

TOPOLOGICAL CYCLIC HOMOLOGY, II

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We calculate $\mathrm{TC}(\mathbb{F}_p)$, exhibit a presentation of reduced TC emphasized by Drinfeld and Bhatt–Lurie, and review the Dundas–Goodwillie–McCarthy theorem(s).

1. CALCULATIONS FOR $B = \mathbb{F}_p$

Let us calculate $\pi_*\mathrm{TC}(\mathbb{F}_p)$, for which the published reference is [HM97]. Recall that $\pi_*\mathrm{THH}(\mathbb{F}_p) = \mathbb{F}_p[\mu]$ where $\mu = \mu_0$ has degree 2 and Hurewicz image $\sigma\bar{\tau}_0$ (or $\sigma\bar{\zeta}_1$, when $p = 2$).

Lemma 1.1. *The S^1 -homotopy fixed point spectral sequence*

$$\mathcal{E}^2(S^1) = H^{-*}(S^1, \pi_*\mathrm{THH}(\mathbb{F}_p)) = \mathbb{F}_p[t, \mu] \implies \pi_*\mathrm{TC}^-(\mathbb{F}_p)$$

collapses. The image of $p \in \pi_0(\mathbb{S})$ under the unit map $\mathbb{S} \rightarrow \mathrm{TC}^-(\mathbb{F}_p)$ is detected by $t\mu \in \mathcal{E}_{-2,2}^\infty$. Hence

$$\pi_*\mathrm{TC}^-(\mathbb{F}_p) \cong \mathbb{Z}_p[t, \mu]/(t\mu = p)$$

with $|t| = -2$ and $|\mu| = 2$.

Proof. Let X/p denote the cofiber of $p: X \rightarrow X$. In the mod p spectral sequence

$$H^{-*}(S^1, \pi_*\mathrm{THH}(\mathbb{F}_p)/p) \cong \mathbb{F}_p[t, \mu] \otimes \Lambda(\bar{\tau}_0) \implies \pi_*\mathrm{TC}^-(\mathbb{F}_p)/p$$

we have $d^2(\bar{\tau}_0) = t \cdot \sigma\bar{\tau}_0 = t\mu$, so $t\mu$ maps to 0 mod p . Hence $t\mu$ detects a unit times p in the integral spectral sequence $\mathcal{E}^*(S^1)$, and we can normalize t (or μ) to make this unit equal to +1. $\square$

Lemma 1.2. *The S^1 -Tate spectral sequence*

$$\hat{\mathcal{E}}^2(S^1) = \hat{H}^{-*}(S^1, \pi_*\mathrm{THH}(\mathbb{F}_p)) = \mathbb{F}_p[t^{\pm 1}, \mu] \implies \pi_*\mathrm{TP}(\mathbb{F}_p)$$

collapses. Hence

$$\pi_*\mathrm{TP}(\mathbb{F}_p) \cong \mathbb{Z}_p[t^{\pm 1}]$$

with $|t| = -2$. The canonical map induces

$$\mathrm{can}_*: \mathbb{Z}_p[t, \mu]/(t\mu = p) \longrightarrow \mathbb{Z}_p[t^{\pm 1}]$$

given by $t \mapsto t$ and $\mu \mapsto pt^{-1}$.

Proof. This follows by naturality along can from the previous lemma. $\square$

Lemma 1.3. *The C_p -Tate spectral sequence*

$$\hat{\mathcal{E}}^2(C_p) = \hat{H}^{-*}(C_p, \pi_*\mathrm{THH}(\mathbb{F}_p)) = \Lambda(u_1) \otimes \mathbb{F}_p[t_1^{\pm 1}, \mu] \implies \pi_*\mathrm{THH}(\mathbb{F}_p)^{tC_p}$$

has a differential $d^3(u_1 t_1^{-1}) \doteq t_1 \mu$ and collapses at $\hat{\mathcal{E}}^4(C_p) = \mathbb{F}_p[t_1^{\pm 1}]$. Hence

$$\pi_*\mathrm{THH}(\mathbb{F}_p)^{tC_p} \cong \mathbb{F}_p[t_1^{\pm 1}]$$

with $|t_1| = -2$.

Proof. The composite $\mathbb{S} \rightarrow \mathrm{TP}(\mathbb{F}_p) \xrightarrow{F} \mathrm{THH}(\mathbb{F}_p)^{tC_p}$ factors through $\mathrm{THH}(\mathbb{F}_p)$, where $p = 0$ in $\pi_0 \mathrm{THH}(\mathbb{F}_p)$, so $t_1 \mu \in \hat{\mathcal{E}}^2(C_p)$ must be a boundary. The only possible source of this differential is $u_1 t_1^{-1} \in \mathcal{E}_{1,0}^2$. $\square$

Lemma 1.4. *The cyclotomic structure map induces*

$$\varphi_{p*} : \mathbb{F}_p[\mu] \longrightarrow \mathbb{F}_p[t_1^{\pm 1}]$$

given by $\mu \mapsto t_1^{-1}$ (up to a unit in $\mathbb{F}_p$), hence factors as the localization inverting μ , followed by an isomorphism

$$\mathbb{F}_p[\mu^{\pm 1}] = \pi_* \mathrm{THH}(\mathbb{F}_p)[\mu^{-1}] \cong \pi_* \mathrm{THH}(\mathbb{F}_p)^{tC_p}.$$

Proof. We know that $G_* \circ \mathrm{can}_*$ and $\varphi_{p*}^{hS^1}$ agree on the image of $p \in \pi_0(\mathbb{S})$. Since

$$\mathbb{Z}_p \cong \pi_0 \mathrm{TC}^-(\mathbb{F}_p) \xrightarrow{\mathrm{can}_*} \pi_0 \mathrm{TP}(\mathbb{F}_p) \xrightarrow{G_*} \pi_0(\mathrm{THH}(\mathbb{F}_p)^{tC_p})^{hS^1}$$

are isomorphisms, the image of p in the target must generate a subgroup of index p . Since the image of p under $\varphi_{p*}^{hS^1}$ is detected by $t\varphi_p(\mu)$ in the S^1 -homotopy fixed point spectral sequence for $\mathrm{THH}(\mathbb{F}_p)^{tC_p}$ (which collapses at $\mathcal{E}^2$, see the next lemma), this class must be nonzero, so $\varphi_p(\mu)$ is a unit times t_1^{-1} . $\square$

Lemma 1.5. *The S^1 -homotopy fixed point spectral sequence*

$$\mu^{-1} \mathcal{E}^2(S^1) = H^{-*}(S^1, \pi_* \mathrm{THH}(\mathbb{F}_p)^{tC_p}) \cong \mathbb{F}_p[t, \mu^{\pm 1}] \implies \pi_*(\mathrm{THH}(\mathbb{F}_p)^{tC_p})^{hS^1}$$

collapses. Hence

$$\pi_*(\mathrm{THH}(\mathbb{F}_p)^{tC_p})^{hS^1} \cong \mathbb{Z}_p[\mu^{\pm 1}]$$

with $|\mu| = 2$. The map $\varphi_p^{hS^1}$ induces

$$\varphi_{p*}^{hS^1} : \mathbb{Z}_p[t, \mu]/(t\mu = p) \longrightarrow \mathbb{Z}_p[\mu^{\pm 1}]$$

given by $t \mapsto p\mu^{-1}$ and $\mu \mapsto \mu$.

Proof. This follows by naturality along $\varphi_p^{hS^1}$ from the previous lemma. $\square$

Lemma 1.6. *The equivalence G induces the isomorphism*

$$G_* : \mathbb{Z}_p[t^{\pm 1}] \xrightarrow{\cong} \mathbb{Z}_p[\mu^{\pm 1}]$$

given by $t \mapsto \mu^{-1}$, up to a unit in $\mathbb{Z}_p$. Hence

$$(\mathrm{can} - \varphi)_* : \mathbb{Z}_p[t, \mu]/(t\mu = p) \longrightarrow \mathbb{Z}_p[t^{\pm 1}]$$

is given by

$$\begin{cases} t^i \mapsto t^i \mod p & \text{in degree } -2i < 0, \\ \mu^j \mapsto -t^{-j} \mod p & \text{in degree } 2j > 0, \text{ up to a unit in } \mathbb{Z}_p, \\ 1 \mapsto 1 - 1 = 0 & \text{in degree } 0. \end{cases}$$

Proof. Being a graded ring isomorphism, G_* must take t to a unit times μ^{-1} . $\square$

Theorem 1.7 ([HM97]).

$$\pi_* \mathrm{TC}(\mathbb{F}_p)_p^\wedge \cong \Lambda_{\mathbb{Z}_p}(\partial) = \mathbb{Z}_p\{1, \partial\}$$

with $|\partial| = -1$.

Proof. This follows from the previous lemma, using the long exact sequence associated to

$$\Sigma^{-1}\mathrm{TP}(B) \xrightarrow{\partial} \mathrm{TC}(B) \xrightarrow{\pi} \mathrm{TC}^-(B) \xrightarrow{\mathrm{can}^{-\varphi}} \mathrm{TP}(B).$$

□

Since φ_p for $B = \mathbb{F}_p$ is an equivalence above degree $k = -2$, these calculations also apply for the original construction of TC , in degrees $* \geq -1$. On the other hand, each $\mathrm{THH}(\mathbb{F}_p)^{C_{p^n}}$ is connective, and it is easy to show from this that $\mathrm{TC}(\mathbb{F}_p)$ is (-1) -connective, recovering the statement in all degrees.

Since $\mathbb{F}_p$ is an $\mathbb{E}_\infty$ -ring, $\mathrm{TC}(\mathbb{F}_p)$ is also an $\mathbb{E}_\infty$ -ring. While it is equivalent to $\mathbb{Z}_p \oplus \Sigma^{-1}\mathbb{Z}_p$ as a spectrum, the $\mathbb{E}_\infty$ -ring structure is not that of a graded commutative ring. By [BR10], we have $Q_{p-1}(\partial) = Q^0(\partial) = \partial \neq 0$, at least for $p = 2$.

2. A REDUCED DECOMPOSITION

Thinking informally of ring spectra as the rings of functions on suitable geometric objects, the definition of $\mathrm{TC}(B)$ as the equalizer of the two localization maps

$$\mathrm{TC}^-(B) \begin{array}{c} \xrightarrow{\varphi} \\ \xrightarrow{\mathrm{can}} \end{array} \mathrm{TP}(B)$$

can be interpreted geometrically as the formation of the coequalizer $\mathrm{Spec} \mathrm{TC}(B) \sim B^{\mathrm{syn}}$ of two open immersions

$$B^\Delta \sim \mathrm{Spec} \mathrm{TP}(B) \begin{array}{c} \xrightarrow{\varphi^{\mathrm{op}}} \\ \xrightarrow{\mathrm{can}^{\mathrm{op}}} \end{array} \mathrm{Spec} \mathrm{TC}^-(B) \sim B^{\mathrm{Nyg}}.$$

The reduction $\mathrm{TC}(B) \rightarrow \mathrm{TC}(B)/p$ is then dual to the inclusion of a divisor

$$B_{p=0}^{\mathrm{syn}} \sim \mathrm{Spec} \mathrm{TC}(B)/p \subset \mathrm{Spec} \mathrm{TC}(B) \sim B^{\mathrm{syn}}$$

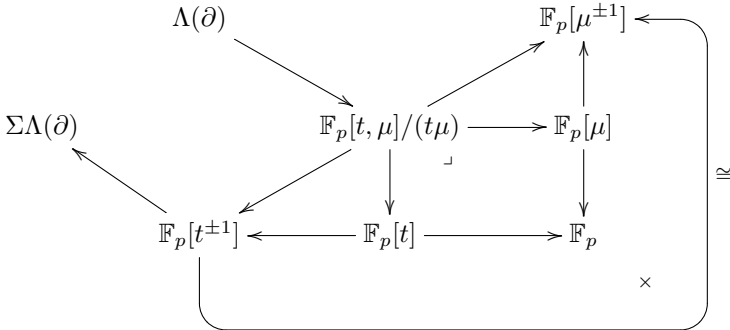
given by the vanishing locus of p , and similarly for $\mathrm{TC}^-(B)/p$ and $\mathrm{TP}(B)/p$. In the context of thinking about prismatic and syntomic cohomology for $B = \mathbb{Z}_p$ as coherent cohomology over suitable stacks, Drinfeld [Dri24] observed that the reduced stack corresponding to $\mathrm{TC}^-(\mathbb{Z}_p)/p, v_1$ decomposes as a union of two closed substacks, related to so-called Hodge–Tate (HT) and de Rham (dR) realization functors. As observed by Bhatt and Lurie [Bha22], this leads to an alternative view of $\mathbb{Z}_{p=v_1=0}^{\mathrm{syn}} \sim \mathrm{Spec} \mathrm{TC}(\mathbb{Z})/p, v_1$ as the sum of two Poincaré pairs. In the simpler case of $B = \mathbb{F}_p$, this corresponds to a decomposition of $\mathrm{TC}(\mathbb{F}_p)/p$, which we now outline. Both cases are generalized in current work with Devalapurkar and Hahn, concerning Poincaré duality in $\mathrm{TC}(\mathrm{BP}\langle n-1 \rangle)/p, \dots, v_n$ and related examples.

The relation $p = t\mu$ in $\pi_* \mathrm{TC}^-(\mathbb{F}_p)$ leads to a pullback square

$$\begin{array}{ccc} \mathrm{TC}^-(\mathbb{F}_p)/p & \xrightarrow{\mathrm{HT}} & \mathrm{TC}^-(\mathbb{F}_p)/t \\ \mathrm{dR} \downarrow & \lrcorner & \downarrow i \\ \mathrm{TC}^-(\mathbb{F}_p)/\mu & \xrightarrow{F} & \mathrm{TC}^-(\mathbb{F}_p)/(t, \mu). \end{array}$$

$$\begin{array}{ccccc}
\mathrm{TC}(\mathbb{F}_p)/p & & & \mathrm{THH}(\mathbb{F}_p)^{tC_p} & \leftarrow \\
\searrow \pi/p & & \nearrow \varphi_p^{hS^1}/p & \uparrow \varphi_p & \\
\Sigma\mathrm{TC}(\mathbb{F}_p)/p & \mathrm{TC}^-(\mathbb{F}_p)/p & \xrightarrow{|_{\mathrm{HT}}} & \mathrm{THH}(\mathbb{F}_p) & \\
& \searrow \mathrm{can}/p & \downarrow |_{\mathrm{dR}} \lrcorner & \downarrow i & \\
& \mathrm{TP}(\mathbb{F}_p)/p & \xleftarrow{j} \mathrm{TC}^-(\mathbb{F}_p)/\mu & \xrightarrow{F} \mathrm{THH}(\mathbb{F}_p)/\mu & \\
& & & & \times \\
& & & & \nearrow G/p \simeq
\end{array}$$

At the level of homotopy groups, this appears as follows.


$$\begin{array}{ccc}
\mathrm{THH}(\mathbb{F}_p) & & \mathrm{TC}^-(\mathbb{F}_p)/\mu \\
\downarrow i & \begin{array}{c} \nearrow \varphi_p \\ \searrow F \end{array} & \downarrow G/p \circ j \\
\mathrm{THH}(\mathbb{F}_p)/\mu & & \mathrm{THH}(\mathbb{F}_p)^{tC_p}
\end{array}$$
$$\mathrm{THH}(\mathbb{F}_p) \oplus \mathrm{TC}^-(\mathbb{F}_p)/\mu \xrightarrow{\begin{bmatrix} -i & F \\ -\varphi_p & G/p\circ j \end{bmatrix}} \mathrm{THH}(\mathbb{F}_p)/\mu \oplus \mathrm{THH}(\mathbb{F}_p)^{tC_p}.$$

Geometrically, we are gluing two lines $\mathrm{Spec} \mathbb{F}_p[\mu]$ and $\mathrm{Spec} \mathbb{F}_p[t]$ along the closed point $\mathrm{Spec} \mathbb{F}_p$ where $\mu = t = 0$, as well as along the open complements $\mathrm{Spec} \mathbb{F}_p[\mu^{\pm 1}] \cong \mathrm{Spec} \mathbb{F}_p[t^{\pm 1}]$. This presentation is convenient for exhibiting duality in $\mathrm{TC}(B)$ or syntomic cohomology for more general B .

3. CALCULATIONS FOR $B = \mathbb{Z}$

Let us also calculate $\pi_*\mathrm{TC}(\mathbb{Z})/p$ for p odd, recovering the result from [BM94] and [BM95]. (Both references are relevant, since the result was conjectural in the first paper, but was resolved in the second paper, using Tsalidis's theorem [Tsa97].)

Lemma 3.1. $\pi_*\mathrm{THH}(\mathbb{Z})/p = \pi_*\mathrm{THH}(\mathbb{Z}, \mathbb{F}_p) = \Lambda(\lambda_1) \otimes \mathbb{F}_p[\mu_1]$, where $|\lambda_1| = 2p - 1$ and $|\mu_1| = 2p$.

Proof. This follows from the Bökstedt spectral sequence, using that $H_*(\mathbb{Z}) = \Lambda(\bar{\tau}_i \mid i \geq 1) \otimes \mathbb{F}_p[\bar{\xi}_i \mid i \geq 1] \subset \mathcal{A}_*$, or from the Brun spectral sequence, using that $\pi_*(\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p) = \Lambda(dp) \cong \Lambda(\bar{\tau}_0)$. $\square$

Lemma 3.2. *In the S^1 -homotopy fixed point spectral sequence*

$$\mathcal{E}^2(S^1) = H^{-*}(S^1; \pi_*\mathrm{THH}(\mathbb{Z})/p) = \mathbb{F}_p[t, \mu_1] \otimes \Lambda(\lambda_1) \implies \pi_*\mathrm{TC}^-(\mathbb{Z})/p$$

the images under $\mathbb{S} \rightarrow \mathbb{S}/p \rightarrow \mathrm{TC}^-(\mathbb{Z})/p$ of $\alpha_1 \in \pi_{2p-3}(\mathbb{S})$ and $v_1 \in \pi_{2p-2}(\mathbb{S}/p)$ are detected by $t\lambda_1$ and $t\mu_1$, respectively.

Proof. This follows as in [AR02] from the long exact sequence in homology associated to the cofiber sequence

$$\Sigma^{-2}X \longrightarrow F(S_+^3, X)^{S^1} \longrightarrow F(S_+^1, X)^{S^1} = X \xrightarrow{\sigma} \Sigma^{-1}X,$$

with $X = \mathrm{THH}(\mathbb{Z})/p$, and an Adams spectral sequence argument, or by use of the Hahn–Wilson double suspension [HW22], with $\sigma^2\alpha_1 = \lambda_1$ and $\sigma^2v_1 = \mu_1$. $\square$

Lemma 3.3. *The C_p -Tate spectral sequence*

$$\hat{\mathcal{E}}^2(C_p) = \hat{H}^{-*}(C_p, \pi_*\mathrm{THH}(\mathbb{Z})/p) = \Lambda(u_1, \lambda_1) \otimes \mathbb{F}_p[t_1^{\pm 1}, \mu_1] \implies \pi_*\mathrm{THH}(\mathbb{Z})^{tC_p}/p$$

has differentials generated by $d^{2p}(t_1^{1-p}) \doteq t_1\lambda_1$ and $d^{2p+1}(u_1t_1^{-p}) \doteq t_1\mu_1$, leaving

$$\hat{\mathcal{E}}^\infty(C_p) = \mathbb{F}_p[t_1^{\pm p}] \otimes \Lambda(\lambda_1).$$

Proof. Since $\mathbb{S} \rightarrow \mathrm{TC}^-(\mathbb{Z}) \rightarrow \mathrm{TP}(\mathbb{Z}) \rightarrow \mathrm{THH}(\mathbb{Z})^{tC_p}$ factors through $\mathrm{THH}(\mathbb{Z})$, where α_1 and v_1 map to 0, the detecting classes $t\lambda_1$ and $t\mu_1$ must map to boundaries in the C_p -Tate spectral sequence. Hence $d^{2p}(t_1^{1-p}) \doteq t_1\lambda_1$ and $d^{2p+1}(u_1t_1^{-p}) \doteq t_1\mu_1$. $\square$

Definition 3.4. Let $\lambda = \mathbb{C}(1)$ denote the standard S^1 -representation, and let $a_\lambda \in \pi_{-\lambda}^{hS^1}(\mathbb{S})$ be the stable class of the S^1 -equivariant map $S^{-\lambda} \rightarrow S^0$, dual to $S^0 \rightarrow S^\lambda$. Hence the cofiber a_λ is dual to $S(\lambda)_+ = S_+^1$.

Define $\tilde{\mu}_1 \in \pi_{2p-2+\lambda}^{hS^1}\mathrm{THH}(\mathbb{Z})/p$ so that $a_\lambda\tilde{\mu}_1 = v_1 \in \pi_{2p-2}\mathrm{TC}^-(\mathbb{Z})/p$. Then $\tilde{\mu}_1$ maps to μ_1 under the map forgetting S^1 -equivariance.

The $\mathbb{Z}\{1, \lambda\} \subset \mathrm{RO}(S^1)$ -graded factorization of v_1 yields the lower right-hand pullback square in the following diagram. Here we use that $(X/a_\lambda)^{hS^1} \simeq X$. The notation $\mathrm{TC}^-(\mathbb{Z})/p, \tilde{\mu}_1$ should be interpreted as $(\mathrm{THH}(\mathbb{Z})/p, \tilde{\mu}_1)^{hS^1}$. The canonical image of the class a_λ is a unit, so $\mathrm{TP}(\mathbb{Z})/p, v_1 = \mathrm{TP}(\mathbb{Z})/p, \tilde{\mu}_1$, which should be interpreted as $(\mathrm{THH}(\mathbb{Z})/p, \tilde{\mu}_1)^{tS^1}$.

In the following, we write ‘ f ’ in place of ‘ $f/p, v_1$ ’ for various maps.

$$\begin{array}{ccccc}
 & \text{TC}(\mathbb{Z})/p, v_1 & & \text{THH}(\mathbb{Z})^{tC_p}/p & \leftarrow \\
 & \searrow \pi & & \nearrow \varphi_p^{hS^1} & \\
 \Sigma \text{TC}(\mathbb{Z})/p, v_1 & & \text{TC}^-(\mathbb{Z})/p, v_1 & \xrightarrow{|\text{HT}|} & \text{THH}(\mathbb{Z})/p \\
 & \nearrow \partial & \searrow \text{can} & \downarrow i & \uparrow \varphi_p \\
 & \text{TP}(\mathbb{Z})/p, v_1 & \xleftarrow{j} \text{TC}^-(\mathbb{Z})/p, \tilde{\mu}_1 & \xrightarrow{F} & \text{THH}(\mathbb{Z})/p, \mu_1 \\
 & & \downarrow \text{dR} & & \downarrow i \\
 & & & & \times
 \end{array}
 \quad G \simeq$$

Lemma 3.5. $\pi_* \text{THH}(\mathbb{Z})/p, \mu_1 = \Lambda(\lambda_1)$. The S^1 -Tate spectral sequence

$$\hat{\mathcal{E}}^2(S^1) = \hat{H}^{-*}(S^1, \pi_* \text{THH}(\mathbb{Z})/p, \mu_1) = \mathbb{F}_p[t^{\pm 1}] \otimes \Lambda(\lambda_1) \implies \pi_* \text{TP}(\mathbb{Z})/p, v_1$$

has differentials $d^{2p}(t^i) \doteq t^{i+p} \lambda_1$ for $p \nmid i$, leaving

$$\hat{\mathcal{E}}^\infty(S^1) = \mathbb{F}_p[t^{\pm p}] \otimes \Lambda(\lambda_1).$$

Proof. The d^{2p} -differentials for $\text{THH}(\mathbb{Z})^{tC_p}/p$ lift to differentials for $\text{THH}(\mathbb{Z})^{tS^1}/p = \text{TP}(\mathbb{Z})/p$ and push forward to differentials for $\text{TP}(\mathbb{Z})/p, v_1$. $\square$

Lemma 3.6. The S^1 -homotopy fixed point spectral sequence

$$\mathcal{E}^2(S^1) = H^{-*}(S^1, \pi_* \text{THH}(\mathbb{Z})/p, \mu_1) = \mathbb{F}_p[t] \otimes \Lambda(\lambda_1) \implies \pi_* \text{TC}^-(\mathbb{Z})/p, \tilde{\mu}_1$$

has differentials $d^{2p}(t^i) \doteq t^{i+p} \lambda_1$ for $p \nmid i \geq 0$, leaving

$$\mathcal{E}^\infty(S^1) = \mathbb{F}_p[t^p] \otimes \Lambda(\lambda_1) \oplus \mathbb{F}_p\{t^i \lambda_1 \mid 0 < i < p\}.$$

The homomorphism j_* is injective on the first summand and zero on the second summand.

Proof. This follows by naturality along j . $\square$

Corollary 3.7.

$$\pi_* \text{TC}^-(\mathbb{Z})/p, v_1 \cong \mathbb{F}_p[t^p]\{1, \lambda_1\} \oplus \mathbb{F}_p\{t^i \lambda_1 \mid 0 < i < p\} \oplus \mathbb{F}_p[\mu_1]\{\mu_1, \lambda_1 \mu_1\}.$$

The homomorphism can_* is injective on the first summand and zero on the second and third summands.

Proof. This is a calculation of the pullback of

$$\pi_* \text{TC}^-(\mathbb{Z})/p, \tilde{\mu}_1 \longrightarrow \pi_* \text{THH}(\mathbb{Z})/p, \mu_1 \longleftarrow \pi_* \text{THH}(\mathbb{Z})/p.$$

Alternatively, one can prove this without the use of $\text{RO}(S^1)$ -graded classes and the pullback decomposition, by fully calculating the S^1 -homotopy fixed point spectral sequence in Lemma 3.2, and then reducing the result modulo v_1 (detected by $t\mu_1$). $\square$

Lemma 3.8. can and φ agree on $\lambda_1 \in \pi_{2p-1} \text{TC}^-(\mathbb{Z})/p, v_1$, which therefore admits a lift to $\pi_{2p-1} \text{TC}(\mathbb{Z})/p, v_1$.

Proof. Either use results of Waldhausen and Bökstedt on the fiber of $K(\mathbb{S}) \rightarrow K(\mathbb{Z})$ to find a class $\lambda_1^K \in K_{2p-1}(\mathbb{Z})$ with trace image λ_1 , or show that the transfer map $\text{trf}: \text{TC}(\mathbb{F}_p)/p, v_1 \rightarrow \text{TC}(\mathbb{Z})/p, v_1$ takes $\bar{\epsilon}_1 \in \pi_{2p-1}\text{TC}(\mathbb{F}_p)/p, v_1$ to the required lift of λ_1 . $\square$

Lemma 3.9. *The cyclotomic structure map induces*

$$\varphi_{p*}: \Lambda(\lambda_1) \otimes \mathbb{F}_p[\mu_1] \longrightarrow \mathbb{F}_p[t^{\pm p}] \otimes \Lambda(\lambda_1)$$

given by $\mu_1 \mapsto t_1^{-p}$ (up to a unit in $\mathbb{F}_p$) and $\lambda_1 \mapsto G(\lambda_1)$, hence factors as the localization inverting μ_1 , followed by an isomorphism

$$\Lambda(\lambda_1) \otimes \mathbb{F}_p[\mu_1^{\pm 1}] = \pi_* \text{THH}(\mathbb{Z})/p[\mu_1^{-1}] \cong \pi_* \text{THH}(\mathbb{Z})^{t_{C_p}}/p.$$

Proof. Being a ring homomorphism, φ_{p*} must take the unit μ_1 to a unit, i.e., to a nonzero scalar times t_1^{-p} . Moreover, $\varphi_p(\lambda_1) = \varphi_p^{hS^1}(\lambda_1) = (G \circ \text{can})(\lambda_1) = G(\lambda_1)$. $\square$

Theorem 3.10 (Bökstedt–Madsen, etc.). *There is a short exact sequence*

$$0 \rightarrow \Sigma^{-1}\Lambda(\lambda_1) \xrightarrow{\partial} \pi_* \text{TC}(\mathbb{Z})/p, v_1 \xrightarrow{\pi} \mathbb{F}_p\{t^i \lambda_1 \mid 0 < i < p\} \oplus \Lambda(\lambda_1) \rightarrow 0,$$

so that

$$\pi_* \text{TC}(\mathbb{Z})/p, v_1 \cong \Lambda(\partial, \lambda_1) \oplus \mathbb{F}_p\{t^i \lambda_1 \mid 0 < i < p\}$$

with $|\partial| = -1$, $|\lambda_1| = 2p - 1$ and $|t| = -2$. The relations $t^i \lambda_1 \cdot t^{p-i} \lambda_1 = \partial \lambda_1$ hold, making this a Poincaré duality algebra.

$$\begin{array}{ccccccc}
 & & & & & \partial \lambda_1 & \\
 & & & & & & \\
 \partial & & t^{p-1} \lambda_1 & \dots & t \lambda_1 & & \lambda_1 \\
 & & & & & & \\
 \dots & 1 & \dots & & & & \\
 & & & & & & \\
 -1 & 0 & 1 & \dots & 2p-3 & 2p-2 & 2p-1
 \end{array}$$

Proof. To calculate the kernel and cokernel of $(\text{can} - \varphi)_*: \pi_* \text{TC}^-(\mathbb{Z})/p, v_1 \rightarrow \pi_* \text{TP}(\mathbb{Z})/p, v_1$, we express it as the direct sum of the following four homomorphisms.

$$\begin{aligned}
 \mathbb{F}_p[t^p]\{t^p, t^p \lambda_1\} &\xrightarrow{\cong} \mathbb{F}_p[t^p]\{t^p, t^p \lambda_1\} \\
 \mathbb{F}_p\{t^i \lambda_1 \mid 0 < i < p\} &\longrightarrow 0 \\
 \Lambda(\lambda_1) &\xrightarrow{0} \Lambda(\lambda_1) \\
 \mathbb{F}_p[\mu_1]\{\mu_1, \lambda_1 \mu_1\} &\xrightarrow{\cong} \mathbb{F}_p[t^{-p}]\{t^{-p}, t^{-p} \lambda_1\}
 \end{aligned}$$

The multiplicative relation only makes sense at the level of spectra when $\mathbb{S}/p, v_1 = V(1)$ is a ring spectrum, i.e., for $p \geq 5$, but works at all primes in the associated graded for the Hahn–Raksit–Wilson motivic filtration. It can be deduced from

the Browder/Wall sum theorem for the two Poincaré pairs given by (i, φ_p) and (j, F) . $\square$

Corollary 3.11. *The v_1 -Bockstein spectral sequence*

$$\mathcal{E}^1 = \pi_* \mathrm{TC}(\mathbb{Z})/p, v_1 \otimes \mathbb{F}_p[v_1] \implies \pi_* \mathrm{TC}(\mathbb{Z})/p$$

collapses, and

$$\pi_* \mathrm{TC}(\mathbb{Z})/p \cong \mathbb{F}_p[v_1] \otimes \left(\Lambda(\partial, \lambda_1) \oplus \mathbb{F}_p\{t^i \lambda_1 \mid 0 < i < p\} \right).$$

Proof. This uses that λ_1 (and $\partial \lambda_1$) admit mod p (and integral) lifts, hence cannot support v_1 - (or p -)Bockstein differentials. $\square$

((ETC: Relate $\mathrm{TC}(\mathbb{Z})/p$ to $K(\mathbb{Z}_p)/p$, $K(\mathbb{Q}_p)/p$, $K^{\acute{e}t}(\mathbb{Q}_p)/p$ and $H_{\acute{e}t}^*(\mathbb{Q}_p; \mathbb{Z}/p(*))$.)

4. THE DUNDAS–GOODWILLIE–MCCARTHY THEOREM

For (possibly simplicial) rings R , Goodwillie [Goo86] introduced negative cyclic homology, $\mathrm{HC}^-(R)$, and constructed a lift $K(R) \rightarrow \Sigma \mathrm{HC}^-(R)$ of the Dennis trace map to $\mathrm{HH}(R)$. He then proved that for any homomorphism $R \rightarrow T$ such that $\pi_0(R) \rightarrow \pi_0(T)$ is surjective with nilpotent kernel, there is a Cartesian square

$$\begin{array}{ccc} K(R) \otimes \mathbb{Q} & \longrightarrow & \Sigma \mathrm{HC}(R) \otimes \mathbb{Q} \\ \downarrow & & \downarrow \\ K(T) \otimes \mathbb{Q} & \longrightarrow & \Sigma \mathrm{HC}(T) \otimes \mathbb{Q}. \end{array}$$

This allowed him to calculate $A(X) \otimes \mathbb{Q}$ for simply-connected spaces $X \simeq BG$, since rationally $\mathbb{S}[G] \rightarrow \mathbb{S}$ can be modeled by a map $\mathbb{Z}[G] \rightarrow \mathbb{Z}$ of simplicial rings that induces an isomorphism on π_0 . This generalized related work by Burghelea, Dwyer, Hsiang and Staffeldt, some of which predated Connes’s cyclic (co-)homology.

With the advent of the cyclotomic trace map $\mathrm{trc}: K(B) \rightarrow \mathrm{TC}(B)$ lifting the Bökstedt trace map to $\mathrm{THH}(B)$, Goodwillie conjectured that there should be a similar Cartesian square

$$\begin{array}{ccc} K(A)_p^\wedge & \xrightarrow{\mathrm{trc}_A} & \mathrm{TC}(A)_p^\wedge \\ \downarrow & & \downarrow \\ K(B)_p^\wedge & \xrightarrow{\mathrm{trc}_B} & \mathrm{TC}(B)_p^\wedge, \end{array}$$

for maps $A \rightarrow B$ of connective $\mathbb{E}_1$ -rings such that $\pi_0(A) \rightarrow \pi_0(B)$ is surjective with nilpotent kernel. This was proven in the case of simplicial rings by McCarthy [McC97], and then extended to the case of $\mathbb{E}_1$ -rings by Dundas [Dun97].

The three of them thereafter used an arithmetic square to merge the rational and p -complete calculations, so as to obtain an integral definition of TC , and to prove that the square above is Cartesian, without p -completion, for $A \rightarrow B$ as above. This eventually appeared in [DGM13]. In other words, relative algebraic K -theory, i.e., the fiber of $K(A) \rightarrow K(B)$, is equivalent to relative topological cyclic homology, i.e., the fiber of $\mathrm{TC}(A) \rightarrow \mathrm{TC}(B)$, for maps $A \rightarrow B$ of connective $\mathbb{E}_1$ -rings such that $\pi_0(A) \rightarrow \pi_0(B)$ is surjective with nilpotent kernel. In yet another formulation, $\mathrm{fib}(\mathrm{trc}_A) \simeq \mathrm{fib}(\mathrm{trc}_B)$ for such maps $A \rightarrow B$.

Example 4.1. Simple examples of ring surjections with nilpotent kernel are the homomorphisms $\mathbb{Z}/p^n \rightarrow \mathbb{Z}/p = \mathbb{F}_p$, with limit $\mathbb{Z}_p \rightarrow \mathbb{F}_p$. By McCarthy's theorem, we therefore have one commutative and two cartesian squares

$$\begin{array}{ccc}
 K(\mathbb{Z}_p)_p^\wedge & \longrightarrow & \mathrm{TC}(\mathbb{Z}_p)_p^\wedge \\
 \downarrow & \circ & \downarrow \\
 \lim_n K(\mathbb{Z}/p^n)_p^\wedge & \longrightarrow & \lim_n \mathrm{TC}(\mathbb{Z}/p^n)_p^\wedge \\
 \downarrow & \lrcorner & \downarrow \\
 K(\mathbb{Z}/p^n)_p^\wedge & \longrightarrow & \mathrm{TC}(\mathbb{Z}/p^n)_p^\wedge \\
 \downarrow & \lrcorner & \downarrow \\
 K(\mathbb{F}_p)_p^\wedge & \longrightarrow & \mathrm{TC}(\mathbb{F}_p)_p^\wedge .
 \end{array}$$

As an application of results of Suslin and coworkers, Hesselholt–Madsen [HM97] show that for a perfect field k of characteristic p and a finite algebra A over its ring of Witt vectors $W(k)$, the canonical map

$$K(A)_p^\wedge \xrightarrow{\sim} \lim_n K(A/p^n)_p^\wedge$$

is an equivalence. Here ‘finite algebra’ means that A is finitely generated as a $W(k)$ -module. Recall that when $k = \mathbb{F}_q$ is a finite field with $q = p^f$ elements, the ring $W(k)$ is the unramified degree f extension of $\mathbb{Z}_p$. In particular, $W(\mathbb{F}_p) = \mathbb{Z}_p$. Moreover, they show that for these finite algebras over $W(k)$, also the canonical map

$$\mathrm{TC}(A)_p^\wedge \xrightarrow{\sim} \lim_n \mathrm{TC}(A/p^n)_p^\wedge$$

is an equivalence. In particular, in the example above, the vertical maps in the top square are p -adic equivalences, so that square is also cartesian after p -completion.

Quillen [Qui72] proved that $K(\mathbb{F}_q)_p^\wedge \simeq \mathbb{Z}_p$ for each finite field $\mathbb{F}_q$ with $q = p^f$ elements, and we have calculated that $\mathrm{TC}(\mathbb{F}_p)_p^\wedge \simeq \mathbb{Z}_p \oplus \Sigma^{-1}\mathbb{Z}_p$, so $K(\mathbb{F}_p)_p^\wedge \simeq \tau_{\geq 0}\mathrm{TC}(\mathbb{F}_p)_p^\wedge$. More generally, $K(k)_p^\wedge \simeq \tau_{\geq 0}\mathrm{TC}(k)_p^\wedge$ for perfect fields k as above, and moreover $K(A/p)_p^\wedge \simeq \tau_{\geq 0}\mathrm{TC}(A/p)_p^\wedge$ for algebras A as above, by another application of McCarthy's theorem. In combination, this proves the following useful result.

Theorem 4.2 ([HM97, Thm. D]). *Let A be a finite algebra over the Witt vectors of a perfect field of characteristic p . Then the cyclotomic trace map induces an equivalence*

$$K(A)_p^\wedge \simeq \tau_{\geq 0}\mathrm{TC}(A)_p^\wedge .$$

By Dundas's theorem, the same conclusion holds if A is a connective $\mathbb{E}_1$ -ring with $\pi_0(A)$ a finite $W(k)$ -algebra, for k as above. This includes the cases when $\pi_0(A)$ is $\mathbb{Z}_p$, or more generally the valuation ring $\mathcal{O}_L$ in any p -adic number field L .

Remark 4.3. Since the calculation of $\pi_*\mathrm{TC}(A)_p^\wedge$ is to some extent accessible by the methods of equivariant homotopy theory, it is no longer accurate to claim that algebraic K -theory is ‘notoriously hard to compute’, at least when only considering p -complete K -theory for many p -complete rings or ring spectra.

The invariance of $\text{fib}(\text{trc}_A: K(A)_p^\wedge \rightarrow \text{TC}(A)_p^\wedge)$ can be extended beyond the nilpotent extensions $A \rightarrow B$, or the examples $A = \lim_n A/p^n \rightarrow A/p$ discussed above, at least in the case of commutative rings.

Definition 4.4. A Henselian pair (R, I) is a commutative ring R and an ideal I such that Hensel’s lemma holds along $R \rightarrow R/I$, i.e., given any polynomial $f \in R[x]$ and a root $\bar{\alpha} \in R/I$ of its reduction $\bar{f} \in R/I[x]$, satisfying that $\bar{f}'(\alpha)$ is a unit in R/I , then there exists a lift $\alpha \in R$ of $\bar{\alpha}$ that is a root of f .

For example, for a pair (R, I) with residue ring R/I its I -adic completion $(R_I^\wedge, I_I^\wedge)$ is Henselian, but there also exist smaller Henselian pairs with the same residue ring, including the so-called Henselization (R^h, I^h) with $R \rightarrow R^h \rightarrow R_I^\wedge$ over R/I .

Theorem 4.5 ([CMM21]). *Let (R, I) be any Henselian pair. Then*

$$\begin{array}{ccc} K(R)_p^\wedge & \xrightarrow{\text{trc}_R} & \text{TC}(R)_p^\wedge \\ \downarrow & & \downarrow \\ K(R/I)_p^\wedge & \xrightarrow{\text{trc}_{R/I}} & \text{TC}(R/I)_p^\wedge \end{array}$$

is Cartesian.

5. TR REVISITED

((ETC: Formula for TR. Antieau–Nikolaus t -structure [AN21].))

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