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Trace maps from the algebraic K-theory of the integers (after Marcel Bökstedt)

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Abstract

Let p be any prime. We consider Bökstedt's topological refinement $K(\mathbb{Z}) \to T(\mathbb{Z}) = \mathrm{THH}(\mathbb{Z})$ of the Dennis trace map from algebraic K-theory of the integers to topological Hochschild homology of the integers. This trace map is shown to induce a surjection on homotopy in degree $2p-1$, onto the first p-torsion in the target. Furthermore, Bökstedt's map factors through the S^1-homotopy fixed points $T(\mathbb{Z})^{hS^1}$ of $T(\mathbb{Z})$, and it is shown that the first p-torsion element in degree $2p-3$ of the stable homotopy groups of spheres is detected in the homotopy of $T(\mathbb{Z})^{hS^1}$. Both results are due to Bökstedt, but have remained unpublished. © 1998 Elsevier Science B.V.

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1. Introduction

The purpose of this paper is to provide a reference for two theorems due to Marcel Bökstedt.

Let $K(\mathbb{Z})$ be the K-theory spectrum, and $T(\mathbb{Z}) = \mathrm{THH}(\mathbb{Z})$ the topological Hochschild homology spectrum of the integers. We write $K_i(\mathbb{Z}) = \pi_i K(\mathbb{Z})$ and $T_i(\mathbb{Z}) = \pi_i T(\mathbb{Z})$. The trace map $\mathrm{tr}: K(\mathbb{Z}) \to T(\mathbb{Z})$ is the map constructed by Bökstedt in [1], which strengthens the Dennis trace map to ordinary Hochschild homology. By the calculations of [2], reproduced in [7], $T_0(\mathbb{Z}) = \mathbb{Z}$ and $T_{2i-1}(\mathbb{Z}) \cong \mathbb{Z}/i$ for all $i \in \mathbb{N}$, while the remaining groups are zero.

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Theorem 1.1 (Bökstedt). *Let p be any prime. The trace map induces a surjection*

$$\pi_{2p-1}(\mathrm{tr}) : K_{2p-1}(\mathbb{Z}) \to T_{2p-1}(\mathbb{Z}) \cong \mathbb{Z}/p$$

onto the first p-torsion in $T_*(\mathbb{Z})$.

Bökstedt's proof appears in an unpublished Bielefeld preprint [3]. Another proof is given in Section 10 of [5], but that proof apparently assumes p is odd. We give a proof in Section 2, taking special care to cover the case $p=2$.

The topological Hochschild homology spectrum admits the structure of an S^1-spectrum, and there is a compatible family of factorizations of tr

$$K(\mathbb{Z}) \xrightarrow{\mathrm{tr}_{p^n}} T(\mathbb{Z})^{C_{p^n}} \subseteq T(\mathbb{Z}),$$

for a fixed prime p and for all $n \geq 0$. Hence C_{p^n} is the cyclic subgroup of S^1 with p^n elements. See [4] or [7] for more on this and the following material. These factorizations, composed with the standard maps

$$\Gamma : T(\mathbb{Z})^{C_{p^n}} \to T(\mathbb{Z})^{hC_{p^n}}$$

from fixed points to homotopy fixed points, induce a map of homotopy limits

$$K(\mathbb{Z}) \to \operatorname*{holim}_n T(\mathbb{Z})^{C_{p^n}} \to \operatorname*{holim}_n T(\mathbb{Z})^{hC_{p^n}}.$$

After p-adic completion (denoted in this paper by a subscript p) there is a natural homotopy equivalence

$$T(\mathbb{Z})_p^{hS^1} \xrightarrow{\simeq} \operatorname*{holim}_n T(\mathbb{Z})_p^{hC_{p^n}}$$

determining a map

$$\mathrm{tr}_{S^1} : K(\mathbb{Z})_p \to T(\mathbb{Z})_p^{hS^1},$$

which we call the *circle trace map*. The cyclotomic trace map $\mathrm{trc} : K(\mathbb{Z})_p \to \mathrm{TC}(\mathbb{Z}, p)$ of [4] is a further refinement of this map.

There is a second quadrant spectral sequence E^r_{**} with $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$, converging to

$$\pi_{s+t} T(\mathbb{Z})_p^{hS^1} = \pi_{s+t} \mathrm{Map}(ES^1_+, T(\mathbb{Z}))_p^{S^1}$$

and having

$$E^2_{s,t} = H^{-s}(BS^1; T_t(\mathbb{Z})_p). \tag{1.2}$$

The spectral sequence arises from the skeleton filtration of a standard model for ES^1, a contractible space with a free action of S^1, and the cohomology groups arise as the cohomology of the topological group S^1 acting on $T_*(\mathbb{Z})$. Since S^1 is a path-connected group the action is trivial, and hence

$$E^2_{s,t} = \begin{cases} T_t(\mathbb{Z})_p & \text{when } s \leq 0 \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

The edge homomorphism

$$\pi_* T(\mathbb{Z})_p^{hS^1} \to E^\infty_{0,*} \to T_*(\mathbb{Z})_p$$

is induced by the natural map

$$T(\mathbb{Z})_p^{hS^1} = \mathrm{Map}(ES^1_+, T(\mathbb{Z}))_p^{S^1} \to \mathrm{Map}(S^1_+, T(\mathbb{Z}))_p^{S^1} \cong T(\mathbb{Z})_p$$

given by restriction over any choice of S^1-equivalent imbedding $S^1_+ \subset ES^1_+$. This spectral sequence may be derived from the spectral sequence of a tower of fibrations constructed by Bousfield and Kan in [6, p. 258].

Hence Theorem 1.1 has the following corollary.

Corollary 1.3. *Let p be any prime. There is a class $\lambda_{2p-1} \in K_{2p-1}(\mathbb{Z})_p$ such that $\mathrm{tr}_{S^1}(\lambda_{2p-1}) \in \pi_{2p-1}T(\mathbb{Z})_p^{hS^1}$ is detected on a permanent cycle surviving to E^∞ in bidegree $(0, 2p-1)$ of the spectral sequence (1.2). When $p = 2$, the class $\lambda = \lambda_3 \in K_3(\mathbb{Z})_2 \cong \mathbb{Z}/16$ is a generator.*

The second theorem concerns the class $\alpha_1 \in \pi_{2p-3}Q(S^0)_p$ generating the first p-torsion in the stable homotopy groups of spheres. When $p = 2$ this is the stable class of the Hopf map $\eta : S^3 \to S^2$.

Theorem 1.4 (Bökstedt). *The composite*

$$Q(S^0)_p \to K(\mathbb{Z})_p \xrightarrow{\mathrm{tr}_{S^1}} T(\mathbb{Z})_p^{hS^1}$$

maps $\alpha_1 \in \pi_{2p-3}Q(S^0)_p$ to an element of $\pi_{2p-3}T(\mathbb{Z})_p^{hS^1}$ which is detected on a permanent cycle which survives to E^∞ in bidegree $(-2, 2p-1)$ of the spectral sequence (1.2).

We give a proof in Section 3. [1]

2. The trace map $K(\mathbb{Z}) \to T(\mathbb{Z})$

The proof of Theorem 1.1 depends on Waldhausen's Corollary 3.7 of [8], and on Bökstedt and Madsen's Lemma 10.5 of [5].

Let F be a functor with smash product (FSP). See [1] or [4] for the definition of this notion, and for the construction of the K-theory $K(F)$ and topological Hochschild homology $T(F)$ of such a functor, together with the trace map $\mathrm{tr} : K(F) \to T(F)$.

Let F^s be the underlying ring spectrum of F, associated to the prespectrum $\{F(S^n)\}_n$, and let $M_1(F)$ be its zeroth space. $\pi_0 M_1(F) = \pi_0 F^s$ is a ring, and $\mathrm{GL}_1(F) \subset M_1(F)$

[1] I thank Marcel Bökstedt for explaining these results, and many others, to me.

is defined as the union of the components corresponding to units in $\pi_0 M_1(F)$. Then $\mathrm{GL}_1(F)$ is an associative topological monoid. Let $F_{(k)}$ be the $k \times k$ matrix FSP with

$$F_{(k)}(X) = \mathrm{Map}([k], [k] \wedge F(X))$$

(based maps) where $[k] = \{0, 1, \dots, k\}$. Indeed, $\pi_* F^s_{(k)}$ is the $k \times k$ matrix algebra over $\pi_* F^s$. Write $M_k(F) = M_1(F_{(k)})$ and $\mathrm{GL}_k(F) = \mathrm{GL}_1(F_{(k)})$.

Let $\mathrm{BGL}_k(F)$ and $N^{\mathrm{cy}}\mathrm{GL}_k(F)$ be the classifying space and the cyclic nerve of $\mathrm{GL}_k(F)$, respectively. There is a natural projection $\pi : N^{\mathrm{cy}}\mathrm{GL}_k(F) \to \mathrm{BGL}_k(F)$, with a (weak homotopy) section $i : \mathrm{BGL}_k(F) \to N^{\mathrm{cy}}\mathrm{GL}_k(F)$. The K-theory $K(F)$ is constructed as a group completion of the topological monoid $\coprod_{k \geq 0} \mathrm{BGL}_k(F)$. Let the cyclic K-theory $K^{\mathrm{cy}}(F)$ be likewise constructed from the topological monoid $\coprod_{k \geq 0} N^{\mathrm{cy}}\mathrm{GL}_k(F)$.

There is a natural projection $\pi : K^{\mathrm{cy}}(F) \to K(F)$, with a section $i : K(F) \to K^{\mathrm{cy}}(F)$. The trace map $\mathrm{tr} : K(F) \to T(F)$ factors though i by construction. A standard inclusion $\mathrm{GL}_1(F) \to \mathrm{GL}_k(F)$ induces maps $\mathrm{BGL}_1(F) \to K(F)$ and $N^{\mathrm{cy}}\mathrm{GL}_1(F) \to K^{\mathrm{cy}}(F)$, compatible with the projections and sections π and i.

The composite

$$s : N^{\mathrm{cy}}\mathrm{GL}_1(F) \to K^{\mathrm{cy}}(F) \to T(F)$$

is given in simplicial degree q by

$$(f_0, \dots, f_q) \mapsto f_0 \wedge \cdots \wedge f_q.$$

Here each $f_i : S^{n_i} \to F(S^{n_i})$ is assumed to stabilize to a class in $\pi_0 \mathrm{GL}_1(F) \subset \pi_0 M_1(F)$ as $n_i \to \infty$. Clearly the map s may also be factorized as

$$N^{\mathrm{cy}}\mathrm{GL}_1(F) \to N^{\mathrm{cy}} M_1(F) \to T(F).$$

Let $\lambda : S^1_+ \wedge M_1(F) \to T(F)$ be given by the S^1-action on $T(F)$ combined with the inclusion of $M_1(F)$ as the zero-simplices $T(F)_0 = \mathrm{hocolim}_{n \in I} \mathrm{Map}(S^n, F(S^n))$ into $T(F)$. In simplicial degree q the map λ identifies $(C_{q+1})_+ \wedge M_1(F)$ with the maps

$$f_0 \wedge \cdots \wedge f_q : S^{n_0} \wedge \cdots \wedge S^{n_q} \to F(S^{n_0}) \wedge \cdots \wedge F(S^{n_q})$$

in $T(F)_q$ where all but one of the f_i equal a unit map $1_{S^{n_i}} : S^{n_i} \to F(S^{n_i})$. Here C_{q+1} is the cyclic group with $(q+1)$ elements, viewed as the q-simplices in a simplicial model for S^1.

Restricting λ over $S^1_+ \wedge \mathrm{GL}_1(F) \to S^1_+ \wedge M_1(F)$ we get a factorization through $s : N^{\mathrm{cy}}(\mathrm{GL}_1(F)) \to T(F)$:

$$(C_{q+1})_+ \wedge \mathrm{GL}_1(F) \to N^{\mathrm{cy}}(\mathrm{GL}_1(F))_q$$

$$\tau^i_{q+1} \wedge f \mapsto (1, \dots, 1, f, 1, \dots, 1)$$

with f in the ith position, for $i \in [q]$. Here τ_{q+1} is a generator of C_{q+1}.

Hence we have the following commutative diagram, natural in F:

$$\begin{array}{ccccccc}
\mathrm{BGL}_1(F) & \xrightarrow{i} & N^{\mathrm{cy}}\mathrm{GL}_1(F) & & \longleftarrow & & S^1_+\wedge \mathrm{GL}_1(F) \\
\downarrow & & \downarrow & \searrow^{s} & & & \downarrow \\
K(F) & \xrightarrow{i} & K^{\mathrm{cy}}(F) & \longrightarrow & T(F) & \xleftarrow{\lambda} & S^1_+\wedge M_1(F)
\end{array}$$

Let F_1 be the identity FSP with $F_1(X)=X$, and let F_2 be the Eilenberg–Mac Lane FSP of the integers, with $F_2(S^n)=K(\mathbb{Z},n)$. There is a linearization morphism $\ell : F_1 \to F_2$ of FSPs, inducing a π_0-isomorphism on underlying ring spectra

$$\ell : F_1^s = S^0 \to F_2^s = H\mathbb{Z}.$$

Let $\mathrm{SG} \subset G$ be the identity component and the homotopy units of $Q(S^0)$, respectively. We have $M_1(F_1) \simeq Q(S^0)$, $M_1(F_2) \simeq \mathbb{Z}$, $\mathrm{GL}_1(F_1) \simeq G$ and $\mathrm{GL}_1(F_2) \simeq \{\pm 1\} \cong \mathbb{Z}/2$. We identify $N^{\mathrm{cy}}\mathrm{SG}$ with the free loop space $\Lambda\mathrm{BSG}$ as usual. Consider the diagram of homotopy fibers of maps induced by ℓ in the diagram above:

$$\begin{array}{ccccccc}
\mathrm{BSG} & \xrightarrow{i} & \Lambda\mathrm{BSG} & & \longleftarrow & & S^1_+\wedge \mathrm{SG} \\
\downarrow & & \downarrow & \searrow^{s} & & & \downarrow \simeq \\
K(F_1\to F_2) & \xrightarrow{i} & K^{\mathrm{cy}}(F_1\to F_2) & \longrightarrow & T(F_1\to F_2) & \xleftarrow{\lambda} & S^1_+\wedge \mathrm{SG}
\end{array} \qquad (2.1)$$

Here $K(F_1 \to F_2) = \mathrm{hofib}(\ell : K(F_1) \to K(F_2))$, and so on.

The map $\ell : F_1^s \to F_2^s$ is $r = (2p-3)$-connected when localized at p. We need the following two lemmas.

Lemma 2.2. *Let F_1 be the identity* FSP, *and F_2 the Eilenberg–Mac Lane* FSP *of the integers, as above. Then*

$$\lambda : S^1_+ \wedge \mathrm{SG}_{(p)} \to T(F_1 \to F_2)_{(p)}$$

is $(2r+1) = (4p-5)$-connected.

Proof. Let $F_0(X) = \mathrm{hofib}(\ell : F_1(X) \to F_2(X))$ for all X. Then F_0 is a $F_1 - F_1$-bimodule FSP. Let $T(F_1,F_0)$ be the topological Hochschild homology space of F_1 with coefficients in F_0, as defined in Section 10 of [5]. $T(F_1,F_0)$ is the geometric realization of a simplicial space with q-simplices

$$T(F_1,F_0)_q = \operatorname*{hocolim}_{(n_i)_i \in I^{q+1}} \mathrm{Map}(S^{n_0}\wedge\cdots\wedge S^{n_q}, F_0(S^{n_0})\wedge S^{n_1}\wedge\cdots\wedge S^{n_q}).$$

Here we are using the assumption that F_1 is the identity FSP.

The inclusion of the zero-simplices

$$T(F_1,F_0)_0 = \operatorname*{hocolim}_{n\in I} \operatorname{Map}(S^n, F_0(S^n)) \to T(F_1,F_0)$$

is a weak homotopy equivalence, because for $n \in \mathbb{N}$ the map

$$\Omega^n F_0(S^n) \to \Omega^n Q(F_0(S^n))$$

is $(n+1)$-connected. Thus, if we identify $M_1(F_0)$ with the zero-simplices $T(F_1,F_0)_0$, we obtain a homotopy equivalence

$$\mathrm{SG} = M_1(F_0) \to T(F_1,F_0).$$

In [5, p. 130–134], there is constructed a map $S^1_+ \wedge T(F_1,F_0) \to T(F_1 \to F_2)$, and it is easy to see that there is a factorization of λ as

$$S^1_+ \wedge M_1(F_0) \to S^1_+ \wedge T(F_1,F_0) \to T(F_1 \to F_2).$$

Lemma 10.5 of [5] states that the second map in this factorization is $(2r)$-connected, and in fact their proof shows that the map is $(2r+1)$-connected. (The map $S^1_+ \wedge T(F_1,F_0) \to T(F_1 \to F_2)$ is the geometric realization of a map of simplicial spaces which is a homotopy equivalence in simplicial degree zero, and $(2r)$-connected in all other degrees. The results follows).

Thus λ is the composite of a weak homotopy equivalence and a $(2r+1)$-connected map. This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let F_1 and F_2 be as above. Localized at p,*

$$\pi_{2p-2}T(F_1 \to F_2)_{(p)} \cong \begin{cases} \mathbb{Z}/p & \text{if } p \text{ is odd,} \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } p = 2, \end{cases}$$

and likewise

$$\pi_{2p-2}\Lambda\mathrm{BSG}_{(p)} \cong \begin{cases} \mathbb{Z}/p & \text{if } p \text{ is odd,} \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } p = 2. \end{cases}$$

Proof. The inclusion of the zero-simplices $Q(S^0) \simeq T(F_1)_0 \to T(F_1)$ is a homotopy equivalence, so the map $\ell : T(F_1) \to T(F_2)$ factors up to homotopy through the zero-simplices $\mathbb{Z} \simeq T(F_2)_0 \to T(F_2)$. Thus ℓ induces an inessential map on connected components, and so $\pi_{2p-2}T(F_1 \to F_2) \cong T_{2p-1}(\mathbb{Z}) \oplus \pi_{2p-2}Q(S^0)$.

The fiber sequence $\Omega\mathrm{BSG} \to \Lambda\mathrm{BSG} \to \mathrm{BSG}$ has a section, so $\pi_{2p-2}\Lambda\mathrm{BSG} \cong \pi_{2p-2}\Omega\mathrm{BSG} \oplus \pi_{2p-2}\mathrm{BSG}$. Now $\pi_{2p-3}\mathrm{SG}_{(p)} \cong \mathbb{Z}/p$ for all p, while $\pi_{2p-2}\mathrm{SG}_{(p)} = 0$ for p odd and $\pi_2\mathrm{SG} \cong \mathbb{Z}/2$. □

We return to the proof of the theorem. Consider the diagram of homotopy fibers (2.1). We implicitly localize at the prime p. The map λ is $(4p-5) > (2p-2)$-connected, so $\pi_{2p-2}(\lambda)$ is a surjection (in fact an isomorphism). Thus $\pi_{2p-2}(s)$ is a split surjection of isomorphic finite groups, and therefore, injective. $\pi_{2p-2}(i)$ is a split injection, so the composite $\mathbb{Z}/p \cong \pi_{2p-2}\mathrm{BSG} \to \pi_{2p-2}T(F_1 \to F_2)$ is also injective, and is in particular nonzero. Hence the relative trace map $\pi_{2p-2}K(F_1 \to F_2) \to \pi_{2p-2}T(F_1 \to F_2)$ is nonzero.

Now consider the following diagram, where the vertical maps are boundary maps in the fiber sequences induced by $\ell : F_1 \to F_2$, and the top horizontal map is the map we wish to show induces a surjection on π_{2p-2}.

$$\begin{array}{ccc} \Omega K(F_2) & \xrightarrow{\Omega \mathrm{tr}} & \Omega T(F_2) \\ \downarrow & & \downarrow \\ K(F_1 \to F_2) & \longrightarrow & T(F_1 \to F_2) \end{array}$$

By Waldhausen's Corollary 3.7 of [8], the map $\Omega K(F_2) \to K(F_1 \to F_2)$ induces a surjection on π_{2p-2}. Hence the composite $\Omega K(F_2) \to T(F_1 \to F_2)$ induces a nonzero map on π_{2p-2}, and it follows that

$$\pi_{2p-2}(\Omega \mathrm{tr}) : \pi_{2p-2}\Omega K(F_2) \to \pi_{2p-2}\Omega T(F_2) \cong \mathbb{Z}/p$$

is nonzero, and thus surjective. This completes the proof of Bökstedt's Theorem 1.1. $\square$

3. The circle trace map

We now turn to the proof of Theorem 1.4.

Let $E = ES^1$ be a contractible S^1-space with free S^1-action. We will use as a concrete model for E the (thin) geometric realization of the usual simplicial space $[q] \mapsto (S^1)^{q+1}$. Let $\bar{E}$ be the corresponding thick realization, where the degenerate simplices are not collapsed. There is a natural S^1-homotopy equivalence $\bar{E} \to E$ induced by collapsing degenerate simplices. Let $\bar{E}^{(k)}$ and $E^{(k)}$ denote the respective k-skeleta.

Then $\bar{E}^{(0)} = E^{(0)} = S^1$. $\bar{E}^{(1)}$ can be described as the quotient space

$$S^1 \cup (S^1 \times S^1 \times I)/\sim$$

with $(g_0, g_1, 0) \sim g_0$ and $(g_0, g_1, 1) \sim g_1$. $E^{(1)}$ is the further quotient space where we also identify $(g, g, t) \sim g$ for all $t \in I$.

So $\bar{E}^{(1)}$ is the equalizer of the two projection maps $\mathrm{pr}_1, \mathrm{pr}_2 : S^1 \times S^1 \to S^1$. The map $\bar{E}^{(1)} \to E^{(1)}$ identifies a diagonal torus to a circle by a projection map $\Delta S^1 \times (I/\partial I) \to S^1$ onto the first factor. Here $\Delta S^1 \subset S^1 \times S^1$ is the diagonal circle.

We remark that $E^{(1)} \cong S^3$, and the skeleton filtration $E^{(0)} \subset E^{(1)} \subset \cdots$ of $E = ES^1$ agrees with the unit sphere filtration $S^1 = S(\mathbb{C}^1) \subset S^3 = S(\mathbb{C}^2) \subset \cdots$ of $S^\infty = S(\mathbb{C}^\infty) \cong ES^1$. Let $\Sigma_+(X) = \Sigma(X_+) = X_+ \wedge S^1$.

Lemma 3.1. *There is a map of Puppe cofibration sequences*

$$\begin{array}{ccccccccc}
(S^1\times S^1)_+ & \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} & S^1_+ & \longrightarrow & \bar{E}^{(1)}_+ & \longrightarrow & \Sigma_+(S^1\times S^1) & \xrightarrow{a} & \Sigma_+ S^1 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle c} & & \| \\
 & & S^1_+ & \longrightarrow & E^{(1)}_+ & \longrightarrow & \Sigma(S^1\times S^1/\Delta S^1) & \xrightarrow{b} & \Sigma_+ S^1
\end{array}$$

where a is homotopic to $\Sigma_+(\mathrm{pr}_2) - \Sigma_+(\mathrm{pr}_1)$, and c is the suspension of the collapse map $(S^1 \times S^1)_+ \to S^1 \times S^1/\Delta S^1$.

Proof. The diagram is induced by the skeleton-preserving map $\bar{E} \to E$. The claim about a follows from making the obvious choice of homotopy inverse to the collapse map

$$\bar{E}^{(1)}_+ \cup C(S^1_+) \xrightarrow{\simeq} \bar{E}^{(1)}_+/S^1_+ \cong \Sigma_+(S^1 \times S^1). \qquad \square$$

There is an S^1-homeomorphism $h : S^1_+ \wedge S^1_+ \to (S^1 \times S^1)_+$ given by $h(g,s) = (g, gs)$, which descends over c to another S^1-homeomorphism $S^1_+ \wedge S^1 \to (S^1 \times S^1)/\Delta S^1$. Hence we can make compatible identifications

$$\begin{aligned}
&\mathrm{Map}(S^1_+, T(\mathbb{Z}))^{S^1} \cong T(\mathbb{Z}),\\
&\mathrm{Map}(\Sigma_+(S^1 \times S^1), T(\mathbb{Z}))^{S^1} \cong \Omega\Lambda T(\mathbb{Z}),\\
&\mathrm{Map}(\Sigma(S^1 \times S^1/\Delta S^1), T(\mathbb{Z}))^{S^1} \cong \Omega^2 T(\mathbb{Z}).
\end{aligned} \tag{3.2}$$

For example, an S^1-map $f : S^1_+ \to T(\mathbb{Z})$ is identified with $f(1) \in T(\mathbb{Z})$.

Lemma 3.3. *There is a map of Puppe fiber sequences*

$$\begin{array}{ccccccc}
\Omega T(\mathbb{Z}) & \xrightarrow{\alpha} & \Omega\Lambda T(\mathbb{Z}) & \longrightarrow & \mathrm{Map}(\bar{E}^{(1)}_+, T(\mathbb{Z}))^{S^1} & \longrightarrow & T(\mathbb{Z}) \\
\| & & \uparrow{\scriptstyle \gamma} & & \uparrow & & \| \\
\Omega T(\mathbb{Z}) & \xrightarrow{\beta} & \Omega^2 T(\mathbb{Z}) & \longrightarrow & \mathrm{Map}(E^{(1)}_+, T(\mathbb{Z}))^{S^1} & \longrightarrow & T(\mathbb{Z})
\end{array}$$

where α is the looped difference of the adjoints to the circle action map $\mu : S^1_+ \wedge T(\mathbb{Z}) \to T(\mathbb{Z})$ and the trivial action map $\nu : S^1_+ \wedge T(\mathbb{Z}) \to T(\mathbb{Z})$. γ is the usual looped inclusion $\Omega(\Omega T(\mathbb{Z})) \to \Omega(\Lambda T(\mathbb{Z}))$.

Proof. We apply $\mathrm{Map}(-, T(\mathbb{Z}))^{S^1}$ to the map of Puppe cofibration sequences in Lemma 3.1, and make the identification of (3.2). Then γ is induced by the collapse map $S^1_+ \to S^1$ taking $1_+ \subset S^1_+$ to the base point. Finally, pr_2 corresponds under (3.2) to the circle action map μ, and pr_1 to the trivial action map ν which forgets the S^1_+-factor. The lemma follows. $\square$

We momentarily change to spectrum level notation. Recall the splitting from [2]

$$T(\mathbb{Z}) \simeq H\mathbb{Z} \vee \bigvee_{i\geq 2} \Sigma^{2i-1} H\mathbb{Z}/i.$$

Here the inclusion of the zero-simplices $\iota : H\mathbb{Z} \to T(\mathbb{Z})$ gives the map to the first summand.

Let $\mathscr{A}_* = H_*(H\mathbb{Z}/p;\mathbb{Z}/p)$ be the dual of the Steenrod algebra, with polynomial generators $(\xi_i)_{i\geq 1}$ and exterior generators $(\tau_i)_{i\geq 0}$ when p is odd, and polynomial generators $(\zeta_i)_{i\geq 1}$ when $p = 2$. Let χ denote the canonical anti-involution on $\mathscr{A}_*$. Then $H_*(H\mathbb{Z};\mathbb{Z}/p)$ is the subalgebra of $\mathscr{A}_*$ generated by $(\xi_i)_{i\geq 1}$ and $(\chi\tau_i)_{i\geq 1}$ when p is odd, and by $(\zeta_1^2, \chi\zeta_2, \chi\zeta_3, \dots)$ when $p=2$. For p odd, $\xi_1 \in \mathscr{A}_{2p-2}$ is dual to the Steenrod pth power operation P^1, while for $p = 2$ the class $\zeta_1^2 \in \mathscr{A}_2$ is dual to Sq^2. (We are following Milnor in writing ζ_i rather than ξ_i for the polynomial generators in the case $p = 2$, to better distinguish between the even and odd cases.)

Let $X = \mathrm{Map}(E_+^{(1)}, T(\mathbb{Z}))_p^{S^1}$ be the p-completed mapping spectrum, and let $X[0,\infty)$ be its connective cover. From the bottom fibration sequence in Lemma 3.3 it is clear that the first nonzero homotopy groups of $X[0,\infty)$ are $\pi_0 X = \hat{\mathbb{Z}}_p$, and $\pi_{2p-3}X \cong \mathbb{Z}/p$.

Lemma 3.4. *The first k-invariant of the connective cover of* $\mathrm{Map}(E_+^{(1)}, T(\mathbb{Z}))_p^{S^1}$ *is the Steenrod pth power operation*

$$P^1 : H\hat{\mathbb{Z}}_p \xrightarrow{\iota} T(\mathbb{Z})_p \xrightarrow{\beta} \Sigma^{-1}T(\mathbb{Z})_p \to \Sigma^{2p-2}H\mathbb{Z}/p$$

when p is odd, respectively, the Steenrod squaring operation $Sq^2 : H\hat{\mathbb{Z}}_2 \to \Sigma^2 H\mathbb{Z}/2$ when $p = 2$.

Proof. The maps μ and $\nu : S^1_+ \wedge T(\mathbb{Z}) \to T(\mathbb{Z})$ restrict over $\iota : H\mathbb{Z} \to T(\mathbb{Z})$ to give maps λ and $\nu\circ\iota : S^1_+ \wedge H\mathbb{Z} \to T(\mathbb{Z})$, which agree on $1_+ \wedge H\mathbb{Z} \subset S^1_+ \wedge H\mathbb{Z}$. Their difference thus extends over $S^1 \wedge H\mathbb{Z} \to T(\mathbb{Z})$, and induces the derivation

$$\sigma : H_*(H\mathbb{Z};\mathbb{Z}/p) \to H_{*+1}(T(\mathbb{Z});\mathbb{Z}/p)$$

given by $\sigma(x) = \lambda_*([S^1]\otimes x)$, where $[S^1] \in H_1(S^1_+;\mathbb{Z}/p)$ is the fundamental class.

By the calculations of [2], σ maps $\xi_1 \in H_{2p-2}(H\mathbb{Z};\mathbb{Z}/p)$ to the spherical element $e_{2p-1} \in H_{2p-1}(T(\mathbb{Z});\mathbb{Z}/p)$ for p odd, while σ maps $\zeta_1^2 \in H_2(H\mathbb{Z};\mathbb{Z}/2)$ to the spherical element $e_3 \in H_3(T(\mathbb{Z});\mathbb{Z}/2)$ when $p=2$. So the k-invariant $H\hat{\mathbb{Z}}_p \to \Sigma^{2p-2}H\mathbb{Z}/p$ maps ξ_1 or ζ_1^2 to the fundamental class of $\Sigma^{2p-2}H\mathbb{Z}/p$, and is therefore equal to the dual cohomology operation, namely P^1 or Sq^2, respectively. □

We may now prove Bökstedt's Theorem 1.4. We return to space level notation (see Fig. 1).

Here the vertical maps are part of the bottom fiber sequence of Lemma 3.3, and ρ is given by restriction over the S^1-inclusion $E_+^{(1)} \subset E_+ = ES^1_+$. On the level of spectral sequences, ρ induces the natural map from (1.2) to its two rightmost nonzero columns,

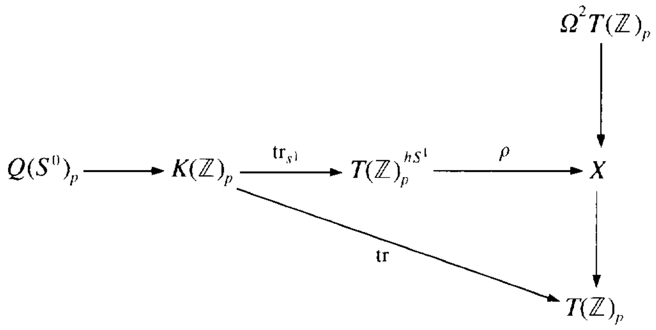

Fig. 1.

where $s = 0$ or $s = -2$. The resulting two-column spectral sequence is simply the long exact homotopy sequence of the cited fiber sequence.

Recall that the first nonzero homotopy groups of $Q(S^0)_p$ are $\pi_0 Q(S^0)_p \cong \hat{\mathbb{Z}}_p$ and $\pi_{2p-3} Q(S^0)_p \cong \mathbb{Z}/p$, and the first k-invariant is P^1 detecting α_1 in the odd primary case, and Sq^2 detecting η in the case $p = 2$.

The composite $Q(S^0)_p \to X[0, \infty)$ induces a π_0-isomorphism, and by Lemma 3.4 the first k-invariants of these spaces agree. Hence the induced map on connected components induces a π_{2p-3}-isomorphism, taking α_1 to the generator of $\pi_{2p-3} X$.

Thus, $\alpha_1 \in \pi_{2p-3} Q(S^0)$ is detected in the rightmost two nonzero columns of the spectral sequence (1.2), where the only nonzero summand in total degree $2p - 3$ is in bidegree $(-2, 2p - 1)$. Thus a generator in this bidegree is hit. This completes the proof of Theorem 1.4. □

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