ALGEBRAIC K-THEORY OF REAL TOPOLOGICAL K-THEORY

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ABSTRACT. We determine the $A(1)$-homotopy of the topological cyclic homology of the connective real K-theory spectrum ko. The answer has an associated graded that is a free $\mathbb{F}_2[v_2^4]$-module of rank 52, on explicit generators in stems $-1 \le * \le 30$. The calculation is achieved by using prismatic and syntomic cohomology of ko as introduced by Hahn–Raksit–Wilson, extending work of Bhatt–Morrow–Scholze from the case of classical commutative rings to $\mathbb{E}_\infty$ rings. A new feature in our case is that there are nonzero differentials in the motivic spectral sequence from syntomic cohomology to topological cyclic homology.

CONTENTS

1. INTRODUCTION

Work of Hahn–Raksit–Wilson [HRW] extends the notions of prismatic cohomology and syntomic cohomology to the setting of $\mathbb{E}_\infty$ rings. This produces a new tool for computing topological cyclic homology and, consequently, algebraic K-theory. In the present paper, we use this tool to compute the $A(1)$-homotopy (cf. Notation 2.15) of the algebraic K-theory of the $\mathbb{E}_\infty$ ring ko, known as connective real K-theory. Throughout, we work at the prime $p = 2$.

This paper continues the program from [AR02], examining the arithmetic of ring spectra through the lens of telescopically localized algebraic K-theory. In particular, the second and third authors put forward, in a bundle of predictions known as the redshift conjectures [AR08], the assertion that algebraic K-theory increases chromatic complexity by one. This has now been proven in a qualitative form, for all $\mathbb{E}_\infty$ rings, in a *tour de force* by Burklund–Schlank–Yuan [BSY], building on [Yua], [CMNN] and [LMMT].

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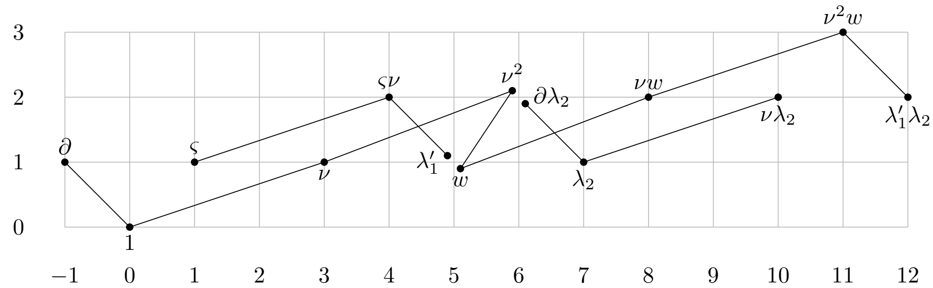

Figure 1.1. $\mathbb{F}_2[v_2]$-basis for the syntomic cohomology modulo $(2, \eta, v_1)$ of ko, with lines of slope -1, 1 and $1/3$ indicating multiplication by ∂, η and ν, respectively

To better understand the arithmetic of $\mathbb{E}_\infty$ rings, it is still desirable to prove more quantitative forms of the redshift conjectures, such as the one originally appearing as the "chromatic redshift problem" in [Rog00]. In the present paper, we solve this problem in the case of ko. Explicitly, we prove the following theorem.

Theorem A (Theorem 6.2)**.** *The $A(1)$-homotopy $A(1)_* \operatorname{TC}(\mathrm{ko})$ of the topological cyclic homology of* ko *is a $\mathbb{Z}/4[v_2^{32}]$-module. The associated graded*

$$\begin{aligned}\operatorname{Gr}^*_{\mathrm{mot}} A(1)_* \operatorname{TC}(\mathrm{ko}) \cong\ & \mathbb{F}_2\{v_2^i \mid 0 \le i \equiv 0,1 \mod 4\} \\ & \oplus \mathbb{F}_2[v_2]\{\partial, \varsigma, \nu, \lambda_1', w, \lambda_2\} \\ & \oplus \mathbb{F}_2[v_2]\{\varsigma\nu, \nu^2, \partial\lambda_2, \nu w, \nu\lambda_2, \lambda_1'\lambda_2\} \\ & \oplus \mathbb{F}_2\{v_2^j \nu^2 w \mid 0 \le j \equiv 2,3 \mod 4\}\end{aligned}$$

of its descending motivic filtration $\operatorname{Fil}^\star_{\mathrm{mot}} A(1)_* \operatorname{TC}(\mathrm{ko})$ is a finitely generated and free $\mathbb{F}_2[v_2^4]$-module of rank 52. Here $|\partial| = -1$, $|\varsigma| = 1$, $|\nu| = 3$, $|\lambda_1'| = |w| = 5$, $|v_2| = 6$ and $|\lambda_2| = 7$.

This calculation is carried out by first computing the syntomic cohomology with $A(1)$-coefficients, *alias* modulo $(2, \eta, v_1)$, of connective real K-theory.

Theorem B (Theorem 5.12)**.** *The syntomic cohomology modulo $(2, \eta, v_1)$ of* ko *is*

$$\begin{aligned}\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \operatorname{TC}(\mathrm{ko}) &:= \pi_*(\overline{A}(1) \otimes \operatorname{gr}^*_{\mathrm{mot}} \operatorname{TC}(\mathrm{ko})) \\ &\cong \mathbb{F}_2[v_2]\{1, \partial, \varsigma, \nu, \lambda_1', w, \lambda_2, \varsigma\nu, \nu^2, \partial\lambda_2, \nu w, \nu\lambda_2, \lambda_1'\lambda_2, \nu^2 w\}\,,\end{aligned}$$

where the (stem, motivic filtration) bidegrees of the $\mathbb{F}_2[v_2]$-module generators are as displayed in Figure 1.1 and Table 5.1.

See Notation 6.1 for the algebraic filtration $\operatorname{Fil}^\star_{\mathrm{mot}}(-)$ on $A(1)_* \operatorname{TC}(\mathrm{ko})$ and its associated graded $\operatorname{Gr}^*_{\mathrm{mot}}(-)$, see Definition 2.14 for the spectrum level filtration $\operatorname{fil}^\star_{\mathrm{mot}}(-)$ on $\operatorname{TC}(\mathrm{ko})$ and its associated graded $\operatorname{gr}^*_{\mathrm{mot}}(-)$, and see Construction 2.24 for the meaning of $\overline{A}(1) \otimes (-) := \operatorname{gr}^*_{\mathrm{ev}} A(1) \otimes_{\operatorname{gr}_{\mathrm{ev}} \mathbb{S}} (-)$.

As a consequence, we determine the $A(1)$-homotopy of the algebraic K-theory of ko and of its 2-completion $\mathrm{ko}_2^\wedge$.

Theorem C (Theorems 6.4 and 6.5)**.** *There are exact sequences of* $\mathbb{Z}/4[v_2^{32}]$*-modules*

$$0 \to \Sigma^3\mathbb{F}_2 \longrightarrow A(1)_* \operatorname{K}(\mathrm{ko}) \xrightarrow{\mathrm{trc}} A(1)_* \operatorname{TC}(\mathrm{ko}) \longrightarrow \mathbb{F}_2\{\partial, \varsigma\} \to 0$$

and

$$0 \to \Sigma^1\mathbb{F}_2 \oplus \Sigma^3\mathbb{F}_2 \longrightarrow A(1)_* \operatorname{K}(\mathrm{ko}_2^\wedge) \xrightarrow{\mathrm{trc}} A(1)_* \operatorname{TC}(\mathrm{ko}) \longrightarrow \mathbb{F}_2\{\partial, \varsigma\} \to 0\,,$$

with $|\partial| = -1$ *and* $|\varsigma| = 1$*.*

Using this, we show in Corollary 6.6 that there is no $\mathbb{E}_0$ ring map $\mathrm{K}(\mathrm{ko}) \to \mathrm{tmf}$, i.e., no spectrum map inducing an isomorphism on π_0. In contrast, an $\mathbb{E}_\infty$ ring map $\mathrm{K}(\mathrm{ko}) \to E_2$ is guaranteed to exist by [BSY]. Here tmf denotes the topological modular forms spectrum, and E_2 is a suitable height 2 Lubin–Tate theory.

As usual, we let $K(n)$ denote the (2-primary) height n Morava K-theory, with coefficient ring $K(n)_* = \mathbb{F}_2[v_n^{\pm 1}]$, and let L_n and L_n^f denote Bousfield localization at $K(0) \oplus \cdots \oplus K(n)$ and at $T(0) \oplus \cdots \oplus T(n)$, respectively, where $T(m) = v_m^{-1}F(m)$ is the telescope of any v_m-self map of a type m finite 2-local spectrum $F(m)$.

We say that a spectrum X satisfies the (strong, 2-primary) *height n telescope conjecture* if the canonical map $L_n^f X \to L_n X$ is an equivalence. By recent ground-breaking work of Burklund–Hahn–Levy–Schlank [BHLS], we know for each $n \geq 2$ that not all spectra have this property. However, it is still interesting to consider the question of which spectra do satisfy this conjecture, see for example [MR99, Conjecture 7.3]. In Theorem 6.7 we deduce from [HRW] and [CMNN20] that the spectra $\mathrm{K}(\mathrm{ko})$, $\mathrm{K}(\mathrm{ko}_2^\wedge)$ and $\mathrm{TC}(\mathrm{ko})$ all satisfy the height 2 telescope conjecture.

We say that a spectrum X has *telescopic complexity n* if the map $X \to L_n^f X$ is an equivalence in all sufficiently large degrees. In [AR08], the redshift conjecture was phrased in terms of this property. In Theorem 6.8 we apply our Theorems A and C to confirm that the spectra $\mathrm{K}(\mathrm{ko})_{(2)}$, $\mathrm{K}(\mathrm{ko}_2^\wedge)_{(2)}$ and $\mathrm{TC}(\mathrm{ko})_{(2)}$ (as well as their 2-completions) have telescopic complexity 2, as predicted.

Organization. In Section 2 we give a presentation for the associated graded $\mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})$ of the motivic filtration on $\mathrm{THH}(\mathrm{ko})$, and calculate its bigraded homotopy modulo $(2, \eta, v_1)$ and modulo $(2, v_1)$. In Section 3 we show how some homotopy classes from $A(1)$ are detected in homotopy modulo $(2, \eta, v_1)$ of $\mathrm{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})$. In Section 4 we study the $\mathbb{T}$-Tate spectral sequence calculating prismatic cohomology, i.e., the homotopy modulo $(2, v_1, v_2)$ and $(2, \eta, v_1, v_2)$ of $\mathrm{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko})$. In Section 5 we study the $\mathbb{T}$-homotopy fixed point spectral sequence calculating the homotopy of $\mathrm{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})$ modulo $(2, v_1, v_2)$ and $(2, \eta, v_1, v_2)$, and use this to calculate syntomic cohomology, i.e., the homotopy of $\mathrm{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$ modulo $(2, v_1)$ and $(2, \eta, v_1)$. Finally, in Section 6 we determine the differential pattern in the motivic spectral sequence calculating $A(1)_* \mathrm{TC}(\mathrm{ko})$, and use known facts about the cyclotomic trace map to deduce corresponding results for $A(1)_* K(\mathrm{ko}_2^\wedge)$ and $A(1)_* K(\mathrm{ko})$.

Conventions. We let $\mathcal{A}$ and $\mathcal{A}^\vee$ denote the mod 2 Steenrod algebra and its dual, respectively. We set $H_*(X) := H_*(X; \mathbb{F}_2)$ and write

$$\nu \colon H_*(X) \longrightarrow \mathcal{A}^\vee \otimes H_*(X)$$

for the $\mathcal{A}^\vee$-coaction. We write $\mathcal{A}(1)$ for the subalgebra of $\mathcal{A}$ generated by Sq^1 and Sq^2, and note that $\mathcal{A}(1)^\vee = \mathbb{F}_2[\xi_1, \xi_2]/(\xi_1^4, \xi_2^2)$.

We say that a spectrum X is *even* if its homotopy groups $\pi_*(X) = X_*$ are concentrated in even degrees.

Let $\mathbb{Z}^{\mathrm{op}}$ be the category whose objects are integers, such that $\mathrm{Hom}_{\mathbb{Z}^{\mathrm{op}}}(n,m) = *$ if $n \geq m$ and $= \varnothing$ otherwise. Let $\mathbb{Z}^{\delta}$ be the integers as a discrete category. Given a presentably symmetric monoidal stable ∞-category $\mathcal{C}$, we write

$$\mathcal{C}^{\mathrm{fil}} := \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C}) \qquad \text{and} \qquad \mathcal{C}^{\mathrm{gr}} := \mathrm{Fun}(\mathbb{Z}^{\delta}, \mathcal{C})\,.$$

Given $I^\star \in \mathcal{C}^{\mathrm{fil}}$ and $J^* \in \mathcal{C}^{\mathrm{gr}}$, we write $I^w := I(w)$ and $J^w := J(w)$. We recall that there is a monoidal functor

$$\mathrm{gr}^* \colon \mathcal{C}^{\mathrm{fil}} \longrightarrow \mathcal{C}^{\mathrm{gr}}$$

defined on an object $I^\star$ by $\mathrm{gr}^w I = I^w/I^{w+1}$. As in the notation above, for consistency we use the superscript $\star$ as in $I^\star$ for a filtered object, the superscript $*$ as in J^* for a graded object, and a superscript $\bullet$ as in $K^\bullet$ for a cosimplicial object.

We use the terminology from [Isa19, Definition 4.1.2] for (hidden) extensions in spectral sequences.

In Sections 2–5, we use the following conventions: We write $\mathrm{THH}(A/R) = |[q] \mapsto A \otimes_R \cdots \otimes_R A|$ for the relative topological Hochschild homology of an R-algebra A, with $q+1$ copies of A in simplicial degree q, and simply write $\mathrm{THH}(A)$ when $R = \mathbb{S}$. We implicitly 2-complete each of the following invariants: $\mathrm{THH} := \mathrm{THH}(-)^{\wedge}_2$, $\mathrm{TC}^- := \big(\mathrm{THH}(-)^{h\mathbb{T}}\big)^{\wedge}_2$, $\mathrm{TP} := \big(\mathrm{THH}(-)^{t\mathbb{T}}\big)^{\wedge}_2$, $\mathrm{TC} := \mathrm{TC}(-)^{\wedge}_2$ and $\mathrm{K} := \mathrm{K}(-)^{\wedge}_2$. We write ko, ku, KU, MU, MUP, $\mathbb{S}$ and BP for the 2-completions of connective real K-theory, connective complex K-theory, periodic complex K-theory, complex cobordism, periodic complex cobordism, the sphere spectrum and the Brown–Peterson spectrum, respectively. We will simply write $\mathbb{Z}$ for the 2-adic integers. We write Sp_2 for the ∞-category of 2-complete spectra with symmetric monoidal product $\otimes$. Note that our smash product is implicitly 2-completed in Sections 2–5, so that $\otimes_R := (- \otimes_R -)^{\wedge}_2$ for any 2-complete $\mathbb{E}_\infty$ ring R, and we omit R from the notation when R is the 2-complete sphere spectrum. We also write $\otimes$ for the (underived) tensor product over the 2-adic integers and expect the intended meaning to be clear from context. We write $\mathbb{T}$ for the circle regarded as the group of complex numbers of modulus 1, and set $\mathcal{C}^{B\mathbb{T}} := \mathrm{Fun}(B\mathbb{T}, \mathcal{C})$.

In Section 6, as in the present Section 1, we explicitly include notation for 2-completion, especially in the argument of $\mathrm{K}(-)$. Note that the canonical map $\mathrm{TC}(\mathrm{ko})^{\wedge}_2 \to \mathrm{TC}(\mathrm{ko}^{\wedge}_2)^{\wedge}_2$ is an equivalence by [Mad94, pp. 274–275] (cf. [NS18, pp. 351–352]), so we can omit 2-completion in the argument of $\mathrm{TC}(-)^{\wedge}_2$.

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2. Hochschild homology and motivic filtrations

We first introduce filtrations on THH, TC^-, TP and TC. The reader is encouraged to read [BHM93, BM94, BM95, HN20] for background on these invariants and [HRW] for a thorough account of the filtrations that we use in this paper.

Definition 2.1. Given a map $f\colon A \to B$ of 2-complete $\mathbb{E}_\infty$ rings, we write $C^\bullet(B/A)$ for the associated cosimplicial *Amitsur complex* with $C^q(B/A) = B^{\otimes_A q+1}$.

Recall that there is a map of $\mathbb{E}_\infty$ rings $c\colon \mathrm{ko} \to \mathrm{ku}$ called the *complexification map*, where $\mathrm{ko}_* = \mathbb{Z}[\eta, A, B]/(2\eta, \eta^3, \eta A, A^2 - 4B)$, $\mathrm{ku}_* = \mathbb{Z}[u]$, $A \mapsto 2u^2$ and $B \mapsto u^4$, with $|\eta| = 1$, $|A| = 4$, $|B| = 8$ and $|u| = 2$.

Lemma 2.2. *There exists an $\mathbb{E}_\infty$ ring map $\mathrm{MU} \to \mathrm{ku}$. We can choose the generators of $\mathrm{MU}_* = \pi_*(\mathrm{MU}) = \mathbb{Z}[x_i \mid i \geq 1]$ so that $x_1 \mapsto u$ and $x_i \mapsto 0$ for each $i \geq 2$. Here $|x_i| = 2i$.*

Proof. The proof of [HY20, Theorem 4.3] shows the existence of $\mathbb{E}_\infty$ ring maps $\mathrm{MU} \to \mathrm{MUP} \to \mathrm{KU}$. Passing to connective covers gives the factorization through ku. Both MU and ku have first Postnikov k-invariant the generator of $H^3(H\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}/2$, so $\mathbb{Z}\{x_1\} = \pi_2(\mathrm{MU}) \to \pi_2(\mathrm{ku}) = \mathbb{Z}\{u\}$ must be an isomorphism. Dividing the first choice of x_1 by a unit, we can assume that $x_1 \mapsto u$. For each $i \geq 2$ we can then subtract a multiple of x_1^i from the first choice of x_i to ensure that $x_i \mapsto 0$. $\square$

We hereafter fix $\mathrm{MU} \to \mathrm{ku}$ and the x_i as in the lemma above. This provides $\mathbb{E}_\infty$ ring maps $C^q(\mathrm{ko}/\mathbb{S}) \to C^q(\mathrm{ku}/\mathrm{MU})$, compatibly for all $q \geq 0$. We write

$$\tau_{\geq\star}\colon \mathrm{Sp}_2 \longrightarrow \mathrm{Sp}_2^{\mathrm{fil}}$$

for the monoidal Whitehead filtration [Lur17, Proposition 1.4.3.6, Example 2.2.1.10].

Definition 2.3. For $F \in \{\mathrm{THH}, \mathrm{TC}^-, \mathrm{TP}, \mathrm{THH}^{tC_2}\}$ we define $\mathbb{E}_\infty$ algebras

$$\mathrm{fil}^\star_{\mathrm{mot}} F(\mathrm{ko}) := \mathrm{Tot}\left(\tau_{\geq 2\star} F(C^\bullet(\mathrm{ku}/\mathrm{MU})/C^\bullet(\mathrm{ko}/\mathbb{S}))\right)$$
$$\mathrm{gr}^*_{\mathrm{mot}} F(\mathrm{ko}) := \mathrm{gr}^*\left(\mathrm{fil}^\star_{\mathrm{mot}} F(\mathrm{ko})\right)$$

in $\mathrm{Sp}_2^{\mathrm{fil}}$ and $\mathrm{Sp}_2^{\mathrm{gr}}$, respectively. Here Tot denotes the totalization of a cosimplicial object. We refer to $\mathrm{gr}^w_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko})$ as the weight w *prismatic cohomology* spectrum of ko, and $\pi_* \mathrm{gr}^w_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko})$ as its the weight w prismatic cohomology groups.

Remark 2.4. We will show in Proposition 2.11 that $\mathrm{THH}(C^q(\mathrm{ku}/\mathrm{MU})/C^q(\mathrm{ku}/\mathbb{S}))$ is even, for each $q \geq 0$, from which the corresponding statements with F in place of THH will follow. Hence the double-speed Whitehead filtration in the definition of $\mathrm{fil}^\star_{\mathrm{mot}} F(\mathrm{ko})$ will be well-behaved. As a consequence, we will verify in Proposition 2.13 that $\pi_* \mathrm{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko})$ agrees with prismatic cohomology in the sense of [HRW, Definition 1.2.4], building on [BMS19].

We also fix terminology for gradings.

Definition 2.5. Given $M^* \in \mathrm{Sp}_2^{\mathrm{gr}}$ and $x \in \pi_n M^w$ we say that x has *stem* n, *weight* w and *motivic filtration* $2w - n$, and write $\|x\| = (n, 2w - n)$. In [HRW, Definition 1.3.2], stem and motivic filtration are called degree and Adams weight, respectively. We refer to the (Adams indexed, Bousfield–Kan) spectral sequences

$$E_2^{n,2w-n} = \pi_n \mathrm{gr}^w_{\mathrm{mot}} F(\mathrm{ko}) \Longrightarrow \pi_n F(\mathrm{ko})$$

for $F \in \{\mathrm{THH}, \mathrm{TC}^-, \mathrm{TP}, \mathrm{THH}^{tC_2}\}$ as the *motivic spectral sequences.*

We plot each motivic spectral sequence with stem on the horizontal axis and motivic filtration on the vertical axis. Given this convention, if $\|x\| = (n, 2w - n)$ then $\|d_r(x)\| = (n - 1, 2w - n + r)$. Note that we write $E_r^{n,2w-n}$ where it is also standard to write $E_r^{2w-n,2w}$ in the literature. With these conventions, the motivic spectral sequence $E_r^{n,s}$-term is concentrated in internal degrees $n + s = 2w$ and, consequently, $d_r = 0$ for all even integers $r \geq 2$.

Our first aim is to show that the filtrations from Definition 2.3 agree with the motivic filtrations considered in [HRW, Variant 4.2.2].

Definition 2.6 ([HRW, Definition 2.3.1])**.** A map $A \to B$ of 2-complete $\mathbb{E}_\infty$ rings is *evenly free* if for each nonzero even $\mathbb{E}_\infty$ A-algebra C the pushout $B \otimes_A C$ is equivalent as a C-module to a nonzero wedge sum of even suspensions of C.

Remark 2.7. This condition on $B \otimes_A C$ is equivalent to $\pi_*(B \otimes_A C)$ being a nonzero free C_*-module on generators in even stems. An evenly free map $A \to B$ of 2-complete $\mathbb{E}_\infty$ rings is eff (= evenly faithfully flat), in the sense of [HRW, Definition 2.2.13], as per [HRW, Remark 2.3.2].

Lemma 2.8. *The complexification map* $\mathrm{ko} \to \mathrm{ku}$ *is evenly free. Moreover, for each even* $\mathbb{E}_\infty$ ko*-algebra* C *there is an isomorphism of* C_**-algebras*

$$\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C) \cong C_*[b_1]/(\hat{b}_1^2)\,,$$

where $\hat{b}_1^2 = b_1^2 + c_2 b_1 + c_4$ *for some* $c_2, c_4 \in C_*$ *with* $|c_2| = 2$ *and* $|c_4| = 4$.

Proof. Recall the Wood (homotopy) cofiber sequence

$$\Sigma\mathrm{ko} \xrightarrow{\eta} \mathrm{ko} \xrightarrow{c} \mathrm{ku} \xrightarrow{R} \Sigma^2\mathrm{ko} \tag{2.1}$$

of ko-modules, where we write R for the map satisfying $R \circ (u \cdot -) = \Sigma^2 r$, with $r\colon \mathrm{ku} \to \mathrm{ko}$ the realification map. The resulting cofiber sequence

$$\Sigma C \xrightarrow{\eta} C \longrightarrow \mathrm{ku} \otimes_{\mathrm{ko}} C \xrightarrow{R} \Sigma^2 C$$

induces a short exact sequence of C_*-modules

$$0 \to C_* \longrightarrow \pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C) \xrightarrow{R} \Sigma^2 C_* \to 0\,, \tag{2.2}$$

since C is even so that $\eta\colon \Sigma C_* \to C_*$ is zero. We conclude that $\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C)$ is concentrated in even degrees.

A choice of a class $b_1 \in \pi_2(\mathrm{ku} \otimes_{\mathrm{ko}} C)$ with $R(b_1) = \Sigma^2 1$ is equivalent to a choice of a splitting of (2.2), so that $\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C) \cong C_*\{1, b_1\}$ is free over C_* and nonzero if $C \neq 0$. Writing $\hat{b}_1^2 = b_1^2 + c_2 b_1 + c_4$ for the monic polynomial that vanishes in $\pi_4(\mathrm{ku} \otimes_{\mathrm{ko}} C)$, there is an isomorphism

$$\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C) \cong C_*[b_1]/(\hat{b}_1^2)$$

of C_*-algebras. □

Remark 2.9. Note that c_2 and c_4 in the statement of Lemma 2.8 need not be zero. For example, when $C = \mathrm{ku}$ then b_1 can be chosen as in [DLR22, Lemma 5.1] so that $\hat{b}_1^2 = b_1^2 - u b_1$.

Proposition 2.10. *The map* $\mathrm{THH}(\mathrm{ko}) \to \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$*, induced by the complexification map* $c\colon \mathrm{ko} \to \mathrm{ku}$ *and the unit map* $\mathbb{S} \to \mathrm{MU}$*, is evenly free.*

Proof. Let $C \neq 0$ be an even $\mathbb{E}_\infty$ THH(ko)-algebra. Then $C = \mathrm{THH}(C/C)$ and

$$\begin{aligned}\mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \otimes_{\mathrm{THH}(\mathrm{ko})} C &= \mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \otimes_{\mathrm{THH}(\mathrm{ko})} \mathrm{THH}(C/C) \\ &\simeq \mathrm{THH}(\mathrm{ku} \otimes_{\mathrm{ko}} C/\mathrm{MU} \otimes C)\,,\end{aligned}$$

where the last equivalence holds because THH commutes with pushouts of $\mathbb{E}_\infty$ rings. Since C is an even $\mathbb{E}_\infty$ ring, the Atiyah–Hirzebruch spectral sequence

$$E^2 = H_*(\mathrm{MU}; C_*) \Longrightarrow \pi_*(\mathrm{MU} \otimes C)$$

collapses, and we can choose generators $b_k \in \pi_{2k}(\mathrm{MU} \otimes C)$ giving an isomorphism $C_*[b_k \mid k \geq 1] \cong \pi_*(\mathrm{MU} \otimes C)$ of C_*-algebras. We claim that these generators may be chosen so that $\pi_*(\mathrm{MU} \otimes C) \to \pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C) = C_*[b_1]/(\hat{b}_1^2)$ satisfies $b_1 \mapsto b_1$ and $b_k \mapsto 0$ for each $k \geq 2$, where b_1 and $\hat{b}_1^2$ in the target are fixed as in Lemma 2.8.

To see this, consider the commutative diagram

$$\begin{tikzcd}
\pi_2(\mathrm{MU} \otimes \tau_{\geq 0}C) \arrow[r, "\cong"] \arrow[d] & \pi_2(\mathrm{ku} \otimes_{\mathrm{ko}} \tau_{\geq 0}C) \arrow[d, "R"] \\
\pi_2(\overline{\mathrm{MU}} \otimes \tau_{\geq 0}C) \arrow[r, "\cong"] & \pi_2(\Sigma^2 \tau_{\geq 0}C)\,,
\end{tikzcd}$$

where $\overline{\mathrm{MU}}$ denotes the cofiber of the unit map $\mathbb{S} \to \mathrm{MU}$. The horizontal maps are isomorphisms because $\mathrm{MU} \to \mathrm{ku}$ is 4-connected and $\mathbb{S} \to \mathrm{ko}$ is 3-connected. We know that $\pi_*(\mathrm{MU} \otimes \tau_{\geq 0}C) \cong \pi_*(\tau_{\geq 0}C)[b_k \mid k \geq 1]$, because $\tau_{\geq 0}C$ is an even $\mathbb{E}_\infty$ ring. The vertical maps both take b_1 to a $\pi_0(C)$-module generator of their respective targets. Consequently, we can adjust our choice of generator $b_1 \in \pi_2(\mathrm{MU} \otimes \tau_{\geq 0}C)$ by dividing by a unit in $\pi_0(C)$ so as to ensure that it maps to the previously chosen $b_1 \in \pi_2(\mathrm{ku} \otimes_{\mathrm{ko}} \tau_{\geq 0}C)$. We then let $b_1 \in \pi_2(\mathrm{MU} \otimes C)$ be the image of $b_1 \in \pi_2(\mathrm{MU} \otimes \tau_{\geq 0}C)$ via $\tau_{\geq 0}C \to C$. Thereafter, we adjust the algebra generators b_k for $k \geq 2$ so that they map to zero in $\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C)$, by subtracting a suitable degree one polynomial in b_1 from each original choice of generator.

To determine THH of $\mathrm{ku} \otimes_{\mathrm{ko}} C$ relative to $\mathrm{MU} \otimes C$, we first study the Künneth (or Tor) spectral sequence

$$\begin{aligned} E^2 &= \mathrm{Tor}_*^{C_*[b_k|k\geq 1]}(C_*[b_1]/(\hat{b}_1^2), C_*[b_1]/(\hat{b}_1^2)) \\ &\Longrightarrow \pi_*((\mathrm{ku} \otimes_{\mathrm{ko}} C) \otimes_{(\mathrm{MU}\otimes C)} (\mathrm{ku} \otimes_{\mathrm{ko}} C))\,. \end{aligned}$$

Since $\hat{b}_1^2$ and b_k for $k \geq 2$ act trivially on $C_*[b_1]/(\hat{b}_1^2)$, we compute that

$$E^2 \cong C_*[b_1]/(\hat{b}_1^2) \otimes \Lambda(\sigma\hat{b}_1^2, \sigma b_k \mid k \geq 2)$$

with $\sigma\hat{b}_1^2, \sigma b_k \in \mathrm{Tor}_1$, where the triviality of the square $(\sigma\hat{b}_1^2)^2$ follows from the fact that the product in Tor is given by the shuffle product. Since the algebra generators are all in filtration ≤ 1, the Künneth spectral sequence collapses at the E^2-term. Note that this is a homological spectral sequence associated to an increasing filtration. Since the E^∞-term is a non-zero free C_*-module and the abutment is also a C_*-module, we conclude that the abutment is a free and non-zero C_*-module.

We can rule out all potential hidden multiplicative extensions, because Tor_0 splits off from the abutment and all the classes in Künneth filtration 1 are in odd total degree. Therefore, we have an algebra isomorphism

$$D_* := \pi_*((\mathrm{ku} \otimes_{\mathrm{ko}} C) \otimes_{(\mathrm{MU}\otimes C)} (\mathrm{ku} \otimes_{\mathrm{ko}} C)) \cong C_*[b_1]/(\hat{b}_1^2) \otimes \Lambda(\sigma\hat{b}_1^2, \sigma b_k \mid k \geq 2)\,.$$

Second, we apply the Künneth spectral sequence

$$E^2 = \mathrm{Tor}_*^{D_*}(C_*[b_1]/(\hat{b}_1^2), C_*[b_1]/(\hat{b}_1^2)) \Longrightarrow \pi_* \mathrm{THH}(\mathrm{ku} \otimes_{\mathrm{ko}} C/\mathrm{MU} \otimes C)\,.$$

The E^2-term of this spectral sequence, namely

$$E^2 = C_*[b_1]/(\hat{b}_1^2) \otimes \Gamma(\sigma^2\hat{b}_1^2, \sigma^2 b_k \mid k \geq 2)\,,$$

is concentrated in even degrees, so the spectral sequence collapses at the E^2-term and the abutment is also concentrated in even degrees. Since the E^∞-term is a non-zero free C_*-module and the abutment is also a C_*-module, we conclude that the abutment is a free and non-zero C_*-module. □

Proposition 2.11. *For each $q \geq 0$, the spectrum $C^q(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\mathrm{THH}(\mathrm{ko}))$ is an even $\mathbb{E}_\infty$ ku-algebra whose homotopy groups are free as a ku_*-module.*

Proof. By Proposition 2.10 and induction, it suffices to show that $\pi_* \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$ is free as a ku_*-module and concentrated in even degrees. The Künneth spectral sequence

$$\begin{aligned} E^2 &= \mathrm{Tor}_*^{\mathrm{MU}_*}(\mathrm{ku}_*, \mathrm{ku}_*) = \mathrm{Tor}_*^{\mathbb{Z}[x_i \mid i \geq 1]}(\mathbb{Z}[u], \mathbb{Z}[u]) \\ &= \mathbb{Z}[u] \otimes \Lambda(\sigma x_i \mid i \geq 2) \Longrightarrow \pi_*(\mathrm{ku} \otimes_{\mathrm{MU}} \mathrm{ku}) \end{aligned}$$

collapses at the E^2-term. Again, we can rule out all hidden multiplicative extensions, because Tor_0 splits off from the abutment and all classes in Künneth filtration 1 are in odd total degrees. Next we apply the Künneth spectral sequence

$$\begin{aligned} E^2 &= \mathrm{Tor}_*^{\pi_*(\mathrm{ku} \otimes_{\mathrm{MU}} \mathrm{ku})}(\mathrm{ku}_*, \mathrm{ku}_*) = \mathrm{Tor}_*^{\mathbb{Z}[u] \otimes \Lambda(\sigma x_i \mid i \geq 2)}(\mathbb{Z}[u], \mathbb{Z}[u]) \\ &= \mathbb{Z}[u] \otimes \Gamma(\sigma^2 x_i \mid i \geq 2) \Longrightarrow \pi_* \mathrm{THH}(\mathrm{ku}/\mathrm{MU})\,. \end{aligned}$$

This E^2-term is also concentrated in even degrees, and is free over $\mathbb{Z}[u] = \mathrm{ku}_*$. Hence $E^2 = E^\infty$, and the abutment is free over ku_* on even degree generators. □

The following fact is stated without proof in [HRW, Example 4.1.5]. We follow this reference and write $L^{\mathrm{alg}}_{B/A}$ for the algebraic cotangent complex, with homological grading, of a homomorphism $A \to B$ of (ungraded) commutative rings.

Proposition 2.12. $\mathrm{ko}_*\mathrm{MU} \cong \mathbb{Z}[u, b_1^2, \bar{b}_k \mid k \geq 2]$ *is a polynomial ring on generators in dimensions 2, 4 and $2k$ for $k \geq 2$. Hence $L^{\mathrm{alg}}_{\mathrm{ko}_*\mathrm{MU}/\mathrm{MU}_*}$ has (2-complete)* Tor*-amplitude contained in* $[0, 1]$*, so that* $\mathrm{MU} \to \mathrm{ko} \otimes \mathrm{MU}$ *is (2-)quasi-lci.*

Proof. Since MU_* is free over BP_*, we have a ring isomorphism $\mathrm{ko}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathrm{MU}_* \cong \mathrm{ko}_*\mathrm{MU}$. Here $H_*(\mathrm{ko}) = \mathcal{A}^\vee \,\square_{\mathcal{A}(1)^\vee}\, \mathbb{F}_2$, so $H_*(\mathrm{ko} \otimes \mathrm{BP}) \cong \mathcal{A}^\vee \,\square_{\mathcal{A}(1)^\vee}\, H_*(\mathrm{BP})$ and the E_2-term of the Adams spectral sequence

$${}^{\mathrm{Ad}}E_2 = \mathrm{Ext}_{\mathcal{A}^\vee}(\mathbb{F}_2, H_*(\mathrm{ko} \otimes \mathrm{BP})) \Longrightarrow \pi_*(\mathrm{ko} \otimes \mathrm{BP})$$

can be rewritten as $\mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2, H_*(\mathrm{BP}))$. Since $H_*(\mathrm{BP}) = \mathbb{F}_2[\xi_k^2 \mid k \geq 1]$ is concentrated in even degrees, its $\mathcal{A}(1)^\vee$-coaction factors as

$$\begin{aligned} H_*(\mathrm{BP}) &\xrightarrow{\nu} \Lambda(\xi_1^2) \otimes H_*(\mathrm{BP}) \subset \mathcal{A}(1)^\vee \otimes H_*(\mathrm{BP}) \\ \xi_k^2 &\longmapsto \begin{cases} 1 \otimes \xi_1^2 + \xi_1^2 \otimes 1 & \text{for } k = 1, \\ 1 \otimes \xi_k^2 & \text{for } k \geq 2. \end{cases} \end{aligned}$$

To calculate the Adams E_2-term we use the Cartan–Eilenberg spectral sequence

$${}^{\mathrm{CE}}E_2 = \mathrm{Ext}_{\Lambda(\xi_1^2)}(\mathbb{F}_2, \mathrm{Ext}_{\Lambda(\xi_1, \xi_2)}(\mathbb{F}_2, H_*(\mathrm{BP}))) \Longrightarrow \mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2, H_*(\mathrm{BP}))$$

(cf. [CE56, Theorem XVI.6.1]) for the Hopf algebra extension $\Lambda(\xi_1^2) \to \mathcal{A}(1)^\vee \to \Lambda(\xi_1, \xi_2)$. Its E_2-term

$${}^{\mathrm{CE}}E_2 \cong \mathrm{Ext}_{\Lambda(\xi_1^2)}(\mathbb{F}_2, \mathbb{F}_2[v_0, v_1] \otimes H_*(\mathrm{BP})) = \mathbb{F}_2[v_0, v_1] \otimes \mathbb{F}_2[\xi_1^4, \xi_k^2 \mid k \geq 2]$$

is concentrated in filtration degree 0, so ${}^{\mathrm{CE}}E_2 = {}^{\mathrm{CE}}E_\infty$. Hence the Adams E_2-term is concentrated in even stems, so also ${}^{\mathrm{Ad}}E_2 = {}^{\mathrm{Ad}}E_\infty$. Thus $\pi_*(\mathrm{ko} \otimes \mathrm{BP}) \cong \mathbb{Z}[u, t_1^2, \bar{t}_k \mid k \geq 2]$, with 2 detected by v_0, u detected by v_1, t_1^2 detected by ξ_1^4 and

$\bar{t}_k$ detected by ξ_k^2. Base change along $\mathrm{BP} \to \mathrm{MU}$ replaces the degree $2(2^k-1)$ generators $\bar{t}_k$ for $k \neq 1$ with degree $2k$ polynomial generators $\bar{b}_k$ for $k \neq 1$, while t_1^2 is replaced by b_1^2.

Since MU_* and $\mathrm{ko}_*\mathrm{MU}$ are polynomial rings, the algebraic cotangent complexes $L^{\mathrm{alg}}_{\mathrm{MU}_*/\mathbb{Z}}$ and $L^{\mathrm{alg}}_{\mathrm{ko}_*\mathrm{MU}/\mathbb{Z}}$ are free modules over MU_* and $\mathrm{ko}_*\mathrm{MU}$, respectively. By the transitivity cofiber sequence

$$\mathrm{ko}_*\mathrm{MU} \otimes_{\mathrm{MU}_*} L^{\mathrm{alg}}_{\mathrm{MU}_*/\mathbb{Z}} \longrightarrow L^{\mathrm{alg}}_{\mathrm{ko}_*\mathrm{MU}/\mathbb{Z}} \longrightarrow L^{\mathrm{alg}}_{\mathrm{ko}_*\mathrm{MU}/\mathrm{MU}_*}$$

associated to the ring homomorphisms $\mathbb{Z} \to \mathrm{MU}_* \to \mathrm{ko}_*\mathrm{MU}$, cf. [Qui70, 5.1], it follows that $L^{\mathrm{alg}}_{\mathrm{ko}_*\mathrm{MU}/\mathrm{MU}_*}$ has (2-complete) Tor-amplitude concentrated in homological degrees $[0,1]$. By definition [HRW, Definition 4.1.1(2)], this means that $\mathrm{MU} \to \mathrm{ko} \otimes \mathrm{MU}$ is (2-)quasi-lci. □

Proposition 2.13. *The filtrations from Definition 2.3 agree with the motivic filtrations considered in* [HRW, Definition 4.2.1, Variant 4.2.2]*. Moreover there exist maps*

$$\mathrm{can}\,,\, \varphi\colon\ \mathrm{fil}^\star_{\mathrm{mot}}\, \mathrm{TC}^-(\mathrm{ko}) \longrightarrow \mathrm{fil}^\star_{\mathrm{mot}}\, \mathrm{TP}(\mathrm{ko})$$

of $\mathbb{E}_\infty$ algebras in $\mathrm{Sp}_2^{\mathrm{fil}}$, which converge to the canonical map and Frobenius map from [NS18]*.*

Proof. By Proposition 2.10 we can apply [HRW, Corollary 2.2.14] to $A = M = \mathrm{THH}(\mathrm{ko})$ and $B = \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$. In this case $M \otimes_A B^{\otimes_A q+1} = B^{\otimes_A q+1} = C^q(B/A)$, which is even by Proposition 2.11, so the totalization of the double-speed Whitehead filtration agrees with the even filtration.

The map of connective $\mathbb{E}_\infty$ rings $\mathbb{S} \to \mathrm{ko}$ is chromatically 2-quasi-lci in the sense of [HRW, Definition 4.1.4], because by Proposition 2.12 we know that both MU and $\mathrm{ko} \otimes \mathrm{MU}$ are even, and the map $\mathrm{MU} \to \mathrm{ko} \otimes \mathrm{MU}$ is 2-quasi-lci. Hence the even filtrations also define the motivic filtrations, as per [HRW, Variant 4.2.2], and this confirms the first statement.

The second statement then follows directly from [HRW, Theorem 4.2.10]. □

Definition 2.14. In light of Proposition 2.13, we define $\mathbb{E}_\infty$ algebras

$$\begin{aligned} \mathrm{fil}^\star_{\mathrm{mot}}\, \mathrm{TC}(\mathrm{ko}) &:= \mathrm{eq}(\mathrm{can}\,,\, \varphi\colon\ \mathrm{fil}^\star_{\mathrm{mot}}\, \mathrm{TC}^-(\mathrm{ko}) \longrightarrow \mathrm{fil}^\star_{\mathrm{mot}}\, \mathrm{TP}(\mathrm{ko})) \\ \mathrm{gr}^*_{\mathrm{mot}}\, \mathrm{TC}(\mathrm{ko}) &:= \mathrm{gr}^*\,(\mathrm{fil}^\star_{\mathrm{mot}}\, \mathrm{TC}(\mathrm{ko})) \end{aligned}$$

in $\mathrm{Sp}_2^{\mathrm{fil}}$ and $\mathrm{Sp}_2^{\mathrm{gr}}$, respectively. By taking the (homotopy) equalizer of can and φ we retain the multiplicative structure, but additively $\mathrm{fil}^\star_{\mathrm{mot}}\, \mathrm{TC}(\mathrm{ko})$ is also the (homotopy) fiber of $\mathrm{can} - \varphi$. In light of [HRW, §5] and [BMS19] we refer to $\mathrm{gr}^w_{\mathrm{mot}}\, \mathrm{TC}(\mathrm{ko})$ and $\pi_*\, \mathrm{gr}^w_{\mathrm{mot}}\, \mathrm{TC}(\mathrm{ko})$ as the weight w *syntomic cohomology* spectrum and syntomic cohomology groups of ko, refer to the spectral sequence

$$E_2^{n,2w-n} = \pi_n\, \mathrm{gr}^w_{\mathrm{mot}}\, \mathrm{TC}(\mathrm{ko}) \Longrightarrow \pi_n\, \mathrm{TC}(\mathrm{ko})$$

as the motivic spectral sequence, and follow the same grading conventions as in Definition 2.5.

We now introduce the relevant type 2 finite coefficient spectra. Let $V(0) = C2$ denote the cofiber of the map $2\colon \mathbb{S} \to \mathbb{S}$ and let $C\eta$ denote the cofiber of the Hopf map $\eta\colon \Sigma\mathbb{S} \to \mathbb{S}$. By [DM81, Proposition 2.1], there are precisely four equivalence classes of finite spectra $X = (V(0) \otimes C\eta)/v_1$ with the property that $H^*(X) \cong \mathcal{A}(1)$ as an $\mathcal{A}(1)$-module, each characterized by the Sq^4-action in its mod 2 cohomology.

Notation 2.15. Following [BEM17, §1], we write $A(1)[ij]$ with $i, j \in \{0,1\}$ for the spectrum $(V(0) \otimes C\eta)/v_1$ where Sq^4 on the cohomology generator in degree 0 is i times the generator in degree 4, and Sq^4 on the cohomology generator in degree 2 is j times the generator in degree 6. We write $A(1)$ in place of $A(1)[ij]$ when making statements that hold for any choice of $i, j \in \{0,1\}$. The identity map $A(1) \to A(1)$ has additive order 4, see Lemma 3.5, and [BEM17, Main Theorem] proves that $A(1)$ admits a v_2^{32}-self-map. Hence the $A(1)$-homotopy $A(1)_*Y = \pi_*(A(1) \otimes Y)$ of any spectrum Y is naturally a $\mathbb{Z}/4[v_2^{32}]$-module.

In view of Proposition 2.11 and the fact that $A(1) \otimes \mathrm{ku}$ is even, we know that $A(1) \otimes C^\bullet(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko}))$ is even. This motivates the following definition.

Definition 2.16. Let

$$\mathrm{fil}^\star_{\mathrm{mot}}(A(1) \otimes \mathrm{THH}(\mathrm{ko})) := \mathrm{Tot}\big(\tau_{\geq 2\star}\big(A(1) \otimes C^\bullet(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko}))\big)\big)\,,$$

and write $\mathrm{gr}^*_{\mathrm{mot}}(A(1) \otimes \mathrm{THH}(\mathrm{ko})) := \mathrm{gr}^*\big(\mathrm{fil}^\star_{\mathrm{mot}}(A(1) \otimes \mathrm{THH}(\mathrm{ko}))\big)$ for the associated graded spectrum.

Remark 2.17. The spectrum $A(1) \otimes \mathrm{ko} \simeq H\mathbb{F}_2$ admits a unique $\mathbb{E}_\infty$ ko-algebra structure. Hence, for each $\mathbb{E}_\infty$ ko-algebra R the identification

$$A(1) \otimes R = A(1) \otimes \mathrm{ko} \otimes_{\mathrm{ko}} R \simeq H\mathbb{F}_2 \otimes_{\mathrm{ko}} R$$

lets us regard $A(1) \otimes R$ as an $\mathbb{E}_\infty$ $H\mathbb{F}_2$-algebra. For example, we will identify $\mathrm{THH}(\mathrm{ko}, H\mathbb{F}_2) := H\mathbb{F}_2 \otimes_{\mathrm{ko}} \mathrm{THH}(\mathrm{ko})$ with $A(1) \otimes \mathrm{THH}(\mathrm{ko})$, and may therefore consider the latter as an $\mathbb{E}_\infty$ $H\mathbb{F}_2$-algebra.

Lemma 2.18. *Let $A := A(1) \otimes \mathrm{THH}(\mathrm{ko})$ and $B := A(1) \otimes \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$. Then there is an isomorphism*

$$A_* \cong \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[\mu]$$

of graded $\mathbb{F}_2$-algebras, where $|\lambda_1'| = 5$, $|\lambda_2| = 7$ and $|\mu| = 8$. There is also an isomorphism

$$B_* \cong \Lambda(\hat{\xi}_1^2) \otimes \mathbb{F}_2[\mu] \otimes P$$

of A_*-algebras, where $|\hat{\xi}_1^2| = 2$, $|\mu| = 8$ and P is a polynomial algebra with generators in even degrees. The A_*-algebra structure is determined by the map $A_* \to B_*$ sending λ_1' and λ_2 to zero and mapping μ to μ.

Proof. The complexification map $c\colon \mathrm{ko} \to \mathrm{ku}$ and the unique $\mathbb{E}_\infty$ ring map $\mathrm{ku} \to H\mathbb{F}_2$ induce monomorphisms

$$H_*(\mathrm{ko}) = \mathbb{F}_2[\xi_1^4, \bar{\xi}_2^2, \bar{\xi}_k \mid k \geq 3] \longrightarrow H_*(\mathrm{ku}) = \mathbb{F}_2[\xi_1^2, \bar{\xi}_2^2, \bar{\xi}_k \mid k \geq 3] \longrightarrow \mathcal{A}^\vee$$

of $\mathcal{A}^\vee$-comodule algebras. By Milnor's construction of the ξ_i, the composite map $\mathrm{MU} \to \mathrm{ku} \to H\mathbb{F}_2$ induces the homomorphism $H_*(\mathrm{MU}) = \mathbb{F}_2[b_i \mid i \geq 1] \to H_*(H\mathbb{F}_2) = \mathcal{A}^\vee$ given by $b_{2^i-1} \mapsto \xi_i^2$ and $b_j \mapsto 0$ for $j \neq 2^i - 1$. Hence these formulas also hold in $H_*(\mathrm{ku})$. Let $\bar{b}_i = \chi b_i$ denote the conjugate classes in $H_*(\mathrm{MU})$, so that $\bar{b}_{2^i-1} \mapsto \bar{\xi}_i^2$ and $\bar{b}_j \mapsto 0$ for $j \neq 2^i - 1$. Standard Hochschild homology computations (cf. [MS93, Proposition 2]) yield

$$\mathrm{HH}_*(H_*(\mathrm{ko})) = H_*(\mathrm{ko}) \otimes \Lambda(\sigma\xi_1^4, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_k \mid k \geq 3)$$
$$\mathrm{HH}_*(H_*(\mathrm{ku})) = H_*(\mathrm{ku}) \otimes \Lambda(\sigma\xi_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_k \mid k \geq 3)\,.$$

The usual argument for hidden extensions in the Bökstedt spectral sequence (see e.g. [AR05, Theorem 6.2]) implies that

$$(\sigma\bar{\xi}_3)^{2^{k-3}} = \sigma\bar{\xi}_k\,,$$

and produces identifications

$$\begin{aligned} H_*(\mathrm{THH}(\mathrm{ko})) &\cong H_*(\mathrm{ko}) \otimes \Lambda(\sigma\xi_1^4, \sigma\bar{\xi}_2^2) \otimes \mathbb{F}_2[\sigma\bar{\xi}_3] \\ H_*(\mathrm{THH}(\mathrm{ku})) &\cong H_*(\mathrm{ku}) \otimes \Lambda(\sigma\xi_1^2, \sigma\bar{\xi}_2^2) \otimes \mathbb{F}_2[\sigma\bar{\xi}_3]\,. \end{aligned}$$

As noted above, $H_*(A(1) \otimes \mathrm{ko}) \cong \mathcal{A}^\vee$. By Remark 2.17 and the evident collapsing Künneth spectral sequence we have

$$H_*(A(1) \otimes \mathrm{ku}) \cong H_*(H\mathbb{F}_2 \otimes_{\mathrm{ko}} \mathrm{ku}) \cong \mathcal{A}^\vee \otimes_{H_*(\mathrm{ko})} H_*(\mathrm{ku}) \cong \mathcal{A}^\vee \otimes \Lambda(\hat{\xi}_1^2)\,,$$

where $\hat{\xi}_1^2 := 1 \otimes \xi_1^2 + \xi_1^2 \otimes 1$ denotes the $\mathcal{A}^\vee$-comodule primitive class, so that $A(1)_*(\mathrm{ku}) = \pi_*(A(1) \otimes \mathrm{ku}) = \Lambda(\hat{\xi}_1^2)$. We conclude that

$$\begin{aligned} A_* = A(1)_*\,\mathrm{THH}(\mathrm{ko}) &= \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[\mu] \\ A(1)_*\,\mathrm{THH}(\mathrm{ku}) &= \Lambda(\hat{\xi}_1^2, \lambda_1, \lambda_2) \otimes \mathbb{F}_2[\mu]\,, \end{aligned}$$

with λ_1', λ_2 and μ in $A(1)_*\,\mathrm{THH}(\mathrm{ko})$ having Hurewicz images $\sigma\xi_1^4$, $\sigma\bar{\xi}_2^2 + \xi_1^2 \cdot \sigma\xi_1^4$ and $\sigma\bar{\xi}_3 + \xi_1 \cdot \sigma\bar{\xi}_2^2 + \xi_2 \cdot \sigma\xi_1^4$, respectively, while λ_1, λ_2 and μ in $A(1)_*\,\mathrm{THH}(\mathrm{ku})$ have Hurewicz images $\sigma\xi_1^2$, $\sigma\bar{\xi}_2^2$ and $\sigma\bar{\xi}_3 + \xi_1 \cdot \sigma\bar{\xi}_2^2$, cf. [AR05, Proposition 8.7]. Note that $\sigma\xi_1^4 = \sigma\xi_1^2 \cdot \xi_1^2 + \xi_1^2 \cdot \sigma\xi_1^2 = 0$ in the ku-case. Hence the induced homomorphism $A(1)_*\,\mathrm{THH}(\mathrm{ko}) \to A(1)_*\,\mathrm{THH}(\mathrm{ku})$ is given by $\lambda_1' \mapsto 0$, $\lambda_2 \mapsto \lambda_2$ and $\mu \mapsto \mu$.

Next, we compute $A(1)_*\,\mathrm{THH}(\mathrm{ku}/\mathrm{MU})$, using the fact that

$$A(1) \otimes \mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \simeq (A(1) \otimes \mathrm{THH}(\mathrm{ku})) \otimes_{\mathrm{THH}(\mathrm{MU})} \mathrm{MU}\,. \tag{2.3}$$

We know that

$$\pi_*\,\mathrm{THH}(\mathrm{MU}) \cong \mathrm{MU}_* \otimes \Lambda(\sigma\bar{b}_i \mid i \geq 1)$$

by [MS93, Remark 4.3], cf. [Rog20, Proposition 4.5]. We expand

$$\mathrm{THH}(\mathrm{MU}) \longrightarrow \mathrm{THH}(\mathrm{ku}) \longrightarrow A(1) \otimes \mathrm{THH}(\mathrm{ku})$$

as the composite

$$\begin{aligned} \mathrm{MU} \otimes_{\mathrm{MU}\otimes\mathrm{MU}} \mathrm{MU} &\longrightarrow \mathrm{ku} \otimes_{\mathrm{ku}\otimes\mathrm{ku}} \mathrm{ku} \\ &\longrightarrow A(1) \otimes \mathrm{ku} \otimes_{A(1)\otimes\mathrm{ku}\otimes\mathrm{ku}} A(1) \otimes \mathrm{ku}\,, \end{aligned}$$

where

$$\begin{aligned} \mathrm{MU}_*[\bar{b}_i \mid i \geq 1] = \pi_*(\mathrm{MU} \otimes \mathrm{MU}) &\longrightarrow \pi_*(\mathrm{ku} \otimes \mathrm{ku}) \\ &\longrightarrow \pi_*(A(1) \otimes \mathrm{ku} \otimes \mathrm{ku}) = \Lambda(\hat{\xi}_1^2) \otimes H_*(\mathrm{ku}) \end{aligned}$$

takes $\bar{b}_{2^i-1}$ to $\bar{\xi}_i^2$ for $i \geq 1$ and $\bar{b}_j$ to 0 for $j \neq 2^i - 1$. Hence $\pi_*\,\mathrm{THH}(\mathrm{MU}) \to A(1)_*\,\mathrm{THH}(\mathrm{ku})$ is given by $\sigma\bar{b}_1 \mapsto \sigma\xi_1^2 = \lambda_1$, $\sigma\bar{b}_3 \mapsto \sigma\bar{\xi}_2^2 = \lambda_2$, and $\sigma\bar{b}_i \mapsto 0$ for $i \notin \{1, 3\}$. (This uses that $\sigma\bar{b}_{2^i-1} \mapsto \sigma\bar{\xi}_i^2 = \sigma\bar{\xi}_i \cdot \bar{\xi}_i + \bar{\xi}_i \cdot \sigma\bar{\xi}_i = 0$ for $i \geq 3$, while ξ_1 and $\bar{\xi}_2$ do not exist in $H_*(\mathrm{ku})$.) The $\pi_*\,\mathrm{THH}(\mathrm{MU})$-algebra structure on MU_* is given by mapping x_i to x_i and mapping $\sigma\bar{b}_i$ trivially for all $i \geq 1$. The Künneth spectral sequence associated to (2.3) therefore has E^2-term

$$\Lambda(\hat{\xi}_1^2) \otimes \mathbb{F}_2[\mu] \otimes \Gamma(\sigma^2\bar{b}_{2^i-1} \mid i \geq 3) \otimes \Gamma(\sigma^2\bar{b}_j \mid j \neq 2^i - 1)\,.$$

This spectral sequence is concentrated in even total degrees and therefore collapses at the E^2-term. We resolve the hidden multiplicative extensions using Steinberger's computation [BMMS86, III.2] of the Dyer–Lashof operations on $H_*(H\mathbb{F}_2) = \mathcal{A}^\vee$ and Kochman's computation [Koc73, Theorem 6] of the Dyer–Lashof operations on $H_*(\mathrm{BU}) \cong H_*(\mathrm{MU})$, as in the proof of [HW22, Lemma 2.4.1]. (Note that Steinberger's result is used for the $\sigma^2\bar{b}_{2^i-1}$, while Kochman's theorem is used for the remaining $\sigma^2\bar{b}_j$.) This produces the identification

$$B_* = A(1)_* \,\mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \cong \Lambda(\hat{\xi}_1^2) \otimes \mathbb{F}_2[\mu] \otimes P\,,$$

where

$$P := \mathbb{F}_2[w_i \mid i \geq 0] \otimes \mathbb{F}_2[y_{j,i} \mid j \geq 2 \text{ even}, \, i \geq 0]$$

is a polynomial algebra with algebra generators in even degrees. Here w_i is any choice of lift of $\gamma_{2^i}(\sigma^2\bar{b}_7)$ and $y_{j,i}$ is any choice of lift of $\gamma_{2^i}(\sigma^2\bar{b}_j)$. We conclude that B_* is an even $\mathbb{E}_\infty$ ring, and the A_*-algebra structure is determined by λ_1' and λ_2 mapping trivially for degree reasons and μ mapping to μ by the first half of this proof. □

Corollary 2.19. *The map $A(1) \otimes \mathrm{THH}(\mathrm{ko}) \to A(1) \otimes \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$, induced by the complexification map $c\colon \mathrm{ko} \to \mathrm{ku}$ and the unit map $\mathbb{S} \to \mathrm{MU}$, is evenly free.*

Proof. This can be proven directly using Lemma 2.18, but instead we simply point out that it follows from Proposition 2.10 and Remark 2.17 by base change along $\mathrm{ko} \to H\mathbb{F}_2$, using the following three pushout squares of $\mathbb{E}_\infty$ rings.

$$\begin{tikzcd}
\mathrm{ko} \arrow[r] \arrow[d] & \mathrm{THH}(\mathrm{ko}) \arrow[r] \arrow[d] & \mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \arrow[d] \\
H\mathbb{F}_2 \arrow[r] & A(1) \otimes \mathrm{THH}(\mathrm{ko}) \arrow[r] \arrow[d] & A(1) \otimes \mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \arrow[d] \\
 & C \arrow[r] & \mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \otimes_{\mathrm{THH}(\mathrm{ko})} C
\end{tikzcd}$$

□

Convention 2.20. To be consistent with our implicit 2-completion, we write $\mathrm{fil}^\star_{\mathrm{ev}}$ for the functor denoted $\mathrm{fil}^\star_{\mathrm{ev},2}$ in [HRW, Variant 2.1.7].

Remark 2.21. By [HRW, Corollary 2.2.14], Remark 2.17, and Corollary 2.19, we can identify

$$\mathrm{fil}^\star_{\mathrm{mot}}(A(1) \otimes \mathrm{THH}(\mathrm{ko})) \simeq \mathrm{fil}^\star_{\mathrm{ev}}(A(1) \otimes \mathrm{THH}(\mathrm{ko}))\,,$$

in the sense of [HRW, Remark 2.1.4].

Theorem 2.22. *There is an isomorphism*

$$\pi_* \,\mathrm{gr}^*_{\mathrm{mot}}(A(1) \otimes \mathrm{THH}(\mathrm{ko})) \cong \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[\mu]$$

of bigraded $\mathbb{F}_2$-algebras, with $\|\lambda_1'\| = (5,1)$, $\|\lambda_2\| = (7,1)$ and $\|\mu\| = (8,0)$.

Proof. We closely follow [HW22] and [HRW]. Starting with the proof of [HW22, Proposition 6.1.6], let $A := A(1) \otimes \mathrm{THH}(\mathrm{ko})$ and $B := A(1) \otimes \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$, so that $A_* \cong \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[\mu]$ and $B_* \cong \Lambda(\hat{\xi}_1^2) \otimes \mathbb{F}_2[\mu] \otimes P$ by Lemma 2.18. The descent spectral sequence associated to the cosimplicial Amitsur resolution $C^\bullet(B/A) = B^{\otimes_A \bullet+1}$ for $A \to B$ has E_1-term

$$E_1^q(B/A) = \pi_*(B^{\otimes_A q+1})$$

for $q \geq 0$, and converges to A_*. Since B_* is concentrated in even stems, Corollary 2.19 implies that $\Sigma := \pi_*(B \otimes_A B)$ is even and free over B_*, so that (B_*, Σ) is a flat Hopf algebroid. Let $C^*_\Sigma(B_*, B_*)$ denote the associated cobar complex. It follows by induction on q that the natural homomorphism

$$\begin{aligned} C^q_\Sigma(B_*, B_*) = \Sigma \otimes_{B_*} \cdots \otimes_{B_*} \Sigma \xrightarrow{\cong} \pi_*((B \otimes_A B) \otimes_B \cdots \otimes_B (B \otimes_A B)) \\ \cong \pi_*(B \otimes_A \cdots \otimes_A B) = E_1^q(B/A) \end{aligned}$$

is an isomorphism for each $q \geq 0$, since the relevant Künneth spectral sequences collapse. Passing to cohomology, we obtain an isomorphism

$$\mathrm{Ext}^*_\Sigma(B_*, B_*) \cong E_2^*(B/A)\,,$$

identifying the descent spectral sequence E_2-term with the Hopf algebroid cohomology of (B_*, Σ). We claim that in each stem this E_2-term has the same finite order as A_*, so that the descent spectral sequence for $A \to B$ must collapse at $E_2 = E_\infty$.

By convergence, the descent E_2-term is an upper bound for A_*. To show that the bound is exact, we consider the multiplicative Whitehead filtrations $\tau_{\geq\star}A$ and $\tau_{\geq\star}B$ of A and B, respectively. For each $q \geq 0$ we equip $B^{\otimes_A q+1}$ with the relative convolution filtration

$$\mathrm{fil}^\star\, B^{\otimes_A q+1} = (\tau_{\geq\star}B)^{\otimes_{(\tau_{\geq\star}A)} q+1}\,,$$

having associated graded $\mathbb{E}_\infty$ ring

$$\mathrm{gr}^*\, B^{\otimes_A q+1} = (H\pi_*B)^{\otimes_{(H\pi_*A)} q+1}\,.$$

Here $H\pi_*A$ and $H\pi_*B$ are to be interpreted as the graded $\mathbb{E}_\infty$ rings $\mathrm{gr}^*(\tau_{\geq\star}A)$ and $\mathrm{gr}^*(\tau_{\geq\star}B)$, respectively. We proved in Lemma 2.18 that $A_* \to B_*$ is given by $\lambda_1' \mapsto 0$, $\lambda_2 \mapsto 0$ and $\mu \mapsto \mu$, so that

$$\overline{\Sigma} := \pi_*(H\pi_*B_* \otimes_{H\pi_*A} H\pi_*B) \cong \Gamma(\sigma\lambda_1', \sigma\lambda_2) \otimes \Lambda(\hat{\xi}_1^2) \otimes P \otimes \Lambda(\hat{\xi}_1^2) \otimes P \otimes \mathbb{F}_2[\mu]$$

is even and free over B_*. Hence $(B_*, \overline{\Sigma})$ is a flat Hopf algebroid, and as above we have compatible isomorphisms

$$C^q_{\overline{\Sigma}}(B_*, B_*) = \overline{\Sigma} \otimes_{B_*} \cdots \otimes_{B_*} \overline{\Sigma} \xrightarrow{\cong} (H\pi_*B)^{\otimes_{(H\pi_*A)} q+1}$$

for all $q \geq 0$. Since these bigraded groups are concentrated in even stems, and differentials reduce the stem by one, the convolution filtration spectral sequence

$$\pi_*((H\pi_*B)^{\otimes_{(H\pi_*A)} q+1}) \Longrightarrow \pi_*(B^{\otimes_A q+1})$$

collapses at this term. This proves that $\pi_*(B^{\otimes_A q+1}) = E_1^q(B/A)$ has a descending filtration with associated graded given by $C^q_{\overline{\Sigma}}(B_*, B_*)$. These filtrations are compatible for varying $q \geq 0$, so the descent E_1-term is a filtered differential graded algebra with associated graded $E_1 = C^*_{\overline{\Sigma}}(B_*, B_*)$. Passing to cohomology, we obtain the May–Ravenel spectral sequence

$$E_2 = \mathrm{Ext}_{\overline{\Sigma}}(B_*, B_*) \Longrightarrow \mathrm{Ext}_\Sigma(B_*, B_*)$$

converging to the descent E_2-term, cf. [Rav86, Theorem A1.3.9].

We now view the Hopf algebroid $(B_*, \overline{\Sigma})$ as the tensor product of the three Hopf algebroids

$$(\mathbb{F}_2, \Gamma(\sigma\lambda_1', \sigma\lambda_2)) \quad , \quad (\Lambda(\hat{\xi}_1^2) \otimes P, \Lambda(\hat{\xi}_1^2) \otimes P \otimes \Lambda(\hat{\xi}_1^2) \otimes P) \quad \text{and} \quad (\mathbb{F}_2[\mu], \mathbb{F}_2[\mu])\,.$$

These have cohomology algebras $\Lambda(\lambda_1', \lambda_2)$, $\mathbb{F}_2$ and $\mathbb{F}_2[\mu]$, respectively, with $\lambda_1', \lambda_2 \in \mathrm{Ext}^1$ and $\mu \in \mathrm{Ext}^0$. This confirms that the May–Ravenel E_2-term

$$\mathrm{Ext}_{\overline{\Sigma}}(B_*, B_*) \cong \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[\mu]$$

has the same finite order in each stem as A_*, which implies that the May–Ravenel spectral sequence and the descent spectral sequence both collapse at their E_2-terms. Moreover, there is no room for hidden multiplicative extensions, since λ_1' and λ_2 both square to zero in A_*.

We have now established that the descent spectral sequence

$$E_1^q(B/A) = \pi_*(B^{\otimes_A q+1}) \Longrightarrow A_*$$

is concentrated in even internal degrees $n + q = 2w$, having E_2-term

$$E_2(B/A) \cong \mathrm{Ext}_\Sigma(B_*, B_*) \cong \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[\mu]$$

with (n, q)-bidegrees $\|\lambda_1'\| = (5, 1)$, $\|\lambda_2\| = (7, 1)$ and $\|\mu\| = (8, 0)$. Following [HRW, Example 4.2.3] we apply [HRW, Corollary 2.2.14(1)] to the evenly free map $A \to B$, to see that

$$\mathrm{fil}^\star_{\mathrm{ev}} A \xrightarrow{\simeq} \mathrm{Tot}\big(\mathrm{fil}^\star_{\mathrm{ev}} B^{\otimes_A \bullet+1}\big) = \mathrm{Tot}\big(\tau_{\ge 2\star}(B^{\otimes_A \bullet+1})\big)$$

is an equivalence. For each integer weight w there is a spectral sequence converging to $\pi_* \mathrm{Tot}\big(\tau_{\ge 2w}(B^{\otimes_A \bullet+1})\big)$, with E_1-term given by the part of the descent spectral sequence $E_1(B/A)$ that is located in internal degrees $n+q \ge 2w$. The d_1-differential preserves this part, so the E_2-term for weight w is given by the part of $E_2(B/A)$ in the same range of internal degrees. By naturality, this spectral sequence must collapse at the E_2-term, since the full descent spectral sequence does so. It follows that

$$\pi_* \mathrm{fil}^w_{\mathrm{ev}} A \longrightarrow A_*$$

maps the source isomorphically to the subgroup of classes in internal degree $\ge 2w$, and $\pi_* \mathrm{gr}^w_{\mathrm{ev}} A$ is isomorphic to the summand in A_* consisting of classes in internal degree $= 2w$. Hence

$$\pi_* \mathrm{gr}^*_{\mathrm{ev}} A \cong \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[\mu]$$

as bigraded algebras, with $(n, 2w - n)$-bidegrees $\|\lambda_1'\| = (5, 1)$, $\|\lambda_2\| = (7, 1)$ and $\|\mu\| = (8, 0)$. □

As shown in the proof of [HRW, Corollary 2.2.17], the map $\mathbb{S} \to \mathrm{MU}$ is evenly free and MU is even, so that $\mathrm{fil}^\star_{\mathrm{ev}} \mathbb{S} \simeq \mathrm{Tot}\big(\tau_{\ge 2\star} C^\bullet(\mathrm{MU}/\mathbb{S})\big)$ and

$$\mathrm{gr}^*_{\mathrm{ev}} \mathbb{S} = \mathrm{Tot}\, H\pi_{2*} C^\bullet(\mathrm{MU}/\mathbb{S})\,.$$

Note that the even filtration is symmetric monoidal, so $\mathrm{gr}^*_{\mathrm{ev}} \mathbb{S}$ is an $\mathbb{E}_\infty$ algebra in graded 2-complete spectra, and $\mathrm{gr}^*_{\mathrm{ev}}$ is a lax symmetric monoidal functor from 2-complete spectra to $\mathrm{gr}^*_{\mathrm{ev}} \mathbb{S}$-modules.

Convention 2.23. We will simply write $\otimes$ for $\otimes_{\mathrm{gr}^*_{\mathrm{ev}} \mathbb{S}}$ when it is clear from the context that we are in the category of modules over $\mathrm{gr}^*_{\mathrm{ev}} \mathbb{S}$.

Construction 2.24. For finite spectra V with $\mathrm{MU}_*(V)$ concentrated in even degrees, we shall write $\overline{V}$ for the $\mathrm{gr}^*_{\mathrm{ev}} \mathbb{S}$-module $\mathrm{gr}^*_{\mathrm{ev}} V \simeq \mathrm{Tot}\, H\pi_{2*}(C^\bullet(\mathrm{MU}/\mathbb{S}) \otimes V)$. In particular, this defines the modules $\overline{V}(0)$, $\overline{C}\eta$ and $\overline{A}(1)$.

By [GIKR22], we can identify $\mathrm{gr}^*_{\mathrm{ev}} \mathbb{S}$ with $C\tau$ in the $\mathbb{C}$-motivic homotopy category $\mathrm{SH}(\mathbb{C})$ (cf. [HRW, Remark 1.1.7]). Consequently, the $\mathrm{gr}^*_{\mathrm{ev}} \mathbb{S}$-module $\overline{V}$ corresponds to the even $\mathrm{MU}_*\mathrm{MU}$-comodule $\mathrm{MU}_*(V)$ under the equivalence of [GWX21, Theorem 1.13(2)]. Moreover, by [GWX21, Remark 4.15], this equivalence is symmetric monoidal.

Hence, for each $k \geq 0$ there is an essentially unique $\mathbb{E}_\infty$ $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-algebra $\overline{V}(k)$ that corresponds to the commutative $\mathrm{MU}_*\mathrm{MU}$-comodule algebra $\mathrm{MU}_*/(2, v_1, \dots, v_k)$. For $k = 0$ this gives the previously defined $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-module $\overline{V}(0)$ an $\mathbb{E}_\infty$ algebra structure, even though $V(0) = C2$ is not a ring spectrum, and for $k \geq 1$ it defines the $\mathbb{E}_\infty$ algebras $\overline{V}(k)$, in spite of $V(k)$ not existing as a spectrum. Furthermore, the isomorphism

$$\mathrm{MU}_*(A(1)) \cong \mathrm{MU}_*/(2, v_1) \otimes_{\mathrm{MU}_*} \mathrm{MU}_*(C\eta) \tag{2.4}$$

of $\mathrm{MU}_*\mathrm{MU}$-comodules exhibits $\mathrm{MU}_*(A(1))$ as an $\mathrm{MU}_*/(2, v_1)$-module in that category. It follows that we have an equivalence $\overline{A}(1) \simeq \overline{V}(1) \otimes \overline{C}\eta$, exhibiting $\overline{A}(1)$ as a $\overline{V}(1)$-module in the category of $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-modules. Hence we have a cofiber sequence

$$\Sigma^{1,1}\overline{V}(1) \xrightarrow{\eta} \overline{V}(1) \xrightarrow{i} \overline{A}(1) \xrightarrow{j} \Sigma^{2,0}\overline{V}(1) \tag{2.5}$$

of $\overline{V}(1)$-modules, mapping to the cofiber sequence

$$\Sigma^{1,1}\overline{V}(2) \xrightarrow{\eta} \overline{V}(2) \xrightarrow{i} \overline{V}(2) \otimes \overline{C}\eta \xrightarrow{j} \Sigma^{2,0}\overline{V}(2) \tag{2.6}$$

of $\overline{V}(2)$-modules.

When $\overline{V}$ and M^* are $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-modules, we shall write

$$\overline{V}_* M^* := \pi_*(\overline{V} \otimes M^*)$$

for the homotopy groups of the graded spectrum $\overline{V} \otimes M^*$, keeping in mind that this is a bigraded abelian group. Note that we are applying Convention 2.23 throughout this construction.

Lemma 2.25. *The* $\mathrm{MU}_*\mathrm{MU}$*-comodules* $\mathrm{MU}_*(C\eta)$ *and* $\mathrm{MU}_*(A(1))$ *do not admit* $\mathrm{MU}_*\mathrm{MU}$*-comodule algebra structures.*

Proof. In view of (2.4), it suffices to prove this for $\mathrm{MU}_*(A(1))$. Writing $\mathrm{MU}_*(C\eta) = \mathrm{MU}_*\{1, b_1\}$, so that $\mathrm{MU}_*(A(1)) \cong \mathrm{MU}_*/(2, v_1)\{1, b_1\}$, the coaction

$$\nu \colon \mathrm{MU}_*(A(1)) \longrightarrow \mathrm{MU}_*\mathrm{MU} \otimes_{\mathrm{MU}_*} \mathrm{MU}_*(A(1))$$

satisfies $\nu(b_1) = b_1 \otimes 1 + 1 \otimes b_1$. If $\mathrm{MU}_*(A(1))$ were an $\mathrm{MU}_*\mathrm{MU}$-comodule algebra, we would have $\nu(b_1^2) = (b_1 \otimes 1 + 1 \otimes b_1)^2 = b_1^2 \otimes 1 + b_1 \otimes 2b_1 + 1 \otimes b_1^2$. Since $2b_1 = 0$ and $b_1^2 = 0$ in $\mathrm{MU}_*(A(1))$, this amounts to the contradiction $0 = \nu(0) = b_1^2 \otimes 1 \neq 0$ in $\mathrm{MU}_*\mathrm{MU} \otimes_{\mathrm{MU}_*} \mathrm{MU}_*(A(1)) \cong \mathrm{MU}_*\mathrm{MU}/(2, v_1)\{1, b_1\}$. □

Lemma 2.26. *There is an equivalence*

$$\overline{A}(1) \otimes \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko}) \simeq \mathrm{gr}^*_{\mathrm{mot}}(A(1) \otimes \mathrm{THH}(\mathrm{ko}))$$

of $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$*-modules.*

Proof. Let $v_1 \colon \Sigma^2 V(0) \otimes C\eta \to V(0) \otimes C\eta$ be one of the (eight) v_1-maps with cofiber one of the four spectra $A(1)$. Since 2 and η come from $\pi_*\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$, the structure map

$$\overline{V}(0) \otimes \overline{C}\eta \otimes \mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ku} \xrightarrow{\simeq} \mathrm{gr}^*_{\mathrm{ev}}(V(0) \otimes C\eta \otimes \mathrm{ku})$$

is an equivalence. The cofiber of $\mathrm{gr}^*_{\mathrm{ev}}(v_1) \otimes 1$ acting on the left-hand side is $\overline{A}(1) \otimes \mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ku}$, and the cofiber of $v_1 \otimes 1$ acting on $V(0) \otimes C\eta \otimes \mathrm{ku}$ is $A(1) \otimes \mathrm{ku}$. Since $\mathrm{MU}_*(V(0) \otimes C\eta \otimes \mathrm{ku})$ is concentrated in even degrees, it follows as in [GIKR22, Proposition 3.18] that the cofiber of $\mathrm{gr}^*_{\mathrm{ev}}(v_1 \otimes 1)$ acting on the right-hand side is $\mathrm{gr}^*_{\mathrm{ev}}(A(1) \otimes \mathrm{ku})$. Hence

$$\overline{A}(1) \otimes \mathrm{gr}^*_{\mathrm{ev}} \mathrm{ku} \xrightarrow{\simeq} \mathrm{gr}^*_{\mathrm{ev}}(A(1) \otimes \mathrm{ku})$$

is an equivalence. Using Proposition 2.11, it follows that there are equivalences

$$\begin{aligned}&\overline{A}(1) \otimes \mathrm{gr}^*_{\mathrm{ev}} C^q(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko}))\\ &\qquad\qquad\xrightarrow{\simeq} \mathrm{gr}^*_{\mathrm{ev}}\big(A(1) \otimes C^q(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko}))\big)\end{aligned}$$

for all $q \geq 0$, compatible with the cosimplicial structure maps. Passing to totalizations, and using that $\overline{A}(1)$ is a finite $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-module, we obtain

$$\begin{aligned}&\overline{A}(1) \otimes \mathrm{Tot}\big(\mathrm{gr}^*_{\mathrm{ev}} C^\bullet(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko}))\big)\\ &\qquad \simeq \mathrm{Tot}\big(\overline{A}(1) \otimes \mathrm{gr}^*_{\mathrm{ev}} C^\bullet(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko}))\big)\\ &\qquad\qquad \simeq \mathrm{Tot}\,\mathrm{gr}^*_{\mathrm{ev}}\big(A(1) \otimes C^\bullet(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko}))\big)\,.\end{aligned}$$

In view of Definitions 2.3 and 2.16, this establishes the asserted equivalence. $\square$

Remark 2.27. A consequence of Lemma 2.25 is that $\overline{C}\eta$ and $\overline{A}(1)$ are not $\mathbb{E}_1$ algebras in $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-modules. However, by Lemma 2.26, there is an identification of $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-modules

$$\overline{A}(1) \otimes \mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \simeq \mathrm{gr}^*_{\mathrm{mot}}(A(1) \otimes \mathrm{THH}(\mathrm{ko}))\,,$$

where the right-hand side is an $\mathbb{E}_\infty$ $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-algebra. We therefore use this to equip the left-hand side with an $\mathbb{E}_\infty$ $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-algebra structure. Note that the left-hand side also has a canonical action of the circle $\mathbb{T}$, but this $\mathbb{T}$-action is not an action through $\mathbb{E}_\infty$ ring maps, because the right-hand side is not equipped with a compatible $\mathbb{T}$-action. See Remark 4.7 for an algebraic incarnation of this.

Lemma 2.28. *Let M^* be a $\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}$-module. Then $\overline{C}\eta \otimes M^* \simeq \mathrm{gr}^*_{\mathrm{ev}} \mathrm{ku} \otimes_{\mathrm{gr}^*_{\mathrm{ev}} \mathrm{ko}} M^*$ and there is a natural, trigraded, η-Bockstein spectral sequence*

$$E_1 = (\overline{C}\eta_* M^*)\,[\eta] \Longrightarrow \pi_*(M^*)\,. \tag{2.7}$$

If M^* is uniformly bounded below, then this spectral sequence is conditionally convergent. If M^* is a $\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}$-algebra, then this is an algebra spectral sequence.

Proof. The Wood cofiber sequence (2.1) induces a cofiber sequence

$$\Sigma^{1,1}\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko} \xrightarrow{\eta} \mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko} \xrightarrow{c} \mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ku} \xrightarrow{R} \Sigma^{2,0}\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}$$

in the category of $\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}$-modules, where c is a map of $\mathbb{E}_\infty$ algebras. This follows as in [GIKR22, Proposition 3.18], since $\mathrm{MU}_*(\mathrm{ko})$ is concentrated in even degrees by Proposition 2.12. Hence $\overline{C}\eta \otimes M^*$ and $\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ku} \otimes_{\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}} M^*$ are both the cofiber of $\eta\colon \Sigma^{1,1} M^* \to M^*$.

The Bousfield–Kan homotopy (= descent) spectral sequence for the cosimplicial $\mathbb{E}_\infty$ $\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}$-algebra

$$C^\bullet(\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ku}/\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko})$$

is well-known to be multiplicative, and converges conditionally (and strongly) to $\pi_*\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}$. The normalized Tot-filtration is the same as the η-adic tower

$$\dots \xrightarrow{\eta} \Sigma^{2,2}\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko} \xrightarrow{\eta} \Sigma^{1,1}\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko} \xrightarrow{\eta} \mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}\,,$$

so we can equally well call this the η-Bockstein spectral sequence. In particular, its E_1-term is

$$E_1^q = \Sigma^{q,q}\pi_* \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ku} \cong (\pi_* \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ku})\{\eta^q\}$$

for each $q \geq 0$, and is concentrated in even internal degrees (= integer weights).

Tensoring over $\operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko}$ with M^*, the Bousfield–Kan spectral sequence for the cosimplicial $\operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko}$-module

$$C^\bullet(\operatorname{gr}^*_{\mathrm{ev}} \mathrm{ku}/ \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko}) \otimes_{\operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko}} M^*$$

has abutment $\pi_* M^*$, and is multiplicative if M^* is a $\operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko}$-algebra. The normalized Tot-filtration is the same as the η-adic tower

$$\dots \xrightarrow{\eta} \Sigma^{2,2} M^* \xrightarrow{\eta} \Sigma^{1,1} M^* \xrightarrow{\eta} M^* ,$$

and the E_1-term is

$$E_1^* \cong \pi_*(\overline{C}\eta \otimes M^*)\,[\eta]\,.$$

If M^* is uniformly bounded below, then its η-adic tower has trivial (homotopy) limit, which ensures conditional convergence. □

Example 2.29. The η-Bockstein spectral sequence

$$E_1 = (\pi_* \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ku})\,[\eta] \Longrightarrow \pi_* \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko}$$

has $E_1 = \mathbb{Z}[\eta, u]$, $d_1(u) = 2\eta$ and $E_2 = E_\infty = \mathbb{Z}[\eta, u^2]/(2\eta) \cong \pi_* \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko}$. The motivic (= Novikov) spectral sequence

$$E_2 = \pi_* \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko} \Longrightarrow \pi_* \mathrm{ko}$$

has $E_2 = \mathbb{Z}[\eta, u^2]/(2\eta)$ with $\|\eta\| = (1,1)$, $\|u^2\| = (4,0)$, $d_3(u^2) = \eta^3$ and $E_4 = E_\infty = (\mathbb{Z}\{1, 2u^2\} \oplus \mathbb{Z}/2\{\eta, \eta^2\}) \otimes \mathbb{Z}[u^4]$. Here $A \in \pi_4(\mathrm{ko})$ and $B \in \pi_8(\mathrm{ko})$ are detected by $2u^2$ and u^4, respectively. The bigraded homotopy rings of the $\mathbb{E}_\infty$ $\operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko}$-algebras

$$\begin{tikzcd}
\operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko} \arrow[r, "i"] \arrow[d, "i_0"] & \overline{C}\eta \otimes \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko} \simeq \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ku} \arrow[d, "i_0"] \\
\overline{V}(0) \otimes \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko} \arrow[r, "i"] \arrow[d, "i_1"] & \overline{V}(0) \otimes \overline{C}\eta \otimes \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko} \simeq \overline{V}(0) \otimes \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ku} \arrow[d, "i_1"] \\
\overline{V}(1) \otimes \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko} \arrow[r, "i"] & \overline{A}(1) \otimes \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko} \simeq \overline{V}(1) \otimes \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ku}
\end{tikzcd}$$

are thus

$$\begin{tikzcd}
\mathbb{Z}[\eta, u^2]/(2\eta) \arrow[r] \arrow[d] & \mathbb{Z}[u] \arrow[d] \\
\mathbb{Z}/2[\eta, u] \arrow[r] \arrow[d] & \mathbb{Z}/2[u] \arrow[d] \\
\mathbb{Z}/2[\eta] \arrow[r] & \mathbb{Z}/2\,.
\end{tikzcd}$$

Definition 2.30. Let $v_2\colon \Sigma^{6,0}\overline{V}(1) \to \overline{V}(1)$ be the $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-module map corresponding to the $\mathrm{MU}_*\mathrm{MU}$-comodule homomorphism $v_2\colon \Sigma^6\mathrm{MU}_*/(2,v_1) \to \mathrm{MU}_*/(2,v_1)$, so that there is a cofiber sequence

$$\Sigma^{6,0}\overline{V}(1) \xrightarrow{v_2} \overline{V}(1) \xrightarrow{i_2} \overline{V}(2) \xrightarrow{j_2} \Sigma^{7,-1}\overline{V}(1)$$

of $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-modules. The induced map

$$v_2\colon \Sigma^{6,0}\overline{V}(1)\otimes \mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko} \longrightarrow \overline{V}(1)\otimes \mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}$$

of $\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}$-modules is null-homotopic, and there is a unique class

$$\varepsilon_2 \in \overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}$$

in bidegree $\|\varepsilon_2\| = (7,-1)$ with $j_2(\varepsilon_2) = \Sigma^{7,-1}1$. We have $\varepsilon_2^2 = 0$, since the group in bidegree $(14,-2)$ is trivial. Then $\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko} \cong \Lambda(\varepsilon_2)\otimes\overline{V}(1)_*\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}$, and in general we have a natural algebra isomorphism

$$\Lambda(\varepsilon_2)\otimes\overline{V}(1)_*M^* \cong \overline{V}(2)_*M^*$$

for any $\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}$-algebra M^*.

Corollary 2.31. *There are preferred isomorphisms of bigraded $\mathbb{F}_2$-algebras*

$$\overline{A}(1)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko}) \cong \Lambda(\lambda_1',\lambda_2)\otimes\mathbb{F}_2[\mu]$$

and

$$(\overline{V}(2)\otimes\overline{C}\eta)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko}) \cong \Lambda(\varepsilon_2,\lambda_1',\lambda_2)\otimes\mathbb{F}_2[\mu]\,.$$

Proof. The first isomorphism is a direct consequence of Theorem 2.22, Lemma 2.26 and Remark 2.27. The second isomorphism arises as in Definition 2.30 with $M^* = \overline{C}\eta\otimes\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})$. □

We now apply Lemma 2.28 with $M^* = \overline{V}(1)\otimes\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})$ and $\overline{C}\eta\otimes M^* \cong \overline{A}(1)\otimes\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})$.

Proposition 2.32. *The η-Bockstein spectral sequence*

$$E_1 = \overline{A}(1)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})\,[\eta] \Longrightarrow \overline{V}(1)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko}) \tag{2.8}$$

has differentials

$$d_1(\lambda_2) = \eta\lambda_1'$$
$$d_3(\lambda_1'\lambda_2) = \eta^3\mu$$

and no further differentials besides those generated by the Leibniz rule. Moreover, there is no room for η-extensions. Consequently, we can identify

$$\overline{V}(1)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko}) \cong \frac{\mathbb{F}_2[\eta,\lambda_1',\mu]}{(\eta\lambda_1', (\lambda_1')^2 = c\cdot\eta^2\mu, \eta^3\mu)}$$

as a bigraded $\mathbb{F}_2$-algebra, where $\|\eta\| = (1,1)$, $\|\mu\| = (8,0)$ and $\|\lambda_1'\| = (5,1)$ and $c\in\mathbb{F}_2$.

Proof. We deduce these differentials using a small part of the known (implicitly 2-complete) computation of $\pi_*\,\mathrm{THH}(\mathrm{ko})$ from [AHL10, §7]. The unit $\mathrm{ko}\to\mathrm{THH}(\mathrm{ko})$ and augmentation $\epsilon\colon \mathrm{THH}(\mathrm{ko})\to\mathrm{ko}$ exhibit ko as a retract of THH(ko) in the category of $\mathbb{E}_\infty$ rings. We write THH(ko)/ko for the complementary summand in ko-modules. In degrees $* < 12$ we have $H_*(\mathrm{ko})\{\sigma\bar\xi_1^4, \sigma\bar\xi_2^2, \sigma\bar\xi_3\} \cong H_*(\mathrm{THH}(\mathrm{ko})/\mathrm{ko})$, so there is an 11-connected map $\Sigma^5\mathrm{ksp} \simeq \mathrm{ko}\otimes(S^5\cup_\eta e^7\cup_2 e^8) \to \mathrm{THH}(\mathrm{ko})/\mathrm{ko}$.

By [AHL10, Corollary 7.3, Figure 5], the η^2-multiple in $\pi_6\mathrm{ksp} \cong \pi_{11}\Sigma^5\mathrm{ksp}$ maps to zero in THH, so $\pi_*(\mathrm{THH(ko)/ko}) \cong (\mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}/2, 0)$ for $5 \le * \le 11$.

We consider the η-Bockstein spectral sequence

$$E_1 = \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[\eta, \mu] \Longrightarrow \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)}$$

with $\|\lambda_1'\| = (5, 1)$, $\|\lambda_2\| = (7, 1)$, $\|\mu\| = (8, 0)$ and $\|\eta\| = (1, 1)$, the v_1-Bockstein spectral sequence

$$E_1 = \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)}\,[v_1] \to \overline{V}(0)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)}$$

with $\|v_1\| = (2, 0)$, the v_0-Bockstein spectral sequence

$$E_1 = \overline{V}(0)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)}\,[v_0] \Longrightarrow \pi_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)}$$

with $\|v_0\| = (0, 0)$, and the motivic spectral sequence

$$E_2 = \pi_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)} \Longrightarrow \pi_* \mathrm{THH(ko)}\,.$$

In each case the spectral sequence for ko splits off as a direct summand. Taking this into account, there is no possible source or target for a differential affecting λ_1' in any of these spectral sequences. Hence λ_1' survives in bidegree $(5, 1)$ to detect the generator of $\pi_5(\mathrm{THH(ko)/ko}) \cong \mathbb{Z}$. Since $\pi_6(\mathrm{THH(ko)/ko}) = 0$, it follows that $\eta\lambda_1'$ in bidegree $(6, 2)$ is an infinite cycle that detects zero, i.e., a boundary in one of these spectral sequences. Since $\eta\lambda_1'$ is not a v_1- or v_0-multiple, it cannot be a v_1-Bockstein or v_0-Bockstein boundary. Since the motivic E_2-term is readily seen to be zero in bidegree $(7, 0)$, it can also not be a motivic boundary. Hence $d_1(\lambda_2) = \eta\lambda_1'$ in the η-Bockstein spectral sequence is the only remaining possibility.

There is no room for other η-Bockstein d_1-differentials, so the next differential to be determined is $d_3(\lambda_1'\lambda_2) \in \mathbb{F}_2\{\eta^3\mu\}$. On one hand, if $d_3(\lambda_1'\lambda_2) = \eta^3\mu$ then the η-Bockstein E_∞-term (modulo the summand for ko) will be

$$\mathbb{F}_2\{\lambda_1', \mu, \eta\mu, \eta^2\mu\}$$

in stems ≤ 12. On the other hand, if $d_3(\lambda_1'\lambda_2) = 0$ then it will be

$$\mathbb{F}_2\{\lambda_1', \mu, \eta\mu, \eta^2\mu, \eta^3\mu\}$$

in stems ≤ 11, with the 12-stem concentrated in motivic filtrations ≥ 2. In either case this determines $\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)}$ in these stems.

The first nonzero v_1-Bockstein differential is $d_1(\mu) = v_1\lambda_1'$. If it were not there, then $v_1\lambda_1'$ would survive to $\overline{V}(0)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)}$ and $\pi_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)}$ to detect a nonzero class in $\pi_7(\mathrm{THH(ko)/ko}) = 0$, which is impossible. There is no room for other v_1-Bockstein differentials affecting stems ≤ 11, so if $d_3(\lambda_1'\lambda_2) = \eta^3\mu$ then the v_1-Bockstein E_∞-term (modulo the summand for ko) will be

$$\mathbb{F}_2\{\lambda_1', \eta\mu, \eta^2\mu, v_1\eta\mu\}$$

in stems ≤ 11, while if $d_3(\lambda_1'\lambda_2) = 0$ then it will be

$$\mathbb{F}_2\{\lambda_1', \eta\mu, \eta^2\mu, \eta^3\mu, v_1\eta\mu\}$$

in these stems. In either case the 12-stem is concentrated in motivic filtrations ≥ 2, and these expressions determine $\overline{V}(0)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)}$ in this range of stems.

In the v_0-Bockstein spectral sequence, there is no room for differentials on (λ_1' and) $\eta\mu$. Multiplying by η^2, it follows that $\eta^3\mu$ is an infinite cycle (but possibly zero). Since it is not a v_0-multiple, it cannot be a v_0-Bockstein boundary, and since it is in motivic filtration 3, and the motivic E_2-term is now known to be zero in bidegrees $(12, 0)$ and $(12, 1)$, it cannot be a motivic d_r-boundary for $r \ge 2$.

Hence if $d_3(\lambda_1'\lambda_2)$ were zero, then $\eta^3\mu$ would survive to $\overline{V}(0)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})$ and $\pi_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})$ to detect a nonzero class in $\pi_{11}(\mathrm{THH}(\mathrm{ko})/\mathrm{ko}) = 0$, which is impossible.

This contradiction shows that $d_3(\lambda_1'\lambda_2) = \eta^3\mu$, as claimed. This leaves the η-Bockstein E_4-term

$$\frac{\Lambda(\lambda_1') \otimes \mathbb{F}_2[\eta, \mu]}{(\eta\lambda_1', \eta^3\mu)} .$$

There is no room for further differentials, so this is also the E_∞-term. The only possible multiplicative extension in the abutment $\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})$ is the one stated, with $\lambda_1' \cdot \lambda_1' \in \mathbb{F}_2\{\eta^2\mu\}$. □

Corollary 2.33. *We can identify*

$$\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \cong \Lambda(\varepsilon_2) \otimes \frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 = c \cdot \eta^2\mu, \eta^3\mu)}$$

as a bigraded $\mathbb{F}_2$-algebra, for some $c \in \mathbb{F}_2$.

Proof. Here ε_2 is chosen as in Definition 2.30 with $M^* = \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})$. □

Remark 2.34. We will show in Proposition 4.11 that, in fact, $c = 1$ and $\lambda_1' \cdot \lambda_1' = \eta^2\mu$. We can therefore give the complete computation of $\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})$, including its multiplicative structure, in Corollary 4.12.

3. Detection

The classical mod 2 Adams spectral sequence

$${}^{\mathrm{Ad}}E_2(X) = \mathrm{Ext}_{\mathcal{A}^\vee}(\mathbb{F}_2, H_*(X)) \Longrightarrow \pi_*(X_2^\wedge)$$

is strongly convergent for bounded below spectra X with $H_*(X)$ of finite type. Its E_2-term can be calculated as the cohomology of the normalized cobar complex

$$0 \longrightarrow H_*(X) \xrightarrow{d_1^0} \bar{\mathcal{A}}^\vee \otimes H_*(X) \xrightarrow{d_1^1} \bar{\mathcal{A}}^\vee \otimes \bar{\mathcal{A}}^\vee \otimes H_*(X) \longrightarrow \dots .$$

Here $\bar{\mathcal{A}}^\vee = \mathrm{cok}(\mathbb{F}_2 \to \mathcal{A}^\vee)$, and we will use the notation $[a]m = a \otimes m \in \bar{\mathcal{A}}^\vee \otimes H_*(X)$. Recall that d_1^0 is given by the normalized $\mathcal{A}^\vee$-coaction on $H_*(X)$, while d_1^1 also involves the coproduct $\psi \colon \mathcal{A}^\vee \to \mathcal{A}^\vee \otimes \mathcal{A}^\vee$.

When $X = A(1)[ij]$ as in Notation 2.15, the Adams E_2-terms

$${}^{\mathrm{Ad}}E_2(A(1)[ij]) = \mathrm{Ext}_{\mathcal{A}}(H^*(A(1)[ij]), \mathbb{F}_2) \Longrightarrow \pi_* A(1)[ij]$$

are readily calculated in a finite range using Bruner's `ext` software [Bru93, BR]). The results in stems $* \le 28$ are shown in Figure 3.1, with the usual (stem, Adams filtration) bigrading. Lines of bidegree $(0, 1)$, $(1, 1)$ and $(3, 1)$ (dashed) indicate multiplications by h_0, h_1 and h_2, respectively. In each case, the three 1-cochains

$$[\xi_1^4]1 \quad , \quad [\xi_2^2]1 + [\xi_1^4]\xi_1^2 \quad \text{and} \quad [\xi_3]1 + [\xi_2^2]\xi_1 + [\xi_1^4]\xi_2 \tag{3.1}$$

in $\bar{\mathcal{A}}^\vee \otimes \mathcal{A}(1)^\vee$ are cocycles, but not coboundaries, hence represent nonzero classes in ${}^{\mathrm{Ad}}E_2(A(1)[ij])$ in bidegrees $(3, 1)$, $(5, 1)$ and $(6, 1)$, respectively. For sparsity reasons, these survive to ${}^{\mathrm{Ad}}E_\infty(A(1)[ij])$, and detect nonzero homotopy classes in stems 3, 5 and 6, denoted

$$\nu \quad , \quad w \quad \text{and} \quad v_2$$

in $\pi_* A(1)[ij]$, for each $i, j \in \{0, 1\}$. Observe that v_2 is only defined modulo ν^2, and $2v_2 = 0$ if $i = 0$ while $2v_2 = \nu^2$ if $i = 1$.

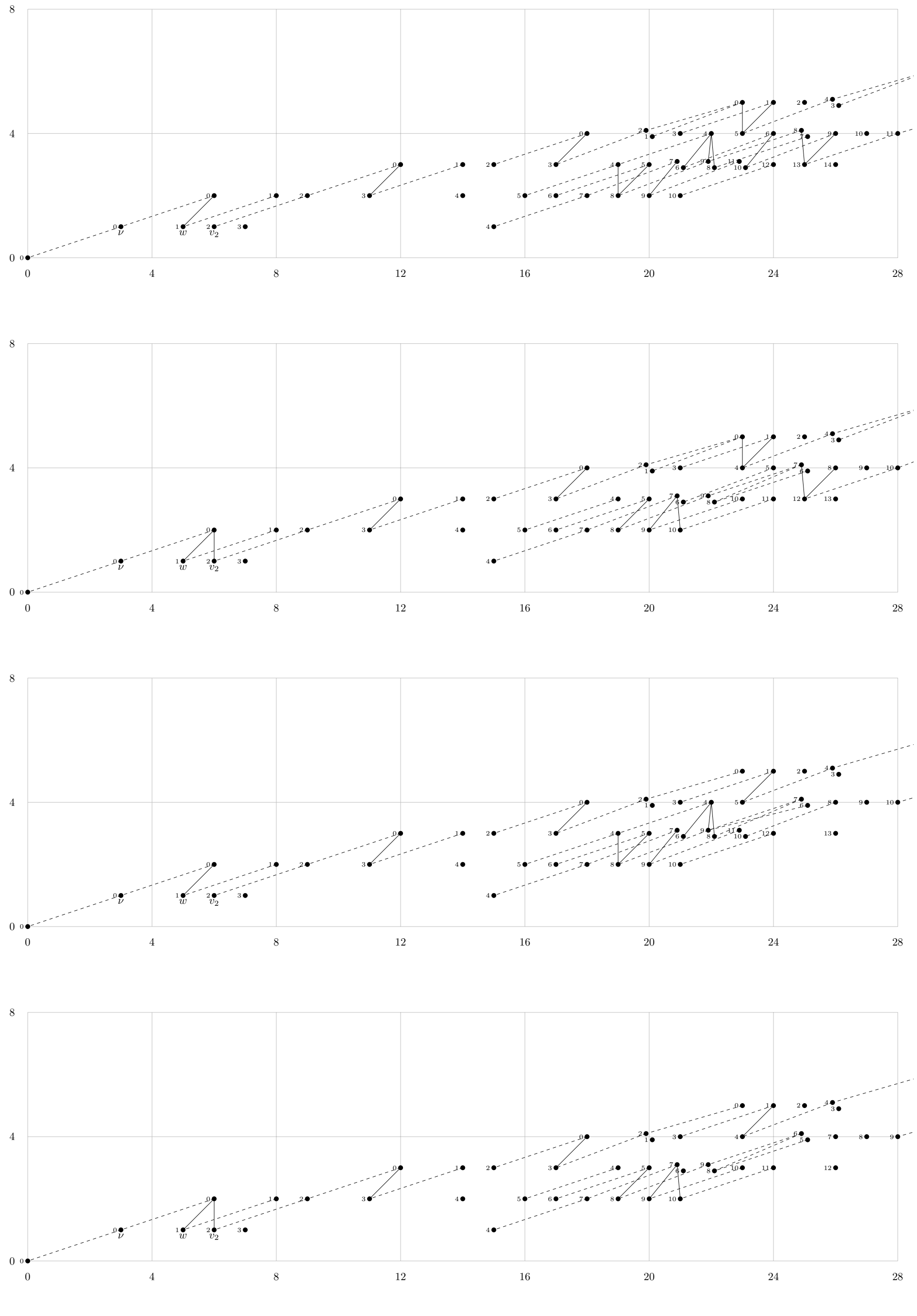

FIGURE 3.1. Adams E_2-terms for $A(1)[00]$, $A(1)[10]$, $A(1)[01]$ and $A(1)[11]$ (from top to bottom)

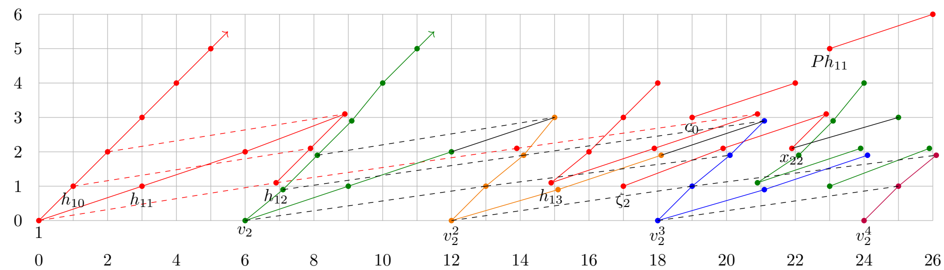

FIGURE 3.2. v_2-Bockstein $E_\infty \Longrightarrow \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_2)$

Lemma 3.1. *In the Adams spectral sequences for the $A(1)[ij]$ the differentials originating in stems $* \le 24$ are all zero. The class $\nu\bar{\kappa} \in \pi_{23}(\mathbb{S})$ maps to zero in $\pi_{23}A(1)[ij]$.*

Proof. This mostly follows from sparsity and the module structure over the Adams spectral sequence for $\mathbb{S}$, using that $d_2(h_4) = h_0h_3^2$ maps to zero under $\mathbb{S} \to A(1)$. Only the Adams d_2-differential from bidegree $(t-s, s) = (19, 2)$ requires special attention, but the Novikov E_2-term (see Figure 3.4) shows that $\pi_{19}A(1)$ has order $2^2 = 4$, so there is no room for such an Adams differential.

In each case, the map ${}^{\mathrm{Ad}}E_2(\mathbb{S}) \to {}^{\mathrm{Ad}}E_2(A(1))$ of Adams E_2-terms takes the bidegree $(23, 5)$ class h_2g detecting $\nu\bar{\kappa}$ to zero, as can be checked with `ext`, and the target has no classes in stem 23 and Adams filtration ≥ 6. It follows that $\nu\bar{\kappa} \mapsto 0$. □

To calculate the Novikov E_2-term

$${}^{\mathrm{Nov}}E_2(A(1)) = \mathrm{Ext}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_*A(1)) \cong \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*A(1))$$

for these spectra, we can note that $\mathrm{BP}_*A(1) = \mathrm{BP}_*/I_2\{1, t_1\}$ and use the long exact sequence obtained by applying $\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, -)$ to the $\mathrm{BP}_*\mathrm{BP}$-comodule extension

$$0 \longrightarrow \mathrm{BP}_*/I_2 \longrightarrow \mathrm{BP}_*A(1) \longrightarrow \Sigma^2\mathrm{BP}_*/I_2 \to 0$$

classified by

$$h_{10} = [t_1] \in \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*)$$

in (stem, Novikov filtration) bidegree $(1, 1)$. The groups

$$\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_2)$$

are calculated in a range as in [Rav86, §4.4, p. 162], starting with the isomorphism

$$\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_\infty)$$

that doubles internal degrees, followed by the v_n-Bockstein spectral sequences

$$E_1 = \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_{n+1})\,[v_n] \Longrightarrow \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_n)$$

for descending $n \ge 2$. The v_2-Bockstein spectral sequence E_∞-term for BP_*/I_2 in stems $* \le 26$ is shown in Figure 3.2, corresponding to [Rav86, Fig. 4.4.23(c)]. Lines of bidegree $(1, 1)$, $(3, 1)$ and $(7, 1)$ (dashed) indicate multiplications by $h_{10} = [t_1]$, $h_{11} = [t_1^2]$ and $h_{12} = [t_1^4]$, respectively. (Some) hidden extensions are shown in black. We emphasize the relation

$$v_2h_{11}^3 = v_2^2h_{10}^3\,, \tag{3.2}$$

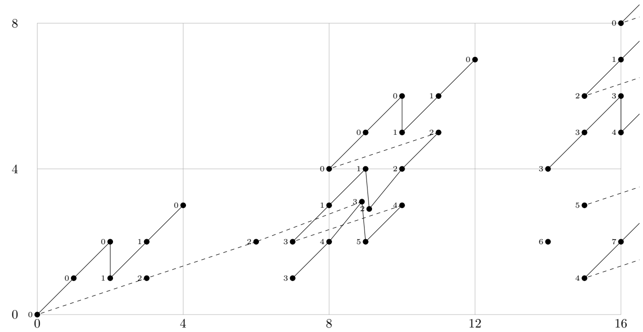

FIGURE 3.3. Adams E_2-term for $C2$

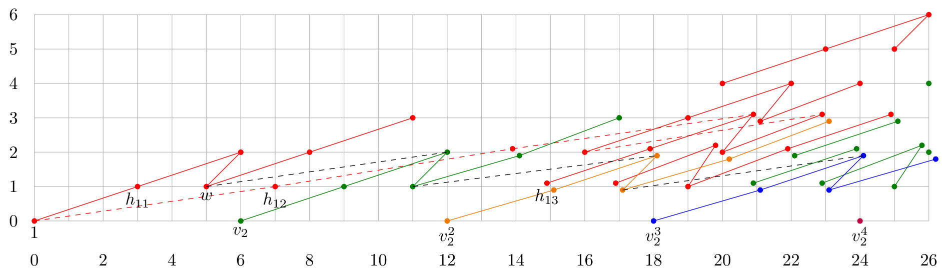

FIGURE 3.4. v_2-Bockstein $E_\infty \Longrightarrow \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*A(1))$

which follows from $v_2 h_{12} = v_2^2 h_{10}$, which in turn follows from the formula $\eta_R(v_3) \equiv v_3 + v_2 t_1^4 + v_2^2 t_1 \mod I_2$ for the right unit in $\mathrm{BP}_*\mathrm{BP}$, see [Rav86, 4.3.1].

Alternatively, one can start with the internal-degree-doubling isomorphism

$$\mathrm{Ext}_{\mathcal{A}}(H^*(C2), \mathbb{F}_2) \cong \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_\infty\{1, t_1\})$$

and calculate the v_n-Bockstein spectral sequences

$$E_1 = \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_{n+1}\{1, t_1\})\,[v_n] \Longrightarrow \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_n\{1, t_1\})$$

for descending $n \geq 2$. The Adams E_2-term for $C2$ in stems $* \leq 16$ is shown in Figure 3.3, and the resulting v_2-Bockstein E_∞-term for $\mathrm{BP}_*/I_2\{1, t_1\} = \mathrm{BP}_*A(1)$ in stems $* \leq 26$ is shown in Figure 3.4. Again, (some) hidden extensions are shown in black.

Lemma 3.2. *In the Novikov spectral sequences for the $A(1)[ij]$ the nonzero differentials originating in stems $* \leq 22$ are*

$$d_3(v_2^2) = h_{11}^2 w \qquad \text{and} \qquad d_3(v_2^3) = v_2 h_{11}^2 w\,.$$

In the cases $A(1)[10]$ and $A(1)[11]$ there is a nonzero d_3 from bidegree $(t-s, s) = (23, 1)$.

In every case $d_3(v_2^4) = 0$ and $d_5(v_2^4) \neq 0$.

Proof. This follows by comparison of the order in each stem of the Adams E_∞-term, which equals that of the abutment $\pi_*A(1)[ij]$, with the order in each stem of the Novikov E_2-term. In particular, $\pi_{12}A(1) = \mathbb{Z}/2$ implies that v_2^2 must support a nonzero differential. Similarly, the group $\pi_{18}A(1)$ has order 2^2, so v_2^3 must support a nonzero differential. The groups $\pi_{22}A(1)[ij]$ have order $2^3 = 8$ for $i = 0$ and order $2^2 = 4$ for $i = 1$, while the groups $\pi_{23}A(1)[ij]$ have order 2^4 for $i = 0$ and 2^3 for $i = 1$. To account for this, the Novikov differential d_3 from bidegree $(t-s, s) = (23, 1)$ to $(22, 4)$ must be nonzero when $i = 1$. Moreover, there must be a rank 1 Novikov differential from the 24-stem to the 23-stem. By h_{11}-linearity, it cannot originate in bidegree $(24, 2)$, hence it is either a d_3 or a d_5 starting on v_2^4.

Inspection of the Novikov E_2-term for $\mathbb{S}$ in [Rav86, Figure 4.4.45] shows that $\nu\bar{\kappa} \in \pi_{23}(\mathbb{S})$ is detected by a generator x of the $\mathbb{Z}/8$ in (stem, Novikov filtration) bidegree $(23, 5)$ of ${}^{\mathrm{Nov}}E_2(\mathbb{S})$. The unit map $\mathbb{S} \to A(1)$ takes this generator x to the generator y of the $\mathbb{Z}/2$ in the same bidegree of ${}^{\mathrm{Nov}}E_2(A(1))$, see Figure 3.4. Since $\nu\bar{\kappa}$ maps to zero in $\pi_{23}A(1)$ (by Lemma 3.1) it follows that this nonzero class y is a boundary, and so $d_5(v_2^4) = y \neq 0$ is the only possibility. In particular, we must have $d_3(v_2^4) = 0$. □

The circle group $\mathbb{T}$ acts freely on $S^1 \subset S^3 \subset \cdots \subset S^\infty = E\mathbb{T}$, and we can form the "approximate homotopy fixed point" spectrum $F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$, as in [BR05, §2] and [AKAC$^+$, §7]. There is a cofiber sequence

$$\Sigma^{-2}\,\mathrm{THH}(\mathrm{ko}) \xrightarrow{i} F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \xrightarrow{p} \mathrm{THH}(\mathrm{ko}) \xrightarrow{\sigma} \Sigma^{-1}\,\mathrm{THH}(\mathrm{ko})\,, \tag{3.3}$$

where σ is induced by the $\mathbb{T}$-action on THH(ko), and a commutative diagram

$$\begin{tikzcd}
\mathbb{S} \arrow[r] \arrow[d] & \mathrm{TC}^-(\mathrm{ko}) \arrow[r, "q"] \arrow[dr] & F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \arrow[d, "p"] \\
\mathrm{ko} \arrow[rr] & & \mathrm{THH}(\mathrm{ko})\,.
\end{tikzcd}$$

By truncating the homotopy fixed point spectral sequence

$$E^2 = A(1)_*\,\mathrm{THH}(\mathrm{ko})\,[t] \Longrightarrow A(1)_*\,\mathrm{TC}^-(\mathrm{ko})\,,$$

where t is in stem -2, we obtain a two-column approximate homotopy fixed point spectral sequence

$$E^2 = A(1)_*\,\mathrm{THH}(\mathrm{ko})\{1, t\} \Longrightarrow A(1)_* F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}\,, \tag{3.4}$$

which is really just the long exact sequence in $A(1)$-homotopy associated to the cofiber sequence (3.3). We have the following analogue of [AR02, Proposition 4.8].

Proposition 3.3. *The unit images in $A(1)_*\,\mathrm{TC}^-(\mathrm{ko})$ and $A(1)_*F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$ of the classes ν, w and $v_2 \in \pi_*A(1)$ are detected by*

$$t\lambda_1' \quad , \quad t\lambda_2 \quad \text{and} \quad t\mu\,,$$

respectively, in the homotopy fixed point and approximate fixed point spectral sequences.

Proof. By naturality, it suffices to prove this in the approximate fixed point case. The unit map takes the infinite cycles in (3.1), detecting ν, w and v_2 in $\pi_*A(1)$, to the 1-cocycles

$$\begin{aligned}
&[\xi_1^4](1 \otimes 1) \\
&[\xi_2^2](1 \otimes 1) + [\xi_1^4](\xi_1^2 \otimes 1) \\
&[\xi_3](1 \otimes 1) + [\xi_2^2](\xi_1 \otimes 1) + [\xi_1^4](\xi_2 \otimes 1)
\end{aligned} \tag{3.5}$$

in $\bar{\mathcal{A}}^\vee \otimes \mathcal{A}(1)^\vee \otimes H_*(F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}})$. We claim that these are not in the image of the coboundary d_1^0 from the 0-cochains

$$\mathcal{A}(1)^\vee \otimes H_*(F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}),$$

hence represent nonzero classes in ${}^{\mathrm{Ad}}E_\infty(A(1) \otimes F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}})$, detecting the (nonzero) images of ν, w and v_2 in $A(1)_* F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$.

Recall that $H_*(\mathrm{ko}) = \mathbb{F}_2[\bar\xi_1^4, \bar\xi_2^2, \bar\xi_3, \dots]$, $H_* \mathrm{THH}(\mathrm{ko}) = H_*(\mathrm{ko}) \otimes \Lambda(\sigma\bar\xi_1^4, \sigma\bar\xi_2^2) \otimes \mathbb{F}_2[\sigma\bar\xi_3]$ and $\mathcal{A}(1)^\vee = \mathbb{F}_2[\xi_1, \xi_2]/(\xi_1^4, \xi_2^2)$. In the long exact sequence associated to (3.3), the map σ has kernel $\mathbb{F}_2\{1, \sigma\bar\xi_1^4, \sigma\bar\xi_2^2\}$ in degrees ≤ 7, and the image of i consists of t-multiples. In the extension

$$0 \longrightarrow \mathcal{A}(1)^\vee \otimes \mathrm{im}(i) \longrightarrow \mathcal{A}(1)^\vee \otimes H_* F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \longrightarrow \mathcal{A}(1)^\vee \otimes \ker(\sigma) \to 0$$

the coboundaries on classes in $\mathcal{A}(1)^\vee \otimes \mathrm{im}(i)$ will lie in $\bar{\mathcal{A}}^\vee \otimes \mathcal{A}(1)^\vee \otimes \mathrm{im}(i)$, hence do not contribute any terms of the form $[a](m \otimes 1)$. The classes 1, $\sigma\bar\xi_1^4$ and $\sigma\bar\xi_2^2$ are $\mathcal{A}^\vee$-comodule primitive in $\ker(\sigma)$, hence lift to classes in $H_* F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$ that are $\mathcal{A}^\vee$-comodule primitive modulo $\mathrm{im}(i)$, so also the coboundaries on (the lifts of) $\mathcal{A}(1)^\vee \otimes \mathbb{F}_2\{1, \sigma\bar\xi_1^4, \sigma\bar\xi_2^2\}$ do not contain any terms of the form in (3.5). This proves our claim.

It remains to be determined where in (3.4) the (nonzero) unit images of ν, w and v_2 are detected. Recall that $A(1)_* \mathrm{THH}(\mathrm{ko}) = \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[\mu]$ is equal to $\mathbb{F}_2\{1, \lambda_1', \lambda_2, \mu\}$ in stems ≤ 8. The composite map

$$A(1) \longrightarrow A(1) \otimes F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \xrightarrow{p} A(1) \otimes \mathrm{THH}(\mathrm{ko})$$

factors through $A(1) \otimes \mathrm{ko} \simeq \mathbb{F}_2$, so the images of ν, w and v_2 in $A(1)_* \mathrm{THH}(\mathrm{ko})$ are all zero. (This was obvious for ν and v_2.) Hence the nonzero images of ν, w and v_2 must all be detected by t-multiples in the approximate fixed point spectral sequence, and for degree reasons the only possible detecting classes are $t\lambda_1'$, $t\lambda_2$ and $t\mu$, respectively. □

Since $\mathrm{MU}_* A(1)$ is even, the motivic spectral sequence

$$E_2 = \pi_* \overline{A}(1) \Longrightarrow \pi_* A(1) \tag{3.6}$$

can be identified with the Novikov spectral sequence

$${}^{\mathrm{Nov}}E_2(A(1)) = \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_* A(1)) \Longrightarrow \pi_* A(1),$$

as in [HRW, Corollary 2.2.17]. The spectral sequence must collapse in stems ≤ 10, for sparsity reasons, so the three classes denoted h_{11}, w and v_2 in Figure 3.4 must detect ν, w and v_2 in $\pi_* A(1)$, respectively. We can also identify $\pi_* \overline{V}(1)$ with $\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_2)$, with classes h_{11} and v_2 as shown in Figure 3.2, but in this case there is no Novikov spectral sequence to $\pi_* V(1)$, since the spectrum $V(1)$ does not exist.

Corollary 3.4. *The classes h_{11}, w and v_2 in $\pi_* \overline{A}(1)$ map by the unit to classes in*

$$\overline{A}(1)_* \mathrm{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})$$

detected by $t\lambda_1'$, $t\lambda_2$ and $t\mu$, respectively. Likewise, the images of h_{11} and w in $\pi_*(\overline{V}(2) \otimes \overline{C}\eta)$ are detected by $t\lambda_1'$ and $t\lambda_2$ in $(\overline{V}(2) \otimes \overline{C}\eta)_* \mathrm{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})$.

The classes h_{11} and v_2 in $\pi_* \overline{V}(1)$ map by the unit to classes in

$$\overline{V}(1)_* \mathrm{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})$$

detected by $t\lambda_1'$ and $t\mu$, respectively. Likewise, the image of h_{11} in $\pi_*\overline{V}(2)$ in $\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})$ is detected by $t\lambda_1'$.

In each of these cases, for $V \in \{V(1), A(1), V(2), V(2) \otimes C\eta\}$, a unit image detected by a class in $\overline{V}_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})$ is also detected in $\overline{V}_* \operatorname{gr}^*_{\mathrm{mot}} F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$, by the class with the same name.

Proof. Naturality of the motivic spectral sequences with respect to

$$q\colon \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko}) \simeq (\operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}))^{h\mathbb{T}} \longrightarrow F(S^3_+, \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$$

shows that it suffices to prove these assertions in the case of the approximate homotopy fixed points. The motivic spectral sequence

$$\begin{aligned} E_2 = \overline{A}(1)_* F(S^3_+, \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} &= \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[\mu]\,\{1, t\} \\ &\Longrightarrow A(1)_* F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \end{aligned}$$

is concentrated in filtrations $0 \le * \le 2$ and integer weights, hence collapses, and the result follows from Proposition 3.3. The claim with coefficients in $\overline{V}(2) \otimes \overline{C}\eta$ follows by passing to cofibers for multiplication by v_2.

The E_2-term of the motivic spectral sequence

$$\begin{aligned} \overline{V}(1)_* F(S^3_+, \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} &= (\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}))\,\{1, t\} \\ &\Longrightarrow V(1)_* F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \end{aligned}$$

was determined (up to the coefficient $c \in \mathbb{F}_2$) in Proposition 2.32. Since h_{11} and v_2 in $\pi_*\overline{V}(1)$ map to h_{11} and v_2 in $\pi_*\overline{A}(1)$, the claim with $\overline{V}(1)$-coefficients follows from the commuting square

$$\begin{tikzcd} \overline{V}(1) \arrow[r, "i"] \arrow[d] & \overline{A}(1) \arrow[d] \\ \overline{V}(1) \otimes F(S^3_+, \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \arrow[r, "i\otimes 1"] & \overline{A}(1) \otimes F(S^3_+, \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}\,. \end{tikzcd}$$

Again, the claim with $\overline{V}(2)$-coefficients follows by passing to cofibers for multiplication by v_2. □

In Section 5 we shall calculate the syntomic cohomology $\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$, and in Section 6 we shall use the associated motivic spectral sequence to calculate $A(1)_* \mathrm{TC}(\mathrm{ko})$. The following lemma and corollary will be used to show that the d_3-differentials in this spectral sequence propagate in a v_2^4-periodic pattern. Recall from [Lan73, Proposition 2.11] that

$$\mathbb{F}_2[v_2] = \mathrm{Ext}^0_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/(2, v_1)) \subset \mathrm{BP}_*/(2, v_1)\,.$$

Lemma 3.5. *Let $\operatorname{End} A(1) = F(A(1), A(1))$ be the endomorphism $\mathbb{S}$-algebra of any one of the spectra $A(1)[ij]$. The induced $\mathrm{BP}_*\mathrm{BP}$-comodule BP_*-algebra structure on $\mathrm{BP}_* \operatorname{End} A(1)$ descends to a $\mathrm{BP}_*\mathrm{BP}$-comodule $\mathrm{BP}_*/(2, v_1)$-algebra structure. Hence the E_2-term of the Novikov spectral sequence*

$${}^{\mathrm{Nov}}E_2(\operatorname{End} A(1)) = \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_* \operatorname{End} A(1)) \Longrightarrow \pi_* \operatorname{End} A(1)$$

is an algebra over $\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/(2, v_1)) \supset \mathbb{F}_2[v_2]$. In particular,

$$d_2(v_2^2) = 0 \qquad \textit{and} \qquad d_3(v_2^4) = 0$$

in this Novikov spectral sequence. Moreover, $4 \cdot \mathrm{id} = 0$ in $\pi_0 \operatorname{End} A(1)$.

Proof. To see that the BP_*-algebra structure in $\mathrm{BP}_*\mathrm{BP}$-comodules descends to a $\mathrm{BP}_*/(2, v_1)$-algebra structure in that category, it suffices to check that the homomorphism $\mathrm{BP}_* \to \mathrm{BP}_* \operatorname{End} A(1)$, induced by the $\mathbb{S}$-algebra unit map, sends 2 and v_1 to zero. This can be verified using the map of Adams spectral sequences

$$\begin{aligned} {}^{\mathrm{Ad}}E_2(\mathrm{BP}) &= \operatorname{Ext}_{\mathcal{A}}(H^*\mathrm{BP}, \mathbb{F}_2) \\ &\longrightarrow \operatorname{Ext}_{\mathcal{A}}(H^*(\mathrm{BP} \otimes \operatorname{End} A(1)), \mathbb{F}_2) = {}^{\mathrm{Ad}}E_2(\mathrm{BP} \otimes \operatorname{End} A(1)) \,. \end{aligned}$$

Standard minimal resolution calculations, which can be obtained from Bruner's `ext` program [Bru93, BR], show that multiplication by h_0 and $\langle h_0, h_1, -\rangle$ both act trivially from bidegree $(0, 0)$ on the right-hand side, and that there are no classes in stems 0 or 2 that have Adams filtration ≥ 2. The last fact also implies that $4 \cdot \mathrm{id} = 0$ in $\pi_0 \operatorname{End} A(1)$, and a closer inspection shows that $2 \cdot \mathrm{id} \neq 0$ in each case.

The functor $\operatorname{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, -)$ is lax symmetric monoidal, so it follows that ${}^{\mathrm{Nov}}E_2(\operatorname{End} A(1))$ is an $\operatorname{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/(2, v_1))$-algebra. In particular, it is an $\mathbb{F}_2[v_2]$-algebra, with v_2 acting centrally and $1 + 1 = 0$. Hence $d_2(v_2^2) = d_2(v_2) \cdot v_2 + v_2 \cdot d_2(v_2) = 0$ and $d_3(v_2^4) = d_3(v_2^2) \cdot v_2^2 + v_2^2 \cdot d_3(v_2^2) = 0$. □

The tautological left action of $\operatorname{End} A(1)$ on $A(1)$ induces a left action of the Novikov filtration of $\operatorname{End} A(1)$ on the Novikov filtration of $A(1)$. The latter is equivalent to the even filtration $\mathrm{fil}^\star_{\mathrm{ev}} A(1)$, since $MU_*A(1)$ is concentrated in even degrees. Hence $\mathrm{fil}^\star_{\mathrm{Nov}} \operatorname{End} A(1)$ also acts on the convolution product filtration

$$\mathrm{fil}^\star_{\mathrm{ev}} A(1) \otimes_{\mathrm{fil}^\star_{\mathrm{ev}} \mathbb{S}} \mathrm{fil}^\star_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \,,$$

and induces a left action of the Novikov spectral sequence for $\operatorname{End} A(1)$ on the motivic spectral sequence converging to $A(1)_* \mathrm{TC}(\mathrm{ko})$.

Corollary 3.6. *The differentials in the motivic spectral sequence*

$$E_2 = \overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \Longrightarrow A(1)_* \mathrm{TC}(\mathrm{ko})$$

satisfy $d_3(v_2^4 \cdot y) = v_2^4 \cdot d_3(y)$ for all $y \in E_3$.

Proof. This follows from the Leibniz rule $d_3(v_2^4 \cdot y) = d_3(v_2^4) \cdot y + v_2^4 \cdot d_3(y)$ for the pairing of spectral sequences

$${}^{\mathrm{Nov}}E_2(\operatorname{End} A(1)) \otimes \overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \longrightarrow \overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \,,$$

since $d_3(v_2^4) = 0$ by Lemma 3.5. □

4. Prismatic cohomology

We consider the $\overline{V}(2)$-homotopy $\mathbb{T}$-Tate spectral sequence

$$\begin{aligned} \hat{E}^2(\mathbb{T}) &= \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})\,[t^{\pm 1}] \\ &\Longrightarrow \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \end{aligned} \tag{4.1}$$

with t in stem -2, and the $\overline{V}(2) \otimes \overline{C}\eta$-homotopy $\mathbb{T}$-Tate spectral sequence

$$\begin{aligned} \hat{E}^2(\mathbb{T}) &= (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})\,[t^{\pm 1}] \\ &\Longrightarrow (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \,. \end{aligned} \tag{4.2}$$

They can be reindexed as cohomologically graded periodic t-Bockstein spectral sequences, in which case $\hat{E}^{2r}(\mathbb{T})$ and d^{2r} correspond to E_r and d_r. However, we shall need to make a comparison with similar C_2-Tate spectral sequences, for which our indexing is convenient.

Theorem 4.1 (Prismatic cohomology modulo $(2, v_1, v_2)$ of ko)**.** *The $\overline{V}(2)$-homotopy $\mathbb{T}$-Tate spectral sequence (4.1) is an algebra spectral sequence with E^2-term*

$$\hat{E}^2(\mathbb{T}) = \Lambda(\varepsilon_2) \otimes \frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 + \eta^2\mu)} \otimes \mathbb{F}_2[t^{\pm 1}]$$

and differentials

$$d^2(t^{-1}) = \eta \qquad d^2(\varepsilon_2) = t\mu$$
$$d^6(t^{-2}) = t\lambda_1' \qquad d^6(t^{-1}\lambda_1') = t^2(\lambda_1')^2 = t^2\eta^2\mu\,,$$

leading to

$$\hat{E}^\infty(\mathbb{T}) = \mathbb{F}_2\{1, t^2\lambda_1', (t^2\lambda_1')^2, \lambda_1'\} \otimes \mathbb{F}_2[t^{\pm 4}]\,.$$

Hence there is a preferred isomorphism

$$\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \cong \mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \otimes \mathbb{F}_2[t^{\pm 4}]\,,$$

where 1, η, η^2, λ_1' and $t^{\pm 4}$ in bidegrees $(0,0)$, $(1,1)$, $(2,2)$, $(5,1)$ and $(\mp 8, 0)$ are detected by 1, $t^2\lambda_1'$, $(t^2\lambda_1')^2 = t^4\eta^2\mu$, λ_1' and $t^{\pm 4}$, respectively.

Theorem 4.2 (Prismatic cohomology modulo $(2, \eta, v_1, v_2)$ of ko)**.** *The $\overline{V}(2) \otimes \overline{C}\eta$-homotopy $\mathbb{T}$-Tate spectral sequence (4.2) is a module spectral sequence over (4.1), with E^2-term*

$$\hat{E}^2(\mathbb{T}) = \Lambda(\varepsilon_2) \otimes \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \mathbb{F}_2[\mu] \otimes \mathbb{F}_2[t^{\pm 1}]$$

and differentials

$$d^2(\varepsilon_2) = t\mu$$
$$d^6(t^{-1}) = t^2\lambda_1' \qquad d^6(t^{-2}) = t\lambda_1'$$
$$d^6(t^{-1}\lambda_2) = t^2\lambda_1'\lambda_2 \qquad d^6(t^{-2}\lambda_2) = t\lambda_1'\lambda_2$$
$$d^8(t^{-3}) = t\lambda_2 \qquad d^8(t^{-1}\lambda_1') = t^3\lambda_1'\lambda_2$$

leading to

$$\hat{E}^\infty(\mathbb{T}) = \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \mathbb{F}_2[t^{\pm 4}]\,.$$

Hence there is a preferred isomorphism

$$(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \cong \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \mathbb{F}_2[t^{\pm 4}]\,,$$

where 1, λ_1', λ_2, $\lambda_1'\lambda_2$ and $t^{\pm 4}$ in bidegrees $(0,0)$, $(5,1)$, $(7,1)$, $(12,2)$, and $(\mp 8, 0)$ are detected by the classes in the E^∞-term with the same names.

The proofs of these theorems will occupy the remainder of this section. By Corollary 2.33, we can identify the E^2-term in (4.1) as

$$\hat{E}^2(\mathbb{T}) = \Lambda(\varepsilon_2) \otimes \frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 = c \cdot \eta^2\mu, \eta^3\mu)} \otimes \mathbb{F}_2[t^{\pm 1}]\,,$$

where we have yet to determine the coefficient $c \in \mathbb{F}_2$.

Proposition 4.3. *The spectral sequence (4.1) is multiplicative and has differentials*

$$d^2(t^{-1}) = \eta \qquad d^2(\eta) = 0 \qquad d^2(\lambda_1') = 0$$
$$d^2(\varepsilon_2) = t\mu \qquad \textit{and} \qquad d^2(\mu) = t\eta\mu\,.$$

Consequently, we can identify

$$\hat{E}^4(\mathbb{T}) = \mathbb{F}_2\{1, t\lambda_1', \lambda_1', \eta^2\mu\} \otimes \mathbb{F}_2[t^{\pm 2}]$$

with $\eta^2\mu = \eta^3\varepsilon_2$.

Proof. The first claim follows as in [HR, §6.7], because $\overline{V}(1) \otimes \mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})$ is a naively $\mathbb{T}$-equivariant $(\mathbb{E}_\infty)$ $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-algebra. Using the $\mathbb{T}$-equivariant attaching maps of the standard $\mathbb{T}$-CW complex structure on $S^\infty = E\mathbb{T}$, we compute differentials

$$d^2(t^{-1}) = \eta \qquad \text{and} \qquad d^2(\eta) = 0$$

as in [Hes96, Lemma 1.4.2].

We know that $d^2(t\lambda_1') = 0$, because $t\lambda_1'$ detects $\nu \in \{h_{11}\}$ by Corollary 3.4. Consequently, $d^2(\lambda_1') = 0$ by the Leibniz rule and the fact that $\eta \cdot \lambda_1' = 0$.

To show that $d_2(\varepsilon_2) = t\mu$ we apply the graded analogue of [BR22, Proposition 2.3] (a variant of [BG95, Lemma 2.2]) to the smash product of cofiber sequences

$$\begin{tikzcd}
\Sigma^{4,-1}\overline{V}(1) \otimes \mathrm{THH}(\mathrm{ko}) \arrow[r] \arrow[d, "v_2"] & \Sigma^{6,0}\overline{V}(1) \otimes F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \arrow[r, "p"] \arrow[d, "v_2"] & \Sigma^{6,0}\overline{V}(1) \otimes \mathrm{THH}(\mathrm{ko}) \arrow[d, "v_2"] \\
\Sigma^{-2,-1}\overline{V}(1) \otimes \mathrm{THH}(\mathrm{ko}) \arrow[r] \arrow[d, "i_2"] & \overline{V}(1) \otimes F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \arrow[r, "p"] \arrow[d, "i_2"] & \overline{V}(1) \otimes \mathrm{THH}(\mathrm{ko}) \arrow[d, "i_2"] \\
\Sigma^{-2,-1}\overline{V}(2) \otimes \mathrm{THH}(\mathrm{ko}) \arrow[r] & \overline{V}(2) \otimes F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \arrow[r, "p"] & \overline{V}(2) \otimes \mathrm{THH}(\mathrm{ko})\,.
\end{tikzcd}$$

Starting in the upper right-hand corner, the unit $\Sigma^{6,0}1$ lifts over the connecting map j_2 to ε_2 in the lower right-hand corner and maps under the connecting map σ to $d^2(\varepsilon_2)$ in the lower left-hand corner. This is the same as the result of lifting to $\Sigma^{6,0}1$ at the top, mapping to v_2 in the center, lifting to $t\mu$ at the left, and pushing to $t\mu$ in the lower left-hand corner.

In particular, $t\mu$ is a d^2-cycle, and the Leibniz rule implies that $d^2(\mu) = t\eta\mu$. This leads to the E^4-term shown in Figure 4.1. $\square$

Proposition 4.4. *The classes $t^{\pm 4}$ and λ_1' are permanent cycles in the spectral sequence (4.1). Moreover, there are differentials*

$$d^6(t^{-2}) = t\lambda_1' \qquad \textit{and} \qquad d^6(t^{-1}\lambda_1') = c \cdot t^2\eta^2\mu\,,$$

for some $c \in \mathbb{F}_2$. The spectral sequence (4.1) collapses at $\hat{E}^8(\mathbb{T}) = \hat{E}^\infty(\mathbb{T})$.

Proof. We know that $t\lambda_1'$ is an infinite cycle in (4.1), because it detects $\nu \in \{h_{11}\}$ in $\overline{V}(2)_* \,\mathrm{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})$ by Corollary 3.4. There is a commutative square of homotopy fixed point and Tate spectral sequences, for the groups $\mathbb{T}$ and C_2, converging to

$$\begin{tikzcd}
\overline{V}(2)_* \,\mathrm{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko}) \arrow[r, "F^h"] \arrow[d, "\mathrm{can}"'] & \overline{V}(2)_* \,\mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{hC_2} \arrow[d, "\mathrm{can}"] \\
\overline{V}(2)_* \,\mathrm{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \arrow[r, "F^t"] & \overline{V}(2)_* \,\mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2}\,,
\end{tikzcd}$$

with E^2-terms

$$\begin{tikzcd}
\overline{V}(2)_* \,\mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \otimes \mathbb{F}_2[t] \arrow[r] \arrow[d] & \overline{V}(2)_* \,\mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t] \arrow[d] \\
\overline{V}(2)_* \,\mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \otimes \mathbb{F}_2[t^{\pm 1}] \arrow[r] & \overline{V}(2)_* \,\mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 1}]\,.
\end{tikzcd}$$

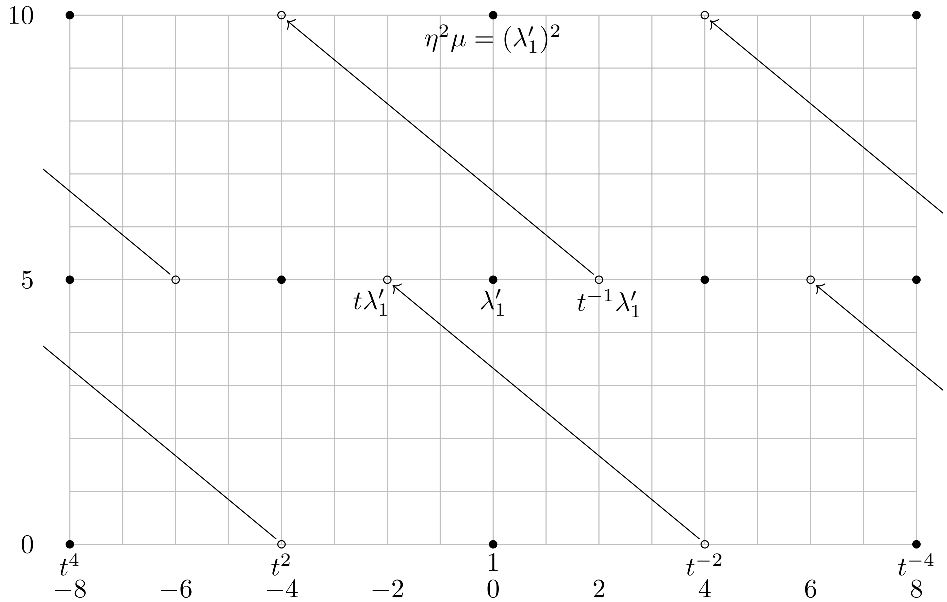

FIGURE 4.1. $\mathbb{T}$-Tate $\hat{E}^4(\mathbb{T}) \Longrightarrow \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko})$

In the two right-hand cases, u_1 has (stem, motivic filtration) bidegree $\|u_1\| = (-1, -1)$.

We know that $\nu \in \{h_{11}\}$ maps to zero in $\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})$, because the target is zero in the relevant bidegree by Corollary 2.33. A chase in the diagram

$$\begin{tikzcd}
 & \mathbb{S} \arrow[d] & \\
\mathrm{TF}(\mathrm{ko}) \arrow[r, "F"] \arrow[d, "\Gamma"'] & \mathrm{THH}(\mathrm{ko})^{C_2} \arrow[r, "R"] \arrow[d, "\Gamma_1"'] & \mathrm{THH}(\mathrm{ko}) \arrow[d, "\hat{\Gamma}_1 = \varphi_2^{h\mathbb{T}}"] \\
\mathrm{TC}^-(\mathrm{ko}) \arrow[r, "F^h"] \arrow[d, "q"'] & \mathrm{THH}(\mathrm{ko})^{hC_2} \arrow[r, "R^h = \mathrm{can}"] \arrow[d, "F^h"'] & \mathrm{THH}(\mathrm{ko})^{tC_2} \\
F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \arrow[r, "p"] & \mathrm{THH}(\mathrm{ko}) \,, &
\end{tikzcd}$$

similar to the proof of [AR02, Theorem 5.5], shows that $t\lambda_1'$ must be boundary in the $\overline{V}(2)$-homotopy C_2-Tate spectral sequence. There is no earlier C_2-Tate d^r-differential hitting $t\lambda_1'$, for $2 \le r \le 5$, since η is an infinite cycle. Consequently,

$$d^6(t^{-2}) = t\lambda_1'$$

in the $\overline{V}(2)$-homotopy C_2-Tate spectral sequence, and therefore also in the $\overline{V}(2)$-homotopy $\mathbb{T}$-Tate spectral sequence, cf. Figure 4.1. To complete the proof, we use the Leibniz rule to deduce that $d^6(t^{-1}\lambda_1') = d^6(t\lambda_1' \cdot t^{-2}) = (t\lambda_1')^2 = c \cdot t^2\eta^2\mu$, that $d^6(t^{-4}) = t\lambda_1' \cdot t^{-2} + t^{-2} \cdot t\lambda_1' = 0$, and that $d^6(t^4) = 0$. All later differentials are zero, because the target groups are trivial. $\square$

Remark 4.5. The coefficients denoted $c \in \mathbb{F}_2$ in Proposition 2.32, Corollary 2.33, and Proposition 4.4 are all the same. We determine that $c = 1$ in Proposition 4.11.

Even with incomplete information about $(\lambda_1')^2$ and $d^6(t^{-1}\lambda_1')$, we can extract the following computation.

Corollary 4.6. *We can identify*

$$\overline{V}(2)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \cong \begin{cases} \mathbb{F}_2\{1\} & \text{for } n = 0, \\ \mathbb{F}_2\{t^2\lambda_1'\} & \text{for } n = 1, \\ \mathbb{F}_2\{t^4\eta^2\mu\} & \text{for } n = 2, \\ 0 & \text{for } n = 3, 4, \\ \mathbb{F}_2\{\lambda_1'\} & \text{for } n = 5. \end{cases}$$

Moreover, if $c = 0$ then

$$\overline{V}(2)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \cong \begin{cases} \mathbb{F}_2\{\eta^2\mu\} & \text{for } n = 6, \\ \mathbb{F}_2\{t^{-1}\lambda_1'\} & \text{for } n = 7, \end{cases}$$

whereas if $c = 1$ then

$$\overline{V}(2)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) = \begin{cases} 0 & \text{for } n = 6, 7. \end{cases}$$

These repeat 8-periodically, via multiplication by $t^{\pm 4}$. □

We now move towards computing the spectral sequence (4.2). By Corollary 2.31, its E^2-term has the form asserted in Theorem 4.2.

Remark 4.7. In general the differentials in (4.2) do not satisfy the Leibniz rule. This is commensurable with Remark 2.27.

Proposition 4.8. *The spectral sequence (4.2) is a module over the spectral sequence (4.1). There are differentials*

$$d^2(\varepsilon_2) = t\mu \qquad , \qquad d^6(t^{-2}) = t\lambda_1' \qquad \text{and} \qquad d^8(t^{-3}) = t\lambda_2$$

in (4.2), and multiplication by $t^{\pm 4}$ and λ_1' commutes with all differentials in this spectral sequence.

Proof. The unit map $\overline{V}(2) \to \overline{V}(2) \otimes \overline{C}\eta$ is a map of $\overline{V}(2)$-modules, so (4.2) is a module spectral sequence over (4.1), and the map from (4.1) to (4.2) respects this module structure. This implies that multiplication by the infinite cycles $t^{\pm 4}$ and λ_1' will commute with each differential in (4.2). The module structure also implies that $d^2(\varepsilon_2 \cdot 1) = t\mu \cdot 1 - \varepsilon_2 \cdot d^2(1)$, so that $d^2(\varepsilon_2) = t\mu$ in (4.2). It follows that

$$\hat{E}^4(\mathbb{T}) = \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[t^{\pm 1}]\,.$$

See Figure 4.2.

We showed in Corollary 3.4 that the images in $(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})$ of $\nu \in \{h_{11}\}$ and w are detected by $t\lambda_1'$ and $t\lambda_2$, respectively, so the same holds in $(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{hC_2}$. We also know that the images of ν and w are trivial in $(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})$, so this means that classes detected by $t\lambda_1'$ and $t\lambda_2$ map trivially to $(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2}$. It follows that $t\lambda_1'$ and $t\lambda_2$ must be hit by differentials in the $\overline{V}(2) \otimes \overline{C}\eta$-homotopy C_2-Tate spectral

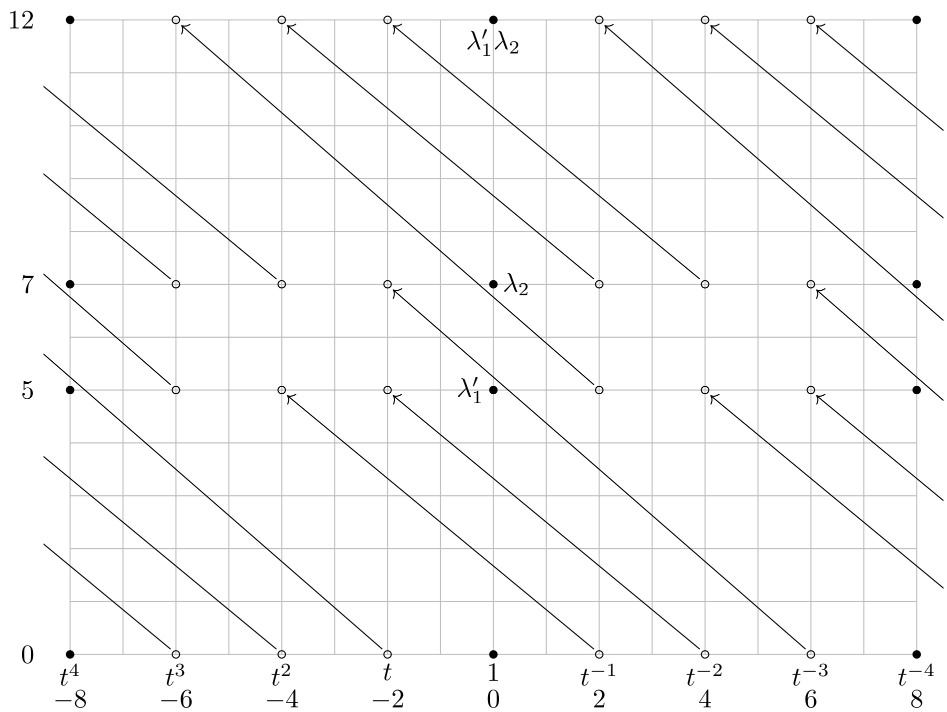

FIGURE 4.2. $\mathbb{T}$-Tate $\hat{E}^4(\mathbb{T}) \Longrightarrow (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko})$

sequence. By examination of bidegrees, the only possibility is that $d^6(t^{-2}) = t\lambda_1'$ and $d^8(t^{-3}) = t\lambda_2$. Since the map of spectral sequences converging to

$$F^t \colon (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \longrightarrow (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2}$$

is injective in the relevant bidegrees, we also have the stated differentials in the spectral sequence (4.2). □

Proposition 4.9. *There are differentials*

$$d^6(t^{-1}) = t^2\lambda_1' \qquad d^6(t^{-1}\lambda_2) = t^2\lambda_1'\lambda_2$$
$$d^6(t^{-2}\lambda_2) = t\lambda_1'\lambda_2 \qquad d^6(\lambda_2) = 0$$

in the spectral sequence (4.2).

Proof. We must have $d^6(t^{-3}) = 0$ in the spectral sequence (4.2), since t^{-3} survives to its E^8-term by Proposition 4.8. By Proposition 4.4 we have $d^6(t^{-2}) = t\lambda_1'$ in the spectral sequence (4.1). Using the module structure of (4.2) over (4.1), we deduce that $d^6(t^{-5}) = d^6(t^{-2} \cdot t^{-3}) = t\lambda_1' \cdot t^{-3} + t^{-2} \cdot 0 = t^{-2}\lambda_1'$ and $d^6(t^{-1}) = t^4 \cdot d^6(t^{-5}) = t^2\lambda_1'$.

Since $t\lambda_2$ is a d^8-boundary by Proposition 4.8 it must be a d^6-cycle, which implies that $d^6(t^{-1}\lambda_2) = d^6(t^{-2} \cdot t\lambda_2) = t\lambda_1' \cdot t\lambda_2 + t^{-2} \cdot 0 = t^2\lambda_1'\lambda_2$.

The fact that $t\lambda_2$ is a d^8-boundary also implies that $\lambda_1' \cdot t\lambda_2 = t\lambda_1'\lambda_2$ must be a d^r-boundary for some $r \leq 8$. Since $t^{-3}\lambda_1' = t^{-4} \cdot t\lambda_1'$ is a d^6-boundary, it cannot be the source of this d^r-differential, so the only remaining possibility is that $d^6(t^{-2}\lambda_2) = t\lambda_1'\lambda_2$.

Using the module structure over the spectral sequence (4.1), we can also conclude that $d^6(\lambda_2) = d^6(t^2 \cdot t^{-2}\lambda_2) = t^5\lambda_1' \cdot t^{-2}\lambda_2 + t^2 \cdot t\lambda_1'\lambda_2 = 0$. □

Corollary 4.10. *There are isomorphisms*

$$(\overline{V}(2) \otimes \overline{C}\eta)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \cong \begin{cases} \mathbb{F}_2\{1\} & \text{for } n = 0, \\ 0 & \text{for } n = 1, 2, 3, \\ \mathbb{F}_2\{t^4\lambda_1'\lambda_2\} & \text{for } n = 4, \\ \mathbb{F}_2\{\lambda_1'\} & \text{for } n = 5, \end{cases}$$

and these repeat 8*-periodically, via multiplication by* $t^{\pm 4}$.

Proof. This follows directly from Proposition 4.8 and Proposition 4.9. □

Proposition 4.11. *We have the following results:*

(a) *The multiplicative relation*

$$(\lambda_1')^2 = \eta^2\mu$$

holds in the abutment $\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})$ *of the* η*-Bockstein spectral sequence* (2.8).

(b) *There is a nonzero differential*

$$d^6(t^{-1}\lambda_1') = t^2\eta^2\mu$$

in the $\overline{V}(2)$*-homotopy* $\mathbb{T}$*-Tate spectral sequence* (4.1)*. Hence*

$$\overline{V}(2)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) = \begin{cases} 0 & \text{for } n \in \{6, 7\}, \end{cases}$$

repeating 8*-periodically via multiplication by* $t^{\pm 4}$.

(c) *The unit images of* η *and* η^2 *are detected by* $t^2\lambda_1'$ *and* $t^4\eta^2\mu$*, respectively, in the spectral sequence* (4.1).

(d) *There is a nonzero differential*

$$d^8(t^{-1}\lambda_1') = t^3\lambda_1'\lambda_2$$

in the $\overline{V}(2) \otimes \overline{C}\eta$*-homotopy* $\mathbb{T}$*-Tate spectral sequence* (4.2)*. Hence*

$$(\overline{V}(2) \otimes \overline{C}\eta)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \cong \begin{cases} 0 & \text{for } n = 6, \\ \mathbb{F}_2\{\lambda_2\} & \text{for } n = 7, \end{cases}$$

repeating 8*-periodically via multiplication by* $t^{\pm 4}$.

Proof. The $\overline{V}(2)$-module cofiber sequence (2.6) induces a long exact sequence

$$\begin{aligned} \dots \longrightarrow (\overline{V}(2) \otimes \overline{C}\eta)_{n+2} \operatorname{gr}^{*+1}_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) &\xrightarrow{j} \overline{V}(2)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \\ \xrightarrow{\eta} \overline{V}(2)_{n+1} \operatorname{gr}^{*+1}_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) &\xrightarrow{i} (\overline{V}(2) \otimes \overline{C}\eta)_{n+1} \operatorname{gr}^{*+1}_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \longrightarrow \dots . \end{aligned}$$

By case $n = 0$ of Corollary 4.6, the cases $n \in \{0, 1\}$ of Corollary 4.10, and the fact that $i(1) = 1$, we deduce from exactness that $\overline{V}(2)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) = 0$ for $n \equiv -1 \mod 8$. Referring back to Proposition 4.4, this implies that $t^{-1}\lambda_1'$ in stem $7 = -1 + 8$ cannot survive to the E^∞-term of (4.1), so the differential $d^6(t^{-1}\lambda_1') = t^2(\lambda_1')^2 = c \cdot t^2\eta^2\mu$ must be nonzero. Hence $c = 1$, which proves that $(\lambda_1')^2 = \eta^2\mu$

and $d^6(t^{-1}\lambda_1') = t^2\eta^2\mu$. This means that the E^∞-term of (4.1), and its abutment, must be trivial in stems 6 and 7.

By the cases $n \in \{1, 2\}$ of Corollary 4.10, and exactness, it also follows that η and η^2 generate $\overline{V}(2)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \cong \mathbb{F}_2$ for $n = 1$ and 2, hence are detected by the only classes in stems 1 and 2, namely $t^2\lambda_1'$ and $t^4\eta^2\mu$, in the E^∞-term of (4.1).

By item (b) and exactness, it follows that $(\overline{V}(2) \otimes \overline{C}\eta)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko})$ is 0 for $n = 6$ and $\mathbb{F}_2$ for $n = 7$. Hence $t^3\lambda_1'\lambda_2$ in stem 6 cannot survive to the E^∞-term of (4.2), and since $d^6(\lambda_2) = 0$ by Proposition 4.9 the only possible source of a differential killing it is $t^{-1}\lambda_1'$. Hence $d^8(t^{-1}\lambda_1') = t^3\lambda_1'\lambda_2$, and the lone surviving class in stem 7 of the E^∞-term of (4.2) is λ_2.

In summary, the homomorphisms in the long exact sequence above are given, at the level of detecting classes, by $\eta\colon 1 \mapsto t^2\lambda_1'$, $\eta\colon t^2\lambda_1' \mapsto t^4\eta^2\mu$, $i\colon 1 \mapsto 1$, $i\colon \lambda_1' \mapsto \lambda_1'$, $j\colon t^4\lambda_1'\lambda_2 \mapsto t^4\eta^2\mu$ and $j\colon \lambda_2 \mapsto \lambda_1'$. □

Corollary 4.12. *We have a preferred isomorphism of bigraded $\mathbb{F}_2$-algebras*

$$\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \cong \frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 + \eta^2\mu)}\,.$$

□

We can now prove Theorems 4.1 and 4.2.

Proof of Theorem 4.1. By Corollary 4.12, the spectral sequence (4.1) has E^2-term:

$$\hat{E}^2(\mathbb{T}) = \Lambda(\varepsilon_2) \otimes \frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 = \eta^2\mu)} \otimes \mathbb{F}_2[t^{\pm 1}]\,. \tag{4.3}$$

The differentials follow from Propositions 4.3, 4.4 and 4.11 leaving

$$\hat{E}^4(\mathbb{T}) = \hat{E}^6(\mathbb{T}) = \mathbb{F}_2\{1, t\lambda_1', \lambda_1', (\lambda_1')^2\} \otimes \mathbb{F}_2[t^{\pm 2}]$$

and

$$\hat{E}^\infty(\mathbb{T}) = \mathbb{F}_2\{1, t^2\lambda_1', (t^2\lambda_1')^2, \lambda_1'\} \otimes \mathbb{F}_2[t^{\pm 4}]\,,$$

with 1, η, η^2, λ_1' and $t^{\pm 4}$ being detected by 1, $t^2\lambda_1'$, $(t^2\lambda_1')^2$, λ_1' and $t^{\pm 4}$ respectively in the E^∞-term. □

Proof of Theorem 4.2. By Corollary 2.31, the spectral sequence (4.2) has E^2-term:

$$\hat{E}^2(\mathbb{T}) = \Lambda(\varepsilon_2) \otimes \frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 + \eta^2\mu)} \otimes \mathbb{F}_2[t^{\pm 1}]\,. \tag{4.4}$$

The differentials follow from Propositions 4.8, 4.9 and 4.11, leaving

$$\hat{E}^4(\mathbb{T}) = \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \mathbb{F}_2[t^{\pm 1}]$$

and

$$\hat{E}^\infty(\mathbb{T}) = \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \mathbb{F}_2[t^{\pm 4}]\,.$$ □

5. Syntomic cohomology

We shall now calculate the syntomic cohomology modulo $(2, v_1)$ and $(2, \eta, v_1)$ of ko (cf. Definition 2.14). We first carry out these computations in $\overline{V}(2)$- and $\overline{V}(2)\otimes\overline{C}\eta$-homotopy, and then use v_2-Bockstein spectral sequences to lift the results to $\overline{V}(1)$- and $\overline{A}(1)$-homotopy.

By restricting the $\mathbb{T}$-Tate spectral sequences (4.1) and (4.2) to the second quadrant, we obtain the $\overline{V}(2)$-homotopy $\mathbb{T}$-homotopy fixed point spectral sequence

$$\begin{aligned} E^2(\mathbb{T}) &= \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})\,[t] \\ &= \Lambda(\varepsilon_2) \otimes \frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 + \eta^2\mu)} \otimes \mathbb{F}_2[t] \\ &\Longrightarrow \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko}) \end{aligned} \tag{5.1}$$

and the $\overline{V}(2) \otimes \overline{C}\eta$-homotopy $\mathbb{T}$-homotopy fixed point spectral sequence

$$\begin{aligned} E^2(\mathbb{T}) &= (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})\,[t] \\ &= \Lambda(\varepsilon_2) \otimes \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \mathbb{F}_2[\mu] \otimes \mathbb{F}_2[t] \\ &\Longrightarrow (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})\,. \end{aligned} \tag{5.2}$$

The former is an algebra spectral sequence, and the latter is a module spectral sequence over it. They can be reindexed as cohomologically graded t-Bockstein spectral sequences, but the current indexing is the one inherited from the homologically graded C_2- and $\mathbb{T}$-Tate spectral sequences. See Figures 5.1. and 5.2.

Proposition 5.1. *There is an isomorphism*

$$\begin{aligned} \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko}) \cong\ & \mathbb{F}_2[t^4]\{1, t^2\lambda_1', \lambda_1', (\lambda_1')^2\} \\ & \oplus \mathbb{F}_2\{t\lambda_1', (t\lambda_1')^2\} \\ & \oplus \mathbb{F}_2[\eta]\{\eta, \eta^4\varepsilon_2\} \\ & \oplus \mathbb{F}_2[\bar\mu]\{\bar\mu, \eta\bar\mu, \eta^2\bar\mu, \lambda_1'\mu\} \end{aligned}$$

with $\bar\mu = \mu + \eta\varepsilon_2$*, where* $(\lambda_1')^2 = \eta^2\mu \neq \eta^2\bar\mu$*,* $\eta \cdot \eta^2\bar\mu = \eta^4\varepsilon_2$ *and* $\bar\mu^2 = \mu^2$*.*

Proof. The map of spectral sequences induced by can: $\mathrm{TC}^-(\mathrm{ko}) \to \mathrm{TP}(\mathrm{ko})$ is given at the E^2-terms by inverting t, so the differentials in (4.1) from Theorem 4.1 lift to differentials

$$\begin{aligned} d^2(t) &= t^2\eta & \qquad d^2(\varepsilon_2) &= t\mu \\ d^6(t^2) &= t^5\lambda_1' & \qquad d^6(t^3\lambda_1') &= t^6(\lambda_1')^2 \end{aligned}$$

in (5.1). Moreover, η, λ_1' and $t\mu$ are infinite cycles. Some bookkeeping shows that

$$\begin{aligned} E^4(\mathbb{T}) = \ & \mathbb{F}_2[t^2]\{1, t\lambda_1', \lambda_1', (\lambda_1')^2\} \\ & \oplus \mathbb{F}_2[\eta]\{\eta, \eta^4\varepsilon_2\} \\ & \oplus \mathbb{F}_2[\bar\mu]\{\bar\mu, \eta\bar\mu, \eta^2\bar\mu, \lambda_1'\mu\} \end{aligned}$$

with $\bar\mu = \mu + \eta\varepsilon_2$ and $\eta \cdot \eta^2\bar\mu = \eta^4\varepsilon_2$, and

$$\begin{aligned} E^8(\mathbb{T}) = E^\infty(\mathbb{T}) = \ & \mathbb{F}_2[t^4]\{1, t^2\lambda_1', \lambda_1', (\lambda_1')^2\} \\ & \oplus \mathbb{F}_2\{t\lambda_1', (t\lambda_1')^2\} \\ & \oplus \mathbb{F}_2[\eta]\{\eta, \eta^4\varepsilon_2\} \\ & \oplus \mathbb{F}_2[\bar\mu]\{\bar\mu, \eta\bar\mu, \eta^2\bar\mu, \lambda_1'\mu\}\,. \end{aligned}$$

□

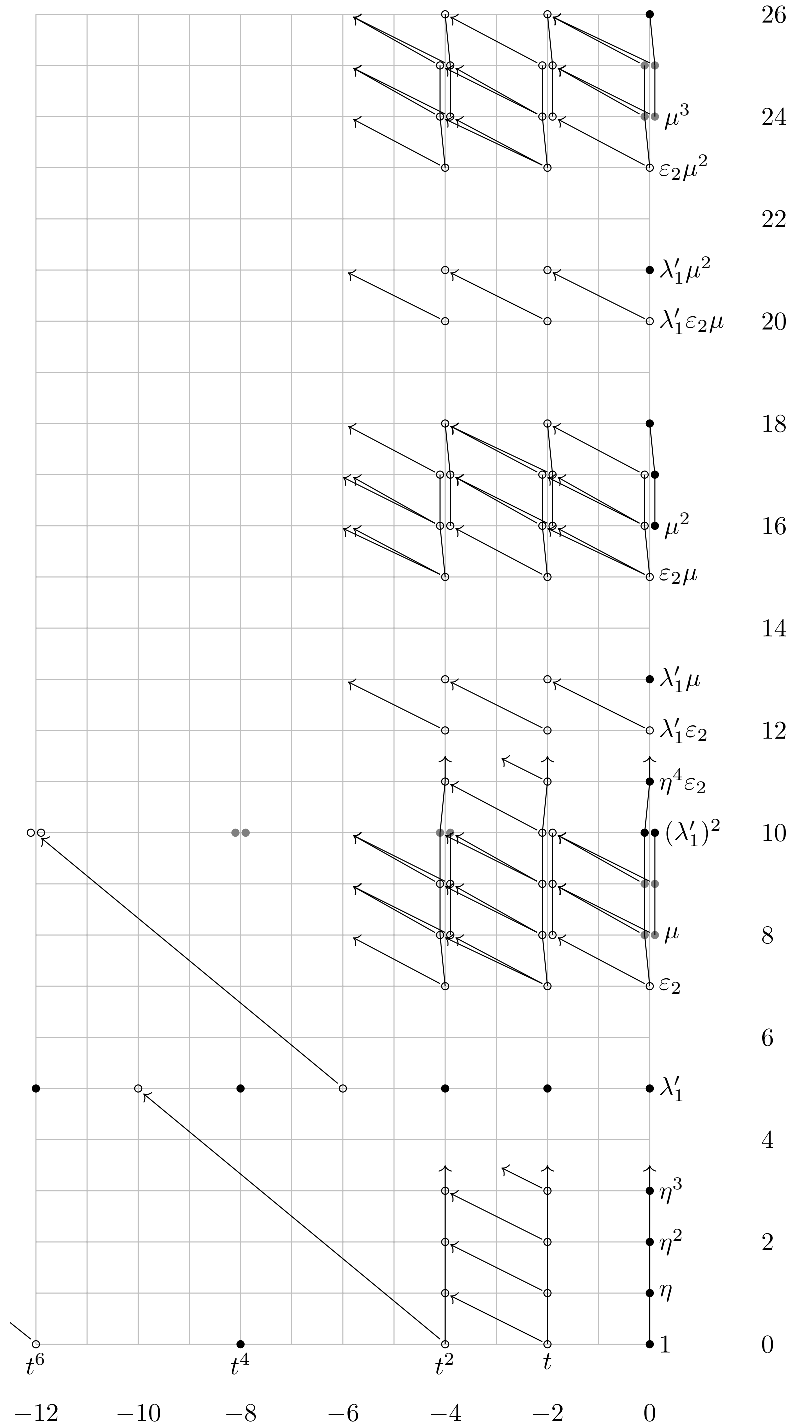

FIGURE 5.1. $\mathbb{T}$-homotopy fixed point spectral sequence converging to $\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})$

Proposition 5.2. *There is an isomorphism*

$$(\overline{V}(2)\otimes\overline{C}\eta)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\mathrm{ko})\cong\frac{\mathbb{F}_2[t^4,\mu]}{(t^4\mu)}\otimes\Lambda(\lambda_1')\{1,\lambda_2\}$$
$$\oplus\,\mathbb{F}_2\{t^2\lambda_1',t\lambda_1',t\lambda_2,t^3\lambda_1'\lambda_2,t^2\lambda_1'\lambda_2,t\lambda_1'\lambda_2\}\,.$$

Proof. The differentials in (4.2) from Theorem 4.2 lift over the canonical map to differentials $d^2(\varepsilon_2)=t\mu$ (repeating t-periodically) and

$$\begin{aligned} d^6(t^2)&=t^5\lambda_1' & d^6(t^3)&=t^6\lambda_1'\\ d^6(t^2\lambda_2)&=t^5\lambda_1'\lambda_2 & d^6(t^3\lambda_2)&=t^6\lambda_1'\lambda_2\\ d^8(t)&=t^5\lambda_2 & d^8(t^3\lambda_1')&=t^7\lambda_1'\lambda_2\end{aligned}$$

(repeating t^4-periodically) in (5.2). It follows that

$$E^4(\mathbb{T})=\mathbb{F}_2[t,\mu]/(t\mu)\otimes\Lambda(\lambda_1')\{1,\lambda_2\}\,,$$

and $E^{10}(\mathbb{T})=E^\infty(\mathbb{T})$ is equal to

$$\frac{\mathbb{F}_2[t^4,\mu]}{(t^4\mu)}\otimes\Lambda(\lambda_1')\{1,\lambda_2\}\,\oplus\,\mathbb{F}_2\{t^2\lambda_1',t\lambda_1',t\lambda_2,t^3\lambda_1'\lambda_2,t^2\lambda_1'\lambda_2,t\lambda_1'\lambda_2\}\,. \qquad\square$$

As discussed in the proof of Proposition 4.4, there is a $\overline{V}(2)$-homotopy C_2-Tate spectral sequence

$$\begin{aligned}\hat{E}^2(C_2)&=\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})\otimes\Lambda(u_1)\otimes\mathbb{F}_2[t^{\pm1}]\\ &=\Lambda(\varepsilon_2)\otimes\frac{\mathbb{F}_2[\eta,\lambda_1',\mu]}{(\eta\lambda_1',(\lambda_1')^2+\eta^2\mu)}\otimes\Lambda(u_1)\otimes\mathbb{F}_2[t^{\pm1}]\\ &\Longrightarrow\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})^{tC_2}\,.\end{aligned}\tag{5.3}$$

Similarly, we have a $\overline{V}(2)\otimes\overline{C}\eta$-homotopy C_2-Tate spectral sequence

$$\begin{aligned}\hat{E}^2(C_2)&=(\overline{V}(2)\otimes\overline{C}\eta)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})\otimes\Lambda(u_1)\otimes\mathbb{F}_2[t^{\pm1}]\\ &=\Lambda(\varepsilon_2)\otimes\Lambda(\lambda_1')\{1,\lambda_2\}\otimes\mathbb{F}_2[\mu]\otimes\Lambda(u_1)\otimes\mathbb{F}_2[t^{\pm1}]\\ &\Longrightarrow(\overline{V}(2)\otimes\overline{C}\eta)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})^{tC_2}\,.\end{aligned}\tag{5.4}$$

There is a map F^t of algebra spectral sequences from (4.1) to (5.3), and (5.4) is a module spectral sequence over (5.3).

Proposition 5.3. *There is an isomorphism*

$$\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})^{tC_2}\cong\mathbb{F}_2\{1,\eta,\eta^2,\lambda_1'\}\otimes\Lambda(u_1)\otimes\mathbb{F}_2[t^{\pm4}]\,,$$

where η, η^2 and λ_1' are detected by $t^2\lambda_1'$, $(t^2\lambda_1')^2$ and λ_1', respectively. Under this correspondence, the cyclotomic structure map

$$\varphi_2\colon\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})\longrightarrow\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})^{tC_2}$$

is given by $\varepsilon_2\mapsto u_1t^{-4}$, $\eta\mapsto\eta$, $\lambda_1'\mapsto\lambda_1'$ and $\mu\mapsto t^{-4}$, hence can be identified with the localization homomorphism

$$\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})\longrightarrow\mu^{-1}\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})$$

that inverts μ.

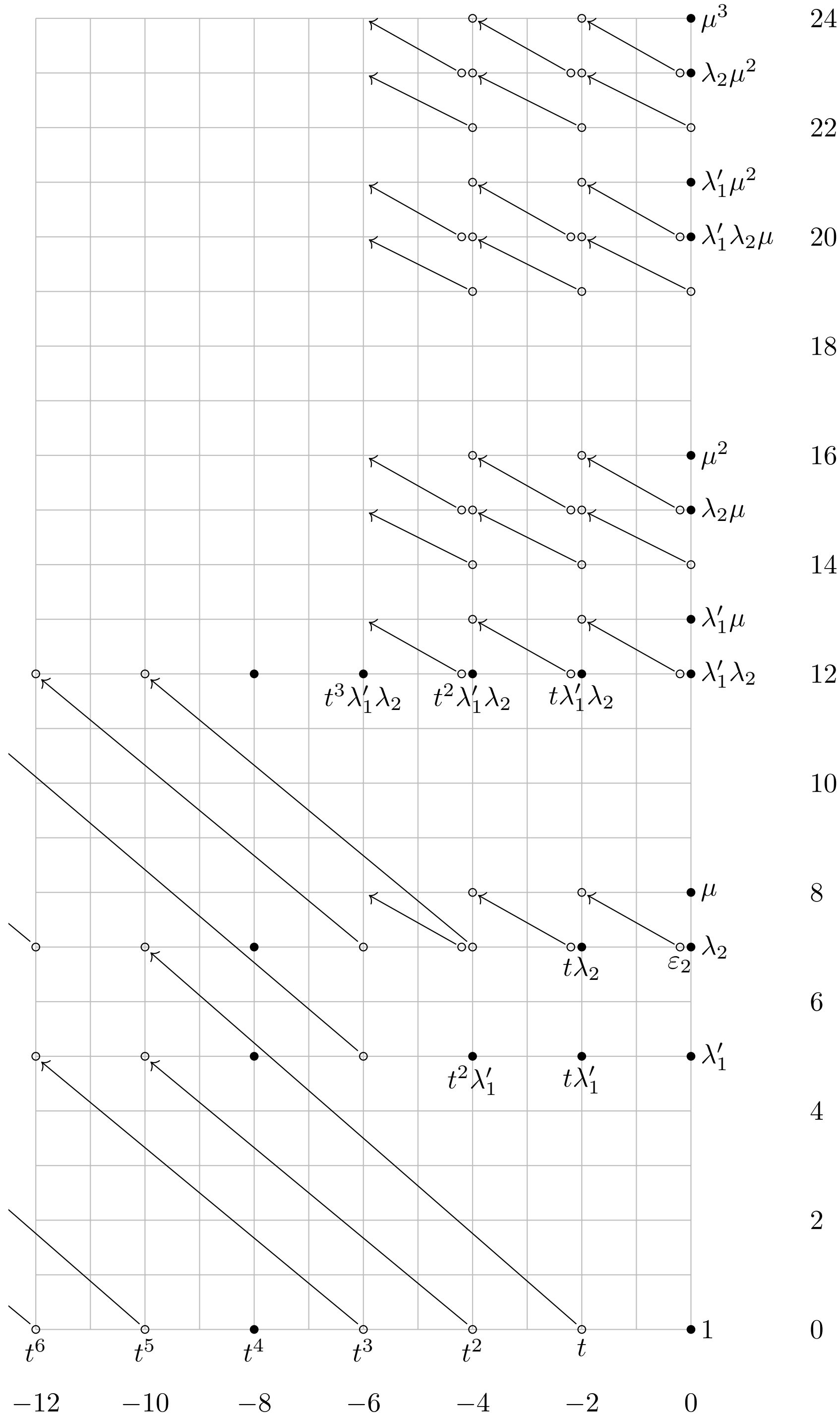

FIGURE 5.2. $\mathbb{T}$-homotopy fixed point spectral sequence converging to $(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})$

Proof. We shall use naturality with respect to the complexification map $c\colon \mathrm{ko} \to \mathrm{ku}$ to access the cyclotomic structure map φ_2 for ko and the differentials in (5.3), so we first review the results of Hahn–Wilson about the complex case.

By [HW22, Proposition 6.1.6], we have $\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku}) \cong \Lambda(\lambda_1, \lambda_2) \otimes \mathbb{F}_2[\mu]$, with $\|\lambda_1\| = (3, 1)$, $\|\lambda_2\| = (7, 1)$ and $\|\mu\| = (8, 0)$. Hence $\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku}) \cong \Lambda(\varepsilon_2, \lambda_1, \lambda_2) \otimes \mathbb{F}_2[\mu]$, with $\|\varepsilon_2\| = (7, -1)$. The $\overline{V}(1)$-homotopy C_2-Tate spectral sequence

$$\begin{aligned}\hat{E}^2(C_2, \mathrm{ku}) &= \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku}) \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 1}] \\ &\Longrightarrow \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku})^{tC_2}\end{aligned}$$

is an algebra spectral sequence with differentials $d^4(t^{-1}) = t\lambda_1$, $d^8(t^{-2}) = t^2\lambda_2$ and $d^9(u_1 t^{-4}) = t\mu$, leaving $\hat{E}^\infty(C_2, \mathrm{ku}) = \Lambda(\lambda_1, \lambda_2) \otimes \mathbb{F}_2[t^{\pm 4}]$, and the cyclotomic structure map is given in $\overline{V}(1)$-homotopy by $\lambda_1 \mapsto \lambda_1$, $\lambda_2 \mapsto \lambda_2$ and $\mu \mapsto t^{-4}$, hence is identified with the ring homomorphism that inverts μ, as in [HRW, Theorem 6.1.2].

It follows that the $\overline{V}(2)$-homotopy C_2-Tate spectral sequence

$$\begin{aligned}\hat{E}^2(C_2, \mathrm{ku}) &= \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku}) \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 1}] \\ &\Longrightarrow \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku})^{tC_2}\end{aligned}$$

has differentials $d^2(\varepsilon_2) = t\mu$, $d^4(t^{-1}) = t\lambda_1$ and $d^8(t^{-2}) = t^2\lambda_2$, leaving behind the E^∞-term $\hat{E}^\infty(C_2, \mathrm{ku}) = \Lambda(\lambda_1, \lambda_2) \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 4}]$. By exactness of localization, the cyclotomic structure map

$$\varphi_2\colon \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku}) \longrightarrow \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku})^{tC_2}$$

must also agree with the ring homomorphism that inverts μ. Hence it is given on λ_1, λ_2 and μ as in the $\overline{V}(1)$-case, while $\varphi_2(\varepsilon_2)$ can only be detected by $u_1 t^{-4}$.

Next we appeal to naturality. We saw in the proof of Lemma 2.18 that

$$c\colon \overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \longrightarrow \overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku})$$

is given by $\lambda_1' \mapsto 0$, $\lambda_2 \mapsto \lambda_2$ and $\mu \mapsto \mu$. It follows by naturality with respect to $i\colon \overline{V}(1) \to \overline{A}(1)$ that $c\colon \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \to \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku})$ is given by $\eta \mapsto 0$, $\lambda_1' \mapsto 0$ and $\mu \mapsto \mu$, and similarly with $\overline{V}(2)$-coefficients, where also $\varepsilon_2 \mapsto \varepsilon_2$. Chasing ε_2 and μ around the commutative diagram

$$\begin{array}{ccc} \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) & \xrightarrow{\varphi_2} & \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2} \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} \\ \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku}) & \xrightarrow{\varphi_2} & \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku})^{tC_2} \,, \end{array}$$

we see that $\varphi_2(\varepsilon_2)$ and $\varphi_2(\mu)$ in the real case must be detected in the same, or higher, Tate filtration as the detecting classes in the complex case, namely $u_1 t^{-4}$ and t^{-4}. There are no classes of higher Tate filtration in the same total degrees, so the only possibility is that $\varphi_2(\varepsilon_2)$ is detected by $u_1 t^{-4}$ and $\varphi_2(\mu)$ is detected by t^{-4}, also in the real case.

In particular, this shows that $u_1 = t^4 \cdot u_1 t^{-4}$ is a permanent cycle in the spectral sequence (5.3). By naturality with respect to the map F^t, we deduce that we have the same differentials in the C_2-Tate spectral sequence as in the $\mathbb{T}$-Tate spectral sequence (4.1), listed in Theorem 4.1. This leaves

$$\hat{E}^\infty(C_2) = \mathbb{F}_2\{1, t^2\lambda_1', (t^2\lambda_1')^2, \lambda_1'\} \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 4}]\,.$$

It is clear that the $\mathbb{S}$-algebra map φ_2 takes η to η, which we saw is detected by $t^2\lambda_1'$. The relation $(\lambda_1')^2 = \eta^2\mu$ now shows that $\varphi_2(\lambda_1')^2$ must be detected by $(t^2\lambda_1')^2 \cdot t^{-4} = (\lambda_1')^2 \neq 0$, which can only happen if $\varphi_2(\lambda_1')$ is detected by λ_1'. Summarizing, we have an isomorphism

$$\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2} \cong \mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 4}]\,,$$

where φ_2 is given by $\varepsilon_2 \mapsto u_1 t^{-4}$, $\eta \mapsto \eta$, $\lambda_1' \mapsto \lambda_1'$ and $\mu \mapsto t^{-4}$. The claim about localization then amounts to the isomorphism

$$\mu^{-1}\frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 + \eta^2\mu)} \cong \mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \otimes \mathbb{F}_2[\mu^{\pm 1}]\,. \qquad \square$$

Proposition 5.4. *There is an isomorphism*

$$(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2} \cong \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 4}]\,,$$

where 1, λ_1', λ_2 *and* $\lambda_1'\lambda_2$ *are detected by classes with the same names. Under this correspondence, the cyclotomic structure map*

$$\varphi_2 \colon (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \longrightarrow (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2}$$

is given by $\varepsilon_2 \mapsto u_1 t^{-4}$, $\lambda_1' \mapsto \lambda_1'$, $\lambda_2 \mapsto \lambda_2$ *and* $\mu \mapsto t^{-4}$, *hence can be identified with the localization homomorphism*

$$(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \longrightarrow \mu^{-1}(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})$$

that inverts μ.

Proof. Naturality with respect to $i \colon \overline{V}(2) \to \overline{V}(2) \otimes \overline{C}\eta$ shows that u_1 is a permanent cycle in the spectral sequence (5.4). When combined with the differentials in (4.2), listed in Theorem 4.2, this shows that

$$\hat{E}^4(C_2) = \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 1}]$$

and

$$\hat{E}^\infty(C_2) = \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \Lambda(u_1) \otimes \mathbb{F}_2[t^{\pm 4}]\,.$$

The detection results then follow from those in Theorem 4.2. The evaluation of φ_2 on ε_2, λ_1' and μ follows from that in Proposition 5.3 by comparison along the same map i.

To show that $\varphi_2(\lambda_2)$ is detected by λ_2, we note that by naturality with respect to $j_2 \colon \overline{V}(2) \otimes \overline{C}\eta \to \Sigma^{7,-1}\overline{A}(1)$ it cannot be detected by $u_1 t^{-4}$. On the other hand, by naturality along $j \colon \overline{V}(2) \otimes \overline{C}\eta \to \Sigma^{2,0}\overline{V}(2)$ it is nonzero, since we saw in the proof of Proposition 4.11 that $j \colon \lambda_2 \mapsto \Sigma^{2,0}\lambda_1'$. Hence $\varphi_2(\lambda_2)$ must be the class detected by λ_2. $\square$

Remark 5.5. The computations in Theorems 4.1 and 4.2 and Propositions 5.3 and 5.4 are consistent with isomorphisms

$$\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \cong \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2}$$
$$(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \cong \overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2}\,,$$

in analogy with [HRW, Theorem 6.2.1].

Theorem 5.6 (Syntomic cohomology modulo $(2, v_1, v_2)$ of ko)**.** *We have an algebra isomorphism*

$$\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \cong \mathbb{F}_2[\eta]\{1, \eta^4\varepsilon_2\} \oplus \mathbb{F}_2\{\partial, \nu, \lambda_1', \partial\lambda_1', \nu^2, (\lambda_1')^2\}\,,$$

with generators in bidegrees $\|\partial\| = (-1, 1)$, $\|\eta\| = (1, 1)$, $\|\nu\| = (3, 1)$, $\|\lambda_1'\| = (5, 1)$ and $\|\eta^4\varepsilon_2\| = (11, 3)$. See Figure 5.3 for a view of the algebra structure of the right-hand side.

Proof. To calculate the effect in $\overline{V}(2)$-homotopy of can: $\mathrm{TC}^-(\mathrm{ko}) \to \mathrm{TP}(\mathrm{ko})$, we use the map of spectral sequences from (5.1) to (4.1), described in Proposition 5.1 and Theorem 4.1, given at the E^2-terms by inverting t. To calculate the effect of $\varphi_2^{h\mathbb{T}}\colon \mathrm{TC}^-(\mathrm{ko}) \to (\mathrm{THH}(\mathrm{ko})^{tC_2})^{h\mathbb{T}}$ we appeal to Proposition 5.3 to see that there is a $\mathbb{T}$-homotopy fixed point spectral sequence

$$\begin{aligned} \mu^{-1}E^2(\mathbb{T}) &= \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2}\,[t] \\ &= \Lambda(\varepsilon_2) \otimes \mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \otimes \mathbb{F}_2[\mu^{\pm 1}] \otimes \mathbb{F}_2[t] \\ &\Longrightarrow \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} (\mathrm{THH}(\mathrm{ko})^{tC_2})^{h\mathbb{T}}\,, \end{aligned} \tag{5.5}$$

and $\varphi_2^{h\mathbb{T}}$ is calculated by the map of spectral sequences from (5.1) to (5.5) that is given at the E^2-terms by inverting μ. The differentials

$$d^2(\varepsilon_2) = t\mu \qquad \text{and} \qquad d^2(\mu) = t\eta\mu$$

carry over from the proof of Proposition 5.1, leaving

$$\mu^{-1}E^4(\mathbb{T}) = \mu^{-1}E^\infty(\mathbb{T}) = \mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \otimes \mathbb{F}_2[\bar{\mu}^{\pm 1}]\,,$$

concentrated on the vertical axis. As before, $\bar{\mu} = \mu + \eta\varepsilon_2$. We know a priori that $G\colon \mathrm{TP}(\mathrm{ko}) \to (\mathrm{THH}(\mathrm{ko})^{tC_2})^{h\mathbb{T}}$ is an equivalence, by [BBLNR14, Proposition 3.8], (cf. [NS18, Lemma II.4.2]). The $\overline{V}(2)$-homotopy isomorphism

$$\mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \otimes \mathbb{F}_2[t^{\pm 4}] \xrightarrow{\cong} \mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \otimes \mathbb{F}_2[\bar{\mu}^{\pm 1}]$$

induced by the equivalence G can then only be given by $\eta \mapsto \eta$, $\lambda_1' \mapsto \lambda_1'$ and $t^{\pm 4} \mapsto \bar{\mu}^{\mp 1}$.

We claim that the map can $- \varphi$ (which is short for $G \circ \mathrm{can} - \varphi_2^{h\mathbb{T}}$) induces isomorphisms

$$\mathbb{F}_2[t^4]\{t^4\} \otimes \mathbb{F}_2\{1, t^2\lambda_1', (t^2\lambda_1')^2, \lambda_1'\} \xrightarrow{\cong} \mathbb{F}_2[\bar{\mu}^{-1}]\{\bar{\mu}^{-1}\} \otimes \mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \tag{5.6}$$

$$\mathbb{F}_2[\bar{\mu}]\{\bar{\mu}\} \otimes \mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \xrightarrow{\cong} \mathbb{F}_2[\bar{\mu}]\{\bar{\mu}\} \otimes \mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \tag{5.7}$$

$$\mathbb{F}_2\{t^2\lambda_1', (t^2\lambda_1')^2\} \xrightarrow{\cong} \mathbb{F}_2\{\eta, \eta^2\} \tag{5.8}$$

and the zero homomorphism

$$\mathbb{F}_2[\eta]\{1, \eta^4\varepsilon_2\} \oplus \mathbb{F}_2\{t\lambda_1', \lambda_1', (t\lambda_1')^2, (\lambda_1')^2\} \xrightarrow{0} \mathbb{F}_2\{1, \lambda_1'\}\,.$$

The isomorphism (5.6) occurs in horizontal degrees (= filtrations) where inverting t (or t^4) is an isomorphism, and $\varphi_2^{h\mathbb{T}}$ is zero. The isomorphism (5.7) occurs in vertical degrees where inverting μ (or $\bar{\mu}$) is an isomorphism, and can is zero. The isomorphism (5.8) uses that η and η^2 in (5.1) are detected by $t^2\lambda_1'$ and $(t^2\lambda_1')^2$ in (4.1), but map to zero in (5.5). The homomorphisms $G \circ \mathrm{can}$ and $\varphi_2^{h\mathbb{T}}$ agree on classes coming from $\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{ev}} \mathbb{S}$, such as 1, η and ν, hence their difference is zero on $\mathbb{F}_2[\eta]\{1\}$ and $\mathbb{F}_2\{t\lambda_1', (t\lambda_1')^2\}$. Both $G \circ \mathrm{can}(\lambda_1')$ and $\varphi_2^{h\mathbb{T}}(\lambda_1')$ are detected by λ_1', hence agree in $\overline{V}(2)$-homotopy since there are no other classes in the same total

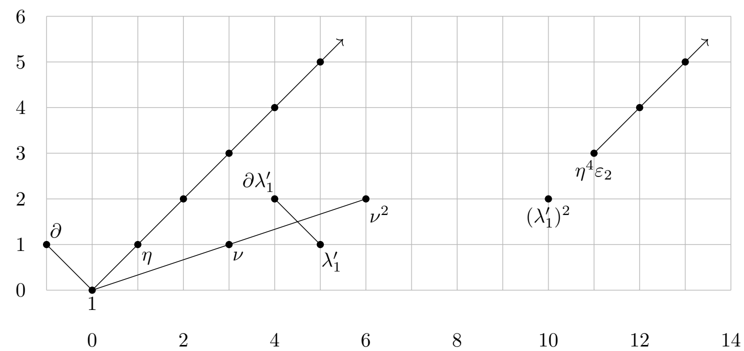

FIGURE 5.3. $\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$, with lines of slope -1, 1 and 1/3 indicating multiplication by ∂, η and ν, respectively

degree, which implies that $G \circ \mathrm{can} - \varphi_2^{h\mathbb{T}}$ is zero on λ_1' and its square. Both $G \circ \mathrm{can}$ and $\varphi_2^{h\mathbb{T}}$ take $\eta^4\varepsilon_2 = \eta^3\bar{\mu}$ to zero, so their difference is zero on $\mathbb{F}_2[\eta]\{\eta^4\varepsilon_2\}$.

Hence we have an isomorphism

$$\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \cong \mathbb{F}_2[\eta]\{1, \eta^4\varepsilon_2\} \oplus \mathbb{F}_2\{\partial, \partial\lambda_1', t\lambda_1', \lambda_1', (t\lambda_1')^2, (\lambda_1')^2\}\,.$$

The classes $t\lambda_1'$ and $(t\lambda_1')^2$ detect ν and ν^2, respectively. The algebra structure is evident from the notation and sparsity, except for the fact that $\eta \cdot \lambda_1' = 0$, which follows from Proposition 5.7 below. □

Next, we compute the v_2-Bockstein spectral sequence

$$E_1 = \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})\,[v_2] \Longrightarrow \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})\,. \tag{5.9}$$

Proposition 5.7. *In the spectral sequence (5.9) there is a d_1-differential*

$$d_1(\eta^4\varepsilon_2) = v_2\eta^4\,,$$

together with its various η- and v_2-power multiples. This produces an algebra isomorphism

$$\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$$
$$\cong \frac{\Lambda(\partial) \otimes \mathbb{F}_2[\eta, \nu, \lambda_1', v_2]}{(\partial\eta, \partial\nu, \eta\nu, \eta\lambda_1', \nu\lambda_1', \nu^3 = v_2\eta^3 = \partial(\lambda_1')^2, (\lambda_1')^3 = d \cdot v_2^2\eta^3)}\,,$$

where $d \in \mathbb{F}_2$ (and we have not resolved this indeterminacy).

Proof. The unit map $\mathbb{S} \to \mathrm{TC}(\mathrm{ko})$ induces a map of v_2-Bockstein spectral sequences, from

$$E_1 = \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_3)\,[v_2] \Longrightarrow \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_2) \tag{5.10}$$

shown in Figure 3.2 to (5.9) shown in Figure 5.4. Since $v_2^2h_{10}^4 = 0$ in the abutment of the former, we must have that $v_2^2\eta^4$ is a boundary in the latter. Considering bidegrees and v_2-powers, this can only happen if $d_1(v_2\eta^4\varepsilon_2) = v_2^2\eta^4$. Hence $d_1(v_2^i\eta^j\varepsilon_2) = v_2^{i+1}\eta^j$ for all $i \geq 0$ and $j \geq 4$, as claimed. There is no room for other v_2-Bockstein differentials, so $E_2 = E_\infty$ in (5.9).

The relation $v_2h_{11}^3 = v_2^2h_{10}^3$ in the abutment of (5.10), see (3.2), also implies that $v_2\nu^3 = v_2^2\eta^3$ in the abutment of (5.9). Hence we have hidden ν-extensions from $v_2^i\nu^2$ to $v_2^{i+1}\eta^3$ for all $i \geq 0$, as shown by black lines of slope 1/3 in Figure 5.4.

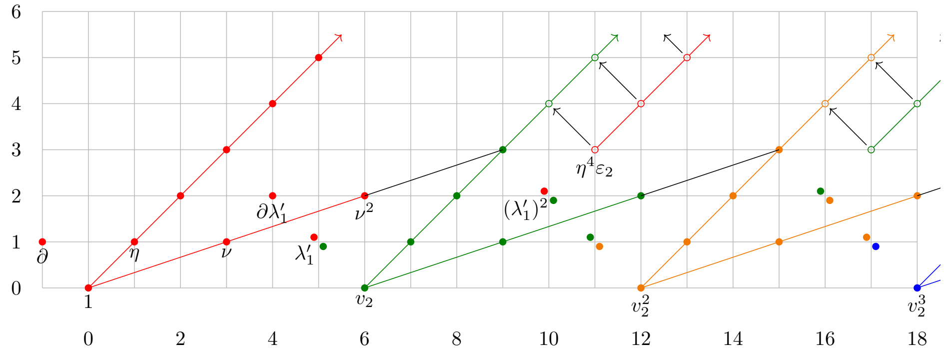

FIGURE 5.4. v_2-Bockstein $E_1 \Longrightarrow \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$

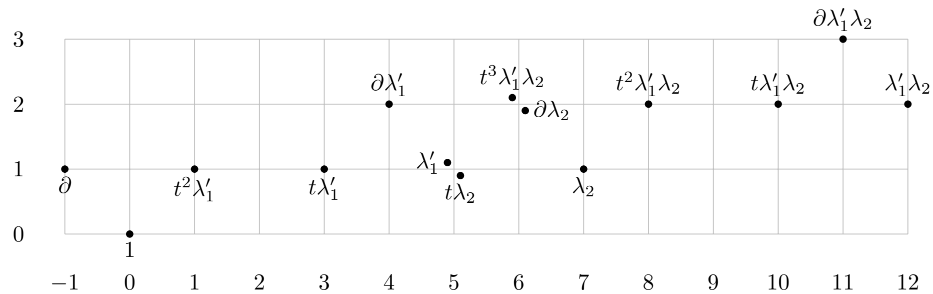

FIGURE 5.5. $\mathbb{F}_2[v_2]$-basis for $\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$

The products ∂^2 and $\partial\eta$ lie in trivial groups. The well-known relation $\eta\nu = 0$ implies the vanishing of $\partial\nu$, $\eta\lambda_1'$ and $\nu\lambda_1'$. We postpone the proof that $\partial(\lambda_1')^2$ is equal to $\nu^3 = v_2\eta^3$ to Remark 5.10. We have not determined whether $(\lambda_1')^3 \in \mathbb{F}_2\{v_2^2\eta^3\}$ is zero or not. □

Proposition 5.8. *We have an isomorphism*

$$\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \cong \mathbb{F}_2[v_2] \otimes \big(\Lambda(\partial, \lambda_1')\{1, \lambda_2\} \oplus \mathbb{F}_2\{t^2\lambda_1', t\lambda_1', t\lambda_2, t^3\lambda_1'\lambda_2, t^2\lambda_1'\lambda_2, t\lambda_1'\lambda_2\}\big)$$

of finitely generated and free $\mathbb{F}_2[v_2]$-modules, where $\|v_2\| = (6, 0)$, $\|\partial\| = (-1, 1)$, $\|\lambda_1'\| = (5, 1)$, $\|\lambda_2\| = (7, 1)$ and $\|t\| = (-2, 0)$. See Figure 5.5.

Proof. This proof is similar to that of Theorem 5.6, to which we refer for a more elaborate review of some of the notations. To calculate the effect of can in $\overline{V}(2) \otimes \overline{C}\eta$-homotopy we use the map of spectral sequences from (5.2) to (4.2), described in Proposition 5.2 and Theorem 4.2, given at the E^2-terms by inverting t. To calculate the effect of $\varphi_2^{h\mathbb{T}}$ we use Corollary 2.31 and Proposition 5.4 to see that there is a

$\mathbb{T}$-homotopy fixed point spectral sequence

$$
\begin{aligned}
\mu^{-1}E^2(\mathbb{T}) &= (\overline{V}(2)\otimes\overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2}\,[t] \\
&= \Lambda(\varepsilon_2)\otimes\Lambda(\lambda_1')\{1,\lambda_2\}\otimes\mathbb{F}_2[\mu^{\pm1}]\otimes\mathbb{F}_2[t] \\
&\Longrightarrow (\overline{V}(2)\otimes\overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} (\mathrm{THH}(\mathrm{ko})^{tC_2})^{h\mathbb{T}}\,,
\end{aligned}
\tag{5.11}
$$

and $\varphi_2^{h\mathbb{T}}$ is given by the map of spectral sequences from (5.2) to (5.11) that is given at the E^2-terms by inverting μ. The differential $d^2(\varepsilon_2) = t\mu$ carries over from the proof of Proposition 5.2, leaving

$$\mu^{-1}E^4(\mathbb{T}) = \mu^{-1}E^\infty(\mathbb{T}) = \Lambda(\lambda_1')\{1,\lambda_2\}\otimes\mathbb{F}_2[\mu^{\pm1}]$$

concentrated on the vertical axis. The $\overline{V}(2)\otimes\overline{C}\eta$-homotopy isomorphism

$$\Lambda(\lambda_1')\{1,\lambda_2\}\otimes\mathbb{F}_2[t^{\pm4}] \xrightarrow{\cong} \Lambda(\lambda_1')\{1,\lambda_2\}\otimes\mathbb{F}_2[\mu^{\pm1}]$$

induced by the equivalence G must thus be given by $\lambda_1'\mapsto\lambda_1'$, $\lambda_2\mapsto\lambda_2$ and $t^{\pm4}\mapsto\mu^{\mp1}$.

The map $G\circ\mathrm{can}-\varphi_2^{h\mathbb{T}}$ induces isomorphisms

$$
\begin{aligned}
\mathbb{F}_2[t^4]\{t^4\}\otimes\Lambda(\lambda_1')\{1,\lambda_2\} &\xrightarrow{\cong} \Lambda(\lambda_1')\{1,\lambda_2\}\otimes\mathbb{F}_2[\mu^{-1}]\{\mu^{-1}\} \\
\mathbb{F}_2[\mu]\{\mu\}\otimes\Lambda(\lambda_1')\{1,\lambda_2\} &\xrightarrow{\cong} \Lambda(\lambda_1')\{1,\lambda_2\}\otimes\mathbb{F}_2[\mu]\{\mu\}
\end{aligned}
$$

and the zero homomorphism

$$\Lambda(\lambda_1')\{1,\lambda_2\}\oplus\mathbb{F}_2\{t^2\lambda_1', t\lambda_1', t\lambda_2, t^3\lambda_1'\lambda_2, t^2\lambda_1'\lambda_2, t\lambda_1'\lambda_2\} \xrightarrow{0} \Lambda(\lambda_1')\{1,\lambda_2\}\,,$$

by the same arguments as in the proof of Theorem 5.6. Hence we have an isomorphism

$$
\begin{aligned}
(\overline{V}(2)\otimes\overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) &\cong \Lambda(\partial,\lambda_1')\{1,\lambda_2\} \\
&\quad\oplus\mathbb{F}_2\{t^2\lambda_1', t\lambda_1', t\lambda_2, t^3\lambda_1'\lambda_2, t^2\lambda_1'\lambda_2, t\lambda_1'\lambda_2\}\,.
\end{aligned}
$$

There is no room for differentials in the v_2-Bockstein spectral sequence

$$E_1 = (\overline{V}(2)\otimes\overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})\,[v_2] \Longrightarrow \overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})\,. \qquad \square$$

Lemma 5.9. *The unit map*

$$\pi_*\overline{A}(1) \longrightarrow \overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$$

sends the classes 1, h_{11}, w, $h_{11}^2 = h_{10}w$, $h_{11}w$ *and* h_{11}^2w *to classes that are detected by* 1, $t\lambda_1'$, $t\lambda_2$, $t^3\lambda_1'\lambda_2 \mod \partial\lambda_2$, $t^2\lambda_1'\lambda_2$ *and* $\partial\lambda_1'\lambda_2$*, respectively. The product* $h_{11}\lambda_2$ *is detected by* $t\lambda_1'\lambda_2$.

Proof. The $\overline{V}(1)$-module cofiber sequence (2.5) induces a long exact sequence

$$
\begin{aligned}
\dots \xrightarrow{\eta} \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) &\xrightarrow{i} \overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \\
&\xrightarrow{j} \Sigma^{2,0}\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \xrightarrow{\eta} \dots
\end{aligned}
$$

of $\mathbb{F}_2[v_2]$-modules, see Figures 5.4 and 5.5. Having chosen $\lambda_1'\in\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$ we choose $\lambda_1', \lambda_2\in\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$ so that $i(\lambda_1') = \lambda_1'$ and $j(\lambda_2) = \Sigma^{2,0}\lambda_1'$. By exactness, i is then given by

$$\begin{aligned} 1 &\longmapsto 1 \\ \partial &\longmapsto \partial \\ \nu &\longmapsto t\lambda_1' \\ \lambda_1' &\longmapsto \lambda_1' \\ \partial\lambda_1' &\longmapsto \partial\lambda_1' \\ \nu^2 &\longmapsto t^3\lambda_1'\lambda_2 \quad \mod \partial\lambda_2 \\ (\lambda_1')^2 &\longmapsto t\lambda_1'\lambda_2 \quad \mod v_2\partial\lambda_1' \,, \end{aligned}$$

while j is given by

$$\begin{aligned} t^2\lambda_1' &\longmapsto \Sigma^{2,0}\partial \\ t\lambda_2 &\longmapsto \Sigma^{2,0}\nu \\ \lambda_2 &\longmapsto \Sigma^{2,0}\lambda_1' \\ \partial\lambda_2 &\longmapsto \Sigma^{2,0}\partial\lambda_1' \\ t^2\lambda_1'\lambda_2 &\longmapsto \Sigma^{2,0}\nu^2 \\ \lambda_1'\lambda_2 &\longmapsto \Sigma^{2,0}(\lambda_1')^2 \\ \partial\lambda_1'\lambda_2 &\longmapsto \Sigma^{2,0}\nu^3 \,. \end{aligned}$$

The formulas for i imply the claims for 1, $\nu = h_{11}$ and $\nu^2 = h_{11}^2$. We know from Corollary 3.4 that w is detected by $t\lambda_2$, so the formulas for j imply the claims for $\nu w = h_{11}w$ and $\nu^2 w = h_{11}^2 w$.

The $\overline{V}(1)$-module action on $\overline{A}(1)$ shows that $\nu \cdot \lambda_2 = h_{11}\lambda_2$ is detected by $t\lambda_1' \cdot \lambda_2$, since the latter product is nonzero in $\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})$. □

Remark 5.10. We can now complete the unfinished business in the proof of Proposition 5.7. Since $\nu^2 w$ is detected by $\partial\lambda_1'\lambda_2$, and j maps w to $\Sigma^{2,0}\nu$ and λ_2 to $\Sigma^{2,0}\lambda_1'$, it follows that $\Sigma^{2,0}\nu^3$ is detected by $\Sigma^{2,0}\partial(\lambda_1')^2$, so $\partial(\lambda_1')^2$ is equal to $\nu^3 = v_2\eta^3$ in $\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$.

Lemma 5.11. *Let $\varsigma \in \overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$ be the class in bidegree $(1,1)$ detected by $t^2\lambda_1'$. Then $\varsigma\nu$ is the class in bidegree $(4,2)$ detected by $\partial\lambda_1'$.*

Proof. By [BHM93, Theorem 5.17], [Rog02, Corollary 1.21] there is a 2-complete equivalence $\mathrm{TC}(\mathbb{S}) \simeq \mathbb{S} \oplus \Sigma\mathbb{C}P^\infty_{-1}$, and by [BM94, Proposition 10.9], [Dun97, Main Theorem] the 3-connected map $\mathbb{S} \to \mathrm{ko}$ induces a 4-connected map $\mathrm{TC}(\mathbb{S}) \to \mathrm{TC}(\mathrm{ko})$. For each $i \ge -1$ let $\Sigma\beta_i \in H_{2i+1}(\Sigma\mathbb{C}P^\infty_{-1})$ denote the generator. The Atiyah–Hirzebruch spectral sequence

$$E^2 = H_*(\Sigma\mathbb{C}P^\infty_{-1}; \pi_* A(1)) \Longrightarrow A(1)_*(\Sigma\mathbb{C}P^\infty_{-1})$$

has nonzero differentials $d^4(\Sigma\beta_1) = \nu\Sigma\beta_{-1}$ and $d^6(\Sigma\beta_2) = w\Sigma\beta_{-1}$. This follows from [Mos68, Proposition 5.2, Proposition 5.4], using that $w \in \langle \nu, \eta, \iota \rangle$ in $\pi_* A(1)$, where ι is the class of $\mathbb{S} \to A(1)$. Hence

$$A(1)_* \mathrm{TC}(\mathbb{S}) \cong \mathbb{F}_2\{\Sigma\beta_{-1}, \iota, \Sigma\beta_0, \nu\iota, \nu\Sigma\beta_0\}$$

in stems $-1 \le * \le 4$, mapping isomorphically to

$$A(1)_* \mathrm{TC}(\mathrm{ko}) \cong \mathbb{F}_2\{\partial, 1, t^2\lambda_1', t\lambda_1', \partial\lambda_1'\}$$

in this range. It follows that $\Sigma\beta_0$ maps to the class ς detected by $t^2\lambda_1'$ and $\nu\Sigma\beta_0$ to the class detected by $\partial\lambda_1'$, which must therefore be equal to $\varsigma\nu$. □

Theorem 5.12 (Syntomic cohomology modulo $(2, \eta, v_1)$ of ko)**.** *We have an isomorphism*

$$\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \cong \mathbb{F}_2[v_2] \otimes \big(\mathbb{F}_2\{1, \partial, \nu, w, \nu^2 = \eta w, \nu w, \lambda_1'\lambda_2, \nu^2 w = \partial\lambda_1'\lambda_2\} \\ \oplus \mathbb{F}_2\{\varsigma, \lambda_1', \varsigma\nu = \partial\lambda_1'\} \oplus \mathbb{F}_2\{\lambda_2, \partial\lambda_2, \nu\lambda_2\}\big)$$

of $\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$-modules, where the (stem, motivic filtration) bidegrees and detecting classes of the $\mathbb{F}_2[v_2]$-module generators are as in Table 5.1. See also Figure 1.1.

Proof. This summarizes the results of Proposition 5.8 and Lemmas 5.9 and 5.11. The lift of $t^3\lambda_1'\lambda_2$ over $\pi\colon \mathrm{TC}(\mathrm{ko}) \to \mathrm{TC}^-(\mathrm{ko})$ is only defined modulo $\partial\lambda_2$ in the image under $\partial\colon \Sigma^{-1}\mathrm{TP}(\mathrm{ko}) \to \mathrm{TC}(\mathrm{ko})$, but the image of ν^2 specifies one such choice of lift. □

generator	bidegree	detecting class
1	$(0,0)$	1
∂	$(-1,1)$	∂
ς	$(1,1)$	$t^2\lambda_1'$
ν	$(3,1)$	$t\lambda_1'$
w	$(5,1)$	$t\lambda_2$
λ_1'	$(5,1)$	λ_1'
λ_2	$(7,1)$	λ_2
$\varsigma\nu$	$(4,2)$	$\partial\lambda_1'$
$\partial\lambda_2$	$(6,2)$	$\partial\lambda_2$
ν^2	$(6,2)$	$t^3\lambda_1'\lambda_2 \mod \partial\lambda_2$
νw	$(8,2)$	$t^2\lambda_1'\lambda_2$
$\nu\lambda_2$	$(10,2)$	$t\lambda_1'\lambda_2$
$\lambda_1'\lambda_2$	$(12,2)$	$\lambda_1'\lambda_2$
$\nu^2 w$	$(11,3)$	$\partial\lambda_1'\lambda_2$

TABLE 5.1. Bidegrees and detecting classes for the $\mathbb{F}_2[v_2]$-module generators of $\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$

6. Topological cyclic homology and algebraic K-theory

We now use the motivic spectral sequence

$$E_2 = \overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \Longrightarrow A(1)_* \mathrm{TC}(\mathrm{ko}) \tag{6.1}$$

to compute the $A(1)$-homotopy of the topological cyclic homology of ko. The E_2-term, given in Theorem 5.12, is concentrated in even total degrees and motivic filtrations $0 \le * \le 3$, so the only possibly nonzero differentials are

$$d_3(v_2^i) \in \mathbb{F}_2\{v_2^{i-2}\nu^2 w\}$$

for $i \ge 2$. We show that some, but not all, of these differentials are nonzero. This contrasts with the motivic spectral sequence converging to $V(1)_* \mathrm{TC}(\ell)$ at odd primes p, which was shown to collapse at the E_2-term by Hahn–Raksit–Wilson in [HRW, Corollary 1.3.3].

Notation 6.1. We equip $A(1) \otimes \mathrm{TC}(\mathrm{ko})$ with the relative convolution filtration

$$\mathrm{fil}^\star_{\mathrm{mot}}(A(1) \otimes \mathrm{TC}(\mathrm{ko})) := \mathrm{fil}^\star_{\mathrm{ev}} A(1) \otimes_{\mathrm{fil}^*_{ev}\mathbb{S}} \mathrm{fil}^\star_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})\,,$$

with associated graded $\mathrm{gr}^*_{\mathrm{mot}}(A(1) \otimes \mathrm{TC}(\mathrm{ko})) \simeq \overline{A}(1) \otimes \mathrm{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$. (In view of Lemma 2.26, this filtration agrees with the motivic filtration on $A(1) \otimes \mathrm{TC}(\mathrm{ko})$ as defined in [HRW, Variants 2.1.9 and 4.2.2] for the 2-cyclotomic module spectrum $A(1) \otimes \mathrm{THH}(\mathrm{ko})$, using [HRW, Corollary 2.2.15].) The motivic spectral sequence (6.1) is the associated spectral sequence, converging to $\pi_*(A(1)\otimes\mathrm{TC}(\mathrm{ko})) = A(1)_* \mathrm{TC}(\mathrm{ko})$.

We write

$$\mathrm{Fil}^w_{\mathrm{mot}} A(1)_* \mathrm{TC}(\mathrm{ko}) = \mathrm{im}(\pi_* \mathrm{fil}^w_{\mathrm{mot}}(A(1) \otimes \mathrm{TC}(\mathrm{ko})) \longrightarrow A(1)_* \mathrm{TC}(\mathrm{ko}))$$
$$\mathrm{Gr}^w_{\mathrm{mot}} A(1)_* \mathrm{TC}(\mathrm{ko}) = \mathrm{Fil}^w_{\mathrm{mot}} A(1)_* \mathrm{TC}(\mathrm{ko}) / \mathrm{Fil}^{w+1}_{\mathrm{mot}} A(1)_* \mathrm{TC}(\mathrm{ko})$$

for the induced (algebraic) filtration on $A(1)_* \mathrm{TC}(\mathrm{ko})$ and its associated graded, so that

$$E_\infty \cong \mathrm{Gr}^*_{\mathrm{mot}} A(1)_* \mathrm{TC}(\mathrm{ko})\,.$$

In each stem n the E_∞-term contains at most two nonzero groups, in motivic filtrations $s \in \{0, 2\}$ or $s \in \{1, 3\}$, according to the parity of n.

Bhattacharya–Egger–Mahowald [BEM17, Main Theorem] proved for each version of $A(1)$ that there exists a v_2^{32}-self map $\Sigma^{192}A(1) \to A(1)$. We noted in Lemma 3.5 that $\mathrm{id}\colon A(1) \to A(1)$ has additive exponent 4. Hence there is a natural $\mathbb{Z}/4[v_2^{32}]$-module structure on (6.1) and its abutment. This factors through a finitely generated and free $\mathbb{F}_2[v_2^4]$-module structure on the associated graded.

Theorem 6.2. *The motivic spectral sequence* (6.1) *has nonzero differentials*

$$d_3(v_2^i) = v_2^{i-2}\nu^2 w$$

for $i \equiv 2, 3 \mod 4$*. The remaining differentials are zero. Hence*

$$\begin{aligned}\mathrm{Gr}^*_{\mathrm{mot}} A(1)_* \mathrm{TC}(\mathrm{ko}) = {} & \mathbb{F}_2\{v_2^i \mid i \equiv 0, 1 \mod 4\} \\ & \oplus \mathbb{F}_2[v_2]\{\partial, \varsigma, \nu, \lambda_1', w, \lambda_2\} \\ & \oplus \mathbb{F}_2[v_2]\{\varsigma\nu, \nu^2, \partial\lambda_2, \nu w, \nu\lambda_2, \lambda_1'\lambda_2\} \\ & \oplus \mathbb{F}_2\{v_2^j \nu^2 w \mid j \equiv 2, 3 \mod 4\}\end{aligned}$$

is a finitely generated and free $\mathbb{F}_2[v_2^4]$*-module of rank* 52*. Here* $|\partial| = -1$*,* $|\varsigma| = 1$*,* $|\nu| = 3$*,* $|\lambda_1'| = |w| = 5$*,* $|v_2| = 6$ *and* $|\lambda_2| = 7$*.*

Proof. The unit map $\mathbb{S} \to \mathrm{TC}(\mathrm{ko})$ induces a map from the Novikov spectral sequence for $A(1)$, as discussed in Lemma 3.2, to the motivic spectral sequence (6.1). By Lemma 5.9 this map of E_2-terms sends v_2^i to v_2^i and $v_2^i h_{11}^2 w$ to $v_2^i \partial\lambda_1'\lambda_2$ for each $i \geq 0$. Since $d_3(1) = d_3(v_2) = 0$, $d_3(v_2^2) = h_{11}^2 w$ and $d_3(v_2^3) = v_2 h_{11}^2 w$ in the Novikov spectral sequence, we must have $d_3(1) = d_3(v_2) = 0$, $d_3(v_2^2) = \partial\lambda_1'\lambda_2$ and $d_3(v_2^3) = v_2\partial\lambda_1'\lambda_2$ in the motivic spectral sequence. This handles the cases $0 \leq i < 4$. By Corollary 3.6, we know that these d_3-differentials propagate v_2^4-periodically, as claimed.

It follows that all classes in motivic filtrations 1 and 2 survive to E_∞. In filtrations 0 and 3, only the classes v_2^i with $0 \le i \equiv 0, 1 \mod 4$ and $v_2^{i-2}\nu^2 w$ with $2 \le i \equiv 0, 1 \mod 4$ survive. Setting $0 \le j = i - 2$ gives the asserted formula. □

Remark 6.3. The additive extensions

$$0 \to E_\infty^{n,s+2} \longrightarrow A(1)_n \operatorname{TC}(\mathrm{ko}) \longrightarrow E_\infty^{n,s} \to 0$$

(with $s = 0$ for n even, $s = 1$ for n odd) are sometimes nontrivial. For example, we see from Figure 3.1 that $2 \cdot v_2 = \nu^2$ in $\pi_6 A(1)[ij]$ if (and only if) $[ij] \in \{[10], [11]\}$, which implies that $A(1)_6 \operatorname{TC}(\mathrm{ko}) \cong \mathbb{Z}/4\{v_2\} \oplus \mathbb{Z}/2\{\partial\lambda_2\}$ in these two cases. We have not carried out a complete analysis of these extension problems.

Theorem 6.2 allows us to determine the $A(1)$-homotopy of the algebraic K-theory spectra of ko and $\mathrm{ko}_2^\wedge$. We begin with the 2-complete case.

Theorem 6.4. *There is an exact sequence of $\mathbb{Z}/4[v_2^{32}]$-modules*

$$0 \to \Sigma^1\mathbb{F}_2 \oplus \Sigma^3\mathbb{F}_2 \longrightarrow A(1)_* \operatorname{K}(\mathrm{ko}_2^\wedge) \xrightarrow{\mathrm{trc}} A(1)_* \operatorname{TC}(\mathrm{ko}) \longrightarrow \mathbb{F}_2\{\partial, \varsigma\} \to 0\,,$$

with $|\partial| = -1$ and $|\varsigma| = 1$.

Proof. Let $\mathbb{Z}_2 = \pi_0(\mathrm{ko}_2^\wedge)$ denote the 2-adic integers. By [HM97, Theorem D] and [Dun97, Main Theorem] (cf. [DGM13, Theorem 7.3.1.8]) applied to the 1-connected $\mathbb{E}_\infty$ ring map $\mathrm{ko}_2^\wedge \to H\mathbb{Z}_2$ there is a cofiber sequence

$$\operatorname{K}(\mathrm{ko}_2^\wedge)_2^\wedge \xrightarrow{\mathrm{trc}} \operatorname{TC}(\mathrm{ko})_2^\wedge \xrightarrow{p} \Sigma^{-1}H\mathbb{Z}_2\,.$$

The associated long exact sequence in $A(1)$-homotopy breaks up into four-term exact sequences, as above.

In more detail, the 3-connected map $A(1) \to H = H\mathbb{F}_2$ identifies $A(1)_*H\mathbb{Z}_2$ with $\mathbb{F}_2\{1, \xi_1^2, \bar\xi_2, \xi_1^2\bar\xi_2\} \subset H_*H\mathbb{Z}_2 \subset \mathcal{A}^\vee$. By [BM94, Proposition 10.9], $\operatorname{K}(\mathrm{ko}_2^\wedge) \to \operatorname{K}(\mathbb{Z}_2)$ is 2-connected, where $\operatorname{K}_0(\mathbb{Z}_2) = \mathbb{Z}$ and $\operatorname{K}_1(\mathbb{Z}_2) = \mathbb{Z}_2^\times$, so that $A(1)_0 \operatorname{K}(\mathrm{ko}_2^\wedge) \cong A(1)_0 \operatorname{K}(\mathbb{Z}_2) = \mathbb{Z}/2$ and $A(1)_1 \operatorname{K}(\mathrm{ko}_2^\wedge) \cong A(1)_1 \operatorname{K}(\mathbb{Z}_2) = \mathbb{Z}_2^\times/(\pm(\mathbb{Z}_2^\times)^2) \cong \mathbb{Z}/2$, generated by any $u \in \mathbb{Z}_2^\times$ congruent to 3 or 5 modulo 8. This uses that $\eta \in \pi_1(\mathbb{S})$ maps to $-1 \in \mathbb{Z}_2^\times \cong \operatorname{K}_1(\mathbb{Z}_2)$. By exactness, we know $p\colon \partial \mapsto \Sigma^{-1}1$ and $p\colon \varsigma \mapsto \Sigma^{-1}\xi_1^2$. Multiplication by ν acts trivially on $H\mathbb{Z}_2$, so $p\colon \varsigma\nu \mapsto 0$ does not hit $\Sigma^{-1}\xi_1^2\bar\xi_2$. There is no class in degree 2 that p could map to $\Sigma^{-1}\bar\xi_2$. Hence these two classes instead appear as $\Sigma^{-2}\bar\xi_2$ and $\Sigma^{-2}\xi_1^2\bar\xi_2$ in $A(1)_* \operatorname{K}(\mathrm{ko}_2^\wedge)$, in degrees 1 and 3, respectively. □

The proof in the integral case relies on the proven Lichtenbaum–Quillen conjecture for $\mathbb{Z}[1/2]$, cf. [Voe03] and [RW00].

Theorem 6.5. *There is an exact sequence of $\mathbb{Z}/4[v_2^{32}]$-modules*

$$0 \to \Sigma^3\mathbb{F}_2 \longrightarrow A(1)_* \operatorname{K}(\mathrm{ko}) \xrightarrow{\mathrm{trc}} A(1)_* \operatorname{TC}(\mathrm{ko}) \longrightarrow \mathbb{F}_2\{\partial, \varsigma\} \to 0\,,$$

with $|\partial| = -1$ and $|\varsigma| = 1$.

Proof. By [Rog02, Theorem 3.13] there are two cofiber sequences

$$\operatorname{K}(\mathrm{ko})_2^\wedge \xrightarrow{\mathrm{trc}} \operatorname{TC}(\mathrm{ko})_2^\wedge \xrightarrow{q} X$$

$$\Sigma^{-2}\mathrm{ku}_2^\wedge \xrightarrow{\delta} \Sigma^4\mathrm{ko}_2^\wedge \longrightarrow X$$

with equivalent third terms. Passing to $A(1)$-homotopy, the second cofiber sequence ensures that $A(1)_*X = \mathbb{F}_2\{x_{-1}, x_1, x_4\}$, where $|x_i| = i$. The long exact sequence associated to the first cofiber sequence then breaks up into four-term exact sequences, as shown.

This time, the details are as follows. The 3-connected $\mathbb{E}_\infty$ ring map $\mathbb{S} \to \mathrm{ko}$ induces a 4-connected map $\mathrm{K}(\mathbb{S}) \to \mathrm{K}(\mathrm{ko})$, where

$$\mathrm{K}(\mathbb{S}) \simeq \mathbb{S} \oplus \mathrm{Wh}^{\mathrm{Diff}}(*)\,.$$

Here $\mathrm{Wh}^{\mathrm{Diff}}(*)$ is 2-connected with $\pi_3 \mathrm{Wh}^{\mathrm{Diff}}(*) = \mathbb{Z}/2$, cf. [Rog02, Theorem 5.8]. Hence $A(1)_0 \mathrm{K}(\mathrm{ko}) \cong A(1)_0 \mathrm{K}(\mathbb{S}) = \mathbb{Z}/2\{1\}$, $A(1)_1 \mathrm{K}(\mathrm{ko}) \cong A(1)_1 \mathrm{K}(\mathbb{S}) = 0$, $A(1)_2 \mathrm{K}(\mathrm{ko}) \cong A(1)_2 \mathrm{K}(\mathbb{S}) = 0$ and $A(1)_3 \mathrm{K}(\mathrm{ko}) \cong A(1)_3 \mathrm{K}(\mathbb{S}) = \mathbb{Z}/2\{\nu\} \oplus \mathbb{Z}/2$. By exactness, we know $q\colon \partial \mapsto x_{-1}$ and $q\colon \varsigma \mapsto x_1$, while x_4 must contribute to $A(1)_3 \mathrm{K}(\mathrm{ko})$ and cannot be in the image of q. (It follows that $\nu x_1 = 0 \neq x_4$.) $\square$

Corollary 6.6. *The unit map $\mathbb{S} \to \mathrm{tmf}$ does not factor through $\mathrm{K}(\mathrm{ko})$.*

Proof. In fact, the unit map $A(1) \to A(1) \otimes \mathrm{tmf}$ does not factor through $A(1) \otimes \mathrm{K}(\mathrm{ko})$, since $\pi_{20}A(1) \to A(1)_{20}(\mathrm{tmf}) \cong (\mathbb{Z}/2)^2$ is surjective, as can be seen using Bruner's `ext` program or from [Pha22, Figure 16], while

$$A(1)_{20}\,\mathrm{K}(\mathrm{ko}) \cong A(1)_{20}\,\mathrm{TC}(\mathrm{ko}) \cong \mathbb{Z}/2\{v_2^2 \nu w\}$$

by Theorems 6.2 and 6.5, cf. Figure 1.1. $\square$

The proof by Hahn–Raksit–Wilson [HRW] of the height 2 telescope conjecture for TC(ku) can be adapted to prove the corresponding statement for TC(ko), using our Proposition 2.10 and Theorem 2.22. However, as was kindly pointed out to us by Ishan Levy, this is also a direct consequence of the descent result of Clausen–Mathew–Naumann–Noel [CMNN20], as we summarize below.

Theorem 6.7. *For each $X \in \{\mathrm{K}(\mathrm{ko}), \mathrm{K}(\mathrm{ko}_2^\wedge), \mathrm{TC}(\mathrm{ko})\}$ the canonical map $L_2^f X \to L_2 X$ is an equivalence. In other words, these spectra all satisfy the height 2 telescope conjecture (at the prime 2).*

Proof. According to [Dun97], [HM97] and [RW00] there are equivalences

$$L_{T(2)}\,\mathrm{K}(\mathrm{ku}) \simeq L_{T(2)}\,\mathrm{K}(\mathrm{ku}_2^\wedge) \simeq L_{T(2)}\,\mathrm{TC}(\mathrm{ku})\,.$$

By [HRW, Theorem 6.6.4], $L_2^f\,\mathrm{TC}(\mathrm{ku}) \simeq L_2\,\mathrm{TC}(\mathrm{ku})$, which by [Hov95, Corollary 2.2] implies that $L_{T(2)}\,\mathrm{TC}(\mathrm{ku}) \simeq L_{K(2)}\,\mathrm{TC}(\mathrm{ku})$ is $K(2)$-local. Applying descent [CMNN20, Theorem 1.8] along $\mathrm{ko} \to \mathrm{ku}$ or $\mathrm{ko}_2^\wedge \to \mathrm{ku}_2^\wedge$, for $E = \mathrm{K}$ or TC, it follows that $L_{T(2)}\,\mathrm{K}(\mathrm{ko})$, $L_{T(2)}\,\mathrm{K}(\mathrm{ko}_2^\wedge)$ and $L_{T(2)}\,\mathrm{TC}(\mathrm{ko})$ are all limits of $K(2)$-local spectra, hence are $K(2)$-local. In particular, $L_{T(2)}X \simeq L_{K(2)}X$ in each case. Standard telescopic and chromatic fracture squares [DFHH14, Proposition 6.2.2], and the known validity of the height 1 telescope conjecture [Mah81], [Bou79, Proposition 4.2], then imply that $L_2^f X \simeq L_2 X$ in each case. $\square$

Our calculations also show that TC(ko) has fp-type 2 in the sense of Mahowald–Rezk [MR99], with the following consequence.

Theorem 6.8. *For $X \in \{\mathrm{K}(\mathrm{ko}), \mathrm{K}(\mathrm{ko}_2^\wedge), \mathrm{TC}(\mathrm{ko})\}$ and $Y \in \{X_{(2)}, X_2^\wedge\}$, the canonical map $Y \to L_2^f Y$ is an equivalence in all sufficiently large degrees.*

Proof. Theorems 6.5, 6.4 and 6.2 show, respectively, that $(A(1)/(v_2^{32}))_* X_2^\wedge$ is finite for each of the three choices for X. This implies that $X_2^\wedge$ has fp-type 2 in the sense of [MR99, p. 5], by [MR99, Proposition 3.2]. According to [MR99, Theorem 8.2], this implies that the Brown–Comenetz dual spectrum $IC_2^f X_2^\wedge$ is bounded below and, consequently, that the fiber $C_2^f X_2^\wedge$ of the map $X_2^\wedge \to L_2^f X_2^\wedge$ is bounded above (cf. [HW22, Theorem 3.1.3]). Using the pullback square

$$\begin{tikzcd}
X_{(2)} \arrow[r] \arrow[d] & X_2^\wedge \arrow[d] \\
X_{(2)}[1/2] \arrow[r] & X_2^\wedge[1/2] ,
\end{tikzcd}$$

and the fact that $X_{(2)}[1/2]$ and $X_2^\wedge[1/2]$ are L_2^f-local, it also follows that $X_{(2)} \to L_2^f X_{(2)}$ is an equivalence in all sufficiently large degrees. □

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