$$P_*^r(\bar{\xi}_k) = \begin{cases} (-1)^r \bar{\xi}_{k-i}^{p^i} & \text{for } r = (p^i - 1)/(p - 1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See [BMMS86, Lem. III.6.1] (with a sign error) or [LNR11, Lem. 6.5]. $\square$

Goro Nishida [Nis68] calculated the action of the Steenrod algebra the mod p cohomology of $\mathrm{Sym}_{\mathbb{E}_\infty}^p(X) = D_p(X) = (X^{\otimes p})_{h\Sigma_p}$, in terms of the action on $H^*(X; \mathbb{F}_p)$. For $\mathbb{E}_\infty$ -rings X , with structure maps $\mathrm{Sym}_{\mathbb{E}_\infty}^p(X) \subset \mathrm{Free}_{\mathbb{E}_\infty}(X) \rightarrow X$, this implies the Nishida relations

$$\mathrm{Sq}_*^r Q^s(x) = \sum_i \binom{s-r}{r-2i} Q^{s-r+i} \mathrm{Sq}_*^i(x)$$

in $H_*(X; \mathbb{F}_2)$, and

$$\begin{aligned} P_*^r Q^s(x) &= \sum_i (-1)^{r+i} \binom{(s-r)(p-1)}{r-pi} Q^{s-r+i} P_*^i(x) \\ P_*^r \beta Q^s(x) &= \sum_i (-1)^{r+i} \binom{(s-r)(p-1)-1}{r-pi} \beta Q^{s-r+i} P_*^i(x) \\ &\quad + \sum_i (-1)^{r+i} \binom{(s-r)(p-1)-1}{r-pi-1} Q^{s-r+i} P_*^i \beta(x) \end{aligned}$$

in $H_*(X; \mathbb{F}_p)$. See also [May70, §9], [CLM76, §I.1, §III.1] and [BMMS86, §III.1] (with a likely sign error). In the sums, only the finitely many terms with $i \geq 0$ and $2i \leq r$, $pi \leq r$, $pi \leq r$ or $pi + 1 \leq r$, respectively, can be nonzero.

We shall only need this for the lowest degree power operation after $Q_0(x) = x^p$, namely Q_1 for $p = 2$ and $Q_{p-1}(x)$ for p and $q = |x|$ odd. When $p = 2$,

$$\mathrm{Sq}_*^1 Q_1(x) = \begin{cases} x^2 & \text{for } q \text{ odd,} \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathrm{Sq}_*^r Q_1(x) = Q_1 \mathrm{Sq}_*^{r/2}(x)$ for r even. When p and q are odd,

$$P_*^r Q_{p-1}(x) = \begin{cases} Q_{p-1} P_*^{r/p}(x) & \text{for } r \equiv 0 \pmod{p}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$P_*^r \beta Q_{p-1}(x) = \begin{cases} \beta Q_{p-1} P_*^{r/p}(x) & \text{for } r \equiv 0 \pmod{p}, \\ -(P_*^{(r-1)/p} \beta(x))^p & \text{for } r \equiv 1 \pmod{p}, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.4. $Q_1(\bar{\zeta}_k) = \bar{\zeta}_{k+1}$ for $k \geq 1$, and $Q_{p-1}(\bar{\tau}_k) = \bar{\tau}_{k+1}$ for $k \geq 0$.

Another formula that is possible to remember is $Q_{2p-2}(\bar{\xi}_k) = \bar{\xi}_{k+1}$ for $k \geq 1$, but we will not need that. These formulas can be rewritten as $Q^{2^k}(\bar{\zeta}_k) = \bar{\zeta}_{k+1}$, $Q^{p^k}(\bar{\tau}_k) = \bar{\tau}_{k+1}$ and $Q^{p^k}(\bar{\xi}_k) = \bar{\xi}_{k+1}$.

Proof. The right $\mathcal{A}$ -action on $\mathcal{A}_*$ is faithful, and it follows from the Adem relations that $\mathcal{A}$ is generated by the Sq^{2^i} with $i \geq 0$ for $p = 2$, and by β and the P^{p^i} with $i \geq 0$ for p odd. For $p = 2$ it therefore suffices to note that

$$Sq_*^{2^i} Q_1(\bar{\zeta}_k) = \begin{cases} \bar{\zeta}_k^2 & \text{for } i = 0, \\ 0 & \text{for } i \geq 1 \end{cases}$$

is equal to $Sq_*^{2^i}(\bar{\zeta}_{k+1})$ for $i \geq 0$. For p odd it suffices to note that

$$P_*^{p^i} Q_{p-1}(\bar{\tau}_k) = \begin{cases} 0 & \text{for } i = 0, \\ Q_{p-1} P_*^{p^{i-1}}(\bar{\tau}_k) = 0 & \text{for } i \geq 1 \end{cases}$$

agrees with $P_*^{p^i}(\bar{\tau}_{k+1})$, and to verify that $\beta Q_{p-1}(\bar{\tau}_k) = \beta(\bar{\tau}_{k+1})$. Since $\beta^2 = 0$ and $\beta(\bar{\tau}_{k+1}) = \bar{\xi}_{k+1}$, it suffices in turn to note that

$$P_*^{p^i} \beta Q_{p-1}(\bar{\tau}_k) = \begin{cases} -(\beta(\bar{\tau}_k))^p = -\bar{\xi}_k^p & \text{for } i = 0, \\ \beta Q_{p-1} P_*^{p^{i-1}}(\bar{\tau}_k) = 0 & \text{for } i \geq 1 \end{cases}$$

is equal to $P_*^{p^i}(\bar{\xi}_{k+1})$ for $i \geq 0$. □

For $p = 2$, the homology isomorphism of the following proof, but not the statement about free algebras, appears in [MM74, p. 50].

Theorem 2.5 ([KN22, Thm. 1.2]). *There is an equivalence $\mathbb{F}_p[\Omega^2 S^3] \simeq \mathbb{F}_p \otimes \mathbb{F}_p$ of $\mathbb{E}_2$ $\mathbb{F}_p$ -algebras, both being free on one generator in degree 1.*

Proof. We view $\mathbb{F}_p \otimes \mathbb{F}_p$ as an $\mathbb{E}_\infty \mathbb{F}_p$ -algebra via the left unit $\eta_L: \mathbb{F}_p \cong \mathbb{F}_p \otimes \mathbb{S} \rightarrow \mathbb{F}_p \otimes \mathbb{F}_p$. Let $\Sigma \mathbb{S} \rightarrow \mathbb{F}_p \otimes \mathbb{F}_p$ in $\mathcal{S}p$ represent $\bar{\zeta}_1$ (for $p = 2$) or $\bar{\tau}_0$ (for p odd) in $\pi_1(\mathbb{F}_p \otimes \mathbb{F}_p) = \mathcal{A} \cong \mathbb{F}_p$. It extends uniquely to an $\mathbb{F}_p$ -module map $\Sigma \mathbb{F}_p \rightarrow \mathbb{F}_p \otimes \mathbb{F}_p$, and to an $\mathbb{E}_2 \mathbb{F}_p$ -algebra map $f: \mathbb{F}_p[\Omega^2 S^3] \rightarrow \mathbb{F}_p \otimes \mathbb{F}_p$ by Proposition 1.2. We claim that the induced algebra homomorphism in homotopy

$$f_*: H_*(\Omega^2 S^3; \mathbb{F}_p) \longrightarrow \mathcal{A}_*$$

is an isomorphism, so that f is an equivalence.

For $p = 2$, f_* takes x to $\bar{\zeta}_1 = \zeta_1$, hence takes $Q_1^{(k-1)}(x)$ to $Q_1^{(k-1)}(\bar{\zeta}_1) = \bar{\zeta}_k$ for all $k \geq 1$ by Proposition 2.4, and therefore induces the isomorphism

$$\mathbb{F}_2[x, Q_1(x), Q_1 Q_1(x), \dots] \xrightarrow{\cong} \mathbb{F}_2[\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3, \dots]$$

by Theorems 1.1 and 2.1.

For p odd, f_* takes x to $\bar{\tau}_0 = -\tau_0$, hence takes $Q_{p-1}^{(k)}(x)$ to $Q_{p-1}^{(k)}(\bar{\tau}_0) = \bar{\tau}_k$ for all $k \geq 0$ by Proposition 2.4, takes $\beta Q_{p-1}^{(k)}(x)$ to $\beta(\bar{\tau}_k) = \bar{\xi}_k$ for all $k \geq 1$ by Lemma 2.3, and therefore induces the isomorphism

$$\begin{aligned} \Lambda_{\mathbb{F}_p}(x, Q_{p-1}(x), Q_{p-1} Q_{p-1}(x), \dots) \otimes \mathbb{F}_p[\beta Q_{p-1}(x), \beta Q_{p-1} Q_{p-1}(x), \dots] \\ \xrightarrow{\cong} \Lambda_{\mathbb{F}_p}(\bar{\tau}_0, \bar{\tau}_1, \bar{\tau}_2, \dots) \otimes \mathbb{F}_p[\bar{\xi}_1, \bar{\xi}_2, \dots] \end{aligned}$$

by Theorems 1.1 and 2.2. □

3. THOM SPECTRA

The group $O(m) \simeq \mathrm{GL}_m(\mathbb{R})$ acts on $\mathbb{R}^m$ through linear isometries. For a vector bundle $E = P \times_{O(m)} \mathbb{R}^m \rightarrow X$, with classifying map $\gamma: X \rightarrow BO(m) \simeq \mathrm{Gr}_m(\mathbb{R}^\infty)$, the associated Thom complex is the based space

$$\mathrm{Th}(\gamma) = \frac{P \times_{O(m)} S^m}{P \times_{O(m)} \{\infty\}} \cong D(\gamma)/S(\gamma),$$

where $S^m = \mathbb{R}^m \cup \{\infty\}$ is the one-point compactification, $D(\gamma) = P \times_{O(m)} D^m$ is the unit disk bundle, and $S(\gamma) = P \times_{O(m)} S^{m-1}$ is the unit sphere bundle.

When $\gamma = \epsilon^m$ is the trivial bundle, $\mathrm{Th}(\gamma) = \Sigma^m(X_+)$ is the unreduced suspension. In general, for R -oriented vector bundles there is a Thom isomorphism $\tilde{H}_{*+m}(\mathrm{Th}(\gamma); R) \cong H_*(X; R)$. Here R denotes a coefficient ring.

The Whitney sum $E' \oplus E'' \rightarrow X$ of vector bundles corresponds to an $\mathbb{E}_\infty$ -space structure on $\coprod_{m \geq 0} BO(m)$, with group completion the infinite loop space $\mathbb{Z} \times BO \simeq \Omega^\infty \mathrm{ko}$ sitting in an extension

$$BO \longrightarrow \mathbb{Z} \times BO \longrightarrow \mathbb{Z}.$$

A map $\gamma: X \rightarrow \mathbb{Z} \times BO$ classifies a virtual vector bundle, with associated Thom spectrum $\mathrm{Th}(\gamma) \in \mathcal{S}p$. The bundle has virtual dimension zero if γ factors through BO . For R -oriented such bundles, there is an equivalence $R \otimes \mathrm{Th}(\gamma) \simeq R \otimes X_+$. In the universal case $X = BO$, one obtains the Thom spectrum MO , which is an $\mathbb{E}_\infty$ -ring with

$$\pi_*(\mathrm{MO}) \cong \mathbb{F}_2[x_n \mid 1 \leq n \neq 2^k - 1] = \mathbb{F}_2[x_2, x_4, x_5, \dots]$$

isomorphic to the bordism ring $\mathcal{N}_*$ of smooth, unoriented manifolds. See [Tho54], [MS74], [Mah79], [BMMS86], [Hus94], [Rud98].

Let $\mathcal{S}p^\simeq$ denote the maximal ∞ -groupoid (= space) contained in $\mathcal{S}p$. A spectrum M is smash invertible if there exists a spectrum N with $M \otimes N \simeq \mathbb{S}$, and up to equivalence these are all of the form $\Sigma^m \mathbb{S}$ for $m \in \mathbb{Z}$. The Picard space $\text{Pic}_{\mathbb{S}}$ is the subspace of $\mathcal{S}p^\simeq$ with objects the smash invertible spectra. We write ι for the composite map/ ∞ -functor

$$\iota: \text{Pic}_{\mathbb{S}} \subset \mathcal{S}p^\simeq \subset \mathcal{S}p.$$

The smash product makes $\text{Pic}_{\mathbb{S}} \simeq \Omega^\infty \text{pic}_{\mathbb{S}}$ a grouplike $\mathbb{E}_\infty$ -space, sitting in an extension

$$BGL_1(\mathbb{S}) \longrightarrow \text{Pic}_{\mathbb{S}} \longrightarrow \mathbb{Z}.$$

Here $GL_1(\mathbb{S}) \simeq \Omega BGL_1(\mathbb{S}) \simeq \Omega \text{Pic}_{\mathbb{S}}$ is the grouplike $\mathbb{E}_\infty$ -space of self-equivalences $\mathbb{S} \xrightarrow{\sim} \mathbb{S}$, denoted $F \simeq \Omega BF$ in [May77], and equivalent to the grouplike $\mathbb{E}_\infty$ -space often denoted $G \simeq \Omega BG$ in the context of surgery theory, for spaces such as G/O , G/PL and G/TOP . A map $\gamma: X \rightarrow \text{Pic}_{\mathbb{S}}$ classifies a stable spherical fibration, which has virtual dimension zero if the map factors through $BGL_1(\mathbb{S})$. Again, for R -oriented such fibrations there is an equivalence $R \otimes \text{Th}(\gamma) \simeq R \otimes X_+$, and this also makes sense for $\mathbb{E}_1$ -rings R .

One-point compactification stabilizes to define an $\mathbb{E}_\infty$ -map $j: \mathbb{Z} \times BO \rightarrow \text{Pic}_{\mathbb{S}}$, restricting to $j_0: BO \rightarrow BGL_1(\mathbb{S})$. The induced homomorphism

$$\pi_*(BO) \longrightarrow \pi_*(BGL_1(\mathbb{S})) \cong \pi_{*-1}(GL_1(\mathbb{S})) \subset \pi_{*-1}(\mathbb{S})$$

was studied by George Whitehead [Whi42], and referred to as the ‘ J -homomorphism’ by Milnor [Mil58b]. Its source is known by Bott periodicity [Bot57], and its precise image is the subject of a series of papers by Frank Adams [Ada63], [Ada65a], [Ada65b], [Ada66], as well as the Adams conjecture, proven by Daniel Quillen [Qui71] and Dennis Sullivan [Sul74].

$$\begin{array}{ccc} BO & \longrightarrow & \mathbb{Z} \times BO \\ j_0 \downarrow & & \downarrow j \\ BGL_1(\mathbb{S}) & \longrightarrow & \text{Pic}_{\mathbb{S}} \xrightarrow{\iota} \mathcal{S}p \end{array}$$

The construction of $\text{Th}(\gamma)$, and the associated orientation theory from [May77], [BMMS86], was reworked in ∞ -categorical terms by Matt Ando, Andrew Blumberg, David Gepner, Mike Hopkins and Charles Rezk. In [ABG⁺14] they simply define

$$\text{Th}(\gamma) := \text{colim}_X \iota \gamma$$

to be the colimit in $\mathcal{S}p$ of the composite map $X \xrightarrow{\gamma} \text{Pic}_{\mathbb{S}} \xrightarrow{\iota} \mathcal{S}p$. Informally, this is obtained from gluing together a spectrum $\gamma(p)$ equivalent to some $\Sigma^m \mathbb{S}$ for each point p in X . The functor

$$\begin{aligned} \text{Th}: \mathcal{S}_{/\text{Pic}_{\mathbb{S}}} &\longrightarrow \mathcal{S}p \\ (\gamma: X \rightarrow \text{Pic}_{\mathbb{S}}) &\longmapsto \text{Th}(\gamma) \end{aligned}$$

is symmetric monoidal. In particular, if $\gamma: X \rightarrow \text{Pic}_{\mathbb{S}}$ is an $\mathbb{E}_k$ -map, then $\text{Th}(\gamma)$ is an $\mathbb{E}_k$ -ring. The functor Th is also a left adjoint. The right adjoint takes a spectrum M to the pullback $\gamma: \Omega^{\mathbb{S}}(M) \rightarrow \text{Pic}_{\mathbb{S}}$ of $\mathcal{S}p/M \rightarrow \mathcal{S}p$ along ι ; see [RSS, §2].

More generally, for an $\mathbb{E}_\infty$ -ring R , the category $(\text{Mod}_R, \otimes_R, R)$ of R -modules is symmetric monoidal, we define $\text{Mod}_{\widetilde{R}} \subset \text{Mod}_R$ to be the maximal ∞ -groupoid, and

we define $\text{Pic}_R \subset \text{Mod}_R^\sim$ to be the subspace of $\otimes_R$ -invertible R -modules. There is then a fiber sequence

$$BGL_1(R) \longrightarrow \text{Pic}_R \longrightarrow \pi_0(\text{Pic}_R)$$

of grouplike $\mathbb{E}_\infty$ -spaces, where $GL_1(R) \subset \Omega^\infty R$ consists of the path components corresponding to the units, i.e., the invertible elements, in $\pi_0(R)$. The Thom R -module $\text{Th}_R(\gamma)$ of a map $\gamma: X \rightarrow \text{Pic}_R$ is then the colimit of the X -shaped diagram in Mod_R defined by the composite

$$X \xrightarrow{\gamma} \text{Pic}_R \subset \text{Mod}_R.$$

(There is a similar theory for $\mathbb{E}_k$ -rings R , at least for $k \geq 2$.) We shall apply this with $R = \mathbb{S}_p$, the p -complete sphere spectrum, with $\pi_0(\mathbb{S}_p) = \mathbb{Z}_p = \lim_n \mathbb{Z}/p^n$ the p -adic integers, and simply write $\text{Th}(\gamma)$ in place of $\text{Th}_{\mathbb{S}_p}(\gamma)$.

4. THEOREMS OF MAHOWALD AND HOPKINS–MAHOWALD

First suppose $p = 2$. The map $j_0: BO \rightarrow BGL_1(\mathbb{S})$ induces an isomorphism

$$\pi_1(BO) \xrightarrow{j_0} \pi_1(BGL_1(\mathbb{S})) \cong \pi_0(GL_1(\mathbb{S})) \cong \mathbb{Z}^\times = \{\pm 1\}.$$

Let

$$\gamma_0: S^1 \longrightarrow BO \longrightarrow BGL_1(\mathbb{S})$$

be the $\mathbb{E}_0$ -map, i.e., map of based spaces, representing the unit -1 in $\pi_1 BGL_1(\mathbb{S}) \cong \{\pm 1\}$. (This map is also known as η , since composition with the unit map $Q(S^0) = \Omega^\infty \mathbb{S} \rightarrow \Omega^\infty \text{ko} = \mathbb{Z} \times BO$ takes the map $S^1 \rightarrow Q(S^0)$ representing the generator $\eta \in \pi_1(\mathbb{S})$ to the map γ_0 . It is also the stable class of the Möbius line bundle over S^1 .)

Now suppose p is odd. There is an isomorphism

$$\pi_1(BGL_1(\mathbb{S}_p)) \cong \pi_0(GL_1(\mathbb{S}_p)) \cong \mathbb{Z}_p^\times.$$

Let $\gamma_0: S^1 \rightarrow BGL_1(\mathbb{S}_p)$ be the $\mathbb{E}_0$ -map representing the p -adic unit $1 - p$ in $\pi_1 BGL_1(\mathbb{S}_p) \cong \mathbb{Z}_p^\times$.

Lemma 4.1. *For $p = 2$, the Thom $\mathbb{E}_0$ -spectrum of γ_0 is the cofiber $\mathbb{S} \rightarrow \mathbb{S}/2 = \text{Th}(\gamma_0)$ of $2: \mathbb{S} \rightarrow \mathbb{S}$.*

For p odd, the Thom $\mathbb{E}_0$ -spectrum of γ_0 is the cofiber $\mathbb{S}_p \rightarrow \mathbb{S}/p = \text{Th}(\gamma_0)$ of $p: \mathbb{S}_p \rightarrow \mathbb{S}_p$.

Proof. First suppose $p = 2$. The colimit over $S^1 \cong I/\partial I$ of ι_{γ_0} is obtained from the colimit $I_+ \wedge \mathbb{S}$ over I by identifying the end copies of $\mathbb{S}$ via $-1: \mathbb{S} \rightarrow \mathbb{S}$. In other words, this is the (homotopy) coequalizer of 1 and $-1: \mathbb{S} \rightarrow \mathbb{S}$. Since $\mathcal{S}p$ is stable, this equals the (homotopy) cofiber of the difference $1 - (-1) = 2: \mathbb{S} \rightarrow \mathbb{S}$, which equals the Moore spectrum $\mathbb{S}/2$. The $\mathbb{E}_0$ structure map $\mathbb{S} \rightarrow \text{Th}(\gamma_0)$ is the structure map from the target $\mathbb{S}$ to the coequalizer.

The same argument applies for p odd, after replacing $\mathbb{S}$ with $\mathbb{S}_p$, -1 with $1 - p$, 2 with p , and $\mathcal{S}p$ with $\text{Mod}_{\mathbb{S}_p}$. $\square$

Corollary 4.2. *The base change to $\mathbb{F}_p$ of $\mathbb{S} \rightarrow \text{Th}(\gamma_0)$ is*

$$\mathbb{F}_p \rightarrow \mathbb{F}_p \otimes \mathbb{S}/p \simeq \mathbb{F}_p \oplus \Sigma \mathbb{F}_p \simeq \mathbb{F}_p[S^1].$$

Proof. Multiplication by p acts trivially on $\mathbb{F}_p$. $\square$

Recall that $\mathrm{Free}_{\mathbb{E}_2/\mathbb{E}_0}(S^1) \simeq \Omega^2 S^3$. For $p = 2$ let

$$\gamma_2: \Omega^2 S^3 \longrightarrow BO \longrightarrow BGL_1(\mathbb{S})$$

be the left adjoint of $\gamma_0: S^1 \rightarrow BO \rightarrow BGL_1(\mathbb{S})$, with respect to the free-forgetful adjunction between $\mathbb{E}_0$ -spaces and $\mathbb{E}_2$ -spaces. The $\mathbb{E}_2$ -map $\gamma_2: \Omega^2 S^3 \rightarrow \mathrm{Pic}_{\mathbb{S}}$ an $\mathbb{E}_2$ -algebra in $\mathcal{S}/\mathrm{Pic}_{\mathbb{S}}$, equal to $\mathrm{Free}_{\mathbb{E}_2/\mathbb{E}_0}$ applied to $\gamma_0: S^1 \rightarrow \mathrm{Pic}_{\mathbb{S}}$ viewed as an $\mathbb{E}_0$ -algebra in the same category.

For p odd, let $\gamma_2: \Omega^2 S^3 \rightarrow BGL_1(\mathbb{S}_p)$ be the left adjoint of $\gamma_0: S^1 \rightarrow BGL_1(\mathbb{S}_p)$, with respect to the same adjunction. Again, $\gamma_2: \Omega^2 S^3 \rightarrow \mathrm{Pic}_{\mathbb{S}_p}$ is $\mathrm{Free}_{\mathbb{E}_2/\mathbb{E}_0}$ applied to $\gamma_0: S^1 \rightarrow \mathrm{Pic}_{\mathbb{S}_p}$ in $\mathcal{S}/\mathrm{Pic}_{\mathbb{S}_p}$.

Lemma 4.3. *$\mathrm{Th}(\gamma_2) = \mathrm{Free}_{\mathbb{E}_2/\mathbb{E}_0} \mathrm{Th}(\gamma_0)$ is a connective and p -complete $\mathbb{E}_2$ -ring.*

Proof. The identity of $\mathbb{E}_2$ -rings holds because Th is symmetric monoidal. Any colimit of connective spectra is connective, and γ_2 has virtual dimension zero, so this Thom spectrum is connective. The unit map $\mathbb{S} \rightarrow \mathrm{Th}(\gamma_2)$ factors through $\mathrm{Th}(\gamma_0) = \mathbb{S}/p$, so $p = 0$ in the ring $\pi_* \mathrm{Th}(\gamma_2)$, implying that $\mathrm{Th}(\gamma_2)$ is p -complete. $\square$

We know that $\mathrm{Th}(\gamma_0) = \mathbb{S}/p$ is connective, with $\pi_0(\mathbb{S}/p) \cong \mathbb{Z}/p$. Hence there exists a 1-connected Hurewicz map $h_0: \mathrm{Th}(\gamma_0) \rightarrow \mathbb{F}_p$ under $\mathbb{S}$, i.e., an $\mathbb{E}_0$ -map that induces an isomorphism on π_0 (and, trivially, a surjection on π_1).

The following result is from the early 1970s, with one proof published in [MM74, Thm. 1.3], and (one of) Mahowald's proof(s) appearing in [Mah77, Cor. 4.5].

Theorem 4.4 (Mahowald). *For $p = 2$, the $\mathbb{E}_2$ -ring map*

$$h_2: \mathrm{Th}(\gamma_2: \Omega^2 S^3 \rightarrow BO) \xrightarrow{\simeq} \mathbb{F}_2,$$

left adjoint to $h_0: \mathrm{Th}(\gamma_0) \simeq \mathbb{S}/2 \rightarrow \mathbb{F}_2$, is an equivalence.

The following complement came roughly 20 years later.

Theorem 4.5 (Hopkins–Mahowald). *For p odd, the $\mathbb{E}_2$ -ring map*

$$h_2: \mathrm{Th}(\gamma_2: \Omega^2 S^3 \rightarrow BGL_1(\mathbb{S}_p)) \xrightarrow{\simeq} \mathbb{F}_p,$$

left adjoint to $h_0: \mathrm{Th}(\gamma_0) \simeq \mathbb{S}/p \rightarrow \mathbb{F}_p$, is an equivalence.

Proof. By the Hurewicz theorem and Lemma 4.3, it suffices to prove that the base change

$$\mathbb{F}_p \otimes h_2: \mathbb{F}_p \otimes \mathrm{Th}(\gamma_2) \longrightarrow \mathbb{F}_p \otimes \mathbb{F}_p$$

is an equivalence. This is a map in $\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Mod}_{\mathbb{F}_p})$, and by Theorem 2.5 the target is the free $\mathbb{E}_2$ $\mathbb{F}_p$ -algebra on one generator in degree 1. Since the relatively free algebra functor commutes with base change, the source is

$$\begin{aligned} \mathbb{F}_p \otimes \mathrm{Free}_{\mathbb{E}_2/\mathbb{E}_0} \mathrm{Th}(\gamma_0) &\simeq \mathrm{Free}_{\mathbb{E}_2/\mathbb{E}_0}^{\mathbb{F}_p}(\mathbb{F}_p \otimes \mathrm{Th}(\gamma_0)) \\ &\simeq \mathrm{Free}_{\mathbb{E}_2/\mathbb{E}_0}^{\mathbb{F}_p}(\mathbb{F}_p \oplus \Sigma \mathbb{F}_p) \simeq \mathrm{Free}_{\mathbb{E}_2}^{\mathbb{F}_p}(\Sigma \mathbb{F}_p), \end{aligned}$$

where we used Corollary 4.2. Hence we have a 1-connected map

$$\mathrm{Free}_{\mathbb{E}_2}^{\mathbb{F}_p}(\Sigma \mathbb{F}_p) \simeq \mathbb{F}_p \otimes \mathrm{Th}(\gamma_2) \xrightarrow{\mathbb{F}_p \otimes h_2} \mathbb{F}_p \otimes \mathbb{F}_p \simeq \mathrm{Free}_{\mathbb{E}_2}^{\mathbb{F}_p}(\Sigma \mathbb{F}_p)$$

of $\mathbb{E}_2$ $\mathbb{F}_p$ -algebras.

These algebra maps are determined by their adjoint $\mathbb{F}_p$ -module maps $\Sigma \mathbb{F}_p \rightarrow \mathrm{Free}_{\mathbb{E}_2}^{\mathbb{F}_p}(\Sigma \mathbb{F}_p)$, i.e., by their effect on π_1 . Since $\pi_1(\mathrm{Free}_{\mathbb{E}_2}^{\mathbb{F}_p}(\Sigma \mathbb{F}_p)) \cong \pi_1(\mathbb{F}_p \otimes \mathbb{F}_p) = \mathbb{F}_p$,

generated by ζ_1 for $p = 2$ and τ_0 for p odd, each such algebra map is of the form $\text{Free}_{\mathbb{E}_2}^{\mathbb{F}_p}$ applied to a multiple e of the identity $\Sigma\mathbb{F}_p \rightarrow \Sigma\mathbb{F}_p$. Since our map is 1-connected, the multiple must be nonzero modulo p , hence e is an equivalence. It follows that $\text{Free}_{\mathbb{E}_2}^{\mathbb{F}_p}(e)$ and $\mathbb{F}_p \otimes h_2$ are equivalences. $\square$

Remark 4.6. Unpublished notes by Mahowald and Alan Unell, from lectures in 1969 and 1977, refer to unpublished work by Michael Barratt, ca. 1967, related to the Thom spectra X_i of $\mathbb{E}_2$ -maps $\Omega^2 S^{i+1} \rightarrow BO$ for $i \in \{2, 3, 5, 9\}$.

Remark 4.7. Let $S^3\langle 3 \rangle = \text{fib}(S^3 \rightarrow K(\mathbb{Z}, 3))$ denote the 3-connected cover of S^3 . Mahowald [Mah79, Prop. 2.8] also proved that the integral Eilenberg–MacLane spectrum $\mathbb{Z}$ is the Thom spectrum of a map $\Omega^2 S^3\langle 3 \rangle \rightarrow BGL_1(\mathbb{S})$. Cohen, May and Taylor [CMT81] gave another proof. These arguments also accounted for the odd-primary completions of $\mathbb{Z}$, but were more complicated than Hopkins’ use of $BGL_1(\mathbb{S}_p)$ to realize $\mathbb{F}_p$ as a Thom $\mathbb{S}_p$ -module.

5. SPLITTING THEOREMS

The $\mathbb{E}_2$ -map $\gamma_2: \Omega^2 S^3 \rightarrow BO$ can be viewed as a morphism in $\mathcal{S}_{/BO}$ or $\mathcal{S}_{/Pic_S}$, and induces an $\mathbb{E}_2$ -map

$$\text{Th}(\gamma_2) \longrightarrow \text{MO}$$

of Thom spectra. The 0-th Postnikov truncation $p_0: \text{MO} \rightarrow \tau_{\leq 0}\text{MO} \simeq \mathbb{F}_2$ is an $\mathbb{E}_\infty$ -map. The composite

$$\text{Th}(\gamma_2) \longrightarrow \text{MO} \longrightarrow \mathbb{F}_2$$

is an equivalence, and shows that $\mathbb{F}_2$ splits off from MO in the category of $\mathbb{E}_2$ -rings.

In his work introducing Thom complexes and their relation to bordism [Tho54], René Thom proved that MO splits (additively, i.e., in $\mathcal{S}p$) as a sum of suspensions of copies of $\mathbb{F}_2$. His proof proceeded by showing that $H^*(\text{MO}; \mathbb{F}_2)$ is free as a module over the Steenrod algebra $\mathcal{A}$, and using this to obtain maps $\text{MO} \rightarrow \Sigma^{n_\alpha} \mathbb{F}_2$ inducing an equivalence

$$\text{MO} \xrightarrow{\simeq} \bigoplus_{\alpha} \Sigma^{n_\alpha} \mathbb{F}_2.$$

It follows that $\mathcal{N}_* \cong \pi_*(\text{MO})$ maps injectively to $H_*(\text{MO}; \mathbb{F}_2) \cong H_*(BO; \mathbb{F}_2)$, leading to the calculation $\mathcal{N}_* \cong \mathbb{F}_2[x_n \mid 1 \leq n \neq 2^k - 1]$, and the corollary that the bordism class $[M]$ of a smooth n -manifold M is completely determined by its Stiefel–Whitney numbers $w_1^{i_1} \cdots w_n^{i_n}(M)$, i.e., the values in $\mathbb{F}_2$ of the degree n monomials $w_1^{i_1} \cdots w_n^{i_n}$ in

$$H^n(BO; \mathbb{F}_2) \subset H^*(BO; \mathbb{F}_2) = \mathbb{F}_2[w_i \mid i \geq 1],$$

evaluated on the Hurewicz image $h([M]) \in H_n(BO; \mathbb{F}_2)$.

The following theorem of Steinberger gives an alternative approach to Thom’s splitting, as well as to analogous splittings, such as those due to Browder–Liulevicius–Peterson [BLP66] for MPL and $\text{MG} = \text{MGL}_1(\mathbb{S})$.

Theorem 5.1 (Steinberger [BMMS86, Thm. III.4.1]). *Let p be a prime and X an $\mathbb{E}_2$ -ring with $p = 0$ in $\pi_0(X)$. Then the underlying spectrum of X admits the structure of an $\mathbb{F}_p$ -module, hence splits as a sum of suspensions of $\mathbb{F}_p$.*

Proof. By assumption, the unit map $\mathbb{S} \rightarrow X$ can be extended over $\mathbb{S} \rightarrow \mathbb{S}/p$, giving an $\mathbb{E}_0$ -map $\mathbb{S}/p \rightarrow X$. By adjunction, it extends to an $\mathbb{E}_2$ -map $\text{Free}_{\mathbb{E}_2/\mathbb{E}_0}(\mathbb{S}/p) \rightarrow X$. Here

$$\text{Free}_{\mathbb{E}_2/\mathbb{E}_0}(\mathbb{S}/p) \simeq \text{Free}_{\mathbb{E}_2/\mathbb{E}_0} \text{Th}(\gamma_0) \simeq \text{Th}(\gamma_2) \simeq \mathbb{F}_p,$$

by Lemmas 4.1 and 4.3 and Theorems 4.4 or 4.5. Hence we have an $\mathbb{E}_2$ -map $\mathbb{F}_p \rightarrow X$, giving X an $\mathbb{F}_p$ -module structure via the product in X . Any such module is free, in the sense that it is equivalent to a sum of suspensions of $\mathbb{F}_p$. To see this, note that $\pi_*(X)$ is a free $\mathbb{F}_p$ -module in the algebraic sense. Choose a basis for this vector space, and represent each basis element by a spectrum map $b_\alpha: \Sigma^{n_\alpha} \mathbb{S} \rightarrow X$. It extends to an $\mathbb{F}_p$ -module map $\bar{b}_\alpha: \Sigma^{n_\alpha} \mathbb{F}_p \rightarrow X$, and the sum

$$\bigoplus_{\alpha} \Sigma^{n_\alpha} \mathbb{F}_p \xrightarrow{\simeq} X$$

is the required equivalence. $\square$

In other words, $\mathbb{F}_p$ is the initial $\mathbb{E}_2$ -ring with a choice of null-homotopy $p \cdot 1 \simeq 0$. This shows that Jack Morava's $\mathbb{E}_1$ -rings $K(n)$, with $\pi_* K(n) = \mathbb{F}_p[v_n^{\pm 1}]$, do not admit any $\mathbb{E}_2$ -ring structure, since $p = 0$ in $\pi_0 K(n)$, while it is known that these spectra do not split as sums of Eilenberg–MacLane spectra (for $0 < n < \infty$), e.g. from the computation of $K(n)^*(BC_p)$.

There are a series of other results analyzing Thom spectra $\text{Th}(\gamma)$ after localization or completion at a prime p , initially proved by first calculating $H^*(\text{Th}(\gamma); \mathbb{F}_p)$ as an $\mathcal{A}$ -module coalgebra, or $H_*(\text{Th}(\gamma); \mathbb{F}_p)$ as an $\mathcal{A}_*$ -comodule algebra, and proceeding from there:

- (1) Milnor and Dan Quillen's p -local splitting of the complex bordism spectrum MU , and of the oriented bordism spectrum MSO for odd p , as sums of Brown–Peterson spectra BP [Mil60], [BP66], [Qui69]. Here $\pi_*(\text{MU}) \cong \mathbb{Z}[x_n \mid n \geq 1]$ with $|x_n| = 2n$, while $\pi_*(\text{BP}) \cong \mathbb{Z}_{(p)}[v_n \mid n \geq 1]$ with $|v_n| = 2p^n - 2$.
- (2) Terry Wall's 2-local splitting of the oriented bordism spectrum MSO as a sum of suspensions of $\mathbb{Z}$ and $\mathbb{Z}/2$ [Wal60].
- (3) Donald Anderson, Ed Brown and Frank Peterson's 2-local splitting of the spin bordism spectrum MSpin as a sum of suspensions of Whitehead covers of ko and $\mathbb{Z}/2$ [ABP67].

Several of these can also be proved by Steinberger's method, see [BMMS86, Thm. 4.2, Thm. 4.3].

See the Master's thesis of Riley Hexeberg Mæhle (Oslo, 2024) for more about obstructions to splitting $\mathbb{Z}$ off from MSO and ko off from MSpin as $\mathbb{E}_k$ -rings, and the open problem about whether tmf splits off from $\text{MString} = \text{MO}\langle 8 \rangle$.

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