

# TOPOLOGICAL CYCLIC HOMOLOGY OF THE INTEGERS, AND RELATED TOPICS

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We calculate  $\mathrm{TC}(\mathbb{Z})$ ,  $K(\mathbb{Z}_p)$ ,  $K(\mathbb{Q}_p)$  and  $K^{\mathrm{\acute{e}t}}(\mathbb{Q}_p) \bmod p$  (and  $v_1$ ), as well as the associated graded syntomic, motivic and Galois cohomology groups, illustrating local Tate duality at the group and spectrum levels, as well as two Lagrangian refinements of this duality.

## 1. A SQUARE

We consider the square

$$\begin{array}{ccc} K(\mathbb{Z}_p)_p & \xrightarrow{j^*} & K(\mathbb{Q}_p)_p \\ \mathrm{trc} \downarrow & & \downarrow \ell \\ \mathrm{TC}(\mathbb{Z})_p & \xrightarrow{m} & K^{\mathrm{\acute{e}t}}(\mathbb{Q}_p)_p \end{array}$$

where  $\mathrm{TC}(\mathbb{Z})_p \simeq \mathrm{TC}(\mathbb{Z}_p)_p$  and  $L_{K(1)}K(\mathbb{Q}_p) \simeq K^{\mathrm{\acute{e}t}}(\mathbb{Q}_p)_p$ . All results generalize to  $p$ -adic number fields, i.e., finite extensions  $L \supset \mathbb{Q}_p$ , and their valuation rings  $\mathcal{O}_L \supset \mathbb{Z}_p$ , but we concentrate on the simplest case, for concreteness.

By Quillen's localization cofiber sequence

$$(1) \quad K(\mathbb{F}_p) \xrightarrow{i_*} K(\mathbb{Z}_p) \xrightarrow{j^*} K(\mathbb{Q}_p)$$

from [Qui73], and his calculation  $K(\mathbb{F}_p)_p = \mathbb{Z}_p$  from [Qui72], we see that also  $L_{K(1)}K(\mathbb{Z}_p) \simeq K^{\mathrm{\acute{e}t}}(\mathbb{Q}_p)_p$ . Hesselholt–Madsen [HM97] deduced from the cofiber sequence

$$K(\mathbb{F}_p)_p \xrightarrow{\mathrm{trc}} \mathrm{TC}(\mathbb{F}_p)_p \longrightarrow \Sigma^{-1}\mathbb{Z}_p$$

and the theorem of (Dundas–Goodwillie–)McCarthy [McC97] that there is a cofiber sequence

$$(2) \quad K(\mathbb{Z}_p)_p \xrightarrow{\mathrm{trc}} \mathrm{TC}(\mathbb{Z})_p \longrightarrow \Sigma^{-1}\mathbb{Z}_p.$$

Hence also  $L_{K(1)}\mathrm{TC}(\mathbb{Z})_p \simeq K^{\mathrm{\acute{e}t}}(\mathbb{Q}_p)_p$ , and we can complete the square with the dashed arrow  $m$ , so that the lower right-hand corner is the Morava  $K(1)$ -localization of each of the three other corners.

After reducing mod  $p$ ,  $K(1)$ -localization is given by inverting  $v_1$ , so we also obtain a square

$$\begin{array}{ccc} K(\mathbb{Z}_p)/p & \xrightarrow{j^*} & K(\mathbb{Q}_p)/p \\ \mathrm{trc} \downarrow & & \downarrow \ell \\ \mathrm{TC}(\mathbb{Z})/p & \xrightarrow{m} & K^{\mathrm{\acute{e}t}}(\mathbb{Q}_p)/p \end{array}$$

with the lower right-hand corner the localization away from  $v_1$  of each of the three other corners.

There is an Anderson self-duality

$$K^{\text{ét}}(\mathbb{Q}_p)_p \simeq \Sigma I_p(K^{\text{ét}}(\mathbb{Q}_p)_p),$$

which reduces mod  $p$  to a Brown–Comenetz self-duality

$$K^{\text{ét}}(\mathbb{Q}_p)/p \simeq I(K^{\text{ét}}(\mathbb{Q}_p)/p),$$

reminiscent of Poincaré duality. Here  $\pi_{-*}I(X) = \text{Hom}(\pi_*(X), \mathbb{Q}/\mathbb{Z})$ .

On one hand, these extend to cofiber sequences

$$K(\mathbb{Q}_p)_p \xrightarrow{\ell} K^{\text{ét}}(\mathbb{Q}_p)_p \simeq \Sigma I_p(K^{\text{ét}}(\mathbb{Q}_p)_p) \xrightarrow{\Sigma I_p(\ell)} \Sigma I_p(K(\mathbb{Q}_p)_p)$$

and

$$(3) \quad K(\mathbb{Q}_p)/p \xrightarrow{\ell} K^{\text{ét}}(\mathbb{Q}_p)/p \simeq I(K^{\text{ét}}(\mathbb{Q}_p)/p) \xrightarrow{I(\ell)} I(K(\mathbb{Q}_p)/p),$$

reminiscent of Lefschetz duality.

On the other hand, they also extend to cofiber sequences

$$\text{TC}(\mathbb{Z})_p \xrightarrow{m} K^{\text{ét}}(\mathbb{Q}_p)_p \simeq \Sigma I_p(K^{\text{ét}}(\mathbb{Q}_p)_p) \xrightarrow{\Sigma I_p(m)} \Sigma I_p(\text{TC}(\mathbb{Z})_p)$$

and

$$(4) \quad \text{TC}(\mathbb{Z})/p \xrightarrow{m} K^{\text{ét}}(\mathbb{Q}_p)/p \simeq I(K^{\text{ét}}(\mathbb{Q}_p)/p) \xrightarrow{I(m)} I(\text{TC}(\mathbb{Z})/p).$$

(However, the corresponding statements with  $K(\mathbb{Z}_p)$  in place of  $K(\mathbb{Q}_p)$  or  $\text{TC}(\mathbb{Z})$  do not hold.)

These spectrum-level results admit filtered refinements, with associated graded statements that can be expressed in terms of the Galois (= étale) cohomology of  $\mathbb{Q}_p$ , the motivic cohomology of  $\mathbb{Q}_p$  and  $\mathbb{Z}_p$ , and the syntomic cohomology of  $\mathbb{Z}_p$ .

## 2. GALOIS COHOMOLOGY FOR $\mathbb{Q}_p$

Let  $\bar{\mathbb{Q}}_p$  be an algebraic closure of  $\mathbb{Q}_p$ , and  $G_{\mathbb{Q}_p} = \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$  its absolute Galois group. This is a profinite group acting on  $K(\bar{\mathbb{Q}}_p)_p$ , which we can identify with the connective cover  $\text{ku}_p$  of  $\text{KU}_p$ . The étale  $K$ -theory spectrum

$$K^{\text{ét}}(\mathbb{Q}_p)_p = (\text{KU}_p)^{hG_{\mathbb{Q}_p}}$$

can be interpreted as the continuous homotopy fixed points for this action [DF85]. Here  $\pi_*\text{KU}_p = \mathbb{Z}_p[\beta^{\pm 1}]$  with  $|\beta| = 2$ . To each element  $g \in G_{\mathbb{Q}_p}$  we can associate a  $p$ -adic unit  $k = \chi(g)$  such that  $g(\zeta) = \zeta^k$  for each  $p$ -th power root of unity  $\zeta \in \bar{\mathbb{Q}}_p$ . Then  $g: \beta \mapsto k\beta$  in  $\pi_2(\text{ku}_p) = \mathbb{Z}_p(1)$ , and more generally  $g: \beta^w \mapsto k^w \beta^w$  in  $\pi_{2w}(\text{KU}_p) = \mathbb{Z}_p(w)$ , i.e.,  $g$  acts as the Adams operation  $\psi^k$  on  $p$ -complete  $K$ -theory.

The double-speed Whitehead filtration on  $\text{KU}_p$  then induces a filtration on  $K^{\text{ét}}(\mathbb{Q}_p)_p$ , with associated graded

$$\text{gr}^w K^{\text{ét}}(\mathbb{Q}_p)_p = (\Sigma^{2w} \mathbb{Z}_p(w))^{hG_{\mathbb{Q}_p}}.$$

This leads to the Adams graded descent spectral sequence

$$\begin{aligned} \mathcal{E}_2^{2w-s,s} &= H_{\text{gp}}^s(G_{\mathbb{Q}_p}; \mathbb{Z}_p(w)) = H_{\text{Gal}}^s(\mathbb{Q}_p; \mathbb{Z}_p(w)) \\ &\implies \pi_{2w-s}(K^{\text{ét}}(\mathbb{Q}_p)_p) = K_{2w-s}^{\text{ét}}(\mathbb{Q}_p; \mathbb{Z}_p). \end{aligned}$$

Here  $H_{\text{gp}}^s(-)$  refers to continuous group cohomology, and  $H_{\text{Gal}}^s(-)$  agrees with the étale cohomology of  $\text{Spec } \mathbb{Q}_p$ . Similarly,

$$K^{\text{ét}}(\mathbb{Q}_p)/p = (\text{KU}/p)^{hG_{\mathbb{Q}_p}}$$

with  $\pi_* \text{KU}/p = \mathbb{F}_p[\beta^{\pm 1}]$ , and there is a mod  $p$  descent spectral sequence

$$\mathcal{E}_2^{2w-s,s} = H_{\text{Gal}}^s(\mathbb{Q}_p; \mathbb{F}_p(w)) \implies \pi_{2w-s}(K^{\text{ét}}(\mathbb{Q}_p)/p).$$

Here  $\mathbb{F}_p(w) = \mu_p^{\otimes w}$  as étale sheaves, where  $\mu_p(F) = \{x \in F \mid x^p = 1\}$  for any field  $F$ .

The cyclotomic character  $\chi: G_{\mathbb{Q}_p} \rightarrow \mathbb{Z}_p^\times$  is surjective, so

$$H_{\text{Gal}}^0(\mathbb{Q}_p; \mathbb{Z}_p(w)) = \begin{cases} \mathbb{Z}_p & \text{for } w = 0, \\ 0 & \text{otherwise.} \end{cases}$$

By Fermat's little theorem,

$$H_{\text{Gal}}^0(\mathbb{Q}_p; \mathbb{F}_p(w)) = \begin{cases} \mathbb{F}_p & \text{for } w \equiv 0 \pmod{p-1}, \\ 0 & \text{otherwise.} \end{cases}$$

The short exact sequence

$$1 \longrightarrow \mu_p \longrightarrow \mathbb{G}_m \xrightarrow{x \mapsto x^p} \mathbb{G}_p \longrightarrow 1$$

of étale sheaves, where  $\mathbb{G}_m(F) = F^\times$  for any field  $F$ , induces a long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\text{Gal}}^0(F; \mu_p) & \longrightarrow & H_{\text{Gal}}^0(F; \mathbb{G}_m) & \xrightarrow{p} & H_{\text{Gal}}^0(F; \mathbb{G}_m) \\ & & & & \searrow & & \\ & & H_{\text{Gal}}^1(F; \mu_p) & \longrightarrow & H_{\text{Gal}}^1(F; \mathbb{G}_m) & \xrightarrow{p} & H_{\text{Gal}}^1(F; \mathbb{G}_m) \\ & & & & \searrow & & \\ & & H_{\text{Gal}}^2(F; \mu_p) & \longrightarrow & H_{\text{Gal}}^2(F; \mathbb{G}_m) & \xrightarrow{p} & H_{\text{Gal}}^2(F; \mathbb{G}_m) \\ & & & & \searrow & & \\ & & \dots & & & & \end{array}$$

in Galois cohomology. Here  $H_{\text{Gal}}^0(F; \mathbb{G}_m) = F^\times$ ,  $H_{\text{Gal}}^1(F; \mathbb{G}_m) = 0$  by a theorem of Noether (1933) (generalizing Hilbert's Theorem 90 or Satz 90, which was due to Kummer), and  $H_{\text{Gal}}^2(F; \mathbb{G}_m) = \text{Br}(F)$  is known to be isomorphic to the Brauer group of  $F$ , i.e., the group of Morita equivalence classes of central simple  $F$ -algebras. Moreover,  $\text{Br}(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}$  via the Hasse invariant. This implies that

$$\begin{aligned} H_{\text{Gal}}^0(\mathbb{Q}_p; \mathbb{F}_p(1)) &= \mu_p(\mathbb{Q}_p) = 0 \\ H_{\text{Gal}}^1(\mathbb{Q}_p; \mathbb{F}_p(1)) &= \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^p \cong (\mathbb{F}_p)^\times \\ H_{\text{Gal}}^2(\mathbb{Q}_p; \mathbb{F}_p(1)) &= \text{Br}(\mathbb{Q}_p)[p] \cong \mathbb{F}_p \end{aligned}$$

for  $p$  odd. For  $p = 2$ , the right-hand groups are  $\mathbb{F}_2$ ,  $(\mathbb{F}_2)^3$  and  $\mathbb{F}_2$ , respectively.

**Theorem 2.1** (Local Tate duality). *The pairing*

$$H_{\text{Gal}}^s(\mathbb{Q}_p; \mathbb{F}_p(w)) \otimes H_{\text{Gal}}^{2-s}(\mathbb{Q}_p; \mathbb{F}_p(1-w)) \xrightarrow{\cup} H_{\text{Gal}}^2(\mathbb{Q}_p; \mathbb{F}_p(1)) \cong \mathbb{F}_p$$

is perfect, inducing isomorphisms

$$H_{\text{Gal}}^{2-s}(\mathbb{Q}_p; \mathbb{F}_p(1-w)) \cong H_{\text{Gal}}^s(\mathbb{Q}_p; \mathbb{F}_p(w))^\vee$$

for all  $s$  and  $w$ .

In particular,  $H_{\text{Gal}}^1(\mathbb{Q}_p; \mathbb{F}_p(0)) \cong (\mathbb{F}_p)^2$  for  $p$  odd and  $(\mathbb{F}_2)^3$  for  $p = 2$ ,

$$H_{\text{Gal}}^2(\mathbb{Q}_p; \mathbb{F}_p(w)) = \begin{cases} \mathbb{F}_p & \text{for } w \equiv 1 \pmod{p-1}, \\ 0 & \text{otherwise,} \end{cases}$$

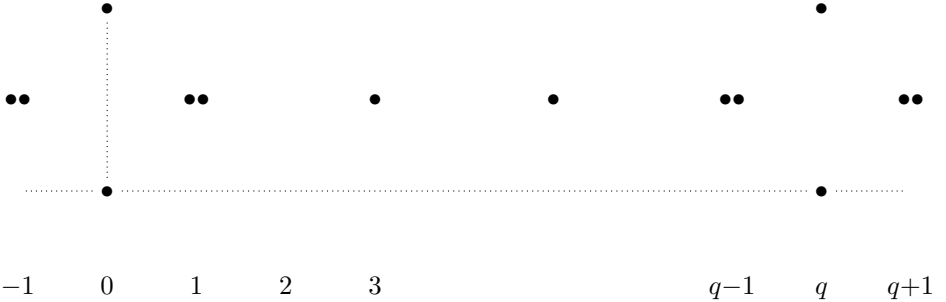
and  $H_{\text{Gal}}^s(\mathbb{Q}_p; \mathbb{F}_p(w)) = 0$  for all  $s \geq 3$ .

Supposing known that  $\pi_* K(\mathbb{Q}_p)_p \cong \pi_* K^{\text{ét}}(\mathbb{Q}_p)_p$  for  $* \geq 1$  (the local Lichtenbaum–Quillen conjecture), that  $\pi_* K(\mathbb{Q}_p)_p$  is a finitely generated  $\mathbb{Z}_p$ -module, and that the  $\mathbb{Q}_p$ -vector space  $\pi_* K(\mathbb{Q}_p)_p[1/p] = \pi_* K(\mathbb{Q}_p)_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  has rank 1 for each odd  $3 \leq * \leq q-3$  (which can be deduced from Wagoner’s paper (1976), following Borel’s treatment (1975) of the number field case), we can determine the missing information, that

$$H_{\text{Gal}}^1(\mathbb{Q}_p; \mathbb{F}_p(w)) \cong \mathbb{F}_p$$

for  $w \neq 0, 1 \pmod{p-1}$ .

For  $p = 5$  the descent spectral sequence has  $\mathcal{E}_2$ -page as below, concentrated in the strip  $0 \leq s \leq 2$ , repeating horizontally in both directions with period  $q := 2(p-1) = |v_1|$ , and converging to  $\pi_* K^{\text{ét}}(\mathbb{Q}_p)/p$ .



Each bullet represents one copy of  $\mathbb{F}_p$ . There is no room for differentials, so  $\mathcal{E}_2 = \mathcal{E}_\infty$ . Local Tate duality thus translates to Brown–Comenetz self-duality for  $K^{\text{ét}}(\mathbb{Q}_p)/p$ , seen as (rotational) symmetry around bidegree  $(0, 1)$  in the chart above.

### 3. MOTIVIC COHOMOLOGY FOR $\mathbb{Q}_p$ AND $\mathbb{Z}_p$

For fields  $F$  (of characteristic  $\neq p$ ), the natural map  $\ell: K(F)_p \rightarrow K^{\text{ét}}(F)_p$  refines to a filtered map, with associated graded a homomorphism

$$\pi_{2w-s} \text{gr}^s(\ell): H_{\text{mot}}^s(F; \mathbb{Z}_p(w)) \longrightarrow H_{\text{Gal}}^s(F; \mathbb{Z}_p(w))$$

from Voevodsky’s motivic cohomology (of  $\text{Spec } F$ ), and similarly for mod  $p$   $K$ -theory and cohomology with  $\mathbb{F}_p(w)$ -coefficients. In particular, there are motivic spectral sequences

$$\mathcal{E}_2^{2w-s, s} = H_{\text{mot}}^s(F; \mathbb{Z}_p(w)) \implies \pi_{2w-s}(K(F)_p)$$

and

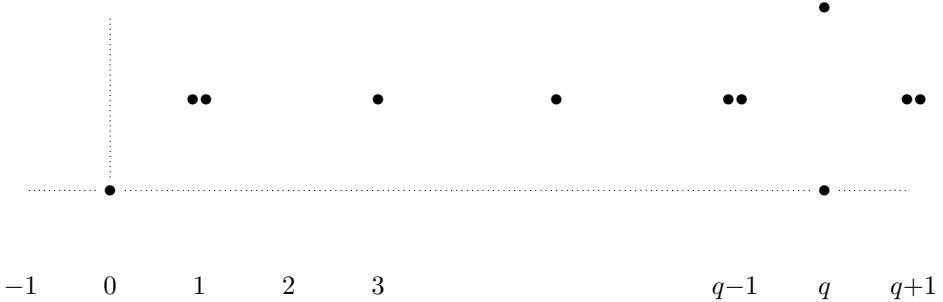
$$\mathcal{E}_2^{2w-s, s} = H_{\text{mot}}^s(F; \mathbb{F}_p(w)) \implies \pi_{2w-s}(K(F)/p),$$

cf. [FS02], and maps from these to the étale descent spectral sequences. Moreover, by the Bloch–Kato conjecture proved by Voevodsky (and Rost, with assistance by Weibel),

$$H_{\text{mot}}^s(F; \mathbb{F}_p(w)) \cong \begin{cases} H_{\text{Gal}}^s(F; \mathbb{F}_p(w)) & \text{for } 0 \leq s \leq w, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the motivic  $\mathcal{E}_2$ -page is obtained from the étale descent  $\mathcal{E}_2$ -page by imposing a slope 1 vanishing line through the center of symmetry. Conversely,  $H_{\text{Gal}}^*(F; \mathbb{F}_p(*))$  is recovered from  $H_{\text{mot}}^*(F; \mathbb{F}_p(*))$  by inverting  $v_1 \in H_{\text{mot}}^0(\mathbb{F}_p(p-1))$ .

In the case  $F = \mathbb{Q}_p$  we obtain the chart below (for  $p = 5$ ), repeating horizontally to the right, but not to the left, and converging to  $\pi_* K(\mathbb{Q}_p)/p$ .



The long exact sequence in homotopy associated to the cofiber sequence (3) breaks up into the short exact sequences

$$0 \rightarrow H_{\text{mot}}^s(\mathbb{Q}_p; \mathbb{F}_p(w)) \xrightarrow{\ell} H_{\text{Gal}}^s(\mathbb{Q}_p; \mathbb{F}_p(w)) \xrightarrow{\ell^\vee} H_{\text{Gal}}^{2-s}(\mathbb{Q}_p; \mathbb{F}_p(1-w))^\vee \rightarrow 0$$

in motivic and Galois cohomology, where we implicitly use the local Tate duality isomorphism in Theorem 2.1.

There is also an extension of motivic cohomology from fields to more general commutative rings and schemes. In particular, there are motivic spectral sequences

$$\mathcal{E}_2^{2w-s,s} = H_{\text{mot}}^s(\mathbb{Z}_p; \mathbb{Z}_p(w)) \implies \pi_{2w-s}(K(\mathbb{Z}_p)_p)$$

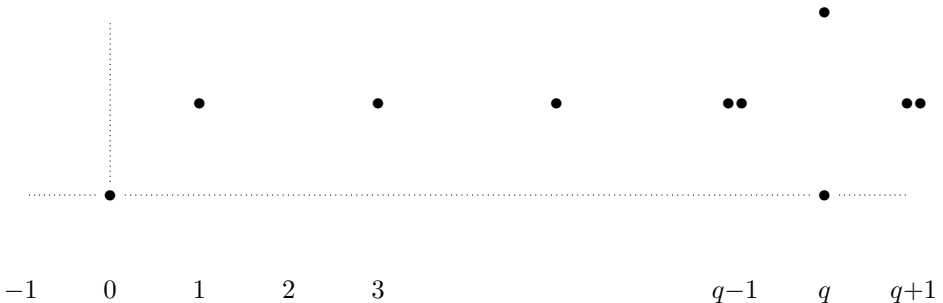
and

$$\mathcal{E}_2^{2w-s,s} = H_{\text{mot}}^s(\mathbb{Z}_p; \mathbb{F}_p(w)) \implies \pi_{2w-s}(K(\mathbb{Z}_p)/p).$$

As is to be expected from Quillen’s localization sequence (1),  $H_{\text{mot}}^s(\mathbb{Z}_p; \mathbb{F}_p(w))$  only differs from  $H_{\text{mot}}^s(\mathbb{Q}_p; \mathbb{F}_p(w))$  in bidegree  $(2w-s, s) = (1, 1)$ , where

$$H_{\text{mot}}^1(\mathbb{Z}_p; \mathbb{F}_p(1)) = (\mathbb{Z}_p^\times)/(\mathbb{Z}_p^\times)^p \subset (\mathbb{Q}_p^\times)/(\mathbb{Q}_p^\times)^p = H_{\text{mot}}^1(\mathbb{Q}_p; \mathbb{F}_p(1)).$$

This leads to the following chart, converging to  $\pi_* K(\mathbb{Z}_p)/p$ .



Thus  $\pi_*K(\mathbb{Q}_p)/p$  and  $\pi_*K(\mathbb{Z}_p)/p$  are both free  $\mathbb{F}_p[v_1]$ -modules of rank  $p+3$ .

For  $p$  odd, the  $p$ -complete connective image-of- $J$  spectrum is defined by the fiber sequence

$$j_p \longrightarrow \mathrm{ku}_p \xrightarrow{\psi^k - 1} \Sigma^2 \mathrm{ku}_p,$$

where  $k \in \mathbb{Z}_p^\times$  is a topological generator. There is a unit map  $e: \mathbb{S} \rightarrow j_p$  inducing the Adams  $e$ -invariant  $\pi_*(\mathbb{S}) \rightarrow \pi_*(j_p)$ . It follows that

$$\pi_*(j/p) = \Lambda(\alpha_1) \otimes \mathbb{F}_p[v_1]$$

where  $\alpha_1$  has degree  $2p-3$  and detects the first  $p$ -torsion in  $\pi_*(\mathbb{S})$ . The unit map  $\mathbb{S} \rightarrow K(\mathbb{Z}_p)_p$  and the map  $\Sigma\mathbb{S} \rightarrow K(\mathbb{Z}_p)_p$  representing  $k \in K_1(\mathbb{Z}_p) \cong \mathbb{Z}_p^\times$  extend over  $e$  and  $\Sigma e$ , and there result splittings

$$K(\mathbb{Z}_p)_p \simeq j_p \oplus \Sigma j_p \oplus \Sigma^3 \mathrm{ku}_p,$$

$$K(\mathbb{Q}_p)_p \simeq j_p \oplus \Sigma j_p \oplus \Sigma \mathrm{ku}_p$$

and

$$K^{\mathrm{ét}}(\mathbb{Q}_p)_p \simeq J_p \oplus \Sigma J_p \oplus \Sigma \mathrm{KU}_p$$

cf. [Rog93]. Here  $J_p$  is the fiber of  $\psi^k - 1: \mathrm{KU}_p \rightarrow \mathrm{KU}_p$ , which is also the  $K(1)$ -localization of  $j_p$ . The case  $p=2$  is a little more complicated [Rog99]. Both for  $p$  odd and for  $p=2$ , these spectrum-level expressions were first obtained by calculating  $\pi_*\mathrm{TC}(\mathbb{Z})/p$ .

#### 4. TOPOLOGICAL CYCLIC HOMOLOGY FOR $\mathbb{Z}$

Bökstedt and Madsen [BM94], [BM95] calculated  $\pi_*\mathrm{TC}(\mathbb{Z})/p$  for  $p$  odd. Both references are relevant, since the result was conjectural in the first paper, but was resolved in the second paper, using Tsalidis's theorem [Tsa97]. We give an updated and simplified presentation.

**Lemma 4.1.**  $\pi_*\mathrm{THH}(\mathbb{Z})/p = \pi_*\mathrm{THH}(\mathbb{Z}, \mathbb{F}_p) = \Lambda(\lambda_1) \otimes \mathbb{F}_p[\mu_1]$ , where  $|\lambda_1| = 2p-1$  and  $|\mu_1| = 2p$ .

*Proof.* This follows from the Bökstedt spectral sequence, using that  $H_*(\mathbb{Z}) = \Lambda(\bar{\tau}_i \mid i \geq 1) \otimes \mathbb{F}_p[\bar{\xi}_i \mid i \geq 1] \subset \mathcal{A}_*$ , or from the Brun spectral sequence, using that  $\pi_*(\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p) = \Lambda(dp) \cong \Lambda(\bar{\tau}_0)$ .  $\square$

**Lemma 4.2.** *In the  $S^1$ -homotopy fixed point spectral sequence*

$$\mathcal{E}^2(S^1) = H^{-*}(S^1; \pi_*\mathrm{THH}(\mathbb{Z})/p) = \mathbb{F}_p[t, \mu_1] \otimes \Lambda(\lambda_1) \Longrightarrow \pi_*\mathrm{TC}^-(\mathbb{Z})/p$$

*the images under  $\mathbb{S} \rightarrow \mathbb{S}/p \rightarrow \mathrm{TC}^-(\mathbb{Z})/p$  of  $\alpha_1 \in \pi_{2p-3}(\mathbb{S})$  and  $v_1 \in \pi_{2p-2}(\mathbb{S}/p)$  are detected by  $t\lambda_1$  and  $t\mu_1$ , respectively.*

*Proof.* This follows as in [AR02] from the long exact sequence in homology associated to the cofiber sequence

$$\Sigma^{-2}X \longrightarrow F(S_+^3, X)^{S^1} \longrightarrow F(S_+^1, X)^{S^1} = X \xrightarrow{\sigma} \Sigma^{-1}X,$$

with  $X = \mathrm{THH}(\mathbb{Z})/p$ , and an Adams spectral sequence argument, or by use of the Hahn–Wilson double suspension [HW22], with  $\sigma^2\alpha_1 = \lambda_1$  and  $\sigma^2v_1 = \mu_1$ .  $\square$

**Lemma 4.3.** *The  $C_p$ -Tate spectral sequence*

$\hat{\mathcal{E}}^2(C_p) = \hat{H}^{-*}(C_p, \pi_* \mathrm{THH}(\mathbb{Z})/p) = \Lambda(u_1, \lambda_1) \otimes \mathbb{F}_p[t_1^{\pm 1}, \mu_1] \implies \pi_* \mathrm{THH}(\mathbb{Z})^{tC_p}/p$   
 has differentials generated by  $d^{2p}(t_1^{1-p}) \doteq t_1 \lambda_1$  and  $d^{2p+1}(u_1 t_1^{-p}) \doteq t_1 \mu_1$ , leaving  
 $\hat{\mathcal{E}}^\infty(C_p) = \mathbb{F}_p[t_1^{\pm p}] \otimes \Lambda(\lambda_1)$ .

*Proof.* Since  $\mathbb{S} \rightarrow \mathrm{TC}^-(\mathbb{Z}) \rightarrow \mathrm{TP}(\mathbb{Z}) \rightarrow \mathrm{THH}(\mathbb{Z})^{tC_p}$  factors through  $\mathrm{THH}(\mathbb{Z})$ , where  $\alpha_1$  and  $v_1$  map to 0, the detecting classes  $t\lambda_1$  and  $t\mu_1$  must map to boundaries in the  $C_p$ -Tate spectral sequence. Hence  $d^{2p}(t_1^{1-p}) \doteq t_1 \lambda_1$  and  $d^{2p+1}(u_1 t_1^{-p}) \doteq t_1 \mu_1$ .  $\square$

**Definition 4.4.** Let  $\lambda = \mathbb{C}(1)$  denote the standard  $S^1$ -representation, and let  $a_\lambda \in \pi_{hS^1}^{-\lambda}(\mathbb{S})$  be the stable class of the  $S^1$ -equivariant map  $S^{-\lambda} \rightarrow S^0$ , dual to  $S^0 \rightarrow S^\lambda$ . The cofiber of  $a_\lambda$  is dual to  $S(\lambda)_+ = S^1_+$ .

Define  $\tilde{\mu}_1 \in \pi_{2p-2+\lambda}^{hS^1} \mathrm{THH}(\mathbb{Z})/p$  so that  $a_\lambda \tilde{\mu}_1 = v_1 \in \pi_{2p-2} \mathrm{TC}^-(\mathbb{Z})/p$ . Then  $\tilde{\mu}_1$  maps to  $\mu_1$  under the map forgetting  $S^1$ -equivariance.

The  $\mathbb{Z}\{1, \lambda\} \subset \mathrm{RO}(S^1)$ -graded factorization of  $v_1$  yields the lower right-hand pullback square in the following diagram. Here we use that  $(X/a_\lambda)^{hS^1} \simeq X$ . The notation  $\mathrm{TC}^-(\mathbb{Z})/p, \tilde{\mu}_1$  should be interpreted as  $(\mathrm{THH}(\mathbb{Z})/p, \tilde{\mu}_1)^{hS^1}$ . The canonical image of the class  $a_\lambda$  is a unit, so  $\mathrm{TP}(\mathbb{Z})/p, v_1 = \mathrm{TP}(\mathbb{Z})/p, \tilde{\mu}_1$ , which should be interpreted as  $(\mathrm{THH}(\mathbb{Z})/p, \tilde{\mu}_1)^{tS^1}$ .

In the following, we write ‘ $f$ ’ in place of ‘ $f/p, v_1$ ’ for various maps.

$$\begin{array}{ccccc}
 \mathrm{TC}(\mathbb{Z})/p, v_1 & & & \mathrm{THH}(\mathbb{Z})^{tC_p}/p & \leftarrow \\
 \searrow \pi & & \nearrow \varphi_p^{hS^1} & \uparrow \varphi_p & \\
 \Sigma \mathrm{TC}(\mathbb{Z})/p, v_1 & \mathrm{TC}^-(\mathbb{Z})/p, v_1 & \xrightarrow{|\mathrm{HT}|} & \mathrm{THH}(\mathbb{Z})/p & \\
 \swarrow \partial & \searrow \mathrm{can} & \downarrow |\mathrm{dR}| & \downarrow i & \\
 & \mathrm{TP}(\mathbb{Z})/p, v_1 & \xleftarrow{j} \mathrm{TC}^-(\mathbb{Z})/p, \tilde{\mu}_1 & \xrightarrow{F} \mathrm{THH}(\mathbb{Z})/p, \mu_1 & \\
 & & & \times & \\
 & & & \text{---} G \simeq \text{---} &
 \end{array}$$

**Lemma 4.5.**  $\pi_* \mathrm{THH}(\mathbb{Z})/p, \mu_1 = \Lambda(\lambda_1)$ . *The  $S^1$ -Tate spectral sequence*

$\hat{\mathcal{E}}^2(S^1) = \hat{H}^{-*}(S^1, \pi_* \mathrm{THH}(\mathbb{Z})/p, \mu_1) = \mathbb{F}_p[t^{\pm 1}] \otimes \Lambda(\lambda_1) \implies \pi_* \mathrm{TP}(\mathbb{Z})/p, v_1$   
 has differentials  $d^{2p}(t^i) \doteq t^{i+p} \lambda_1$  for  $p \nmid i$ , leaving  
 $\hat{\mathcal{E}}^\infty(S^1) = \mathbb{F}_p[t^{\pm p}] \otimes \Lambda(\lambda_1)$ .

*Proof.* The  $d^{2p}$ -differentials for  $\mathrm{THH}(\mathbb{Z})^{tC_p}/p$  lift to differentials for  $\mathrm{THH}(\mathbb{Z})^{tS^1}/p = \mathrm{TP}(\mathbb{Z})/p$  and push forward to differentials for  $\mathrm{TP}(\mathbb{Z})/p, v_1$ .  $\square$

**Lemma 4.6.** *The  $S^1$ -homotopy fixed point spectral sequence*

$\mathcal{E}^2(S^1) = H^{-*}(S^1, \pi_* \mathrm{THH}(\mathbb{Z})/p, \mu_1) = \mathbb{F}_p[t] \otimes \Lambda(\lambda_1) \implies \pi_* \mathrm{TC}^-(\mathbb{Z})/p, \tilde{\mu}_1$   
 has differentials  $d^{2p}(t^i) \doteq t^{i+p} \lambda_1$  for  $p \nmid i \geq 0$ , leaving  
 $\mathcal{E}^\infty(S^1) = \mathbb{F}_p[t^p] \otimes \Lambda(\lambda_1) \oplus \mathbb{F}_p\{t^d \lambda_1 \mid 0 < d < p\}$ .

The homomorphism  $j_*$  is injective on the first summand and zero on the second summand.

*Proof.* This follows by naturality along  $j$ .  $\square$

**Corollary 4.7.**

$$\pi_* \mathrm{TC}^-(\mathbb{Z})/p, v_1 \cong \mathbb{F}_p[t^p]\{1, \lambda_1\} \oplus \mathbb{F}_p\{t^d \lambda_1 \mid 0 < d < p\} \oplus \mathbb{F}_p[\mu_1]\{\mu_1, \lambda_1 \mu_1\}.$$

The homomorphism  $\mathrm{can}_*$  is injective on the first summand and zero on the second and third summands.

*Proof.* This is a calculation of the pullback of

$$\pi_* \mathrm{TC}^-(\mathbb{Z})/p, \tilde{\mu}_1 \longrightarrow \pi_* \mathrm{THH}(\mathbb{Z})/p, \mu_1 \longleftarrow \pi_* \mathrm{THH}(\mathbb{Z})/p.$$

Alternatively, one can prove this without the use of  $\mathrm{RO}(S^1)$ -graded classes and the pullback decomposition, by fully calculating the  $S^1$ -homotopy fixed point spectral sequence in Lemma 4.2, and then reducing the result modulo  $v_1$  (detected by  $t\mu_1$ ).  $\square$

**Lemma 4.8.**  $\mathrm{can}$  and  $\varphi$  agree on  $\lambda_1 \in \pi_{2p-1} \mathrm{TC}^-(\mathbb{Z})/p, v_1$ , which therefore admits a lift to  $\pi_{2p-1} \mathrm{TC}(\mathbb{Z})/p, v_1$ .

*Proof.* Either use results of Waldhausen and Bökstedt on the fiber of  $K(\mathbb{S}) \rightarrow K(\mathbb{Z})$  to find a class  $\lambda_1^K \in K_{2p-1}(\mathbb{Z})$  with trace image  $\lambda_1$  (see also Lee–Levy [LL26, Thm. 7.3] for a stronger result), or show that the transfer map  $\mathrm{trf}: \mathrm{TC}(\mathbb{F}_p)/p, v_1 \rightarrow \mathrm{TC}(\mathbb{Z})/p, v_1$  takes  $\bar{e}_1 \in \pi_{2p-1} \mathrm{TC}(\mathbb{F}_p)/p, v_1$  to the required lift of  $\lambda_1$ .  $\square$

**Lemma 4.9.** The cyclotomic structure map induces

$$\varphi_{p*}: \Lambda(\lambda_1) \otimes \mathbb{F}_p[\mu_1] \longrightarrow \mathbb{F}_p[t^{\pm p}] \otimes \Lambda(\lambda_1)$$

given by  $\mu_1 \mapsto t_1^{-p}$  (up to a unit in  $\mathbb{F}_p$ ) and  $\lambda_1 \mapsto G(\lambda_1)$ , hence factors as the localization inverting  $\mu_1$ , followed by an isomorphism

$$\Lambda(\lambda_1) \otimes \mathbb{F}_p[\mu_1^{\pm 1}] = \pi_* \mathrm{THH}(\mathbb{Z})/p[\mu_1^{-1}] \cong \pi_* \mathrm{THH}(\mathbb{Z})^{tC_p}/p.$$

*Proof.* Being a ring homomorphism,  $\varphi_{p*}$  must take the unit  $\mu_1$  to a unit, i.e., to a nonzero scalar times  $t_1^{-p}$ . Moreover,  $\varphi_p(\lambda_1) = \varphi_p^{hS^1}(\lambda_1) = (G \circ \mathrm{can})(\lambda_1) = G(\lambda_1)$ .  $\square$

**Theorem 4.10** (Bökstedt–Madsen). There is a short exact sequence

$$0 \rightarrow \Sigma^{-1} \Lambda(\lambda_1) \xrightarrow{\partial} \pi_* \mathrm{TC}(\mathbb{Z})/p, v_1 \xrightarrow{\pi} \mathbb{F}_p\{t^d \lambda_1 \mid 0 < d < p\} \oplus \Lambda(\lambda_1) \rightarrow 0,$$

so that

$$\pi_* \mathrm{TC}(\mathbb{Z})/p, v_1 \cong \Lambda(\partial, \lambda_1) \oplus \mathbb{F}_p\{t^d \lambda_1 \mid 0 < d < p\}$$

with  $|\partial| = -1$ ,  $|\lambda_1| = 2p - 1$  and  $|t| = -2$ . The relations  $t^d \lambda_1 \cdot t^{p-d} \lambda_1 \doteq \partial \lambda_1$  hold, making this a Poincaré duality algebra, of total dimension  $p + 3$ .

*Proof.* To calculate the kernel and cokernel of  $(\text{can} - \varphi)_*: \pi_* \text{TC}^-(\mathbb{Z})/p, v_1 \rightarrow \pi_* \text{TP}(\mathbb{Z})/p, v_1$ , we express it as the direct sum of the following four homomorphisms.

$$\begin{array}{ccc} \mathbb{F}_p[t^p]\{t^p, t^p\lambda_1\} & \xrightarrow{\cong} & \mathbb{F}_p[t^p]\{t^p, t^p\lambda_1\} \\ \mathbb{F}_p\{t^d\lambda_1 \mid 0 < d < p\} & \longrightarrow & 0 \\ \Lambda(\lambda_1) & \xrightarrow{0} & \Lambda(\lambda_1) \\ \mathbb{F}_p[\mu_1]\{\mu_1, \lambda_1\mu_1\} & \xrightarrow{\cong} & \mathbb{F}_p[t^{-p}]\{t^{-p}, t^{-p}\lambda_1\} \end{array}$$

The multiplicative relation only makes sense at the level of spectra when  $\mathbb{S}/p, v_1 = V(1)$  is a ring spectrum, i.e., for  $p \geq 5$ , but works at all primes in the associated graded for the Hahn–Raksit–Wilson motivic filtration. It can be deduced from the Browder/Wall sum theorem for the two Poincaré pairs given by  $(i, \varphi_p)$  and  $(F, j)$ , corresponding to the cofiber sequences

$$\mathrm{THH}(\mathbb{Z})/p \xrightarrow{(i, \varphi_p)} \mathrm{THH}(\mathbb{Z})/p, \mu_1 \oplus \mathrm{THH}(\mathbb{Z})^{tC_p}/p \xrightarrow{I^{2p-1}(i, \varphi_p)} I^{2p-1}(\mathrm{THH}(\mathbb{Z})/p)$$

and

$$\mathrm{TC}^-(\mathbb{Z})/p, \tilde{\mu}_1 \xrightarrow{(F,j)} \mathrm{THH}(\mathbb{Z})/p, \mu_1 \oplus \mathrm{TP}(\mathbb{Z})/p, v_1 \xrightarrow{I^{2p-1}(F,j)} I^{2p-1}(\mathrm{TC}^-(\mathbb{Z})/p, \tilde{\mu}_1),$$

keeping in mind the equivalence  $G$ . Alternatively, it can be recovered from local Tate duality for  $\mathbb{Q}_p$  by comparison along  $m$ .  $\square$

We illustrate  $\pi_*\mathrm{TC}(\mathbb{Z})/p, v_1$  below, where the ellipsis indicates the generators  $t^d\lambda_1$ . The vertical direction does not (yet) carry any meaning.

$$\begin{array}{ccccccc}
& & \partial\lambda_1 \\
& & \vdots \\
\partial & t^{p-1}\lambda_1 & \dots & t\lambda_1 & \lambda_1 \\
& \vdots \\
& 1 \\
& \vdots \\
-1 & 0 & 1 & \dots & q-1 & q & q+1
\end{array}$$

**Corollary 4.11.** *The  $v_1$ -Bockstein spectral sequence*

$$\mathcal{E}^1 = \pi_* \mathrm{TC}(\mathbb{Z})/p, v_1 \otimes \mathbb{F}_p[v_1] \implies \pi_* \mathrm{TC}(\mathbb{Z})/p$$

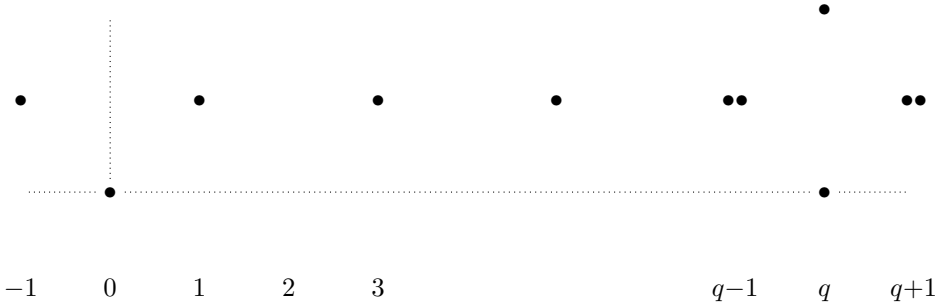
*collapses, and*

$$\pi_* \mathrm{TC}(\mathbb{Z})/p \cong \mathbb{F}_p[v_1] \otimes \left( \Lambda(\partial, \lambda_1) \oplus \mathbb{F}_p\{t^d \lambda_1 \mid 0 < d < p\} \right)$$

is a free  $\mathbb{F}_p[v_1]$ -module of rank  $p + 3$ .

*Proof.* This uses that  $\lambda_1$  (and  $\partial\lambda_1$ ) admit mod  $p$  (and integral) lifts, hence cannot support  $v_1$ - (or  $p$ -)Bockstein differentials.  $\square$

For  $p = 5$ , we have the following chart, continuing  $v_1$ -periodically to the right.



The long exact sequence in homotopy associated to the cofiber sequence (4) realizes the short exact sequences

$$0 \rightarrow \pi_n(\mathrm{TC}(\mathbb{Z})/p) \xrightarrow{m} \pi_n(K^{\mathrm{ét}}(\mathbb{Q}_p)/p) \xrightarrow{m^\vee} \pi_{-n}(\mathrm{TC}(\mathbb{Z})/p)^\vee \rightarrow 0.$$

Next, we indicate how these break up further into short exact sequences in syntomic cohomology.

## 5. THE EVEN FILTRATION FOR $\mathrm{THH}(\mathbb{Z})$

The Hahn–Raksit–Wilson (arXiv:2206.11208) even filtration of an  $\mathbb{E}_\infty$ -ring  $R$  is given by

$$\mathrm{fil}_{\mathrm{ev}}^w R = \lim_{R \rightarrow E} \tau_{\geq 2w} E,$$

where the limit ranges over  $\mathbb{E}_\infty$ -maps  $R \rightarrow E$  with  $E$  even, i.e., with  $\pi_*(E)$  concentrated in even degrees. The ‘even weight’  $w$  associated graded  $\mathrm{gr}_{\mathrm{ev}}^w R$  is the cofiber of  $\mathrm{fil}_{\mathrm{ev}}^{w+1} R \rightarrow \mathrm{fil}_{\mathrm{ev}}^w R$ .

There is a spectral sequence  $\pi_* \mathrm{gr}_{\mathrm{ev}}^* R \implies \pi_* R$ . The even filtration is always complete Hausdorff, but additional hypotheses are needed to ensure that it exhausts  $R$ . For an  $\mathbb{E}_\infty$ -ring  $B$ , the even filtration on  $\mathrm{THH}(B)$  is also called the motivic filtration, and

$$\pi_* \mathrm{gr}_{\mathrm{ev}}^* \mathrm{THH}(B) \implies \pi_* \mathrm{THH}(B)$$

is called the motivic spectral sequence. If  $B$  is ‘chromatically quasisyntomic’, e.g., if  $\mathrm{MU}_*(B)$  is a polynomial ring on even degree generators, possibly reduced modulo a regular sequence, then this filtration exhausts  $\mathrm{THH}(B)$ .

An  $\mathbb{E}_\infty$ -map  $R \rightarrow T$  is evenly faithfully flat (eff) if for each  $\mathbb{E}_\infty$ -map  $R \rightarrow E$  with  $E$  even, the base change  $E \otimes_R T$  is even with  $\pi_*(E \otimes_R T)$  faithfully flat as a  $\pi_*(E)$ -module. For example, this is satisfied if  $E \otimes_R T$  is a sum of one or more even suspensions of  $E$ .

If  $R \rightarrow T$  is eff and  $T$  is even, then

$$\mathrm{fil}_{\mathrm{ev}}^w R \simeq \lim_{[q] \in \Delta} \tau_{\geq 2w} (T^{\otimes_R 1+q})$$

is the totalization of the codegreeewise  $2w$ -connective cover of the Čech (or Amitsur) complex

$$T^{\otimes_R 1+\bullet} = \{T \rightrightarrows T \otimes_R T \rightrightarrows T \otimes_R T \otimes_R T \rightrightarrows \dots\}.$$

It follows that

$$\mathrm{gr}_{\mathrm{ev}}^w R \simeq \lim_{[q] \in \Delta} \Sigma^{2w} \pi_{2w}(T^{\otimes_R 1+q}).$$

Let  $B \rightarrow k$  be an  $\mathbb{E}_\infty$ -map, with  $B$  connective,  $k$  a perfect field of characteristic  $p > 0$ , and  $\pi_0(B) \rightarrow k$  surjective. There is an equivalence  $\mathrm{THH}(B, k) \simeq \mathrm{THH}(k, k \otimes_B k)$ , and two pushout squares

$$\begin{array}{ccccc} k \otimes_B k & \longrightarrow & \mathrm{THH}(B, k) & \longrightarrow & k \\ \downarrow & & \downarrow & & \downarrow \\ k & \longrightarrow & \mathrm{THH}(k) & \longrightarrow & \mathrm{HH}(k/B) \end{array}$$

of  $\mathbb{E}_\infty$ -rings. By Bökstedt's calculation,  $\pi_* \mathrm{THH}(k) = k[\mu_0]$  with  $|\mu_0| = 2$ , so  $\mathrm{THH}(k)$  is even. As observed with Devalapurkar and Hahn, if  $\mathrm{HH}(k/B)$  is even, then by Hurewicz it follows that  $\mathrm{THH}(k)$  can be built as a  $\mathrm{THH}(B, k)$ -module with only even cells. Hence  $\mathrm{THH}(B, k) \rightarrow \mathrm{THH}(k)$  is eff if (and only if)  $\mathrm{HH}(k/B)$  is even.

This applies, in particular, to  $B = \mathbb{Z} \rightarrow \mathbb{F}_p = k$ , in which case  $\pi_*(k \otimes_B k) \cong \Lambda(\tau_0)$  and  $\pi_* \mathrm{HH}(k/B) \cong \Gamma(\mu_0)$  is even. Hence we can calculate  $\mathrm{gr}_{\mathrm{ev}}^* \mathrm{THH}(\mathbb{Z})/p = \mathrm{gr}_{\mathrm{ev}}^* \mathrm{THH}(\mathbb{Z}, \mathbb{F}_p)$  by eff descent along  $R = \mathrm{THH}(\mathbb{Z})/p \rightarrow \mathrm{THH}(\mathbb{F}_p) = T$ .

Here  $\pi_* R = \pi_* \mathrm{THH}(\mathbb{Z})/p \cong \Lambda(\lambda_1) \otimes_{\mathbb{F}_p} [\mu_1]$ , while  $\pi_* T = \pi_* \mathrm{THH}(\mathbb{F}_p) = \mathbb{F}_p[\mu_0]$ .

The bar spectral sequence

$$\mathrm{Tor}_*^{\pi_* (R)}(\pi_* T, \pi_* T) \Longrightarrow \pi_*(T \otimes_R T)$$

has

$$\begin{aligned} \mathcal{E}^2 &= \mathrm{Tor}^{\Lambda(\lambda_1) * \otimes_{\mathbb{F}_p} [\mu_1]}(\mathbb{F}_p[\mu_0], \mathbb{F}_p[\mu_0]) \\ &= \Gamma(\sigma \lambda_1) \otimes (\mathbb{F}_p[\mu_0] \otimes_{\mathbb{F}_p[\mu_1]} \mathbb{F}_p[\mu_0]) \end{aligned}$$

concentrated in even stems, hence collapses. By induction on  $q \geq 0$ , the cosimplicial abelian group

$$[q] \mapsto \pi_*(T^{\otimes_R 1+q})$$

is the tensor product of the cobar constructions for the Hopf algebroids

$$(\mathbb{F}_p, \Gamma(\sigma \lambda_1)) \quad \text{and} \quad (\mathbb{F}_p[\mu_0], \mathbb{F}_p[\mu_0] \otimes_{\mathbb{F}_p[\mu_1]} \mathbb{F}_p[\mu_0]).$$

Passing to associated cochain complexes and their cohomology, using the Eilenberg–Zilber theorem and that  $\mathbb{F}_p[\mu_1] \rightarrow \mathbb{F}_p[\mu_0]$  is faithfully flat, we obtain

$$\begin{aligned} \pi_* \mathrm{gr}_{\mathrm{ev}}^* \mathrm{THH}(\mathbb{Z})/p &\cong \mathrm{Cotor}_{\Gamma(\sigma \lambda_1)}^*(\mathbb{F}_p, \mathbb{F}_p) \otimes \mathrm{Cotor}_{\mathbb{F}_p[\mu_0] \otimes_{\mathbb{F}_p[\mu_1]} \mathbb{F}_p[\mu_0]}^*(\mathbb{F}_p[\mu_0], \mathbb{F}_p[\mu_0]) \\ &\cong \Lambda(\lambda_1) \otimes_{\mathbb{F}_p} [\mu_1], \end{aligned}$$

with  $\lambda_1 \in \mathrm{Cotor}^1$  in stem  $2p - 1$  and weight  $p$  and  $\mu_1 \in \mathrm{Cotor}^0$  in stem  $2p$  and weight  $p$ . Using  $(n, s)$ -coordinates, with the stem  $n = 2w - s$  horizontally and the ‘Adams weight’  $s = 2w - n$  vertically, we obtain the following chart for  $\pi_n \mathrm{gr}_{\mathrm{ev}}^w \mathrm{THH}(\mathbb{Z})/p$ , extending  $\mu_1$ -periodically to the right.

$$\begin{array}{ccccccc} 1 & & & \lambda_1 & & & \lambda_1 \mu_1 \\ & \vdots & & & & & \\ 0 & 1 & \cdots & \mu_1 & \cdots & \cdots & \mu_1^2 \\ & & & & & & \\ & 0 & & 2p-1 & 2p & & 4p-1 & 4p \end{array}$$

In particular, the motivic spectral sequence  $\pi_* \mathrm{gr}_{\mathrm{ev}}^* \mathrm{THH}(\mathbb{Z})/p \implies \pi_* \mathrm{THH}(\mathbb{Z})/p$  collapses.

## 6. SYNTOMIC COHOMOLOGY FOR $\mathbb{Z}_p$

Hahn, Raksit and Wilson also introduce motivic filtrations

$$\mathrm{fil}_{\mathrm{mot}}^w \mathrm{TC}^-(B) = \lim_{\mathrm{THH}(B) \rightarrow E} \tau_{\geq 2w}(E^{hS^1})$$

and

$$\mathrm{fil}_{\mathrm{mot}}^w \mathrm{TP}(B) = \lim_{\mathrm{THH}(B) \rightarrow E} \tau_{\geq 2w}(E^{tS^1}),$$

with  $\mathrm{THH}(B) \rightarrow E$  ranging over  $\mathbb{E}_\infty$ -maps with even target. The homotopy groups of the associated spectra

$$\mathrm{gr}_{\mathrm{mot}}^w \mathrm{TC}^-(B) = \lim_{\mathrm{THH}(B) \rightarrow E} \Sigma^{2w} \pi_{2w}(E^{hS^1})$$

and

$$\mathrm{gr}_{\mathrm{mot}}^w \mathrm{TP}(B) = \lim_{\mathrm{THH}(B) \rightarrow E} \Sigma^{2w} \pi_{2w}(E^{tS^1})$$

define (Nygaard completed) prismatic cohomology. For an even eff-cover  $\mathrm{THH}(B) = R \rightarrow T$ , these can be calculated as

$$\mathrm{fil}_{\mathrm{mot}}^w \mathrm{TC}^-(B) \simeq \lim_{[q] \in \Delta} \tau_{\geq 2w}((T^{\otimes_{R^1+q}})^{hS^1})$$

and

$$\mathrm{fil}_{\mathrm{mot}}^w \mathrm{TP}(B) \simeq \lim_{[q] \in \Delta} \tau_{\geq 2w}((T^{\otimes_{R^1+q}})^{tS^1}),$$

so that

$$\mathrm{gr}_{\mathrm{mot}}^w \mathrm{TC}^-(B) = \lim_{[q] \in \Delta} \Sigma^{2w} \pi_{2w}((T^{\otimes_{R^1+q}})^{hS^1})$$

and

$$\mathrm{gr}_{\mathrm{mot}}^w \mathrm{TP}(B) = \lim_{[q] \in \Delta} \Sigma^{2w} \pi_{2w}((T^{\otimes_{R^1+q}})^{tS^1}).$$

These are compatible under the canonical map from  $S^1$ -homotopy fixed points to  $S^1$ -Tate fixed points. For chromatically quasisyntomic  $B$ , Hahn–Raksit–Wilson also show that  $\varphi: \mathrm{TC}^-(B) \rightarrow \mathrm{TP}(B)$  is filtration-preserving. Hence they can define the motivic filtration of  $\mathrm{TC}(B)$  with

$$\mathrm{fil}_{\mathrm{mot}}^w \mathrm{TC}(B) = \mathrm{eq}(\mathrm{can}, \varphi: \mathrm{fil}_{\mathrm{mot}}^w \mathrm{TC}^-(B) \rightarrow \mathrm{fil}_{\mathrm{mot}}^w \mathrm{TP}(B))$$

(which is then complete (Hausdorff) and exhaustive), and

$$\mathrm{gr}_{\mathrm{mot}}^w \mathrm{TC}(B) = \mathrm{eq}(\mathrm{can}, \varphi: \mathrm{gr}_{\mathrm{mot}}^w \mathrm{TC}^-(B) \rightarrow \mathrm{gr}_{\mathrm{mot}}^w \mathrm{TP}(B)).$$

The homotopy groups

$$H_{\mathrm{syn}}^s(B; \mathbb{Z}_p(w)) := \pi_{2w-s} \mathrm{gr}_{\mathrm{mot}}^w \mathrm{TC}(B)$$

define the syntomic cohomology of  $B$ . When  $B$  is an (ordinary) quasisyntomic ring, these definitions agree with those of Bhatt–Morrow–Scholze [BMS19]. Similar remarks apply with coefficients. There results a motivic spectral sequence

$$\mathcal{E}_2^{2w-s,s} = H_{\mathrm{syn}}^s(B; \mathbb{Z}_p(w)) \implies \pi_{2w-s} \mathrm{TC}(B)_p$$

and the special case

$$\mathcal{E}_2^{2w-s,s} = H_{\mathrm{syn}}^s(\mathbb{Z}_p; \mathbb{F}_p(w)) \implies \pi_{2w-s}(\mathrm{TC}(\mathbb{Z})/p).$$

To calculate  $\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathbb{Z})/p$ , we proceed as for  $\pi_* \mathrm{TC}(\mathbb{Z})/p$ , starting with the bigraded  $\pi_* \mathrm{gr}_{\mathrm{ev}}^* \mathrm{THH}(\mathbb{Z})/p$  in place of  $\pi_* \mathrm{THH}(\mathbb{Z})/p$ . Then we check that all maps and differentials respect the weight grading  $w$ , so that Theorem 4.10 also calculates  $\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathbb{Z})/p, v_1$ , now with  $\partial$  in weight 0,  $\lambda_1$  in weight  $p$  and  $t$  in weight  $-1$ . This gives meaning to the vertical direction in the illustration. The deduction of Corollary 4.11 now becomes easier, since the  $v_1$ -Bockstein differentials have  $(n, s)$ -bidegree  $(-1, 1)$ , and therefore vanish for bidegree reasons. With  $v_1$  in weight  $(p-1)$ , we recover  $H_{\mathrm{syn}}^s(\mathbb{Z}_p; \mathbb{F}_p(w))$ .

As is to be expected from the Hesselholt–Madsen sequence (2),  $H_{\mathrm{syn}}^s(\mathbb{Z}_p; \mathbb{F}_p(w))$  only differs from  $H_{\mathrm{mot}}^s(\mathbb{Z}_p; \mathbb{F}_p(w))$  in bidegree  $(2w-s, s) = (-1, 1)$ , where

$$H_{\mathrm{syn}}^1(\mathbb{Z}_p; \mathbb{F}_p(0)) \cong \mathbb{F}_p\{\partial\}.$$

The long exact sequence in homotopy associated to the cofiber sequence (4) breaks up into the short exact sequences

$$0 \rightarrow H_{\mathrm{syn}}^s(\mathbb{Z}_p; \mathbb{F}_p(w)) \xrightarrow{m} H_{\mathrm{Gal}}^s(\mathbb{Q}_p; \mathbb{F}_p(w)) \xrightarrow{m^\vee} H_{\mathrm{syn}}^{2-s}(\mathbb{Z}_p; \mathbb{F}_p(1-w))^\vee \rightarrow 0$$

in syntomic and Galois cohomology. In this case, the source of  $m$  is not simply given by truncating its target below a vanishing line. On the other hand,  $H_{\mathrm{Gal}}^*(\mathbb{Q}_p; \mathbb{F}_p(*))$  can still be recovered from  $H_{\mathrm{syn}}^*(\mathbb{Z}_p; \mathbb{F}_p(*))$  by inverting  $v_1 \in H_{\mathrm{syn}}^0(\mathbb{Z}_p; \mathbb{F}_p(p-1))$ .

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