



Algebraic K -theory of the two-adic integers

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Abstract

We compute the two-completed algebraic K -groups $K_*(\hat{\mathbb{Z}}_2)^\wedge$ of the two-adic integers, and determine the homotopy type of the two-completed algebraic K -theory spectrum $K(\hat{\mathbb{Z}}_2)^\wedge$. The natural map $K(\mathbb{Z})^\wedge \rightarrow K(\hat{\mathbb{Z}}_2)^\wedge$ is shown to induce an isomorphism modulo torsion in degrees $4k + 1$ with $k \geq 1$. © 1999 Elsevier Science B.V. All rights reserved.

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0. Introduction

This paper concludes a series (see [26–28]) that determines all the higher algebraic K -groups of the ring of two-adic integers $\hat{\mathbb{Z}}_2$, after two-adic completion. This is achieved in Theorem 5.5 below. Explicitly for $* > 0$

$$(0.1) \quad K_*(\hat{\mathbb{Z}}_2)^\wedge = \begin{cases} \hat{\mathbb{Z}}_2 \oplus \mathbb{Z}/2^{v_2(k)+3} & \text{for } * = 4k - 1, \\ \mathbb{Z}/2^{v_2(k)+3} & \text{for } * = 4k, \\ \hat{\mathbb{Z}}_2 \oplus \mathbb{Z}/2 & \text{for } * = 4k + 1, \\ \mathbb{Z}/2 & \text{for } * = 4k + 2. \end{cases}$$

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Here $v_2(k)$ denotes the two-adic valuation of k , i.e., the greatest power of 2 that divides k . We also determine the structure of the two-completed algebraic K -theory spectrum $K(\hat{\mathbb{Z}}_2)$ as a connective spectrum, or infinite loop space, in Theorem 8.1.

By the localization sequence, these results also determine the algebraic K -theory groups and spectrum of the field of two-adic numbers $\hat{\mathbb{Q}}_2$. In both cases the resulting spectra agree with their K -localizations, except in low degrees, as predicted by the Lichtenbaum–Quillen conjectures.

These results extend to the case $p=2$ the odd-primary theorem obtained by Bökstedt and Madsen in [5]. They computed the p -adically completed higher algebraic K -groups and algebraic K -theory spectrum $K(\hat{\mathbb{Z}}_p)^\wedge$ of the ring of p -adic integers for any odd prime p . See also [6], where their calculation is extended to include rings of integers in local number fields which are finite unramified extensions of the p -adic numbers $\hat{\mathbb{Q}}_p$.

The methods used in this series of papers are parallel to those of [5, 6], but modified to account for technical and conceptual difficulties appearing in the even case. The fundamental tool is the *cyclotomic trace map* to topological cyclic homology (see [4]). The significant differences appear first in the spectral sequence calculations of [28], where the algebra structure on the mod p homotopy of a ring spectrum is replaced by a module action of mod four homotopy acting upon the mod two homotopy of the ring spectrum.

Second, our method of identification of the homotopy type of $K(\hat{\mathbb{Z}}_2)^\wedge$ differs from the odd-primary case. At odd primes a comparison is made between algebraic K -theory and Waldhausen's algebraic K -theory of spaces (A -theory), which succeeds because the torsion in $K_*(\hat{\mathbb{Z}}_p)^\wedge$ is tightly related to the Adams e -invariant, or the image of J spectrum, which is essentially the K -localization of the sphere spectrum. When $p=2$ there is a distinction between the real and complex e -invariant, or between the real and complex image of J spectra. It is the former which is essentially the K -localization of the sphere spectrum, while it turns out that $K(\hat{\mathbb{Z}}_2)$ is more closely linked to complex topological K -theory than real topological K -theory, in the sense that it is the complex image of J spectrum which describes the torsion in $K_*(\hat{\mathbb{Z}}_2)^\wedge$. (This may be motivated by noting that $\hat{\mathbb{Z}}_2$ does not have a real embedding.)

We therefore replace the comparison with A -theory with a *Galois reduction map* $\text{red}: K(\hat{\mathbb{Z}}_2) \rightarrow K(\mathbb{F}_3) = \text{Im } J_{\mathbb{C}}$ to the complex image of J , behaving somewhat as if the prime field \mathbb{F}_3 were a residue field of $\hat{\mathbb{Z}}_2$. This map is developed in Section 3, and motivated by a similar odd-primary construction from [10].

We also wish to call attention to Theorem 7.7 below, showing that the natural map $K(\mathbb{Z}) \rightarrow K(\hat{\mathbb{Z}}_2)$ induces an isomorphism modulo torsion on homotopy in all degrees $4k+1$ with $k \geq 1$. At an odd prime p the corresponding map is thought to act as multiplication by specific values of a p -adic L -function. In the case $p=2$ all these values are two-adic units.

We refer to the Introduction in [28] for an overview of the initial papers [26–28]. In [28] we computed the mod two homotopy of the topological cyclic homology of the integers at two, which also determines the mod two homotopy of the algebraic

K -theory of the two-adic integers. Thus the task of the present paper is to erect the two-adic type upon the mod two type.

Here is an overview of the paper. In Section 1 the mod two calculations are reviewed. In Section 2 we review known relationships between algebraic and topological K -theory, including Bökstedt’s model $JK(\mathbb{Z})$ for the analogue of étale K -theory of the integers at two. In Section 3 we construct the Galois reduction map from $K(\hat{\mathbb{Z}}_2)$ to the complex image of J infinite loop space $\text{Im } J_{\mathbb{C}}$, and consider its homotopy fiber, which we call the *reduced K -theory* $K^{\text{red}}(\hat{\mathbb{Z}}_2)$. In Section 4 we compute the K -groups of $\hat{\mathbb{Z}}_2$ and the reduction map in an initial range of degrees. These considerations suffice to determine the two-completed algebraic K -groups of the two-adic integers in Section 5. (Completing at another prime p merely gives the p -completion of $K(\mathbb{F}_2)$.) Then in Section 6 we construct an infinite loop map $B \text{Im } J_{\mathbb{C}} \rightarrow K^{\text{red}}(\hat{\mathbb{Z}}_2)$ mapping isomorphically to the torsion in even degrees. The argument uses obstruction techniques, and a construction from [15] of invertible spectra in the category of K -local, two-complete spectra. In Section 7 we identify the infinite loop space cofiber of this map as BBU , carrying the torsion free part of the K -groups. We obtain the following two fiber sequences:

$$(0.2) \quad \begin{aligned} K^{\text{red}}(\hat{\mathbb{Z}}_2) &\rightarrow K(\hat{\mathbb{Z}}_2) \xrightarrow{\text{red}} \text{Im } J_{\mathbb{C}}, \\ B \text{Im } J_{\mathbb{C}} &\rightarrow K^{\text{red}}(\hat{\mathbb{Z}}_2) \rightarrow BBU. \end{aligned}$$

In Section 8 these sequences are viewed as extensions of infinite loop spaces, and classified up to homotopy equivalence. This determines the infinite loop space structure of $K(\hat{\mathbb{Z}}_2)_2^\wedge$ up to homotopy equivalence. Finally, in Section 9 we discuss the natural map $K(\mathbb{Z}) \rightarrow K(\hat{\mathbb{Z}}_2)$, viewed as a close invariant of the algebraic K -theory of the integers, at two. We note that $\sqrt{-1} \notin \hat{\mathbb{Q}}_2$, so in a sense this invariant is closer to $K(\mathbb{Z})$ than the theories known to satisfy étale descent by Thomason’s theorem [30].

1. Computational input

In this section we review the calculational input from [28] that will be required to determine the completed algebraic K -groups of the two-adic integers. All homotopy groups, spaces and spectra will be implicitly completed at two without further mention.

By Theorem D of Hesselholt and Madsen [14], the cyclotomic trace map induces a two-adic equivalence from $K(\hat{\mathbb{Z}}_2)$ to the connective cover of the topological cyclic homology spectrum $TC(\mathbb{Z})$:

$$\text{trc} : K(\hat{\mathbb{Z}}_2) \xrightarrow{\simeq} TC(\mathbb{Z})[0, \infty)$$

Definition 1.1. The algebraic K -theory spectrum $K(\hat{\mathbb{Z}}_2)$ admits a ring spectrum structure induced by the tensor product of $\hat{\mathbb{Z}}_2$ -modules. Likewise the topological cyclic homology spectrum $TC(\mathbb{Z})$ admits a ring spectrum structure related to the shuffle product ring spectrum structure on topological Hochschild homology, by Proposition 4.6

of [14]. We do not know whether the cyclotomic trace map is multiplicative, i.e., whether the equivalence displayed above also respects these two multiplicative structures.¹ To avoid this problem, we will work with an a priori different multiplicative structure on $K(\hat{\mathbb{Z}}_2)$ than the usual one, namely the one inherited from $TC(\mathbb{Z})[0, \infty)$ by the equivalence induced by trc . For clarity we will refer to this as the *TC-multiplication* on $K(\hat{\mathbb{Z}}_2)$, as opposed to the *K-multiplication* induced by the tensor product.

In this paper we will always use the *TC-multiplication* on $K(\hat{\mathbb{Z}}_2)$, unless the contrary is explicitly stated. All our ring spectra and ring spectrum maps are assumed to be unital.

If the cyclotomic trace map turns out to be multiplicative, then the two ring spectrum structures agree. The choice of ring spectrum structure does not affect the additive structure of $K(\hat{\mathbb{Z}}_2)$ as a spectrum, or infinite loop space. In particular, products of two classes in $K_*(\mathbb{Z}_2)$ under the two ring spectrum structures will always agree when one of the classes is in the image of the unit map from $\pi_*Q(S^0)$. We will often use this observation, also when working with finite coefficients.

Theorem 10.9 of [28] computes the mod two homotopy of $TC(\mathbb{Z})$. Hence it also computes the mod two homotopy of $K(\hat{\mathbb{Z}}_2)$, by restricting to nonnegative degrees. We carry over the notation from Definitions 10.3 and 10.10 of loc. cit. to name classes in $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$, through the equivalence induced by trc :

Theorem 1.2. *There are short exact sequences*

$$0 \rightarrow \mathbb{Z}/2\{\xi_{2r}\} \xrightarrow{\hat{\sigma}} K_{2r-1}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2) \xrightarrow{\pi} \mathbb{Z}/2\{\xi_{2r-1}(e), \xi_{2r-1}\} \rightarrow 0$$

for $r \geq 2$, and

$$0 \rightarrow \mathbb{Z}/2\{\xi_{2r+1}(e)\} \xrightarrow{\hat{\sigma}} K_{2r}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2) \xrightarrow{\pi} \mathbb{Z}/2\{\xi_{2r}\} \rightarrow 0$$

for $r \geq 1$, determining $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ for $* \geq 2$ up to extensions. The same sequences determine $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ in degrees $* = 0, 1$, when the undefined class $\xi_1(e)$ is omitted. Hence

$$\#K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2) = \begin{cases} 2 & \text{for } * = 0, \\ 4 & \text{for } * \geq 2 \text{ even or } * = 1, \\ 8 & \text{for } * \geq 3 \text{ odd.} \end{cases}$$

Definition 1.3. Let $i_k : \pi_*(X) \rightarrow \pi_*(X; \mathbb{Z}/2^k)$ be the mod 2^k coefficient reduction homomorphism, $j_k : \pi_*(X; \mathbb{Z}/2^k) \rightarrow \pi_{*-1}(X)$ the mod 2^k Bockstein homomorphism, and $\rho_\ell^k : \pi_*(X; \mathbb{Z}/2^k) \rightarrow \pi_*(X; \mathbb{Z}/2^\ell)$ the coefficient reduction homomorphism. We briefly write $\rho = \rho_1^2$.

¹Dundas has recently proved [9] that the cyclotomic trace map is multiplicative, hence that the *K-multiplication* and *TC-multiplication* agree.

Let η, ν and σ denote the Hopf classes in $\pi_*Q(S^0)$, as well as their images in $K_*(\mathbb{Z})$ and $K_*(\hat{\mathbb{Z}}_2)$. Let $\mu = \mu_9$ be a class in $\pi_9Q(S^0)$ detected by KO -theory. (The ambiguity in μ consists of classes which vanish in $K_9(\mathbb{Z})$, by [21, 33].) Recall from Notation 0.8 of [28] the classes $\tilde{\nu}_4 \in K_4(\hat{\mathbb{Z}}_2; \mathbb{Z}/4)$ and $\tilde{\eta}_2 \in K_2(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$, together with $\lambda \in K_3(\hat{\mathbb{Z}}_2)$ and $\kappa \in K_5(\hat{\mathbb{Z}}_2)$. These are chosen so that $j_2(\tilde{\nu}_4) = \nu$ and $j_1(\tilde{\eta}_2) = \eta$, while $2\lambda = \nu$ and $i_1(\kappa) = \lambda\tilde{\eta}_2$. (Since $\tilde{\eta}_2$ comes from $\pi_2(Q(S^0); \mathbb{Z}/2)$ the ring spectrum structure is not involved in forming this product.)

The classes ξ_{2r} for $r \geq 0$ are defined in $K_{2r}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ by the formulas $\xi_{4i} = \rho(\tilde{\nu}_4^i)$ and $\xi_{4i+2} = \tilde{\nu}_4^i \cdot \tilde{\eta}_2$. The classes $\xi_{2r-1}(e)$ for $e = [\log_2(r) - 1]$ and $r \geq 2$ are defined in $K_{2r-1}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ by the formulas $\xi_{4i+3}(e) = \rho(\tilde{\nu}_4^i) \cdot \lambda$ and $\xi_{4i+5}(e) = \rho(\tilde{\nu}_4^i) \cdot \kappa$. (In forming products with $\tilde{\nu}_4$, the ring spectrum structure is essentially involved. We are using the TC -multiplication here.)

The further classes $\partial(\xi_{2r})$ and $\partial(\xi_{2r-1}(e))$ are the images of the above classes by the composite $\partial\pi$, where ∂ and π appear in the fiber sequence defining *topological cyclic homology*:

$$\Omega TF(\mathbb{Z}) \xrightarrow{\hat{c}} TC(\mathbb{Z}) \xrightarrow{\pi} TF(\mathbb{Z}) \xrightarrow{R-1} TF(\mathbb{Z})$$

See [28] for further information.

We shall see (in Proposition 4.2) that $i_1(\eta)$ represents ξ_1 and $\eta\tilde{\eta}_2$ represents ξ_3 in $TF_*(\mathbb{Z}, \mathbb{Z}/2)$. Hence we can let $\rho(\tilde{\nu}_4^i) \cdot \eta$ represent ξ_{4i+1} in $K_{4i+1}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ and let $\tilde{\nu}_4^i \cdot \eta\tilde{\eta}_2$ represent ξ_{4i+3} in $K_{4i+3}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$. (This will resolve the ambiguity in Definition 10.10 of [28].)

Next we need the action of $\tilde{\nu}_4$ on these generators, which is clear from Proposition 10.11 of [28].

Proposition 1.4. *TC -multiplication by $\tilde{\nu}_4$ on $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ acts injectively in all degrees, and bijectively in degrees $* \geq 2$. On generators it is given by*

$$\begin{aligned} \tilde{\nu}_4 \cdot \xi_{2r} &= \xi_{2r+4}, & \tilde{\nu}_4 \cdot \xi_{2r-1}(e) &= \xi_{2r+3}(f), \\ \tilde{\nu}_4 \cdot \partial(\xi_{2r}) &= \partial(\xi_{2r+4}), & \tilde{\nu}_4 \cdot \partial(\xi_{2r-1}(e)) &= \partial(\xi_{2r+3}(f)), \\ \tilde{\nu}_4 \cdot \xi_{2r-1} &= \xi_{2r+3} \end{aligned}$$

with $e = [\log_2(r) - 1]$ and $f = [\log_2(r + 2) - 1]$. The classes not in the image of the action are additively generated by $1, \xi_1, \partial(\xi_2), \xi_2, \partial(\xi_3(0)), \xi_3(0), \xi_3, \partial(\xi_4), \partial(\xi_5(0))$ and $\xi_5(0)$.

In addition to the obvious results $K_0(\hat{\mathbb{Z}}_2) = \mathbb{Z}$ and $K_1(\hat{\mathbb{Z}}_2) = \hat{\mathbb{Z}}_2^\times \cong \mathbb{Z}/2 \oplus \hat{\mathbb{Z}}_2$, we have $K_2(\hat{\mathbb{Z}}_2) = \mathbb{Z}/2$ generated by η^2 by [8], and $K_3(\hat{\mathbb{Z}}_2) = \hat{\mathbb{Z}}_2 \oplus \mathbb{Z}/8$ by [17, 19]; see also [16]. Here λ generates the torsion subgroup in $K_3(\hat{\mathbb{Z}}_2)$. We shall not assume the K_3 -result for our proofs.

For units $a, b \in \hat{\mathbb{Z}}_2$ we write $\{a\}$ and $\{a, b\}$ for their symbols in $K_1(\hat{\mathbb{Z}}_2)$ and $K_2(\hat{\mathbb{Z}}_2)$, respectively. Recall that $K_2(\hat{\mathbb{Z}}_2)$ is detected by the *Hilbert symbol* $\{a, b\} \mapsto (a, b)_2 \in \mu_2 = \{\pm 1\}$ (see [20]), given by $(a, b)_2 = +1$ if and only if $ax^2 + by^2 = 1$ has a solution with $x, y \in \mathbb{Q}_2$, which is equivalent to the equation having a solution in $\mathbb{Z}/8$. The pairing $K_1(\hat{\mathbb{Z}}_2) \otimes K_1(\hat{\mathbb{Z}}_2) \rightarrow K_2(\hat{\mathbb{Z}}_2)$ taking $\{a\} \otimes \{b\}$ to $\{a, b\}$ is induced by the K -multiplication on $K(\hat{\mathbb{Z}}_2)$. It therefore agrees with the (default) TC -multiplication when $a = -1$ or $b = -1$, since the symbol $\{-1\}$ is the image of $\eta \in \pi_1 Q(S^0)$.

$K_*(\hat{\mathbb{Z}}_2)$ is degreewise finitely generated as a $\hat{\mathbb{Z}}_2$ -module by [32], and

$$(1.5) \quad K_*(\hat{\mathbb{Z}}_2)/(\text{torsion}) \cong \begin{cases} \hat{\mathbb{Z}}_2 & \text{for } * = 0 \text{ or } * \geq 1 \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Combining (1.1) and (1.2) we find

$$(1.6) \quad K_*(\hat{\mathbb{Z}}_2) \cong \begin{cases} \hat{\mathbb{Z}}_2 & \text{for } * = 0, \\ \hat{\mathbb{Z}}_2 \oplus (\text{cyclic}) & \text{for } * \geq 1 \text{ odd,} \\ (\text{cyclic}) & \text{for } * \geq 2 \text{ even} \end{cases}$$

by a universal coefficient sequence argument. Each group named ‘‘cyclic’’ is a nontrivial cyclic group of order a power of two.

Remark 1.7. The localization sequence

$$H\hat{\mathbb{Z}}_2 \simeq K(\mathbb{F}_2) \rightarrow K(\hat{\mathbb{Z}}_2) \rightarrow K(\hat{\mathbb{Q}}_2)$$

and the mod two calculations for $K(\hat{\mathbb{Z}}_2)$ give a mod two calculation of $K(\hat{\mathbb{Q}}_2)$. The injection $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2) \rightarrow K_*(\hat{\mathbb{Q}}_2; \mathbb{Z}/2)$ is an isomorphism for all $* \neq 1$. In degree one the classes ξ_1 and $\partial(\xi_2)$ generate $K_1(\hat{\mathbb{Z}}_2)/2 \cong \mathbb{Z}/2\{-1, \{5\}\}$. We can adjoin a formal element $\xi_1(e) \in K_1(\hat{\mathbb{Q}}_2; \mathbb{Z}/2)$ with $e = -1$ representing the unit $2 \in \hat{\mathbb{Q}}_2^\times$, so

$$K_1(\hat{\mathbb{Q}}_2; \mathbb{Z}/2) \cong \mathbb{Z}/2\{\xi_1(e), \xi_1, \partial(\xi_2)\}.$$

K -multiplicatively $\xi_1(-1)^2 = \partial(\xi_3(0))$, $\xi_1 \cdot \xi_1(-1) = 0$ and $\partial(\xi_2) \cdot \xi_1(-1) = \partial(\xi_3(0))$ by the Hilbert symbol calculations $(2, 2)_2 = -1$, $(-1, 2)_2 = +1$ and $(5, 2)_2 = -1$.

2. Topological models for algebraic K -theory

In this section we review a variety of results relating algebraic K -theory spectra to spectra built using real and complex topological K -theory. We will generally use infinite loop space notation, for almost all spectra that appear are connective. Recall the implicit completion at two.

Let $\mathbb{Z} \times BO$ and $\mathbb{Z} \times BU$ be the connective real and complex topological K -theory ring spectra. By theorems of Suslin [29] the natural maps $K(\mathbb{R}) \rightarrow \mathbb{Z} \times BO$ and $K(\mathbb{C}) \rightarrow \mathbb{Z} \times BU$ are equivalences of ring spectra, and likewise for an embedding $\hat{\mathbb{Q}}_2 \rightarrow \mathbb{C}$ of the

algebraic closure of the two-adic numbers $\hat{\mathbb{Q}}_2$ the induced maps $K(\hat{\mathbb{Q}}_2) \rightarrow K(\mathbb{C}) \rightarrow \mathbb{Z} \times BU$ are equivalences. We let $v_1 \in \pi_2(\mathbb{Z} \times BU)$ denote the periodicity element. Then $\pi_*(\mathbb{Z} \times BU) \cong \hat{\mathbb{Z}}_2[v_1]$ and $\pi_*(\mathbb{Z} \times BU; \mathbb{Z}/2) \cong \mathbb{Z}/2[i_1(v_1)]$. The unit map $Q(S^0) \rightarrow \mathbb{Z} \times BU$ takes $\tilde{\eta}_2$ to $i_1(v_1)$ on mod two homotopy.

The real and complex image of J -spectra $\text{Im} J_{\mathbb{R}}$ and $\text{Im} J_{\mathbb{C}}$ are ring spectra defined by the fiber sequences

$$\text{Im} J_{\mathbb{R}} \xrightarrow{i} \mathbb{Z} \times BO \xrightarrow{\psi^3-1} BSpin,$$

$$\text{Im} J_{\mathbb{C}} \xrightarrow{i} \mathbb{Z} \times BU \xrightarrow{\psi^3-1} BU,$$

respectively. Here ψ^3 is the Adams operation, $BSpin$ is the three-connected cover of $\mathbb{Z} \times BO$, and BU is the zero-component of $\mathbb{Z} \times BU$. Both maps labeled i are maps of ring spectra. (To see this, note that for a self-map $f : X \rightarrow X$ of a ring spectrum, the homotopy fixed point spectrum X^{hf} of the free monoid action generated by f forms a ring spectrum, which can be identified with the homotopy fiber of $f - 1$. The forgetful map $i : X^{hf} \rightarrow X$ is then a ring spectrum map.)

We let $e_{2n-1} \in \pi_{2n-1}(\text{Im} J_{\mathbb{C}})$ denote the integral generator obtained as the image of v_1^n under the fiber map $\Omega BU \rightarrow \text{Im} J_{\mathbb{C}}$. For n odd and $k = 1$, or n even and $k \leq 3$, we let $i_k(v_1^n) \in \pi_{2n}(\text{Im} J_{\mathbb{C}}; \mathbb{Z}/2^k)$ be the generator mapping under i_* to the class with the same name in $\pi_{2n}(\mathbb{Z} \times BU; \mathbb{Z}/2^k)$. When $k = 1$ we shall briefly write e_1 for $i_1(e_1) \in \pi_1(\text{Im} J_{\mathbb{C}}; \mathbb{Z}/2)$, and v_1 for the lift of $i_1(v_1)$ to $\pi_2(\text{Im} J_{\mathbb{C}}; \mathbb{Z}/2)$. Then $\pi_*(\text{Im} J_{\mathbb{C}}; \mathbb{Z}/2) \cong \mathbb{Z}/2[e_1, v_1]/(e_1^2 = 0)$.

Let L_K denote Bousfield’s K -localization functor [7]. The localization map $Q(S^0) \rightarrow L_K Q(S^0)$ has a factorization

$$Q(S^0) \xrightarrow{e_{\mathbb{R}}} \text{Im} J_{\mathbb{R}} \rightarrow L_K Q(S^0),$$

where the right map is an equivalence above degree one by a theorem of Mahowald (see [7]), and the left map induces (the negative of) Adams’ real e -invariant $e'_{\mathbb{R}}$ on homotopy groups (see [1, 24]). Hence we refer to the unit map $e_{\mathbb{R}}$ as the *real e -invariant*.

In the complex case there is an equivalence of ring spectra $q : \text{Im} J_{\mathbb{C}} \rightarrow K(\mathbb{F}_3)$, by Quillen’s theorem [23]. We identify $\text{Im} J_{\mathbb{C}}$ with $K(\mathbb{F}_3)$ through this equivalence. By the complex version of Quillen’s argument from [24] the unit map $Q(S^0) \rightarrow \text{Im} J_{\mathbb{C}}$ induces (the negative of) Adams’ complex e -invariant $e_{\mathbb{C}}$ on homotopy groups (see [1, 24]). Hence we refer to the unit map $e_{\mathbb{C}}$ as the *complex e -invariant*.

The complexification map $c : \mathbb{Z} \times BO \rightarrow \mathbb{Z} \times BU$ is compatible with the Adams operations, and induces a ring spectrum map $c : \text{Im} J_{\mathbb{R}} \rightarrow \text{Im} J_{\mathbb{C}}$. By the (confirmed) Adams conjecture $e_{\mathbb{R}}$ admits a space level section, which we denote $\alpha : \text{Im} J_{\mathbb{R}} \rightarrow Q(S^0)$. Thus $e_{\mathbb{R}} \circ \alpha \simeq \text{id}$.

The mod three reduction homomorphism $\hat{\pi} : \hat{\mathbb{Z}}_3 \rightarrow \mathbb{F}_3$ induces an equivalence $\hat{\pi} : K(\hat{\mathbb{Z}}_3) \rightarrow K(\mathbb{F}_3)$ by a rigidity theorem of Gabber [13]. Following Friedlander [12] we choose an embedding of fields $\hat{\mathbb{Q}}_3 \rightarrow \mathbb{C}$ such that the composite map

$$K(\hat{\mathbb{Z}}_3) \rightarrow K(\hat{\mathbb{Q}}_3) \rightarrow K(\mathbb{C}) \xrightarrow{\simeq} \mathbb{Z} \times BU$$

agrees with Quillen’s Brauer lifting $b : K(\mathbb{F}_3) \rightarrow \mathbb{Z} \times BU$ from [23] under Gabber’s equivalence, i.e., makes the following diagram homotopy commutative:

$$\begin{array}{ccccc} K(\hat{\mathbb{Z}}_3) & \xrightarrow[\simeq]{\hat{\pi}} & K(\mathbb{F}_3) & \xrightarrow[\simeq]{q} & \text{Im}J_{\mathbb{C}} \\ \downarrow & & \downarrow & \searrow b & \downarrow i \\ K(\hat{\mathbb{Q}}_3) & \longrightarrow & K(\mathbb{C}) & \xrightarrow[\simeq]{} & \mathbb{Z} \times BU \end{array}$$

Here i is the fiber map of $\psi^3 - 1$, as before.

We next review Bökstedt’s model $JK(\mathbb{Z})$ for the étale K -theory of the integers. The ring spectrum $JK(\mathbb{Z})$ is defined in [3] as the homotopy fiber of the composite map

$$c \circ (\psi^3 - 1) : \mathbb{Z} \times BO \xrightarrow{\psi^3 - 1} BSpin \xrightarrow{c} BSU.$$

Hence there is a square of fiber sequences

$$\begin{array}{ccccc} BSO & \longrightarrow & * & \longrightarrow & BBSO \\ \downarrow & & \downarrow & & \downarrow \eta \\ \text{Im}J_{\mathbb{R}} & \xrightarrow{i} & \mathbb{Z} \times BO & \xrightarrow{\psi^3 - 1} & BSpin \\ \downarrow j & & \parallel & & \downarrow c \\ JK(\mathbb{Z}) & \longrightarrow & \mathbb{Z} \times BO & \xrightarrow{c(\psi^3 - 1)} & BSU. \end{array}$$

The unit map of $JK(\mathbb{Z})$ factors as

$$(2.1) \quad Q(S^0) \xrightarrow{e_{\mathbb{R}}} \text{Im}J_{\mathbb{R}} \xrightarrow{j} JK(\mathbb{Z})$$

because the homotopy fiber $\text{Cok}J$ of $e_{\mathbb{R}}$ is five-connected and K -acyclic, and $JK(\mathbb{Z})$ agrees with its K -localization above degree one.

The alternative factorization $c \circ (\psi^3 - 1) = (\psi^3 - 1) \circ c$ gives rise to the following square of fiber sequences:

$$(2.2) \quad \begin{array}{ccccc} BBO & \longrightarrow & B(\mathbb{Z} \times BO) & \longrightarrow & S^1 \\ \downarrow & & \downarrow & & \downarrow \\ JK(\mathbb{Z}) & \longrightarrow & \mathbb{Z} \times BO & \xrightarrow{(\psi^3 - 1)c} & BSU \\ \downarrow & & \downarrow c & & \downarrow \\ \text{Im } J_{\mathbb{C}} & \xrightarrow{i} & \mathbb{Z} \times BU & \xrightarrow{\psi^3 - 1} & BU. \end{array}$$

Hence $JK(\mathbb{Z})$ fits into the following three fiber sequences:

$$(2.3) \quad \begin{array}{l} \text{Im } J_{\mathbb{R}} \xrightarrow{j} JK(\mathbb{Z}) \rightarrow BBSO, \\ BBO \rightarrow JK(\mathbb{Z}) \rightarrow \text{Im } J_{\mathbb{C}}, \\ SU \rightarrow JK(\mathbb{Z}) \rightarrow \mathbb{Z} \times BO. \end{array}$$

There is a commutative square of ring spectra

$$\begin{array}{ccc} K(\mathbb{Z}) & \longrightarrow & K(\mathbb{R}) \\ \downarrow & & \downarrow c \\ K(\hat{\mathbb{Z}}_3) & \longrightarrow & K(\mathbb{C}) \end{array}$$

induced by a corresponding commutative square of rings. By the equivalences $K(\hat{\mathbb{Z}}_3) \simeq \text{Im } J_{\mathbb{C}}$, $K(\mathbb{R}) \simeq \mathbb{Z} \times BO$ and $K(\mathbb{C}) \simeq \mathbb{Z} \times BU$, this square induces a map from $K(\mathbb{Z})$ to the pullback PB in the lower left square of (2.2). The latter square is nearly homotopy Cartesian, since its iterated homotopy fiber $\Omega S^1 \simeq \mathbb{Z}$ has trivial homotopy away from degree zero. By a check on π_1 it follows that the induced map $K(\mathbb{Z}) \rightarrow PB$ has a lift

$$\Phi : K(\mathbb{Z}) \rightarrow JK(\mathbb{Z})$$

which is at least four-connected. Φ a map of ring spectra.

The Lichtenbaum–Quillen conjecture for \mathbb{Z} extended to the prime two (as in [10]) then asserts that the map Φ is an equivalence.² In [3] Bökstedt constructed a space

² After this paper was written, this conjecture was proved by Voevodsky [31]. See also [34, 35].

level section

$$\phi : \Omega JK(\mathbb{Z}) \rightarrow \Omega K(\mathbb{Z})$$

to the map $\Omega\Phi : \Omega K(\mathbb{Z}) \rightarrow \Omega JK(\mathbb{Z})$. Thus $\Omega\Phi \circ \phi \simeq \text{id}$. Hence Φ induces split surjections on all homotopy groups. The space level section ϕ is constructed so as to make the following diagram of spaces homotopy commutative:

$$(2.4) \quad \begin{array}{ccc} \Omega \text{Im } J_{\mathbb{R}} & \xrightarrow{\Omega\alpha} & \Omega Q(S^0) \\ \downarrow \Omega j & & \downarrow \Omega i \\ \Omega JK(\mathbb{Z}) & \xrightarrow{\phi} & \Omega K(\mathbb{Z}) \end{array}$$

Hence the splitting ϕ respects classes in $K_*(\mathbb{Z})$ and $JK_*(\mathbb{Z})$ coming from the (real) image of J in $\pi_* Q(S^0)$. (See [3] or p. 541 of [25] for details. Remember that α is the section to $e_{\mathbb{R}}$ arising from a solution to the Adams conjecture.)

The homotopy groups of $JK(\mathbb{Z})$ can easily be computed from (2.2). We transport the notation for classes in $K_*(\mathbb{Z})$ to $JK_*(\mathbb{Z})$ by Φ , i.e., we briefly write x for $\Phi(x) \in JK_*(\mathbb{Z})$ when $x \in K_*(\mathbb{Z})$.

Lemma 2.5. *The homotopy groups of $JK(\mathbb{Z})$ begin*

$$JK_*(\mathbb{Z}) = (\mathbb{Z}\{1\}, \mathbb{Z}/2\{\eta\}, \mathbb{Z}/2\{\eta^2\}, \mathbb{Z}/16\{\lambda\}, 0, \mathbb{Z}\{\kappa\}, 0, \mathbb{Z}/16\{\sigma\}, 0, \mathbb{Z}\{\kappa_9\} \oplus \mathbb{Z}/2\{\mu\}, \dots).$$

Clearly $\lambda\eta = 0, \kappa\eta = 0$ and $\sigma\eta = 0$ in $JK_*(\mathbb{Z})$. There are relations $\lambda\tilde{\eta}_2 = i_1(\kappa), \kappa\tilde{\eta}_2 = i_1(\sigma)$ and $\sigma\tilde{\eta}_2 = i_1(\kappa_9)$.

For torsion classes $x \in K_*(\mathbb{Z})$ with $* \leq 9$ we have $\phi(\Phi(x)) = x$ in $K_*(\mathbb{Z})$. (Without assuming the Lichtenbaum–Quillen conjecture we do not know if $\phi(\kappa) = \kappa$ in $K_5(\mathbb{Z})$, or if $\phi(\kappa_9) = \kappa_9$ in $K_9(\mathbb{Z})$.)

Hence in $K_*(\mathbb{Z})$ we have $\lambda\eta = 0$ and $\lambda\tilde{\eta}_2 = i_1(\phi(\kappa)), \phi(\kappa)\eta = 0$ and $\phi(\kappa)\tilde{\eta}_2 = i_1(\sigma)$, and $\sigma\eta = 0$ and $\sigma\tilde{\eta}_2 = i_1(\phi(\kappa_9))$.

Proof. The group calculations are easy. Let $\gamma_i \in \pi_i(SU)$ be a generator for $i \geq 3$ odd. Then multiplication by $\tilde{\eta}_2$ acts as multiplication by $i_1(v_1)$ on the homotopy of SU , so $\gamma_3 \cdot \tilde{\eta}_2 = i_1(\gamma_5), \gamma_5 \cdot \tilde{\eta}_2 = i_1(\gamma_7)$ and $\gamma_7 \cdot \tilde{\eta}_2 = i_1(\gamma_9)$. But the natural map $SU \rightarrow JK(\mathbb{Z})$ in (2.3) takes $\gamma_3, \gamma_5, \gamma_7$ and γ_9 to λ, κ, σ and κ_9 , at least mod two.

The remaining claims follow from (2.4) and naturality with respect to the space level map ϕ . (Note that multiplication by $\tilde{\eta}_2$ is first defined unstably on π_3 of a space.) \square

We remark that the following diagram homotopy commutes

$$\begin{array}{ccc}
 K(\mathbb{Z}) & \xrightarrow{\phi} & JK(\mathbb{Z}) \\
 \downarrow \pi & & \downarrow \\
 K(\mathbb{F}_3) & \xrightarrow[\simeq]{q} & \text{Im } J_{\mathbb{C}}
 \end{array}$$

where π is induced by the ring surjection $\mathbb{Z} \rightarrow \mathbb{F}_3$, and the unlabeled map is as in (2.2). Also the composite

$$\text{Im } J_{\mathbb{R}} \xrightarrow{j} JK(\mathbb{Z}) \rightarrow \text{Im } J_{\mathbb{C}}$$

agrees with the complexification map c .

Lemma 2.6. *The natural map $JK(\mathbb{Z}) \rightarrow \text{Im } J_{\mathbb{C}}$ induces a surjection on all homotopy groups. Hence so does the composite map*

$$\pi : K(\mathbb{Z}) \xrightarrow{\phi} JK(\mathbb{Z}) \rightarrow \text{Im } J_{\mathbb{C}} \simeq K(\mathbb{F}_3)$$

induced by the ring surjection $\pi : \mathbb{Z} \rightarrow \mathbb{F}_3$.

Proof. The second claim follows from the first and Bökstedt’s splitting. The first claim is clear in degree zero, so suppose $i \geq 1$. There is a long exact sequence of Mayer–Vietoris type associated to the lower left square in (2.2):

$$\rightarrow JK_i(\mathbb{Z}) \rightarrow \pi_i(\text{Im } J_{\mathbb{C}}) \oplus \pi_i(\mathbb{Z} \times BO) \rightarrow \pi_i(\mathbb{Z} \times BU) \rightarrow$$

Since $\pi_i(\mathbb{Z} \times BU)$ is torsion free and $\pi_i(\text{Im } J_{\mathbb{C}})$ is torsion, the latter group is in the image from $JK_i(\mathbb{Z})$. \square

We translate the named classes in $K_*(\mathbb{Z})$ to $\pi_*(\text{Im } J_{\mathbb{C}})$ by the map induced by π , so that

$$\pi_*(\text{Im } J_{\mathbb{C}}) = (\mathbb{Z}\{1\}, \mathbb{Z}/2\{\eta\}, 0, \mathbb{Z}/8\{\lambda\}, 0, \mathbb{Z}/2\{\kappa\}, 0, \mathbb{Z}/16\{\sigma\}, \dots).$$

So $\eta = e_1$, $\lambda = e_3$, $\kappa = e_5$ and $\sigma = e_7$. Then the complexification map $c : \text{Im } J_{\mathbb{R}} \rightarrow \text{Im } J_{\mathbb{C}}$ satisfies $c(1) = 1$, $c(\eta) = \eta$, $c(\eta^2) = 0$, $c(v) = 2\lambda$ and $c(\sigma) = \sigma$.

Remark 2.7. The pullback PB in (2.2) is the model $JK(\mathbb{Z}[\frac{1}{2}])$ for the étale K -theory of $\mathbb{Z}[\frac{1}{2}]$, and the natural map $JK(\mathbb{Z}) \rightarrow JK(\mathbb{Z}[\frac{1}{2}])$ is a covering, compatible with the localization map $K(\mathbb{Z}) \rightarrow K(\mathbb{Z}[\frac{1}{2}])$. The lift $\Phi' : K(\mathbb{Z}[\frac{1}{2}]) \rightarrow JK(\mathbb{Z}[\frac{1}{2}])$ is a map of ring

spectra. By comparisons across the two homotopy Cartesian squares

$$\begin{array}{ccccc}
 K(\hat{\mathbb{Z}}_2) & \longleftarrow & K(\mathbb{Z}) & \xrightarrow{\phi} & JK(\mathbb{Z}) \\
 \downarrow & & \downarrow & & \downarrow \\
 K(\hat{\mathbb{Q}}_2) & \longleftarrow & K(\mathbb{Z}[\frac{1}{2}]) & \xrightarrow{\phi'} & JK(\mathbb{Z}[\frac{1}{2}])
 \end{array}$$

the symbol $\{2\} \in K_1(\hat{\mathbb{Q}}_2)$ represents the formally adjoined class $\xi_1(-1)$ on mod two homotopy. The analogue of Lemma 2.5 shows that $\{2\}\tilde{\eta}_2 = i_1(\lambda)$ and $\tilde{\nu}_4 \cdot \xi_1(-1) = \xi_5(0)$, extending Proposition 1.4.

3. The Galois reduction map

In this section we construct a Galois reduction map $K(\hat{\mathbb{Z}}_2) \rightarrow K(\mathbb{F}_3)$, behaving somewhat as if \mathbb{F}_3 were a residue field of $\hat{\mathbb{Z}}_2$. We continue to identify the target with $\text{Im } J_{\mathbb{C}}$. Compare Section 13 of [11].

Let $\hat{\mathbb{Q}}_2(\mu_{2^\infty})$ be the field of two-adic numbers with the 2^n th roots of unity adjoined, for all n . Let $\bar{\mathbb{Q}}_2$ be the algebraic closure of $\hat{\mathbb{Q}}_2$, and choose an embedding $\bar{\mathbb{Q}}_2 \rightarrow \mathbb{C}$. (Any field embedding will do.) Hence we have field extensions

$$\hat{\mathbb{Q}}_2 \rightarrow \hat{\mathbb{Q}}_2(\mu_{2^\infty}) \rightarrow \bar{\mathbb{Q}}_2 \rightarrow \mathbb{C}.$$

Let $\theta^3 \in \Gamma' = \text{Gal}(\hat{\mathbb{Q}}_2(\mu_{2^\infty})/\hat{\mathbb{Q}}_2)$ be the Galois automorphism given by $\theta^3(\zeta) = \zeta^3$ for each root of unity ζ . So θ^3 corresponds to the two-adic unit 3 under the isomorphism $\Gamma' \cong \text{Aut}(\mu_{2^\infty}) \cong \hat{\mathbb{Z}}_2^\times$. We will choose an extension ϕ^3 of θ^3 to the algebraic closure $\bar{\mathbb{Q}}_2$, with specified action on $\sqrt{3}$.

The following was pointed out to me by Steffen Bentzen.

Lemma 3.1. *The rational integer 3 does not have a square root in $\hat{\mathbb{Q}}_2(\mu_{2^\infty})$.*

Proof. An odd rational integer n has a square root in $\hat{\mathbb{Q}}_2$ precisely if $n \equiv 1 \pmod{8}$. Hence 3 does not have a square root in $\hat{\mathbb{Q}}_2$. So if 3 has a square root in $\hat{\mathbb{Q}}_2(\mu_{2^\infty})$ then $\hat{\mathbb{Q}}_2(\sqrt{3})$ is a quadratic subfield of $\hat{\mathbb{Q}}_2(\mu_{2^\infty})$. These correspond to index two subgroups of $\Gamma' \cong \hat{\mathbb{Z}}_2^\times$, and are $\hat{\mathbb{Q}}_2(\sqrt{-1})$, $\hat{\mathbb{Q}}_2(\sqrt{2})$ and $\hat{\mathbb{Q}}_2(\sqrt{-2})$. But neither of these contain $\sqrt{3}$, since none of $-1 \cdot 3$, $2 \cdot 3$ and $-2 \cdot 3$ are congruent to 1 mod 8. \square

Corollary 3.2. *There exists an extension $\phi^3 \in G = \text{Gal}(\bar{\mathbb{Q}}_2/\hat{\mathbb{Q}}_2)$ of $\theta^3 \in \Gamma'$ with $\phi^3(\sqrt{3}) = +\sqrt{3}$.*

Proof. The algebraic closure $\bar{\mathbb{Q}}_2$ is a union of finite Galois extensions of $\hat{\mathbb{Q}}_2(\mu_{2^\infty})$, so $G \rightarrow \Gamma'$ is surjective. Hence extensions ϕ^3 of θ^3 exist, and by the lemma above we may choose the sign of $\phi^3(\sqrt{3})$ as we like. \square

Definition 3.3. Hereafter we fix a choice of extension ϕ^3 with $\phi^3(\sqrt{3}) = +\sqrt{3}$. This will make our Galois reduction map a “plus reduction map”. An alternative extension ϕ^3 with $\phi^3(\sqrt{3}) = -\sqrt{3}$ defines a “minus reduction map”, which has different properties, less suited to our argument.

Our choice of ϕ^3 induces a self map of $K(\bar{\mathbb{Q}}_2)$ which is compatible up to homotopy with the self map ψ^3 of $\mathbb{Z} \times BU$ under the equivalence $K(\bar{\mathbb{Q}}_2) \simeq \mathbb{Z} \times BU$. For by [18] the only ring spectrum operations on $\mathbb{Z} \times BU$ (up to homotopy) are the two-adic Adams operations ψ^k , and these are detected by their action on π_2 . There is a natural Bockstein isomorphism $\beta_n : K_2(\bar{\mathbb{Q}}_2; \mathbb{Z}/2^n) \rightarrow {}_2^n K_1(\bar{\mathbb{Q}}_2) = \mu_{2^n}(\bar{\mathbb{Q}}_2)$, and ϕ^3 multiplies the 2^n -torsion in K_1 by three. Hence ϕ^3 also acts on π_2 of the two-adic completion of $K(\bar{\mathbb{Q}}_2)$ as multiplication by three, like ψ^3 does on $\mathbb{Z} \times BU$.

Thus we get a homotopy commutative diagram:

$$\begin{array}{ccc} K(\bar{\mathbb{Q}}_2) & \xrightarrow{\phi^3 - 1} & K(\bar{\mathbb{Q}}_2) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathbb{Z} \times BU & \xrightarrow{\psi^3 - 1} & \mathbb{Z} \times BU \end{array}$$

A choice of commuting homotopy determines an equivalence of the horizontal homotopy fibers $K(\bar{\mathbb{Q}}_2)^{h\phi^3} \xrightarrow{\simeq} (\mathbb{Z} \times BU)^{h\psi^3}$. Here $K(\bar{\mathbb{Q}}_2)^{h\phi^3}$ denotes the homotopy fixed points of the action of (the free monoid generated by) ϕ^3 on $K(\bar{\mathbb{Q}}_2)$, which we identify with the homotopy fiber of $\phi^3 - 1$, and likewise for $(\mathbb{Z} \times BU)^{h\psi^3}$.

Since ϕ^3 fixes the subfield $\hat{\mathbb{Q}}_2$, we have natural maps of spectra

$$K(\hat{\mathbb{Z}}_2) \rightarrow K(\hat{\mathbb{Q}}_2) \rightarrow K(\bar{\mathbb{Q}}_2)^{\phi^3} \rightarrow K(\bar{\mathbb{Q}}_2)^{h\phi^3} \xrightarrow{\simeq} (\mathbb{Z} \times BU)^{h\psi^3}.$$

Here $K(\bar{\mathbb{Q}}_2)^{\phi^3}$ denotes the fixed points of the functorially defined action of ϕ^3 on $K(\bar{\mathbb{Q}}_2)$, and the third map is the natural inclusion of fixed points into homotopy fixed points.

Definition 3.4. The composite above induces a spectrum map on connective covers

$$\text{red} : K(\hat{\mathbb{Z}}_2) \rightarrow (\mathbb{Z} \times BU)^{h\psi^3} [0, \infty) = \text{Im } J_{\mathbb{C}}$$

which we will call the *Galois reduction map*. Following Dwyer and Mitchell [11] we define the *reduced K-theory* $K^{\text{red}}(\hat{\mathbb{Z}}_2)$ by the fiber sequence

$$K^{\text{red}}(\hat{\mathbb{Z}}_2) \rightarrow K(\hat{\mathbb{Z}}_2) \xrightarrow{\text{red}} \text{Im } J_{\mathbb{C}}.$$

Lemma 3.5. *The reduction map is unital, i.e., the composite map*

$$Q(S^0) \xrightarrow{i} K(\mathbb{Z}) \rightarrow K(\hat{\mathbb{Z}}_2) \xrightarrow{\text{red}} \text{Im } J_{\mathbb{C}}$$

is homotopic to the complex e-invariant

$$Q(S^0) \xrightarrow{e_{\mathbb{C}}} \text{Im } J_{\mathbb{C}}.$$

Proof. This is clear, because the sequence of spectrum maps defining red respect the unit maps from $Q(S^0)$. \square

Let us conclude this section by discussing the Adams v_1^4 -action on mod two homotopy, and Bousfield’s characterization of K -local spectra. In keeping with our infinite loop space notation, we write $Q(S^0/q)$ for the mod q Moore spectrum, and similarly $Q(S^n/q)$ for its n th delooping. The infinite loop space smash products satisfy $Q(X) \wedge Q(Y) \cong Q(X \wedge Y)$.

In [1] Adams constructed a spectrum map $A : Q(S^8/2) \rightarrow Q(S^0/2)$ inducing multiplication by v_1^4 (an isomorphism) on complex topological K -theory. A satisfies $j_1 A i_1 = 8\sigma$, and any such A will do, because different choices of such maps A induce the same map on topological K -theory. The degree eight induced action on $\pi_*(X; \mathbb{Z}/2)$ for any spectrum X is called the Adams v_1^4 -action, and denoted as multiplication by v_1^4 . (It is typically not the fourth power of an action by some element v_1 .)

Let us write $\widetilde{4\sigma}_4$ for a class in $\pi_8(Q(S^0); \mathbb{Z}/4)$ with $j_2(\widetilde{4\sigma}_4) = 4\sigma$. Mod four homotopy acts on mod two homotopy by a pairing $m : Q(S^0/4) \wedge Q(S^0/2) \rightarrow Q(S^0/2)$ (see [22, 27]). Then multiplication by $\widetilde{4\sigma}_4$ induces a spectrum map

$$A' : Q(S^8/2) \cong Q(S^8) \wedge Q(S^0/2) \xrightarrow{\widetilde{4\sigma}_4 \wedge 1} Q(S^0/4) \wedge Q(S^0/2) \xrightarrow{m} Q(S^0/2)$$

such that $j_1 A' i_1 = j_1 m(\widetilde{4\sigma}_4 \wedge 1) i_1 = j_1 \rho(\widetilde{4\sigma}_4) = 2j_2(\widetilde{4\sigma}_4) = 8\sigma$. Hence we can take $A = A'$ as our Adams map.

A choice of Adams v_1^4 -action also determines a choice of class $\mu \in \pi_9 Q(S^0)$, by $i_1(\mu) = v_1^4 \cdot i_1(\eta)$.

By a theorem of Bousfield [7], a spectrum X is K -local if and only if the v_1^4 -action on $\pi_*(X; \mathbb{Z}/2)$ is invertible, i.e., if multiplication by v_1^4 is an isomorphism. Furthermore K -localization is “smashing”, so the K -localization map $X \rightarrow L_K X$ induces the localization homomorphism

$$\pi_*(X; \mathbb{Z}/2) \rightarrow \pi_*(X; \mathbb{Z}/2)[(v_1^4)^{-1}] \xrightarrow{\cong} \pi_*(L_K X; \mathbb{Z}/2).$$

Hence if v_1^4 acts as an isomorphism on $\pi_*(X; \mathbb{Z}/2)$ in degrees $* \geq k$, then the localization map $X \rightarrow L_K X$ is an equivalence on $(k - 1)$ -connected covers.

4. Calculations in low degrees

We proceed by computing the two-adic and mod two homotopy of $K_*(\widehat{\mathbb{Z}}_2)$ in degrees zero through nine.

Let β_k denote the k th order (mod 2^k) homotopy Bockstein differential. So $\beta_k(\rho_1^k x) = i_1 j_k(x)$. Let t_n generate the torsion in $K_n(\widehat{\mathbb{Z}}_2)$ for $n \geq 1$, and let g_n generate $K_n(\widehat{\mathbb{Z}}_2)$ modulo torsion for n odd. Then t_n is well defined up to an odd multiple, and g_n is well defined up to a two-adic unit, modulo multiples of t_n .

Definition 4.1. Recall from Lemma 1.11 of [28] the class $\tilde{\sigma}$ defined in $K_7(\hat{\mathbb{Z}}_2)/4$ by $i_2(\tilde{\sigma}) = \lambda\tilde{v}_4$.

Let $\tilde{\sigma}_{16} \in \pi_8(Q(S^0); \mathbb{Z}/16)$ be a class satisfying $j_4(\tilde{\sigma}_{16}) = \sigma$, chosen so that $\rho_2^4(\tilde{\sigma}_{16}) = 4\tilde{\sigma}_4$. Multiplication by $\tilde{\sigma}_{16}$ induces the Adams v_1^4 -action on mod two homotopy. Let $\rho_3^4(\tilde{\sigma}_{16}) = 2\tilde{\sigma}_8$ and $\rho(4\tilde{\sigma}_4) = 8\tilde{\sigma}_2$, so $j_3(2\tilde{\sigma}_8) = 2\sigma$ and $j_1(8\tilde{\sigma}_2) = 8\sigma$.

Let $\tilde{\lambda}_8 \in K_4(\hat{\mathbb{Z}}_2; \mathbb{Z}/8)$ be a class satisfying $j_3(\tilde{\lambda}_8) = \lambda$. Again we may assume $\rho_2^3(\tilde{\lambda}_8) = \tilde{v}_4$.

Proposition 4.2. (a) The two-adic algebraic K-groups of $\hat{\mathbb{Z}}_2$ begin as follows:

n	0	1	2	3	4
$K_n(\hat{\mathbb{Z}}_2)$	$\hat{\mathbb{Z}}_2$	$\mathbb{Z}/2 \oplus \hat{\mathbb{Z}}_2$	$\mathbb{Z}/2$	$\mathbb{Z}/8 \oplus \hat{\mathbb{Z}}_2$	$\mathbb{Z}/8$
gen.	1	$\eta, \{3\}$	η^2	λ, g_3	$\lambda\{3\}$

5	6	7	8	9
$\mathbb{Z}/2 \oplus \hat{\mathbb{Z}}_2$	$\mathbb{Z}/2$	$\mathbb{Z}/16 \oplus \hat{\mathbb{Z}}_2$	$\mathbb{Z}/16$	$\mathbb{Z}/2 \oplus \hat{\mathbb{Z}}_2$
t_5, κ	$\kappa\{3\}$	σ, g_7	$\sigma\{3\}$	μ, g_9

The generators satisfy $\eta = \{-1\}$, $\eta\{3\} = \eta^2$, $i_1(g_3) = \partial(\rho\tilde{v}_4)$, $\lambda\{3\} = \lambda\{5\} = \partial(\kappa)$, $i_1(t_5) = \tilde{v}_4 \cdot i_1(\eta) = \xi_5$, $\kappa\{3\} = \partial(\sigma)$, $i_1(g_7) = \partial(\rho(\tilde{v}_4^2))$ and $i_1(g_9) = \sigma\tilde{\eta}_2$.

There is a TC-multiplicative relation $\lambda \cdot \lambda = 0$.

(b) The class $\tilde{\sigma}$ is represented by the image of $\sigma \in \pi_7 Q(S^0)$.

(c) The mod two algebraic K-groups of $\hat{\mathbb{Z}}_2$ begin as follows:

n	0	1	2	3	4
$K_n(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/4$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2$
gen.	1	$\xi_1, \partial(\xi_2)$	$\xi_2, \partial(\xi_3(0))$	$\xi_3(0), \xi_3, \partial(\xi_4)$	$\xi_4, \partial(\xi_5(0))$

5	6	7	8	9
$(\mathbb{Z}/2)^3$	$\mathbb{Z}/4$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$
$\xi_5(0), \xi_5, \partial(\xi_6)$	$\xi_6, \partial(\xi_7(1))$	$\xi_7(1), \xi_7, \partial(\xi_8)$	$\xi_8, \partial(\xi_9(1))$	$\xi_9(1), \xi_9, \partial(\xi_{10})$

The generators satisfy $i_1(\eta) = i_1(\{-1\}) = \xi_1$, $i_1(\{5\}) = \partial(\xi_2)$, $i_1(\{3\}) = \xi_1 + \partial(\xi_2)$, $\tilde{\eta}_2 = \xi_2$, $i_1(\eta^2) = \partial(\xi_3(0))$, $i_1(\lambda) = \xi_3(0)$, $\eta\tilde{\eta}_2 = \xi_3$, $i_1(g_3) = \partial(\xi_4)$, $\rho\tilde{v}_4 = \xi_4$, $i_1(\partial(\kappa)) = \partial(\xi_5(0))$, $i_1(\kappa) = \xi_5(0)$, $\tilde{v}_4 \cdot i_1(\eta) = \xi_5$, $\tilde{v}_4 \cdot i_1(\{5\}) = \partial(\xi_6)$, $\tilde{v}_4 \cdot \tilde{\eta}_2 = \xi_6$, $i_1(\partial(\sigma)) = \tilde{v}_4 \cdot i_1(\eta^2) = \partial(\xi_7(1))$, $i_1(\sigma) = \xi_7(1)$, $\tilde{v}_4 \cdot \eta\tilde{\eta}_2 = \xi_7$, $i_1(g_7) = \partial(\xi_8)$, $\rho(\tilde{v}_4^2) = \xi_8$, $i_1(\sigma\{3\}) = \partial(\xi_9(1))$, $i_1(g_9) = \sigma\tilde{\eta}_2 = \xi_9(1)$ and $i_1(\mu) = \xi_9$.

(d) There are nontrivial additive extensions $2 \cdot \xi_2 = \partial(\xi_3(0))$ and $2 \cdot \xi_6 = \partial(\xi_7(1))$. There are nonzero Bockstein differentials $\beta_1(\xi_2) = \xi_1$, $\beta_1(\xi_3) = \partial(\xi_3(0))$, $\beta_3(\xi_4) = \xi_3(0)$, $\beta_3(\partial(\xi_6)) = \partial(\xi_5(0))$, $\beta_1(\xi_6) = \xi_5$, $\beta_1(\xi_7) = \partial(\xi_7(1))$, $\beta_4(\xi_8) = \xi_7(1)$, $\beta_4(\partial(\xi_{10})) = \partial(\xi_9(1))$ and $\beta_1(\xi_{10}) = \xi_9$.

Integral lifts of the classes ξ_1 , $\partial(\xi_3(0))$, $\xi_3(0)$, $\partial(\xi_5(0))$, ξ_5 , $\partial(\xi_7(1))$, $\xi_7(1)$, $\partial(\xi_9(1))$ and ξ_9 generate the torsion in $K_*(\hat{\mathbb{Z}}_2)$ in these degrees.

Integral lifts of the classes 1 , $\xi_1 + \partial(\xi_2)$, $\partial(\xi_4)$, $\xi_5(0)$, $\partial(\xi_8)$ and $\xi_9(1)$ generate $K_*(\hat{\mathbb{Z}}_2)$ modulo torsion in these degrees.

The proof uses some notation from [28].

Proof. For the integral calculations, it is clear that the unit 1 generates $K_0(\hat{\mathbb{Z}}_2) = \hat{\mathbb{Z}}_2$, and that $K_1(\hat{\mathbb{Z}}_2) \cong \hat{\mathbb{Z}}_2^\times$ has generators $\{-1\}$ and $\{3\}$, where the former is hit by $\eta \in \pi_1 Q(S^0)$. We also know by the Hilbert symbol that $K_2(\mathbb{Z}_2) \cong \mathbb{Z}/2$ is generated by $\{-1, -1\}$, which is hit by η^2 .

In degree three, the second and third map in the following display

$$e_C: Q(S^0) \xrightarrow{i} K(\mathbb{Z}) \rightarrow K(\hat{\mathbb{Z}}_2) \xrightarrow{\text{red}} \text{Im } J_C$$

take the generator $\lambda \in K_3(\mathbb{Z}) = \mathbb{Z}/16\{\lambda\}$ to the generator $\lambda \in \pi_3(\text{Im } J_C) = \mathbb{Z}/8\{\lambda\}$, modulo 4λ , because $e_C(v) = 2\lambda$ and $i(v) = 2\lambda$. Furthermore $4v = \eta^3 \in \pi_3 Q(S^0)$ maps to zero in $K_3(\hat{\mathbb{Z}}_2)$, so the cyclic torsion subgroup of $K_3(\hat{\mathbb{Z}}_2)$ is $\mathbb{Z}/8\{\lambda\}$.

In degree seven the composite above takes $\sigma \in \pi_7 Q(S^0) = \mathbb{Z}/16\{\sigma\}$ to the generator $\sigma \in \pi_7(\text{Im } J_C) = \mathbb{Z}/16\{\sigma\}$, so the cyclic torsion subgroup of $K_7(\hat{\mathbb{Z}}_2)$ is $\mathbb{Z}/16\{\sigma\}$.

In degree nine the composite takes the order two class $\mu \in \pi_9 Q(S^0)$ to the generator $\mu \in \pi_9(\text{Im } J_C) = \mathbb{Z}/2\{\mu\}$, since the complex e -invariant agrees with the real e -invariant in degrees $* = 8k + 1$ (see Proposition 7.19 of [1]). Hence the cyclic torsion subgroup of $K_9(\hat{\mathbb{Z}}_2)$ is $\mathbb{Z}/2\{\mu\}$.

We turn to the mod two calculations. The unit 1 generates $K_0(\hat{\mathbb{Z}}_2; \mathbb{Z}/2) = \mathbb{Z}/2$. In degree one $i_1(\eta) \in \pi_1(Q(S^0); \mathbb{Z}/2)$ is detected in $TF_1(\mathbb{Z}; \mathbb{Z}/2)$ as the class represented by $\xi_1 = te_3$, so we may choose $i_1(\eta)$ to represent ξ_1 in $K_1(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$. The product $\eta\xi_2$ is represented by $te_3 \cdot te_4 = t^2e_3e_4 = d^4(e_4)$ in $E^*(S^1; \mathbb{Z}/2)$, hence has filtration < -4 . Thus $\eta\xi_2$ lies in the image of $\pi_3(R - 1; \mathbb{Z}/2)$, and so $\eta\partial(\xi_2) = \partial(\eta\xi_2) = 0$ in $\text{cok } \pi_3(R - 1; \mathbb{Z}/2) \subset K_2(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$. The Hilbert symbol calculations $(-1, 3)_2 = -1$ and $(-1, 5)_2 = +1$ show that $\eta\{3\} = \eta^2$ and $\eta\{5\} = 0$, so we must have $\partial(\xi_2) = \{5\}$.

In degree two, the universal coefficient sequence for $K_2(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ is not split, because η times the order two element η in $K_1(\hat{\mathbb{Z}}_2)$ equals η^2 , which is nonzero in $K_2(\hat{\mathbb{Z}}_2)/2$.

Hence $K_2(\hat{\mathbb{Z}}_2; \mathbb{Z}/2) = \mathbb{Z}/4\{\tilde{\eta}_2\}$, with $2 \cdot \tilde{\eta}_2 = i_1(\eta^2)$. In the extension

$$0 \rightarrow \mathbb{Z}/2\{\xi_3(0)\} \xrightarrow{\hat{c}} K_2(\hat{\mathbb{Z}}_2; \mathbb{Z}/2) \xrightarrow{\pi} \mathbb{Z}/2\{\xi_2\} \rightarrow 0$$

we recognize $i_1(\eta^2) = \partial(\xi_3(0))$ as generating the kernel of the circle trace map, and $\tilde{\eta}_2 = \xi_2$ by definition. Then $j_1(\tilde{\eta}_2) = \eta$ as claimed.

In degree three the universal coefficient sequence for $K_3(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ splits, because η times the order two element η^2 in $K_2(\hat{\mathbb{Z}}_2)$ equals η^3 , which is zero in $K_3(\hat{\mathbb{Z}}_2)/2$. In the extension

$$0 \rightarrow \mathbb{Z}/2\{\xi_4\} \xrightarrow{\hat{c}} K_3(\hat{\mathbb{Z}}_2; \mathbb{Z}/2) \xrightarrow{\pi} \mathbb{Z}/2\{\xi_3, \xi_3(0)\} \rightarrow 0$$

we have $\rho\tilde{v}_4 = \xi_4$ and $i_1(\lambda) = \xi_3(0)$. Here λ is integral and $\rho\tilde{v}_4$ has a mod four lift, so both $\xi_3(0)$ and $\partial(\xi_4)$ are integral because $K_2(\hat{\mathbb{Z}}_2)$ has no torsion classes of order four or greater. Since the integral lift λ of $\xi_3(0)$ generates the torsion subgroup of $K_3(\hat{\mathbb{Z}}_2)$ it also follows that an integral lift of $\partial(\xi_4)$ generates $K_3(\hat{\mathbb{Z}}_2)$ modulo torsion.

Furthermore $\eta\tilde{\eta}_2 \in K_3(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ has $j_1(\eta\tilde{\eta}_2) = \eta^2 \neq 0$, hence is nonzero, and generates $K_3(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ modulo the integral classes. Thus its image $\pi(\eta\tilde{\eta}_2)$ must equal either ξ_3 or $\xi_3(0) + \xi_3$. But the latter has filtration zero in $E^*(S^1; \mathbb{Z}/2)$ while $\pi(\eta\tilde{\eta}_2)$ has filtration < -4 in $E^*(S^1; \mathbb{Z}/2)$. Thus we may take $\eta\tilde{\eta}_2$ to represent ξ_3 , as claimed.

In degree four the universal coefficient sequence for $K_4(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ splits, because η times the order two element $2v$ in $K_3(\hat{\mathbb{Z}}_2)$ is zero in $K_4(\hat{\mathbb{Z}}_2)/2$. The class $i_1(\kappa) = \xi_5(0)$ is integral, so also $\partial(\xi_5(0))$ will be integral. Since $8\lambda = \eta^3 = 0$ in $K_3(\hat{\mathbb{Z}}_2)$ there is a class $\tilde{\lambda}_8 \in K_4(\hat{\mathbb{Z}}_2; \mathbb{Z}/8)$ with $i_1 j_3(\tilde{\lambda}_8) = i_1(\lambda) = \xi_3(0)$, and $\rho_1^3 \tilde{\lambda}_8 = \xi_4$, so $\beta_3(\xi_4) = \xi_3(0)$.

There is a nonzero mod eight Bockstein from degree five to degree four. The class $\partial(\xi_6) = \tilde{v}_4 \cdot \partial(\xi_2)$ admits the mod eight lift $\tilde{\lambda}_8 \cdot \partial(\xi_2)$, where $\partial(\xi_2) = i_1(\{5\})$ is integral. Hence $i_1 j_3(\partial(\xi_6)) = i_1 j_3(\tilde{\lambda}_8 \cdot \partial(\xi_2)) = i_1(\lambda \cdot \partial(\xi_2)) = i_1(\partial(\lambda \cdot \tilde{\eta}_2)) = i_1(\partial(\kappa)) = \partial(\xi_5(0))$, which is nonzero. Thus $K_4(\hat{\mathbb{Z}}_2) \cong \mathbb{Z}/8$ is generated by $\partial(\kappa) = \lambda\{5\}$, with mod two reduction $\partial(\xi_5(0))$. Since $\eta\lambda = 0$, we may also take $\lambda\{3\}$ as the integral generator.

In degree five the universal coefficient sequence for $K_5(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ splits because η times the order two element $4\lambda\{3\}$ in $K_4(\hat{\mathbb{Z}}_2)$ is zero in $K_5(\hat{\mathbb{Z}}_2)/2$. There is a nonzero mod two Bockstein from degree six to degree five. Namely, $\xi_6 = \tilde{v}_4 \cdot \tilde{\eta}_2$ has $\beta_1(\xi_6) = \tilde{v}_4 \cdot \beta_1(\tilde{\eta}_2) = \tilde{v}_4 \cdot i_1(\eta) = \xi_5$, by regularity of the action of mod four homotopy upon mod two homotopy (see Lemma 26 of [22]). Hence ξ_5 is integral and generates the torsion subgroup $\mathbb{Z}/2$ in $K_5(\hat{\mathbb{Z}}_2)$. Likewise $i_1(\kappa) = \xi_5(0)$ is integral and lifts to generate $K_5(\hat{\mathbb{Z}}_2)$ modulo torsion.

There is a nonzero mod two differential $\beta_1(\xi_7) = \partial(\xi_7(1))$ from degree seven to degree six. This follows from $\beta_1(\xi_3) = \partial(\xi_3(0))$ by multiplication by the mod four class \tilde{v}_4 , using regularity of the action again. Hence $K_6(\hat{\mathbb{Z}}_2) = \mathbb{Z}/2$, generated by $\partial(\bar{\sigma})$. We can also compute $i_1(\kappa\{3\}) = i_1(\kappa\{5\}) = \xi_5(0) \cdot \partial(\xi_2) = \partial(te_3e_4 \cdot te_4) = \partial(t^2e_3e_4^2) = \partial(\xi_7(1)) = \partial(\bar{\sigma})$. The first equality uses $i_1(\eta\kappa) = \eta\lambda\tilde{\eta}_2 = 0$. Thus $\kappa\{3\}$ generates $K_6(\hat{\mathbb{Z}}_2)$.

In degree six the universal coefficient sequence for $K_6(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ is not split, because η times the order two element t_5 in $K_5(\hat{\mathbb{Z}}_2)$ reduces mod two to $\eta \cdot \xi_5 = \tilde{v}_4 \cdot \partial(\xi_3(0)) =$

$\partial(\xi_7(1))$, which is nonzero. Hence $K_6(\hat{\mathbb{Z}}_2; \mathbb{Z}/2) \cong \mathbb{Z}/4$, generated by a lift of t_5 over j_1 , with $\partial(\bar{\sigma})$ representing two times the generator.

In degree seven, the universal coefficient sequence for $K_7(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ is split, because η times the order two element $\partial(\bar{\sigma}) = \kappa\{3\}$ in $K_6(\hat{\mathbb{Z}}_2)$ reduces mod two to $i_1(\eta\partial(\bar{\sigma})) = \partial(i_1(\eta\bar{\sigma})) = \partial(\tilde{v}_4 \cdot i_1(\eta\lambda)) = 0$. The classes $i_1(\bar{\sigma}) = \xi_7(1)$ and $\partial(\xi_8)$ are integral. For $\bar{\sigma}$ is integral, and $\beta_1(\partial(\xi_8)) = \partial(\beta_1(\rho\tilde{v}_4^2)) = 0$. Since the torsion in $K_6(\hat{\mathbb{Z}}_2)$ has order two, this proves the assertion.

To proceed with the mod two analysis, we will use the construction of the previous section.

The Galois reduction map $\text{red} : K(\hat{\mathbb{Z}}_2) \rightarrow \text{Im } J_{\mathbb{C}}$ takes λ to $\lambda = e_3 \pmod{4\lambda}$ on π_3 , hence it takes $\tilde{\lambda}_8$ to $i_3(v_1^2) \pmod{4i_3(v_1^2)}$ in $\pi_4(\text{Im } J_{\mathbb{C}}; \mathbb{Z}/8)$. There is a unique class $\tilde{v}_8 \in \pi_4(Q(S^0); \mathbb{Z}/8)$ with $j_3(\tilde{v}_8) = v$. Then \tilde{v}_8 maps to $2\tilde{\lambda}_8$ in $K_4(\hat{\mathbb{Z}}_2; \mathbb{Z}/8)$, and to $2i_3(v_1^2)$ in $\pi_4(\text{Im } J_{\mathbb{C}}; \mathbb{Z}/8)$. We compute $2 \cdot \text{red}(\tilde{\lambda}_8^2) = \text{red}(\tilde{v}_8 \cdot \tilde{\lambda}_8) = 2i_3(v_1^2) \cdot \text{red}(\tilde{\lambda}_8) = 2i_3(v_1^4)$. Hence $\text{red}(\tilde{\lambda}_8^2) \equiv i_3(v_1^4) \pmod{4i_3(v_1^4)}$. (This would be obvious in the K -multiplication for which red is a ring spectrum map, but we are using the TC -multiplication to form the product $\tilde{\lambda}_8 \cdot \tilde{\lambda}_8$.)

There is a commutative square

$$\begin{array}{ccc}
 K_8(\hat{\mathbb{Z}}_2; \mathbb{Z}/8) & \xrightarrow{\text{red}} & \pi_8(\text{Im } J_{\mathbb{C}}; \mathbb{Z}/8) \\
 j_3 \downarrow & & j_3 \downarrow \cong \\
 {}_8K_7(\hat{\mathbb{Z}}_2) & \xrightarrow[\cong]{\text{red}} & {}_8\pi_7(\text{Im } J_{\mathbb{C}})
 \end{array}$$

where ${}_8K_7(\hat{\mathbb{Z}}_2) = \mathbb{Z}/8\{2\sigma\}$. The calculation of $\text{red}(\tilde{\lambda}_8^2)$ then implies $j_3(\tilde{\lambda}_8^2) \equiv 2\sigma \pmod{8\sigma}$, since $\text{red}(2\sigma) = j_3(i_3(v_1^4))$. Thus $\tilde{\lambda}_8^2 \equiv 2\tilde{\sigma}_8 \in K_8(\hat{\mathbb{Z}}_2; \mathbb{Z}/8)$ modulo $4 \cdot 2\tilde{\sigma}_8$ and modulo classes in $i_3(K_8(\hat{\mathbb{Z}}_2))$. By mod four and mod two reduction we get $\tilde{v}_4^2 \equiv 4\tilde{\sigma}_4 \pmod{i_2(K_8(\hat{\mathbb{Z}}_2))}$ in $K_8(\hat{\mathbb{Z}}_2; \mathbb{Z}/4)$, and $\rho(\tilde{v}_4^2) \equiv 8\tilde{\sigma}_2 \pmod{i_1(K_8(\hat{\mathbb{Z}}_2))}$ in $K_8(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$.

We defined $\bar{\sigma} \in K_7(\hat{\mathbb{Z}}_2)$ as a class with $i_2(\bar{\sigma}) = \lambda\tilde{v}_4$. By Theorem 2(a) of [22], the operation $\delta_3 = i_3j_3$ acts as a derivation on mod eight homotopy. Thus $i_2(2\bar{\sigma}) = \rho_2^3(2\lambda\tilde{\lambda}_8) = \rho_2^3\delta_3(\tilde{\lambda}_8^2) = i_2(2\sigma)$ in $K_7(\hat{\mathbb{Z}}_2; \mathbb{Z}/4)$, and so $\bar{\sigma} \equiv \sigma \pmod{2K_7(\hat{\mathbb{Z}}_2)}$.

In fact $\lambda^2 = 0$ in $K_6(\hat{\mathbb{Z}}_2)$, with respect to the TC -multiplication. For $\tilde{v}_4\lambda = i_2(\bar{\sigma}) \equiv i_2(\sigma) \pmod{\text{two}}$, so squaring gives $\tilde{v}_4^2\lambda^2 \equiv i_2(\sigma^2) \pmod{\text{two}}$, and σ^2 maps to zero in $K_{14}(\mathbb{Z})$ by [21]. Thus $\tilde{v}_4^2 \cdot i_1(\lambda^2) = 0$ and $i_1(\lambda^2) = 0$ by injectivity of the \tilde{v}_4 -action. Since $K_6(\hat{\mathbb{Z}}_2) = \mathbb{Z}/2$ it follows that $\lambda^2 = 0$.

Thus $\lambda\tilde{\lambda}_8 \in K_7(\hat{\mathbb{Z}}_2; \mathbb{Z}/8)$ is integral, and we may assume $i_3(\bar{\sigma}) = \lambda\tilde{\lambda}_8$. Then $i_3(2\bar{\sigma}) = \delta_3(\tilde{\lambda}_8^2) = i_3(2\sigma)$ and so $\bar{\sigma} \equiv \sigma \pmod{4K_7(\hat{\mathbb{Z}}_2)}$. Since $\bar{\sigma}$ originally was only defined modulo four, this proves that we may take $\bar{\sigma} = \sigma$.

From here on we may and will assume $\bar{\sigma} \in K_7(\hat{\mathbb{Z}}_2)$ is chosen as the image of $\sigma \in \pi_7Q(S^0)$. Thus $i_1(\sigma)$ maps to $\xi_7(1) = t^2e_3e_4^2$, and σ generates the torsion in $K_7(\hat{\mathbb{Z}}_2)$. The integral class $\partial(\xi_8)$ lifts to a generator of $K_7(\hat{\mathbb{Z}}_2)$ modulo torsion.

We can now compute the remaining Bockstein differentials.

There is a nonzero mod two Bockstein differential from degree ten to degree nine, given by $\beta_1(\xi_{10}) = \xi_9$. This follows from $\beta_1(\xi_2) = \xi_1$ by multiplication by \tilde{v}_4^2 and regularity of the action of mod four homotopy upon mod two homotopy. Hence ξ_9 is integral and generates the torsion in $K_9(\hat{\mathbb{Z}}_2)$, i.e., $i_1(\mu) = \xi_9$.

The product $\sigma\tilde{\eta}_2$ is also integral in $K_9(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$, since $j_1(\sigma\tilde{\eta}_2) = \sigma\eta$ maps to zero in $K_8(\mathbb{Z})$ by [33]. It is represented by $\sigma\tilde{\eta}_2 = \tilde{\sigma}\tilde{\eta}_2 = \tilde{v}_4\lambda\tilde{\eta}_2 = \tilde{v}_4i_1(\kappa) = \xi_9(1)$, so $\xi_9(1)$ is integral, and represents a generator of $K_9(\hat{\mathbb{Z}}_2)$ modulo torsion.

There is a nonzero mod sixteen Bockstein differential from degree nine to degree eight, given by $\beta_4(\partial(\xi_{10})) = \partial(\xi_9(1))$. For the product $\sigma\{5\}$ satisfies $i_1(\sigma\{5\}) = \sigma \cdot \partial(\xi_2) = \partial(\sigma \cdot \xi_2) = \partial(\xi_7(1) \cdot \xi_2) = \partial(\xi_9(1)) \neq 0$. Writing $\tilde{\sigma}_{16}$ for the class with $j_4(\tilde{\sigma}_{16}) = \sigma$ we get $i_1j_4(\tilde{\sigma}_{16}\{5\}) = \partial(\xi_9(1))$, and so $\beta_4(y) = \partial(\xi_9(1))$ where $y = \rho_1^4(\tilde{\sigma}_{16}\{5\})$ is the image of $i_1(\{5\}) = \partial(\xi_2)$ under the Adams v_1^4 -action. We know that this agrees with multiplication by \tilde{v}_4^2 , modulo classes in the image of ∂ . Thus $y \equiv \tilde{v}_4^2 \cdot \partial(\xi_2) = \partial(\xi_{10}) \pmod{\partial(\xi_{10})}$. But $y \neq 0$ since $\beta_4(y) \neq 0$, so $\beta_4(\partial(\xi_{10})) = \partial(\xi_9(1))$ as claimed.

Hence $\partial(\xi_9(1))$ admits the integral lift $\sigma\{3\} = \sigma\{5\}$ which generates the torsion in $K_8(\hat{\mathbb{Z}}_2)$.

There is a nonzero mod sixteen differential $\beta_4(\xi_8) = \xi_7(1)$ from degree eight to degree seven. For $i_1j_4(\tilde{\sigma}_{16}) = i_1(\sigma) = \xi_7(1)$, and $z = \rho_1^4(\tilde{\sigma}_{16})$ equals the image of the Adams v_1^4 -action on $i_1(1)$, hence agrees with $\tilde{v}_4^2 \cdot i_1(1) = \xi_8$ modulo classes in the image of ∂ . But $\partial(\xi_9(1))$ is integral, so $\beta_4(\partial(\xi_9(1))) = 0$, and $\beta_4(z) = \beta_4(\xi_8) = \xi_7(1)$, as claimed.

The universal coefficient sequence for $K_8(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ is split, because η times the order two element 8σ in $K_7(\hat{\mathbb{Z}}_2)$ is zero. Likewise the universal coefficient sequence for $K_9(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ is split, because η times the order two element $8\sigma\{3\}$ in $K_8(\hat{\mathbb{Z}}_2)$ is zero. \square

In the proof above, we have also shown most of the following proposition.

Proposition 4.3. (a) *The Galois reduction map $\text{red} : K(\hat{\mathbb{Z}}_2) \rightarrow \text{Im} J_{\mathbb{C}}$ takes $\tilde{\lambda}_8 \in K_4(\hat{\mathbb{Z}}_2; \mathbb{Z}/8)$ to $i_3(v_1^2) \pmod{4i_3(v_1^2)}$ in $\pi_4(\text{Im} J_{\mathbb{C}}; \mathbb{Z}/8)$, and $\tilde{\lambda}_8^2$ to $i_3(v_1^4) \pmod{4i_3(v_1^4)}$ in $\pi_8(\text{Im} J_{\mathbb{C}}; \mathbb{Z}/8)$.*

(b) *The classes \tilde{v}_4^2 and $4\tilde{\sigma}_4$ have the same image in $K_8(\hat{\mathbb{Z}}_2; \mathbb{Z}/4) \pmod{i_2(K_8(\hat{\mathbb{Z}}_2))}$, i.e., modulo integral classes. Their images in $TF_8(\mathbb{Z}; \mathbb{Z}/4)$ agree, so the Adams v_1^4 -action on $TF_*(\mathbb{Z}; \mathbb{Z}/2)$ equals the multiplication by \tilde{v}_4^2 . Hence these actions on $TC_*(\mathbb{Z}; \mathbb{Z}/2)$ and $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ agree modulo classes in $\partial(TF_{*+1}(\mathbb{Z}; \mathbb{Z}/2))$. The difference of the actions is given by TC -multiplication by a multiple of $\sigma\{3\}$.*

(c) *The Adams v_1^4 -actions on $TC_*(\mathbb{Z}; \mathbb{Z}/2)$ and $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ are injective in all degrees, and bijective in degrees $* \geq 2$. Hence $TC(\mathbb{Z})$ and $K(\hat{\mathbb{Z}}_2)$ agree with their K -localizations above degree one.*

(d) *The same Adams v_1^4 -actions take 1 to ξ_8 or $\xi_8 + \partial(\xi_9(1))$, take ξ_1 to $\partial(\xi_9)$, and take $\partial(\xi_2)$ to $\partial(\xi_{10})$.*

Proof. (a) The calculation of the reduction map was part of the proof above.

(b) Since $TF_*(\mathbb{Z})$ is trivial in positive even degrees, the images in $TF_8(\mathbb{Z}; \mathbb{Z}/4)$ of the classes \tilde{v}_4^2 and $4\tilde{\sigma}_4$ differ by a class from $TF_8(\mathbb{Z})=0$, hence agree. The Adams v_1^4 -action on $TF_*(\mathbb{Z}; \mathbb{Z}/2)$ is given by multiplication by the image of $4\tilde{\sigma}_4$ in the ring spectrum structure on $TF(\mathbb{Z})$, hence this is the square of the \tilde{v}_4 -action.

The TC -multiplication by \tilde{v}_4^2 on $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ is therefore TC -multiplication by $4\tilde{\sigma}_4$ plus a multiple of $\sigma\{3\}$. The first part equals multiplication by $4\tilde{\sigma}_4$ using the infinite loop space structure, hence equals the Adams v_1^4 -action. Thus the difference of the actions is at most a multiple of TC -multiplication by $\sigma\{3\}$.

(c) The Adams v_1^4 -action is injective on $\ker \pi_*(R-1; \mathbb{Z}/2) \subset TF_*(\mathbb{Z}; \mathbb{Z}/2)$ and on $\text{cok } \pi_*(R-1; \mathbb{Z}/2)$, by Proposition 1.4, and therefore also on $TC_*(\mathbb{Z}; \mathbb{Z}/2)$. The bijectivity–conclusion and the corresponding result for $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ follow easily.

(d) By part (a) the product $4\tilde{\sigma}_4 \cdot i_1(1) = 8\tilde{\sigma}_2$ maps to ξ_8 modulo classes in the image of ∂ , i.e., modulo $\partial(\xi_9(1))$. We also saw in the proof above that the Adams v_1^4 -action takes $\partial(\xi_2)$ to $\partial(\xi_{10})$. Finally μ is chosen so that the Adams v_1^4 -action takes $\xi_1 = i_1(\eta)$ to $\xi_9 = i_1(\mu)$. The claims follow. \square

We continue by computing the Galois reduction map on mod two homotopy.

Recall that $\pi_*(\text{Im } J_C; \mathbb{Z}/2) = \mathbb{Z}/2[e_1, v_1]/(e_1^2 = 0)$. The classes $i_1(e_{2n+1}) = e_1 v_1^n$ are integral, while the classes v_1^n map to generators $i_1(v_1^n)$ in $\pi_{2n}(\mathbb{Z} \times BU; \mathbb{Z}/2)$. The unit map from $\mathcal{Q}(S^0)$ takes $i_1(\eta)$ and $\tilde{\eta}_2$ to e_1 and v_1 . Multiplication by $\tilde{\eta}_2$ on classes with a mod four lift equals formal multiplication by v_1 , and the Adams v_1^4 -action equals formal multiplication by v_1^4 .

Proposition 4.4. (a) *The mod two (plus) Galois reduction map*

$$\text{red} : K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2) \rightarrow \pi_*(\text{Im } J_C; \mathbb{Z}/2)$$

is a v_1^4 -module homomorphism in the sense that it commutes with the Adams v_1^4 -actions on the source and target.

(b) *The Galois reduction map is given on generators for the Adams v_1^4 -action as follows:*

deg.	0	1	1	2	2	3	3	3	4	4
x	1	ξ_1	$\partial(\xi_2)$	ξ_2	$\partial(\xi_3(0))$	$\xi_3(0)$	ξ_3	$\partial(\xi_4)$	ξ_4	$\partial(\xi_5(0))$
red(x)	1	e_1	e_1	v_1	0	$e_1 v_1$	$e_1 v_1$	$e_1 v_1$	v_1^2	0

deg.	5	5	5	6	6	7	7	7	8	9
x	$\xi_5(0)$	ξ_5	$\partial(\xi_6)$	ξ_6	$\partial(\xi_7(1))$	$\xi_7(1)$	ξ_7	$\partial(\xi_8)$	$\partial(\xi_9(1))$	$\xi_9(1)$
red(x)	$e_1 v_1^2$	$e_1 v_1^2$	$e_1 v_1^2$	v_1^3	0	$e_1 v_1^3$	$e_1 v_1^3$	$e_1 v_1^3$	0	$e_1 v_1^4$

Proof. (a) Any infinite loop map induces a v_1^4 -module homomorphism on mod two homotopy.

(b) The listed classes clearly generate $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ as a v_1^4 -module. The classes $\partial(\xi_3(0))$, $\partial(\xi_5(0))$, $\partial(\xi_7(1))$ and $\partial(\xi_9(1))$ are integral and in even degrees, hence map to zero.

By Lemma 3.5 the reduction map takes classes from $\pi_*(Q(S^0); \mathbb{Z}/2)$ to their complex e -invariant, so $\text{red}(1) = 1$, $\text{red}(\xi_1) = e_1$, $\text{red}(\xi_2) = v_1$, $\text{red}(\xi_3) = e_1 v_1$, $\text{red}(\xi_7(1)) = e_1 v_1^3$ and $\text{red}(8\sigma_2) = v_1^4$. Here $8\sigma_2 = \xi_8$ or $\xi_8 + \partial(\xi_9(1))$. By Proposition 4.3(a) the reduction map satisfies $\text{red}(\xi_3(0)) = e_1 v_1$ and $\text{red}(\xi_4) = v_1^2$.

In degree one we claim $\text{red}(\partial(\xi_2)) = e_1$. By construction, the integral Galois reduction map $\text{red} : K_1(\hat{\mathbb{Z}}_2) \rightarrow \pi_1(\text{Im } J_{\mathbb{C}})$ factors through $K_1(\hat{\mathbb{Q}}_2(\mu_{2^\infty}))^{\phi^3}$, which can be identified with the units of $\hat{\mathbb{Q}}_2(\mu_{2^\infty})$ that are invariant under ϕ^3 . Since $\phi^3(\sqrt{3}) = +\sqrt{3}$ by Definition 3.3, the unit symbol $\{3\}$ is divisible by two in this group, and thus maps to zero in $\pi_1(\text{Im } J_{\mathbb{C}}) \cong \mathbb{Z}/2$. Thus $\text{red}(\{3\}) = 0$, and so $\text{red}(\{5\}) = \text{red}(\eta) = e_1$.

Multiplying the integral class $\partial(\xi_2)$ with $\tilde{\eta}_2$ gives $\text{red}(\partial(\xi_2) \cdot \tilde{\eta}_2) = e_1 v_1$, so $\partial(\xi_2) \cdot \tilde{\eta}_2 \neq 0$ maps under π to $\pi \partial(\xi_2) \cdot \tilde{\eta}_2 = 0$. The only possibility is $\partial(\xi_2) \cdot \tilde{\eta}_2 = \partial(\xi_4)$, and $\text{red}(\partial(\xi_4)) = e_1 v_1$.

Again $\partial(\xi_4) = i_1(g_3)$ is integral, so we can multiply by $\tilde{\eta}_2$ to obtain $\text{red}(\partial(\xi_4) \cdot \tilde{\eta}_2) = e_1 v_1^2$. Thus $\partial(\xi_4) \cdot \tilde{\eta}_2 \neq 0$ and maps to zero under π , hence $\partial(\xi_4) \cdot \tilde{\eta}_2 = \partial(\xi_6)$ and $\text{red}(\partial(\xi_6)) = e_1 v_1^2$.

Now $\partial(\xi_6)$ is not integral, but we know that $\partial(\xi_8) = i_1(g_7)$ is integral, and $\partial(\xi_8) \cdot \tilde{\eta}_2 = \partial(\tilde{v}_4^2) \cdot \tilde{\eta}_2 = \tilde{v}_4^2 \cdot \partial(\tilde{\eta}_2) = \partial(\xi_{10})$. Also $\text{red}(\partial(\xi_{10})) = v_1^4 \cdot \text{red}(\partial(\xi_2)) = e_1 v_1^4$, since $v_1^4 \cdot \partial(\xi_2) = \partial(\xi_{10})$. So $\text{red}(\partial(\xi_8)) \cdot \tilde{\eta}_2 = \text{red}(\partial(\xi_{10})) \neq 0$, and $\text{red}(\partial(\xi_8)) \neq 0$ must equal $e_1 v_1^3$.

From $i_1(\lambda) = \xi_3(0)$ and $\lambda \tilde{\eta}_2 = i_1(\kappa) = \xi_5(0)$ we get $\text{red}(\xi_5(0)) = \text{red}(\xi_3(0)) \tilde{\eta}_2 = e_1 v_1^2$. From $\rho \tilde{v}_4 = \xi_4$ and $\tilde{v}_4 \tilde{\eta}_2 = \xi_6$ we get $\text{red}(\xi_6) = \text{red}(\xi_4) \tilde{\eta}_2 = v_1^3$. Applying the mod two Bockstein β_1 gives $\text{red}(\xi_5) = e_1 v_1^2$, since $\beta_1(\xi_6) = \xi_5$ and $\beta_1(v_1^3) = e_1 v_1^2$. From $\rho \tilde{v}_4 = \xi_4$ and $\tilde{v}_4 \eta \tilde{\eta}_2 = \xi_7$ we get $\text{red}(\xi_7) = \text{red}(\xi_4) \eta \tilde{\eta}_2 = e_1 v_1^3$. From $\sigma \tilde{\eta}_2 = \xi_9(1)$ we get $\text{red}(\xi_9(1)) = \text{red}(\sigma) \tilde{\eta}_2 = e_1 v_1^4$. \square

Proposition 4.5. *The Galois reduction map induces an isomorphism from the torsion in $K_{2r-1}(\hat{\mathbb{Z}}_2)$ onto $\pi_{2r-1}(\text{Im } J_{\mathbb{C}})$, for all $r \geq 1$. Hence there is a split short exact sequence*

$$0 \rightarrow K_*^{\text{red}}(\hat{\mathbb{Z}}_2) \rightarrow K_*(\hat{\mathbb{Z}}_2) \xrightarrow{\text{red}} \pi_*(\text{Im } J_{\mathbb{C}}) \rightarrow 0$$

in every degree, and $K_{2r-1}(\hat{\mathbb{Z}}_2) \cong \hat{\mathbb{Z}}_2 \oplus \pi_{2r-1}(\text{Im } J_{\mathbb{C}})$ for all $r \geq 1$.

Proof. Recall that the torsion subgroup of $K_{2r-1}(\hat{\mathbb{Z}}_2)$ is cyclic, by (1.6).

In degrees one, three and five, the classes $\eta \in K_1(\hat{\mathbb{Z}}_2)$, $\lambda \in K_3(\hat{\mathbb{Z}}_2)$ and $t_5 \in K_5(\hat{\mathbb{Z}}_2)$ generate torsion summands that map isomorphically onto $\pi_1(\text{Im } J_{\mathbb{C}}) \cong \mathbb{Z}/2$, $\pi_3(\text{Im } J_{\mathbb{C}}) \cong \mathbb{Z}/8$ and $\pi_5(\text{Im } J_{\mathbb{C}}) \cong \mathbb{Z}/2$, respectively. There are corresponding nonzero Bockstein differentials $\beta_1(\xi_2) = \xi_1$, $\beta_3(\xi_4) = \xi_3(0)$ and $\beta_1(\xi_6) = \xi_5$.

The Adams v_1^4 -action is induced by multiplication by $\tilde{\sigma}_{16}$, for which $\beta_k(\tilde{\sigma}_{16}) = 0$ when $k \leq 3$, so it follows by regularity (of the product on mod eight homotopy, and of the

action of mod four homotopy on mod two homotopy) that there are nonzero Bockstein differentials $\beta_1(v_1^{4k} \cdot \xi_2) = v_1^{4k} \cdot \xi_1$, $\beta_3(v_1^{4k} \cdot \xi_4) = v_1^{4k} \cdot \xi_3(0)$ and $\beta_1(v_1^{4k} \cdot \xi_6) = v_1^{4k} \cdot \xi_5$ for all $k \geq 0$. Hence there are integral lifts of $v_1^{4k} \cdot \xi_1$, $v_1^{4k} \cdot \xi_3(0)$ and $v_1^{4k} \cdot \xi_5$ that generate torsion summands in $K_*(\hat{\mathbb{Z}}_2)$ which map isomorphically onto $\pi_*(\text{Im } J_C)$, in all degrees $* \equiv 1, 3, 5 \pmod 8$.

The composite $Q(S^0) \rightarrow K(\hat{\mathbb{Z}}_2) \xrightarrow{\text{red}} \text{Im } J_C$ induces the complex e -invariant by Lemma 3.5, which is split surjective in degrees $* \equiv 7 \pmod 8$. Hence the reduction map restricted to the cyclic torsion subgroup of $K_*(\hat{\mathbb{Z}}_2)$ induces a split surjection, and must therefore be an isomorphism. \square

Proposition 4.6. (a) *The Galois reduction map induces a surjection on mod two homotopy, in each degree.*

(b) *The mod two homotopy groups of $K^{\text{red}}(\hat{\mathbb{Z}}_2)$ begin as follows:*

$$K_*^{\text{red}}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2) = \begin{cases} 0 & \text{for } * = 0, \\ \mathbb{Z}/2 & \text{for } * = 1, 2, 4, 6, 8, \\ (\mathbb{Z}/2)^2 & \text{for } * = 3, 5, 7, 9. \end{cases}$$

The generator in degree one is $\xi_1 + \partial(\xi_2)$. In degree $2r - 2$ with $2 \leq 2r - 2 \leq 8$ the generator is $\partial(\xi_{2r-1}(e))$. In degree $2r - 1$ with $3 \leq 2r - 1 \leq 9$ the generators are $\xi_{2r-1}(e) + \partial(\xi_{2r})$ and $\xi_{2r-1} + \partial(\xi_{2r})$.

(c) *The Adams v_1^4 -action on $K_*^{\text{red}}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ is injective in all degrees, and bijective in degrees $* \geq 2$. Hence $K^{\text{red}}(\hat{\mathbb{Z}}_2)$ agrees with its K -localization above degree one.*

Proof. (a) The mod two homotopy of $\text{Im } J_C$ is generated by the Adams v_1^4 -action on the classes in degrees $0 \leq * < 8$. The mod two reduction map surjects onto these classes by Proposition 4.4(b), hence onto all of $\pi_*(\text{Im } J_C; \mathbb{Z}/2)$ by the v_1^4 -module action.

(b) For this calculation, combine Propositions 4.2(c) and 4.4(b). In degree eight, $\text{red}(\xi_8) = v_1^4$, and in degree nine $\text{red}(\xi_9) = \text{red}(\partial(\xi_{10})) = e_1 v_1^4$ by Proposition 4.3(d).

(c) The injective Adams v_1^4 -action on $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ restricts to an injective action on the kernel $K_*^{\text{red}}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ of the v_1^4 -homomorphism induced by red . By counting orders it follows that the action is bijective in degrees $* \geq 2$. Alternatively one may note that the homotopy fiber of any map of K -local spectra is again K -local. \square

Remark 4.7. By Remark 2.7 we see that $K(\hat{\mathbb{Q}}_2)$ agrees with its K -localization on zero-connected covers; one degree better than $K(\hat{\mathbb{Z}}_2)$.

5. The algebraic K -groups of $\hat{\mathbb{Z}}_2$

We proceed to determine the completed algebraic K -groups of $\hat{\mathbb{Z}}_2$. The odd groups were determined in Proposition 4.5.

Lemma 5.1. *There are isomorphisms $K_{8k+2}(\hat{\mathbb{Z}}_2) \cong \mathbb{Z}/2$, $K_{8k+4}(\hat{\mathbb{Z}}_2) \cong \mathbb{Z}/8$ and $K_{8k+6}(\hat{\mathbb{Z}}_2) \cong \mathbb{Z}/2$ for all $k \geq 0$.*

Proof. In Proposition 4.2 we found nonzero Bockstein differentials $\beta_1(\xi_3) = \partial(\xi_3(0))$, $\beta_3(\partial(\xi_6)) = \partial(\xi_5(0))$ and $\beta_1(\xi_7) = \partial(\xi_7(1))$. The integral lifts of $\partial(\xi_3(0))$, $\partial(\xi_5(0))$ and $\partial(\xi_7(1))$ generate $K_*(\hat{\mathbb{Z}}_2)$ in degrees two, four and six, respectively.

The Adams v_1^4 -action is induced by multiplication by $\tilde{\sigma}_{16}$, for which $\beta_k(\tilde{\sigma}_{16}) = 0$ when $k \leq 3$, so it follows by regularity that there are nonzero Bockstein differentials $\beta_1(v_1^{4k} \cdot \xi_3) = v_1^{4k} \cdot \partial(\xi_3(0))$, $\beta_3(v_1^{4k} \cdot \partial(\xi_6)) = v_1^{4k} \cdot \partial(\xi_5(0))$ and $\beta_1(v_1^{4k} \cdot \xi_7) = v_1^{4k} \cdot \partial(\xi_7(1))$.

The listed K -groups are known to be cyclic by (1.6), so this completes the proof. \square

It remains to settle the degrees divisible by eight.

Definition 5.2. The symbol $\{3\} \in K_1(\hat{\mathbb{Z}}_2)$ is represented by a unique homotopy class of infinite loop maps $Q(S^1) \rightarrow K(\hat{\mathbb{Z}}_2)$. Since $\text{red}(\{3\}) = 0$ this map uniquely factors through $K^{\text{red}}(\hat{\mathbb{Z}}_2)$. Since $K^{\text{red}}(\hat{\mathbb{Z}}_2)$ agrees with its K -localization above degree one, a check on π_1 shows that there is a unique extension of this map over $L_K Q(S^1)$, and thus over $B \text{Im} J_{\mathbb{R}}$. We obtain the following homotopy commutative diagram:

$$(5.3) \quad \begin{array}{ccccc} Q(S^1) & & & & \\ \text{Be}_{\mathbb{R}} \downarrow & \searrow \{3\} & & & \\ B \text{Im} J_{\mathbb{R}} & \xrightarrow{f_{\mathbb{R}}} & K^{\text{red}}(\hat{\mathbb{Z}}_2) & \longrightarrow & K(\hat{\mathbb{Z}}_2) \end{array}$$

The extended map $f_{\mathbb{R}} : B \text{Im} J_{\mathbb{R}} \rightarrow K^{\text{red}}(\hat{\mathbb{Z}}_2)$ then induces an isomorphism on π_1 .

Lemma 5.4. *The map $f_{\mathbb{R}} : B \text{Im} J_{\mathbb{R}} \rightarrow K^{\text{red}}(\hat{\mathbb{Z}}_2)$ induces an isomorphism*

$$K_{8k}(\hat{\mathbb{Z}}_2) \cong \pi_{8k}(B \text{Im} J_{\mathbb{R}})$$

for all $k \geq 1$.

Proof. Let $i_1 \in \pi_1 Q(S^1) \cong \pi_1 B \text{Im} J_{\mathbb{R}}$ be the generator. Then $f_{\mathbb{R}}$ maps $i_1(i_1)$ to $i_1(\{3\}) = \xi_1 + \partial(\xi_2)$. Since $f_{\mathbb{R}}$ is an infinite loop map it also maps $i_1(\sigma i_1)$ to $i_1(\sigma\{3\}) = \partial(\xi_9(1))$, and takes $v_1^4 \cdot i_1(i_1) = \tilde{\sigma}_{16} \cdot i_1(i_1)$ to $v_1^4 \cdot (\xi_1 + \partial(\xi_2)) = \xi_9 + \partial(\xi_{10})$. See Propositions 4.2(c) and 4.3(d).

Thus $f_{\mathbb{R}} : B \text{Im} J_{\mathbb{R}} \rightarrow K^{\text{red}}(\hat{\mathbb{Z}}_2)$ induces an injection in mod two homotopy in degrees eight and nine. The mod two homotopy of both the source and target of this map is v_1^4 -periodic above degree two, so the mod two homomorphism is also injective in every degree $8k$ and $8k + 1$ for $k \geq 1$.

By a Bockstein argument, which uses that $K_{8k}^{\text{red}}(\hat{\mathbb{Z}}_2)$ is finite and cyclic, it follows that $\pi_{8k}(B \text{Im} J_{\mathbb{R}})$ maps isomorphically onto $K_{8k}^{\text{red}}(\hat{\mathbb{Z}}_2)$ for all $k \geq 1$. Clearly $K_{8k}(\hat{\mathbb{Z}}_2) \cong K_{8k}^{\text{red}}(\hat{\mathbb{Z}}_2)$. \square

Combined, these lemmas prove the following result.

Theorem 5.5. *The two-adically completed algebraic K-groups of the two-adic integers are given by*

$$K_*(\hat{\mathbb{Z}}_2)^\wedge \cong \pi_*(\text{Im } J_{\mathbb{C}})_2^\wedge \oplus \pi_*(B \text{Im } J_{\mathbb{C}})_2^\wedge \oplus \pi_*(BBU)_2^\wedge.$$

Proof. It only remains to note that $\pi_*(B \text{Im } J_{\mathbb{C}})$ equals $\mathbb{Z}/2$ for $* = 4k + 2$, equals $\mathbb{Z}/8$ for $* = 8k + 4$, and equals $\pi_*(B \text{Im } J_{\mathbb{R}})$ for $* = 8k + 8$ with $k \geq 0$. \square

The following results discuss the relationship of $K(\mathbb{Z})$ and $JK(\mathbb{Z})$, and the multiplicative structure on $K_*(\hat{\mathbb{Z}}_2)$.

Lemma 5.6. *For any section $\phi : \Omega JK(\mathbb{Z}) \rightarrow \Omega K(\mathbb{Z})$ (to $\Omega\Phi$) the images of κ and $\phi(\kappa)$ in $K_5(\hat{\mathbb{Z}}_2)$ agree. So we may take $\kappa = \phi(\kappa)$ in $K_5(\hat{\mathbb{Z}}_2)$.*

Proof. The classes κ and $\phi(\kappa)$ in $K_5(\mathbb{Z})$ agree under Φ , hence differ by a torsion class, which maps to a torsion class in $K_5(\hat{\mathbb{Z}}_2) \cong \hat{\mathbb{Z}}_2 \oplus \mathbb{Z}/2$. The $\mathbb{Z}/2$ is detected by the reduction map to $\pi_5(\text{Im } J_{\mathbb{C}}) = \mathbb{Z}/2\{\kappa\}$. We claim both κ and $\phi(\kappa)$ map to the generator under red . In the case of κ this was proven in Proposition 4.4(b). For the other case, note that $\tilde{\eta}_2\kappa = i_1(\sigma)$ in $JK_7(\mathbb{Z}; \mathbb{Z}/2)$, so $\tilde{\eta}_2\phi(\kappa) = i_1(\sigma)$ in $K_7(\mathbb{Z}; \mathbb{Z}/2)$ by (2.4). Here $\text{red}(i_1(\sigma)) \neq 0$ in $\pi_7(\text{Im } J_{\mathbb{C}}; \mathbb{Z}/2)$, so $\text{red}(\phi(\kappa)) \neq 0$. Thus κ and $\phi(\kappa)$ agree when mapped to $K_5(\hat{\mathbb{Z}}_2)$. \square

Lemma 5.7. *In the homotopy of $K(\hat{\mathbb{Z}}_2)$ we have $\kappa\eta = 0$, $\kappa\tilde{\eta}_2 = i_1(\sigma)$ and $\lambda\sigma = 0$.*

Proof. The first properties follow from Lemmas 2.5 and 5.6. To see that $\lambda\sigma = 0$ in $K_{10}(\hat{\mathbb{Z}}_2) \cong \mathbb{Z}/2$ it suffices to check that $i_1(\lambda\sigma) = 0$. But $\tilde{v}_4 \cdot i_1(\lambda\sigma) = i_1(\sigma^2) = 0$ in $K_{14}(\hat{\mathbb{Z}}_2)$ by Proposition 4.2(b) and [21]. Hence $i_1(\lambda\sigma) = 0$ by Proposition 1.4. \square

We also inspect the classes inducing v_1^4 -periodicity a little more closely.

Lemma 5.8. *The image in $K_8(\hat{\mathbb{Z}}_2; \mathbb{Z}/4)$ of $\widetilde{4\sigma}_4$ does not depend upon the choice of lift over j_2 of 4σ . The classes $\widetilde{4\sigma}_4$ and \tilde{v}_4^2 differ at most by a multiple of $i_2(\sigma\{3\})$ coming from $K_8(\hat{\mathbb{Z}}_2)/4 = \mathbb{Z}/4\{\sigma\{3\}\}$. By altering the choice of lift \tilde{v}_4 of v over j_2 by a multiple of $i_2(\lambda\{3\})$ from $K_4(\hat{\mathbb{Z}}_2)/4 = \mathbb{Z}/4\{\lambda\{3\}\}$ we may arrange that either $\tilde{v}_4^2 = \widetilde{4\sigma}_4$ or $\tilde{v}_4^2 = \widetilde{4\sigma}_4 + i_2(\sigma\{3\})$ (but we do not know which).*

The nonpresence or presence of the term $i_2(\sigma\{3\})$ corresponds to whether the v_1^4 -action on $K(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ admits a square root or not, respectively.

Proof. The choice in $\widetilde{4\sigma}_4$ stems from elements in $\pi_8 Q(S^0) = \mathbb{Z}/2\{\sigma\eta, \tilde{v}\}$ in Toda’s notation, all of which map to zero in $K_8(\mathbb{Z})$ by [33, 21], respectively.

By Proposition 4.3(b) the classes $\widetilde{4\sigma}_4$ and \tilde{v}_4^2 have the same images under j_2 , implying the second claim.

Changing \tilde{v}_4 by multiple of $i_2(\lambda\{3\})$ from $K_4(\hat{\mathbb{Z}}_2)/4$ alters \tilde{v}_4^2 by a multiple of $2\lambda\{3\}\tilde{v}_4 = 2i_2(\sigma\{3\})$ since $\lambda^2 = 0$ in $K_6(\hat{\mathbb{Z}}_2)$. So the indeterminacy in \tilde{v}_4^2 equals even multiples of $i_2(\sigma\{3\})$. \square

6. An extension argument

We now proceed towards our second aim of determining the homotopy type of $K(\hat{\mathbb{Z}}_2)$ as an infinite loop space. First we extend the map $f_{\mathbb{R}}$ from (5.3) over the (delooped) complexification map $Bc : B \operatorname{Im} J_{\mathbb{R}} \rightarrow B \operatorname{Im} J_{\mathbb{C}}$, to obtain a map $f_{\mathbb{C}} : B \operatorname{Im} J_{\mathbb{C}} \rightarrow K^{\operatorname{red}}(\hat{\mathbb{Z}}_2)$ that later will be seen to induce a split injection on homotopy. In Section 7 we will identify the infinite loop space cofiber of $f_{\mathbb{C}}$ as BBU , thus determining $K(\hat{\mathbb{Z}}_2)$ up to extensions. The extension will be characterized in Section 8.

In this section we apply Hopkins, Mahowald and Sadofsky’s paper [15] on constructions of “invertible spectra” in the $K/2$ -local category to recognize the homotopy fiber W of Bc as such an invertible spectrum, and to show that the composite map $W \rightarrow B \operatorname{Im} J_{\mathbb{R}} \rightarrow K^{\operatorname{red}}(\hat{\mathbb{Z}}_2)$ is null homotopic. The existence of the extension then follows.

Definition 6.1. Let W be the homotopy fiber of the complexification map

$$Bc : B \operatorname{Im} J_{\mathbb{R}} \rightarrow B \operatorname{Im} J_{\mathbb{C}}.$$

We use the following commutative diagram:

$$(6.2) \quad \begin{array}{ccccccc} & & & W & & & \\ & & & \downarrow & \searrow & & \\ Q(S^1) & \xrightarrow{Be_{\mathbb{R}}} & B \operatorname{Im} J_{\mathbb{R}} & \xrightarrow{f_{\mathbb{R}}} & K^{\operatorname{red}}(\hat{\mathbb{Z}}_2) & \longrightarrow & K(\hat{\mathbb{Z}}_2) \\ & \searrow^{Be_{\mathbb{C}}} & \downarrow^{Bc} & \nearrow^{f_{\mathbb{C}}} & & & \\ & & B \operatorname{Im} J_{\mathbb{C}} & & & & \end{array}$$

Theorem 6.3. *The composite*

$$W \rightarrow B \operatorname{Im} J_{\mathbb{R}} \xrightarrow{f_{\mathbb{R}}} K^{\operatorname{red}}(\hat{\mathbb{Z}}_2)$$

is null-homotopic as a map of infinite loop spaces. Hence there exists an infinite loop map $f_{\mathbb{C}} : B \operatorname{Im} J_{\mathbb{C}} \rightarrow K^{\operatorname{red}}(\hat{\mathbb{Z}}_2)$ extending $f_{\mathbb{R}}$ over Bc and extending $\{3\} : Q(S^1) \rightarrow K^{\operatorname{red}}(\hat{\mathbb{Z}}_2)$ over $Be_{\mathbb{C}} = Bc \circ Be_{\mathbb{R}}$. The map $f_{\mathbb{C}}$ induces an isomorphism on π_1 .

Proof. We use spectrum notation in this proof.

First note that W is 2-connected, so the homotopy classes of maps $W \rightarrow K^{\text{red}}(\hat{\mathbb{Z}}_2)$ are in bijection with the homotopy classes of maps $W \rightarrow L_K K^{\text{red}}(\hat{\mathbb{Z}}_2)$, since $K^{\text{red}}(\hat{\mathbb{Z}}_2)$ agrees with its K -localization above degree one. Furthermore such maps are clearly in bijection with maps $L_K W \rightarrow L_K K^{\text{red}}(\hat{\mathbb{Z}}_2)$.

Next note that $L_K W$ is equivalent to the spectrum $\Sigma^2 X_{1/3}$ of [15]. These authors define X_λ with $\lambda \in \hat{\mathbb{Z}}_2$ as the homotopy fiber of $\psi^3 - \lambda$ acting on the real periodic topological K -theory spectrum KO

$$X_\lambda \rightarrow KO \xrightarrow{\psi^3 - \lambda} KO.$$

Then $X_{1/3}$ is clearly also equivalent to the homotopy fiber of $3\psi^3 - 1$.

Recall the Puppe (co-)fiber sequence of spectra

$$(6.4) \quad \Sigma KO \xrightarrow{\eta} KO \xrightarrow{c} KU \xrightarrow{\Sigma^2(r)b^{-1}} \Sigma^2 KO$$

where $b : \Sigma^2 KU \rightarrow KU$ is the Bott periodicity equivalence, $c : KO \rightarrow KU$ and $r : KU \rightarrow KO$ denote complexification and realification, and η denotes smashing with $\eta : S^1 \rightarrow S^0$. The real and complex Adams operations ψ^3 are compatibly defined on KO and KU with respect to the maps c and r , while $b \circ \Sigma^2(3\psi^3) \simeq \psi^3 \circ b$. Thus we have the following diagram of horizontal and vertical fiber sequences of spectra:

$$\begin{array}{ccccccc}
 \Sigma^2 X_{1/3} & \longrightarrow & \Sigma JO & \xrightarrow{\Sigma c} & \Sigma JU & \longrightarrow & \Sigma^3 X_{1/3} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \Sigma^2 KO & \xrightarrow{\Sigma \eta} & \Sigma KO & \xrightarrow{\Sigma c} & \Sigma KU & \xrightarrow{\Sigma^3 r \Sigma b^{-1}} & \Sigma^3 KO \\
 \downarrow \Sigma^2 3\psi^3 - 1 & & \downarrow \Sigma \psi^3 - 1 & & \downarrow \Sigma \psi^3 - 1 & & \downarrow \Sigma^3 3\psi^3 - 1 \\
 \Sigma^2 KO & \xrightarrow{\Sigma \eta} & \Sigma KO & \xrightarrow{\Sigma c} & \Sigma KU & \xrightarrow{\Sigma^3 r \Sigma b^{-1}} & \Sigma^3 KO
 \end{array}$$

The fiber sequence $W \rightarrow B \text{Im } J_{\mathbb{R}} \rightarrow B \text{Im } J_{\mathbb{C}}$ maps to the top horizontal sequence, and $L_K B \text{Im } J_{\mathbb{R}} \simeq \Sigma JO$ and $L_K B \text{Im } J_{\mathbb{C}} \simeq \Sigma JU$, so $L_K W \simeq \Sigma^2 X_{1/3}$ as claimed.

Note that

$$\pi_* \Sigma^2 X_{1/3} \cong \begin{cases} \mathbb{Z}/2 & * \equiv 1, 2, 4, 5 \pmod{8}, \\ \mathbb{Z}/4 & * \equiv 3 \pmod{8}, \\ 0 & \text{otherwise} \end{cases}$$

consists of one “lightning flash” in every 8 degrees, which is one half of the v_1^4 -periodic homotopy of a (two-cell) mod two Moore spectrum, namely of $\pi_*(L_K S^1/2)$. Each spectrum X_i behaves like a one-cell spectrum in this respect.

In the proof of Theorem 5.1 of [15], Hopkins, Mahowald and Sadofsky construct a map $S^1/2 \rightarrow \Sigma^2 X_{1/3} = X_0$, whose natural extension f_0 over $L_K S^1/2$ induces a surjection on homotopy. Inductively, for all $i \geq 0$ they construct maps $f_i : L_K S^1/2 \rightarrow X_i$ which are surjective on homotopy, and then define X_{i+1} as the spectrum level cofiber of f_i . Then each cofiber map $g_i : X_i \rightarrow X_{i+1}$ induces the zero map on homotopy, and $\pi_*(\text{hocolim}_i X_i) = \text{colim}_i \pi_*(X_i) = 0$, so $\text{hocolim}_i X_i \simeq *$. Let $h_i : X_{i+1} \rightarrow L_K S^2/2$ be the connecting maps in the Puppe sequence.

Rephrased, this construction eventually annihilates $\Sigma^2 X_{1/3} = X_0$ by inductively attaching cones on K -localized mod two Moore spectra until the remainder in the colimit is contractible. We obtain the following tower of spectra, with cofiber sequences mapping down, across and down again:

$$\begin{array}{ccccccc}
 L_K S^1/2 & & L_K S^1/2 & & L_K S^1/2 & & \\
 \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \\
 \Sigma^2 X_{1/3} & \xrightarrow{g_0} & X_1 & \xrightarrow{g_1} & X_2 & \longrightarrow \cdots & \longrightarrow \text{hocolim}_i X_i \simeq * \\
 \downarrow k_0 & & \downarrow h_0 & & \downarrow h_1 & & \\
 L_K K^{\text{red}}(\hat{\mathbb{Z}}_2) & & L_K S^2/2 & & L_K S^2/2 & &
 \end{array}$$

From part (iii) of the cited theorem of [15] it follows that each composite map

$$\delta_i = h_{i-1} \circ f_i : L_K S^1/2 \rightarrow L_K S^2/2$$

fits into a diagram

$$\begin{array}{ccc}
 L_K S^1 & \xrightarrow{\alpha_{-1}} & L_K S^2 \\
 \downarrow i_1 & & \downarrow i_1 \\
 L_K S^1/2 & \xrightarrow{\delta_i} & L_K S^2/2.
 \end{array}$$

Here α_{-1} generates $\pi_{-1} L_K S^0 \cong \hat{\mathbb{Z}}_2$ and without K -localizing $v_1^4 \alpha_{-1}$ is realized in $\pi_7 Q(S^0)$ as the class of the Hopf map σ .

Now we claim that the given map $k_0 : X_0 = \Sigma^2 X_{1/3} \rightarrow \Sigma J O \rightarrow L_K K^{\text{red}}(\hat{\mathbb{Z}}_2)$ inductively extends over each map $X_i \rightarrow X_{i+1}$, and so in the limit extends over $\text{hocolim}_i X_i \simeq *$, and thus is null-homotopic.

To see this, inductively assume we have a map $k_i : X_i \rightarrow L_K K^{\text{red}}(\hat{\mathbb{Z}}_2)$ such that $k_i \circ f_i$ is null-homotopic. (We will check the case $i = 0$ after the following lemma.) Then k_i extends over g_i to a map $k_{i+1} : X_{i+1} \rightarrow L_K K^{\text{red}}(\hat{\mathbb{Z}}_2)$. Two choices of extensions differ by a map $\ell \circ h_i$ where $\ell : L_K S^1/2 \rightarrow L_K K^{\text{red}}(\hat{\mathbb{Z}}_2)$. We now ask whether the composite $k_{i+1} \circ f_{i+1}$ is null-homotopic.

This map is determined by its restriction to $S^1/2$, which amounts to a class in $K_2^{\text{red}}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$. Likewise ℓ amounts to a class in $K_3^{\text{red}}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$. Altering k_{i+1} by changing ℓ alters $k_{i+1} \circ f_{i+1}$ by the change in the composite $\ell \circ \delta_{i+1}$. Hence if we show that the map induced by precomposition with δ_{i+1} :

$$\delta_{i+1}^* : K_3^{\text{red}}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2) \rightarrow K_2^{\text{red}}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$$

is surjective, then it follows that k_{i+1} may be chosen so that $k_{i+1} \circ f_{i+1}$ is trivial, as required. Here the target group is generated by the integral class $i_1(\eta^2)$, so this follows from the following inserted lemma.

Lemma 6.5. *Let $f_3 \in K_3^{\text{red}}(\hat{\mathbb{Z}}_2)$ be the integral generator represented mod two by $\xi_3(0) + \partial(\xi_4)$, and let $\mu\eta = \mu_{10} \in \pi_{10} Q(S^0)$ be the element detected by KO -theory. Then in $K_{10}^{\text{red}}(\hat{\mathbb{Z}}_2)$*

$$\sigma \cdot f_3 = \mu\eta$$

so after K -localizing we have $\alpha_{-1} \cdot f_3 = \eta^2$. With mod two coefficients this asserts that $\delta_{i+1}^*(i_1(f_3)) = i_1(\eta^2)$.

Proof. We can also write $f_3 = \lambda + g_3$ in $K_3(\hat{\mathbb{Z}}_2)$. Then $\sigma\lambda = 0$ in $K_{10}(\hat{\mathbb{Z}}_2)$ by Lemma 5.6, while σg_3 is represented mod two by

$$\sigma \cdot \partial(\xi_4) = \partial(\sigma \cdot \rho\tilde{v}_4) = \partial(\xi_{11}(1)).$$

We claim this represents $\mu\eta$. For acting by \tilde{v}_4^2 on $\beta_1(\xi_3) = \partial(\xi_3(0))$ yields $\beta_1(\xi_{11}) = \partial(\xi_{11}(1))$. So $\partial(\xi_{11}(1))$ is integral and generates the two-torsion in $K_{10}(\hat{\mathbb{Z}}_2)$, i.e., it is $i_1(\mu\eta)$. This proves the displayed formula. The other two claims follow directly from this. \square

To complete the proof of Theorem 6.3 we must check the initial case of the induction argument, namely that $k_0 \circ f_0 : L_K S^1/2 \rightarrow L_K K^{\text{red}}(\hat{\mathbb{Z}}_2)$ is null-homotopic. Again this amounts to a claim about a map $S^1/2 \rightarrow K^{\text{red}}(\hat{\mathbb{Z}}_2)$ or a class in $K_2^{\text{red}}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2) \cong \mathbb{Z}/2 \{i_1(\eta^2)\}$. But the given map factors through $W \rightarrow K^{\text{red}}(\hat{\mathbb{Z}}_2)$ and W is 2-connected, so the map is inessential.

Hence by induction over i each map $k_i : X_i \rightarrow L_K K^{\text{red}}(\hat{\mathbb{Z}}_2)$ extends over X_{i+1} , and so $\Sigma^2 X_{1/3} = X_0 \rightarrow L_K K^{\text{red}}(\hat{\mathbb{Z}}_2)$ extends over $\text{hocolim}_i X_i \simeq *$, and must be inessential. Thus the given map $W \rightarrow K^{\text{red}}(\hat{\mathbb{Z}}_2)$ is null-homotopic, and this completes the proof of the theorem. \square

7. Two (co)fiber sequences

Now we know that $f_{\mathbb{R}} : B \operatorname{Im} J_{\mathbb{R}} \rightarrow K^{\operatorname{red}}(\hat{\mathbb{Z}}_2)$ extends over Bc . We wish to recognize the infinite loop space cofiber of the extension $f_{\mathbb{C}}$ as BBU . Let $i_1 \in \pi_1 B \operatorname{Im} J_{\mathbb{C}}$ be the generator.

Proposition 7.1. *Any infinite loop map $f : B \operatorname{Im} J_{\mathbb{C}} \rightarrow K^{\operatorname{red}}(\hat{\mathbb{Z}}_2)$ mapping i_1 to $\{3\}$ on π_1 induces an injection on mod two homotopy and a split injection on integral homotopy in all degrees. Explicitly on mod two homotopy*

$$\begin{aligned} f_*(v_1^k i_1) &= \xi_{2k+1} + \partial(\xi_{2k+2}), \\ f_*(e_1 v_1^k i_1) &= \partial(\xi_{2k+3}(e)) \end{aligned}$$

for $0 \leq k \leq 3$. Let X be the infinite loop space cofiber of such a given map, so there is a fiber sequence

$$B \operatorname{Im} J_{\mathbb{C}} \rightarrow K^{\operatorname{red}}(\hat{\mathbb{Z}}_2) \rightarrow X.$$

Then $\pi_*(X; \mathbb{Z}/2) \cong \pi_*(BBU; \mathbb{Z}/2)$ as free v_1^4 -modules of rank four. Hence X agrees with its K -localization above degree one, and has homotopy groups $\pi_* X \cong \pi_* BBU$.

Proof. By assumption $f_*(i_1(i_1)) = i_1(\{3\}) = \xi_1 + \partial(\xi_2)$. Multiplying by η gives $f_*(e_1 i_1) = i_1(\eta\{3\}) = i_1(\eta^2) = \partial(\xi_3(0))$ by Proposition 4.2. Here $e_1 i_1$ admits an integral lift, so we can multiply by $\tilde{\eta}_2$ and obtain $f_*(e_1 v_1 i_1) = \tilde{\eta}_2 \cdot \eta^2 = \xi_2 \cdot \partial(\xi_3(0)) = \partial(\xi_5(0)) = \partial(\kappa) = \lambda\{3\}$. Again $e_1 v_1 i_1$ is integral, so multiplying by $\tilde{\eta}_2$ gives $f_*(e_1 v_1^2 i_1) = \tilde{\eta}_2 \cdot \lambda\{3\} = \xi_2 \cdot \partial(\xi_5(0)) = \partial(\xi_7(1)) = \partial(\sigma) = \kappa\{3\}$. A third multiplication by $\tilde{\eta}_2$, or multiplying the original equation by σ , gives $f_*(e_1 v_1^3 i_1) = \partial(\xi_9(1)) = \sigma\{3\}$.

Multiplying $f_*(i_1) = \{3\}$ by $\tilde{\eta}_2$ gives $f_*(v_1 i_1) = \tilde{\eta}_2\{3\} = x$. Then $\beta_1(x) = \partial(\xi_3(0)) \neq 0$, $\operatorname{red}(x) = 0$ and $\operatorname{tr}(x) = 0$. (Here tr denotes the Bökstedt trace $K(\hat{\mathbb{Z}}_2) \rightarrow T(\hat{\mathbb{Z}}_2) \cong T(\mathbb{Z})$.) The only possibility is $x = \xi_3 + \partial(\xi_4)$.

Let $y = f_*(v_1^2 i_1)$. Then $\beta_3(y) = \partial(\xi_5(0)) \neq 0$ and $\operatorname{red}(y) = 0$, so either $y = \xi_5(0) + \partial(\xi_6)$ or $y = \xi_5 + \partial(\xi_6)$. Let $z = f_*(v_1^3 i_1)$. Then $\beta_1(z) = \partial(\xi_7(1)) \neq 0$ and $\operatorname{red}(z) = 0$, so either $z = \xi_7(1) + \xi_7$ or $z = \xi_7 + \partial(\xi_8)$.

The class y admits a mod eight lift $y' = f_*(\tilde{\lambda}_8 \cdot i_1)$, so multiplying with $\tilde{\eta}_2$ makes sense, and gives $\tilde{\eta}_2 \cdot y' = z$. Then either $y' \equiv \kappa + \tilde{\lambda}_8 \cdot \{5\} \pmod{two}$ with $\tilde{\eta}_2 \cdot y' = \xi_7(1) + \partial(\xi_8)$, or $y' \equiv \tilde{\lambda}_8 \cdot \{3\} \pmod{two}$ with $\tilde{\eta}_2 \cdot y' = \xi_7 + \partial(\xi_8)$. Comparing with the expressions for z , the only consistent possibility is $y = \xi_5 + \partial(\xi_6)$ and $z = \xi_7 + \partial(\xi_8)$.

Hence $\pi_*(f; \mathbb{Z}/2)$ is an injection for all $1 \leq * \leq 8$. By v_1^4 -periodicity, this holds in all degrees. Hence $\pi_*(f)$ induces a split injection in all degrees. \square

Lemma 7.2. *In $K_*^{\operatorname{red}}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ the classes $\xi_3(0) + \partial(\xi_4)$, $\xi_5(0) + \xi_5$, $\xi_7(1) + \partial(\xi_8)$ and $\xi_9(1) + \xi_9$ admit integral lifts. Their products with $\tilde{\eta}_2$ are $\xi_5(0) + \partial(\xi_6)$, $\xi_7(1) + \xi_7$, $\xi_9(1) + \partial(\xi_{10})$ and $\xi_{11}(1) + \xi_{11}$, respectively.*

In $\pi_*(X; \mathbb{Z}/2)$ all classes are integral, and multiplication by $\tilde{\eta}_2$ is an isomorphism in all degrees $* \geq 3$.

Proof. The integral lifts are $\lambda + g_3$, $\kappa + t_5$, $\sigma + g_7$ and $g_9 + \mu$, in the notation of Proposition 4.2. Their products with $\tilde{\eta}_2$ are computed as in the proof of Proposition 7.1. The claims for X then follow, by dividing $K_*^{\text{red}}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ by the image of $\pi_*(f_{\mathbb{C}}; \mathbb{Z}/2)$ to obtain $\pi_*(X; \mathbb{Z}/2)$ in low degrees, and using v_1^4 -periodicity to obtain the general result. \square

Hence the infinite loop space cofiber of $f_{\mathbb{C}}$ satisfies the hypothesis of the following result.

Proposition 7.3. *Let X be a two-complete infinite loop space with $\pi_*X \cong \pi_*BBU$. If multiplication by $\tilde{\eta}_2$*

$$\pi_i(X; \mathbb{Z}/2) \xleftarrow{\cong} \pi_i(X)/2 \xrightarrow{\tilde{\eta}_2} \pi_{i+2}(X; \mathbb{Z}/2)$$

is injective in all degrees, then $X \simeq BBU$ as infinite loop spaces.

Proof. Each “Postnikov segment” $X[i, i + 2]$ of X for $i \geq 3$ odd has nontrivial k -invariant, since $\tilde{\eta}_2 : \pi_i(X[i, i + 2]; \mathbb{Z}/2) \rightarrow \pi_{i+2}(X[i, i + 2]; \mathbb{Z}/2)$ is assumed to be nonzero. Hence there is a fiber sequence

$$K(\mathbb{Z}, i + 2) \rightarrow X[i, i + 2] \rightarrow K(\mathbb{Z}, i) \xrightarrow{\tilde{Q}_1} K(\mathbb{Z}, i + 3),$$

where \tilde{Q}_1 lifts

$$Q_1 = [Sq^1, Sq^2] = Sq^3 + Sq^2Sq^1 : K(\mathbb{Z}/2, i) \rightarrow K(\mathbb{Z}/2, i + 3).$$

Then as in the proof of Proposition 2.1 of [2], the spectrum cohomology of X satisfies $H^*(X; \mathbb{Z}/2) \cong A/(AQ_0 + AQ_1)$ where A is the mod two Steenrod algebra and $Q_0 = Sq^1$. Thus by Theorem 1.1 of loc.cit., there exists a two-adic equivalence $X \simeq BBU$. \square

Hence we have established the following fiber sequences of infinite loop spaces

$$(7.4) \quad \begin{aligned} K^{\text{red}}(\hat{\mathbb{Z}}_2) &\rightarrow K(\hat{\mathbb{Z}}_2) \xrightarrow{\text{red}} \text{Im } J_{\mathbb{C}}, \\ B \text{Im } J_{\mathbb{C}} &\xrightarrow{f_{\mathbb{C}}} K^{\text{red}}(\hat{\mathbb{Z}}_2) \rightarrow BBU. \end{aligned}$$

We can now identify well-defined choices of integral generators for $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$.

Proposition 7.5. *Let t_n generate the cyclic torsion subgroup in $K_n(\hat{\mathbb{Z}}_2)$ for $n \geq 2$, and let f_n generate the free cyclic group $K_n^{\text{red}}(\hat{\mathbb{Z}}_2) \subset K_n(\hat{\mathbb{Z}}_2)$ for $n \geq 3$ odd. The mod two representatives of these classes begin as follows:*

n	2	3	4	5
$i_1(t_n)$	$\partial(\xi_3(0))$	$\xi_3(0)$	$\partial(\xi_5(0))$	ξ_5
$i_1(f_n)$		$\xi_3(0) + \partial(\xi_4)$		$\xi_5(0) + \xi_5$
n	6	7	8	9
$i_1(t_n)$	$\partial(\xi_7(1))$	$\xi_7(1)$	$\partial(\xi_9(1))$	ξ_9
$i_1(f_n)$		$\xi_7(1) + \partial(\xi_8)$		$\xi_9(1) + \xi_9$

In all higher degrees $i_1(t_{8k+n}) = v_1^{4k} \cdot t_n$ and $i_1(f_{8k+n}) = v_1^{4k} \cdot f_n$.

Proof. In even degrees the classes $\partial(\xi_3(0))$, $\partial(\xi_5(0))$, $\partial(\xi_7(1))$ and $\partial(\xi_9(1))$, and their multiples by v_1^{4k} , represent the image from $\pi_*(B \operatorname{Im} J_{\mathbb{C}}; \mathbb{Z}/2)$ in $K_*^{\operatorname{red}}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ and $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$, by Proposition 7.1. Hence these are the integral classes in even degrees of $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$.

In degree $8k+3$ the integral classes are $v_1^{4k} \cdot \xi_3(0)$ and $v_1^{4k} \cdot \partial(\xi_4)$, since $\beta_3(\xi_4) = \xi_3(0)$ and $\beta_1(\partial(\xi_4)) = 0$ while $\beta_1(\xi_3) \neq 0$. These differentials propagate up by multiplication by v_1^{4k} , because multiplication by v_1^4 respects β_k for $k \leq 3$.

In degree $8k+5$ the integral classes are $v_1^{4k} \cdot \xi_5(0)$ and $v_1^{4k} \cdot \xi_5$, since $\beta_1(\xi_4) = \xi_5$ and $\beta_3(\xi_5(0)) = 0$ while $\beta_3(\partial(\xi_6)) \neq 0$. These differentials also propagate up by multiplication by v_1^{4k} .

In degree $8k+7$ the class $v_1^{4k} \cdot \xi_7(1)$ is integral in $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$. This is because $i_1(\sigma) \in \pi_7(Q(S^0); \mathbb{Z}/2)$ maps to $\xi_7(1)$ under the unit map, and its multiples by v_1^{4k} are all integral in $\pi_{8k+7}(Q(S^0); \mathbb{Z}/2)$. Also $v_1^{4k} \cdot \partial(\xi_8)$ is integral, since $\beta_1(\partial(\xi_8)) = 0$ while $\beta_1(\xi_7) \neq 0$. Again the β_1 -differential propagates.

In degree $8k+9$ the integral classes are $v_1^{4k} \cdot \xi_9(1)$ and $v_1^{4k} \cdot \xi_9$. For $\beta_1(\xi_{10}) = \xi_9$ translates up by multiplication by v_1^{4k} . Also $\xi_9(1)$ is the image of $\sigma \tilde{\eta}_2 \in \pi_9(Q(S^0); \mathbb{Z}/2)$, so $v_1^{4k} \cdot \xi_9(1)$ is the image of $\sigma x \in \pi_{8k+9}(Q(S^0); \mathbb{Z}/2)$, where $j_1(x) = \mu_{8k+1} \in \pi_{8k+1}Q(S^0)$ is the Adams μ -element [1]. Then $j_1(\sigma x) = \sigma \mu_{8k+1}$, which maps to zero in $K_*(\mathbb{Z})$, by [33]. Hence $v_1^{4k} \cdot \xi_9(1)$ is integral in $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$.

The given formulas now follow from Proposition 4.6. \square

We can also extend Proposition 4.2(c).

Lemma 7.6. *In the first two cases below assume $k \geq 1$. In the remaining cases assume $k \geq 0$.*

In $K_{8k}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ the class $v_1^{4(k-1)} \cdot \partial(\xi_9(1))$ is integral, while $\beta_v(v_1^{4(k-1)} \cdot \xi_8) = v_1^{4(k-1)} \cdot \xi_7(1)$ with $v = v_2(k) + 4$.

In $K_{8k+1}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ the classes $v_1^{4(k-1)} \cdot \xi_9(1)$ and $v_1^{4(k-1)} \cdot \xi_9$ are integral, while $\beta_v(v_1^{4(k-1)} \cdot \partial(\xi_{10})) = v_1^{4(k-1)} \cdot \partial(\xi_9(1))$ with $v = v_2(k) + 4$. An integral lift of the class $v_1^{4(k-1)} \cdot (\xi_9(1) + \xi_9)$ generates $K_{8k+1}(\hat{\mathbb{Z}}_2)$ modulo torsion.

In $K_{8k+2}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ the class $v_1^{4k} \cdot \partial(\xi_3(0))$ is integral, while $\beta_1(v_1^{4k} \cdot \xi_2) = v_1^{4k} \cdot \xi_1$.

In $K_{8k+3}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ the classes $v_1^{4k} \cdot \xi_3(0)$ and $v_1^{4k} \cdot (\xi_3(0) + \partial(\xi_4))$ are integral, while $\beta_1(v_1^{4k} \cdot \xi_3) = v_1^{4k} \cdot \partial(\xi_3(0))$. An integral lift of the class $v_1^{4k} \cdot \partial(\xi_4)$ generates $K_{8k+3}(\hat{\mathbb{Z}}_2)$ modulo torsion.

In $K_{8k+4}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ the class $v_1^{4k} \cdot \partial(\xi_5(0))$ is integral, while $\beta_3(v_1^{4k} \cdot \xi_4) = v_1^{4k} \cdot \xi_3(0)$.

In $K_{8k+5}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ the classes $v_1^{4k} \cdot \xi_5(0)$ and $v_1^{4k} \cdot \xi_5$ are integral, while $\beta_3(v_1^{4k} \cdot \partial(\xi_6)) = v_1^{4k} \cdot \partial(\xi_5(0))$. An integral lift of the class $v_1^{4k} \cdot (\xi_5(0) + \xi_5)$ generates $K_{8k+5}(\hat{\mathbb{Z}}_2)$ modulo torsion.

In $K_{8k+6}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ the class $v_1^{4k} \cdot \partial(\xi_7(1))$ is integral, while $\beta_1(v_1^{4k} \cdot \xi_6) = v_1^{4k} \cdot \xi_5$.

In $K_{8k+7}(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$ the classes $v_1^{4k} \cdot \xi_7(1)$ and $v_1^{4k} \cdot \partial(\xi_8)$ are integral, while $\beta_1(v_1^{4k} \cdot \xi_7) = v_1^{4k} \cdot \partial(\xi_7(1))$. An integral lift of the class $v_1^{4k} \cdot (\xi_7(1) + \partial(\xi_8))$ generates $K_{8k+7}(\hat{\mathbb{Z}}_2)$ modulo torsion.

Proof. Combine Proposition 4.2 and Theorem 5.5 with Proposition 7.5. \square

The following comparison theorem is useful.

Theorem 7.7. *The natural map $K_*(\mathbb{Z})^\wedge \rightarrow K_*(\hat{\mathbb{Z}}_2)^\wedge$ induces an isomorphism modulo torsion in degrees $* = 4i + 1$ for all $i \geq 1$.*

Proof. Consider the maps

$$SU \rightarrow JK(\mathbb{Z}) \xleftarrow{\phi} K(\mathbb{Z}) \rightarrow K(\hat{\mathbb{Z}}_2)$$

and the space level section $\phi : \Omega JK(\mathbb{Z}) \rightarrow \Omega K(\mathbb{Z})$. The generators γ_{4i-1} and γ_{4i+1} in $\pi_*(SU)$ map to classes λ_{4i-1} and κ_{4i+1} in $JK_*(\mathbb{Z})$. Here λ_{4i-1} generates the finite cyclic group $JK_{4i-1}(\mathbb{Z})$, while κ_{4i+1} is of infinite order and generates the group $JK_{4i+1}(\mathbb{Z})$ modulo torsion. For i even λ_{4i-1} is in the image from $\pi_{4i-1}(\text{Im } J_{\mathbb{R}})$, while for i odd $2\lambda_{4i-1}$ is in this image. The relation $\gamma_{4i-1}\tilde{\eta}_2 = i_1(\gamma_{4i+1})$ in $\pi_*(SU; \mathbb{Z}/2)$ translates to $\phi(\lambda_{4i-1})\tilde{\eta}_2 = i_1(\phi(\kappa_{4i+1}))$ in $K_{4i+1}(\mathbb{Z}; \mathbb{Z}/2)$. For $i \geq 2$ this follows by naturality of multiplication by $\tilde{\eta}_2$ with respect to ϕ , because this multiplication is unstably defined on π_n with $n \geq 3$. The case $i = 1$ was treated separately in Lemma 5.6.

Now we map across to $K(\hat{\mathbb{Z}}_2)$. We claim that λ_{4i-1} maps to a generator of the torsion in $K_{4i-1}(\hat{\mathbb{Z}}_2)$ for all i . For i even this follows from Lemma 3.5, while for i odd the same lemma shows that $2\lambda_{4i-1}$ maps to twice a generator. Since the torsion group has order eight in these degrees, this suffices to prove the claim. Hence $\phi(\lambda_{8k+3})$ is represented mod two by $v_1^{4k} \cdot \xi_3(0)$, and $\phi(\lambda_{8k+7})$ is represented mod two by $v_1^{4k} \cdot \xi_7(1)$, by Proposition 7.5. The relation $\tilde{\eta}_2 \cdot \phi(\lambda_{8k+3}) = i_1(\phi(\kappa_{8k+5}))$ implies that $\phi(\kappa_{8k+5})$ is represented mod two by $v_1^{4k} \cdot \tilde{\eta}_2 \xi_3(0) = v_1^{4k} \cdot \xi_5(0)$, and thus generates $K_{8k+5}(\hat{\mathbb{Z}}_2)$ modulo torsion. Likewise the relation $\tilde{\eta}_2 \cdot \phi(\lambda_{8k+7}) = i_1(\phi(\kappa_{8k+9}))$ implies that $\phi(\kappa_{8k+9})$ is represented mod two by $v_1^{4k} \cdot \tilde{\eta}_2 \xi_7(1) = v_1^{4k} \cdot \xi_9(1)$, and thus generates $K_{8k+9}(\hat{\mathbb{Z}}_2)$ modulo torsion.

Hence $\phi(\kappa_{4i+1})$ generates $K_{4i+1}(\mathbb{Z})$ modulo torsion, and maps to a class that generates $K_{4i+1}(\hat{\mathbb{Z}}_2)$ modulo torsion. This proves the theorem. \square

8. $K(\hat{\mathbb{Z}}_2)$ as an infinite loop space

We use the multiplicative relations $\eta^2 \neq 0$ and $\sigma f_3 = \mu\eta$ to identify the extensions in (7.4). Thus we determine the two-completed algebraic K -theory of the two-adic integers as an infinite loop space.

Theorem 8.1. *The homotopy type of $K(\hat{\mathbb{Z}}_2)$ as an infinite loop space is determined by the following diagram of infinite loop space fiber sequences*

$$\begin{array}{ccccc}
 B \operatorname{Im} J_{\mathbb{C}} & & & & \\
 \downarrow f_{\mathbb{C}} & & & & \\
 K^{\operatorname{red}}(\hat{\mathbb{Z}}_2) & \longrightarrow & K(\hat{\mathbb{Z}}_2) & \xrightarrow{\operatorname{red}} & \operatorname{Im} J_{\mathbb{C}} \\
 \downarrow & & & & \\
 BBU & & & &
 \end{array}$$

where the connecting maps are given as follows:

The group of infinite loop maps $BU \rightarrow B \operatorname{Im} J_{\mathbb{C}}$ that induce the zero homomorphism on homotopy is free on one generator as a module over the two-adic integers. The vertical connecting map $\partial_1 : BU \rightarrow B \operatorname{Im} J_{\mathbb{C}}$ represents a generator of this group.

The group of infinite loop maps $\Omega \operatorname{Im} J_{\mathbb{C}} \rightarrow K^{\operatorname{red}}(\hat{\mathbb{Z}}_2)$ is also free on one generator as a module over the two-adic integers, and the horizontal connecting map $\partial_2 : \Omega \operatorname{Im} J_{\mathbb{C}} \rightarrow K^{\operatorname{red}}(\hat{\mathbb{Z}}_2)$ represents a generator of this group.

Any two generators of either group differ (multiplicatively) by a two-adic unit, hence by a two-adic automorphism of the spaces $\operatorname{Im} J_{\mathbb{C}}$, $B \operatorname{Im} J_{\mathbb{C}}$ or BBU involved. Thus this theorem characterizes $K(\hat{\mathbb{Z}}_2)$ as an infinite loop space, up to homotopy equivalence.

Proof. We begin with the extension $B \operatorname{Im} J_{\mathbb{C}} \rightarrow K^{\operatorname{red}}(\hat{\mathbb{Z}}_2) \rightarrow BBU$, classified by an infinite loop map ∂_1 . We know that ∂_1 induces the zero map on homotopy since $f_{\mathbb{C}}$ induces injections on homotopy. Consider the diagram

$$\begin{array}{ccccccc}
 & & & BU & & & \\
 & & & \downarrow \partial_1 & & & \\
 \mathbb{Z} \times BU & \xrightarrow{\psi^3 - 1} & BU & \xrightarrow{B\hat{c}} & B \operatorname{Im} J_{\mathbb{C}} & \xrightarrow{Bi} & U.
 \end{array}$$

There are no essential (infinite loop) maps $BU \rightarrow U$, so ∂_1 factors over an infinite loop map $BU \rightarrow BU$. By [18] such maps form a suitably completed group-ring $\hat{\mathbb{Z}}_2[[\hat{\mathbb{Z}}_2^\times]]$ and can be written as certain infinite series in the Adams operations ψ^k with $k \in \hat{\mathbb{Z}}_2^\times$. The

maps $BU \rightarrow B \operatorname{Im} J_{\mathbb{C}}$ are thus the quotient of this ring by the closed ideal $(\psi^3 - 1)$ of maps factoring through $\psi^3 - 1$. In this quotient any ψ^k with $k = 3^i$, or more generally $k \in (3) \subset \hat{\mathbb{Z}}_2^\times$, is identified with 1. Since $\hat{\mathbb{Z}}_2^\times \cong \hat{\mathbb{Z}}_2 \oplus \mathbb{Z}/2$ with 3 generating the first summand and -1 generating the second, it follows that the quotient ring in question is $\hat{\mathbb{Z}}_2[\mathbb{Z}/2]$, which is additively free over $\hat{\mathbb{Z}}_2$ on the two generators $1 = \psi^1$ and $t = \psi^{-1}$ (complex conjugation).

We may thus write $\partial_1 = B\partial \circ (u + v\psi^{-1})$ with $u, v \in \hat{\mathbb{Z}}_2$. Now we use that $(\partial_1)_* = 0$. In degree $2n$ this map $(\partial_1)_*$ multiplies by $u + (-1)^n v$, so for n odd (when $\pi_{2n}(B \operatorname{Im} J_{\mathbb{C}}) = \mathbb{Z}/2$) we find $u - v \equiv 0 \pmod{2}$, while for n even (when $\pi_{2n}(B \operatorname{Im} J_{\mathbb{C}}) \cong \mathbb{Z}/2^i$ with $i = v_2(n) + 2$) we find $u + v \equiv 0 \pmod{2^i}$. Hence two-adically $u + v = 0$ and $\partial_1 = u \cdot B\partial \circ (1 - \psi^{-1})$. Thus the extension is determined by the two-adic coefficient u , and we claim that u is a unit. This will determine the extension and the infinite loop space homotopy type of $K^{\operatorname{red}}(\hat{\mathbb{Z}}_2)$, up to equivalence.

We claim that u is not divisible by two. If it were, we could factor ∂_1 as $2y$ with $\pi_*(y) = 0$, and get a map of fiber sequences

$$\begin{array}{ccccccc}
 BU & \xrightarrow{y} & B \operatorname{Im} J_{\mathbb{C}} & \longrightarrow & Y & \longrightarrow & BBU \\
 \parallel & & \downarrow 2 & & \downarrow & & \parallel \\
 BU & \xrightarrow{\partial_1} & B \operatorname{Im} J_{\mathbb{C}} & \longrightarrow & K^{\operatorname{red}}(\hat{\mathbb{Z}}_2) & \longrightarrow & BBU.
 \end{array}$$

Consider the product $\sigma \cdot f_3 = \mu\eta$ in $K_{10}^{\operatorname{red}}(\hat{\mathbb{Z}}_2)$, from Lemma 6.5. Here f_3 is represented on $\pi_3(BBU)$, so $\sigma \cdot f_3$ is defined in $\pi_*(Y)$ and lifts to $\pi_*(B \operatorname{Im} J_{\mathbb{C}})$ in the upper row. Hence $\mu\eta$ should be divisible by two, which it is not. This contradiction shows that u is a unit. (By applying a two-adic automorphism to our identification of the infinite loop space cofiber of $B \operatorname{Im} J_{\mathbb{C}} \rightarrow K^{\operatorname{red}}(\hat{\mathbb{Z}}_2)$ with BBU we may arrange to have $\partial_1 = B\partial \circ (1 - \psi^{-1})$.)

Next consider the extension $K^{\operatorname{red}}(\hat{\mathbb{Z}}_2) \rightarrow K(\hat{\mathbb{Z}}_2) \rightarrow \operatorname{Im} J_{\mathbb{C}}$, classified by an infinite loop map $\partial_2 : \Omega \operatorname{Im} J_{\mathbb{C}} \rightarrow K^{\operatorname{red}}(\hat{\mathbb{Z}}_2)$. Its delooping $B\partial_2$ fits into a diagram

$$\begin{array}{ccc}
 & \operatorname{Im} J_{\mathbb{C}} & \\
 & \downarrow B\partial_2 & \searrow b \\
 B^2 \operatorname{Im} J_{\mathbb{C}} & \xrightarrow{Bf_{\mathbb{C}}} & BK^{\operatorname{red}}(\hat{\mathbb{Z}}_2) \longrightarrow B^3 U.
 \end{array}$$

Consider maps from the defining fiber sequence

$$(8.2) \quad U \xrightarrow{\partial} \operatorname{Im} J_{\mathbb{C}} \xrightarrow{i} \mathbb{Z} \times BU \xrightarrow{\psi^3 - 1} BU$$

to BU . There are no essential infinite loop maps $U \rightarrow BU$. The infinite loop maps $\mathbb{Z} \times BU \rightarrow BU$ are series in the ψ^k with $k \in \hat{\mathbb{Z}}_2^\times$. Modulo those which extend over $\psi^3 - 1$ these can be written as $u + v\psi^{-1}$ for some $u, v \in \hat{\mathbb{Z}}_2$ (typically different from

the u, v above). Furthermore the map to BU must be zero on π_0 , so $u + v = 0$. Hence the infinite loop maps $\text{Im } J_{\mathbb{C}} \rightarrow BU$ are all uniquely expressible as $u \cdot (1 - \psi^{-1})i$ with $u \in \hat{\mathbb{Z}}_2$. We write $[\text{Im } J_{\mathbb{C}}, BU] \cong \hat{\mathbb{Z}}_2\{(1 - \psi^{-1})i\}$.

The unit map $Q(S^0) \rightarrow \text{Im } J_{\mathbb{C}}$ is 2-connected, so $H_{\text{spec}}^*(\text{Im } J_{\mathbb{C}}) = 0$ for $* = 1, 2$, and so $[\text{Im } J_{\mathbb{C}}, B^3U] \cong [\text{Im } J_{\mathbb{C}}, BU]$ by the natural map $B^3U \simeq BSU \rightarrow BU$. We also write $(1 - \psi^{-1})i$ for the generator of $[\text{Im } J_{\mathbb{C}}, B^3U]$. Hence $b = u \cdot (1 - \psi^{-1})i$ for some unique $u \in \hat{\mathbb{Z}}_2$.

We claim that there are no essential maps $a : \text{Im } J_{\mathbb{C}} \rightarrow B^2\text{Im } J_{\mathbb{C}}$, so $B\hat{\partial}_2$ is uniquely determined by b . For consider maps from $\text{Im } J_{\mathbb{C}}$ to the fiber sequence

$$B^2U \xrightarrow{B^2\hat{\partial}} B^2\text{Im } J_{\mathbb{C}} \xrightarrow{B^2i} B^2(\mathbb{Z} \times BU) \xrightarrow{B^2(\psi^3-1)} B^3U.$$

We have seen that $[\text{Im } J_{\mathbb{C}}, B^2(\mathbb{Z} \times BU)] \cong [\text{Im } J_{\mathbb{C}}, BU] \cong \hat{\mathbb{Z}}_2\{(1 - \psi^{-1})i\}$, and $[\text{Im } J_{\mathbb{C}}, B^3U] \cong [\text{Im } J_{\mathbb{C}}, BSU] \cong \hat{\mathbb{Z}}_2\{(1 - \psi^{-1})i\}$. Under these identifications the map $B^2(\psi^3 - 1)$ corresponds to composition with $(\frac{1}{3}\psi^3 - 1) = -\frac{2}{3} + \frac{1}{3}(\psi^3 - 1)$, which acts as multiplication by $-\frac{2}{3}$, and hence is injective.

Thus a factors through a map $\text{Im } J_{\mathbb{C}} \rightarrow B^2U$ and we claim all such maps are inessential. Consider maps from (8.2) to B^2U . There are no essential maps $\mathbb{Z} \times BU \rightarrow B^2U \simeq SU$, because all maps $\mathbb{Z} \times BU \rightarrow U$ are inessential and each map $\mathbb{Z} \times BU \rightarrow \mathbb{Z}$ through the homotopy fiber of $SU \rightarrow U$ factors through the connecting map $\mathbb{Z} \times BU \simeq \Omega U \rightarrow \mathbb{Z}$, and thus gives a trivial composite map.

Next look at maps $U \rightarrow B^2U \simeq SU$ and precomposition with $\Omega(\psi^3 - 1)$. Delooping once we are led to consider the homomorphism

$$[BU, BSU] \xrightarrow{(\psi^3-1)^*} [\mathbb{Z} \times BU, BSU]$$

which clearly is injective. Hence $[\text{Im } J_{\mathbb{C}}, B^2U] = 0$, and thus $[\text{Im } J_{\mathbb{C}}, B^2\text{Im } J_{\mathbb{C}}] = 0$. We conclude that any map $a : \text{Im } J_{\mathbb{C}} \rightarrow B^2\text{Im } J_{\mathbb{C}}$ is inessential, and so $B\hat{\partial}_2 : \text{Im } J_{\mathbb{C}} \rightarrow BK^{\text{red}}(\hat{\mathbb{Z}}_2)$ is uniquely determined as being a lift of $u \cdot (1 - \psi^{-1})i$ over $BK^{\text{red}}(\hat{\mathbb{Z}}_2) \rightarrow B^3U \simeq BSU \rightarrow BU$.

We claim the coefficient u is a two-adic unit. For $\eta \cdot \eta = \eta^2$ in $K_2(\hat{\mathbb{Z}}_2)$ with η detected in $\pi_1(\text{Im } J_{\mathbb{C}})$ and η^2 coming from $K_2^{\text{red}}(\hat{\mathbb{Z}}_2)$. By the same argument as above, if u were divisible by two then η^2 would be divisible by two, which it is not. Hence $[\text{Im } J_{\mathbb{C}}, BK^{\text{red}}(\hat{\mathbb{Z}}_2)]$ maps isomorphically to $[\text{Im } J_{\mathbb{C}}, B^3U] \cong \hat{\mathbb{Z}}_2$, and the fiber sequence $K^{\text{red}}(\hat{\mathbb{Z}}_2) \rightarrow K(\hat{\mathbb{Z}}_2) \rightarrow \text{Im } J_{\mathbb{C}}$ is represented by a generator of this group.

Hence the connecting map $\hat{\partial}_2$ is determined by the diagram

$$\begin{array}{ccc} \Omega\text{Im } J_{\mathbb{C}} & \xrightarrow{u^{-1} \cdot \hat{\partial}_2} & K^{\text{red}}(\hat{\mathbb{Z}}_2) \\ \Omega i \downarrow & & \downarrow \\ \Omega(\mathbb{Z} \times BU) & \xrightarrow{1 - \Omega\psi^{-1}} & U \longleftarrow BBU. \end{array}$$

where u is a two-adic unit, $K^{\text{red}}(\hat{\mathbb{Z}}_2) \rightarrow BBU$ is the infinite loop cofiber map of $f_{\mathbb{C}}$, and $BBU \simeq SU \rightarrow U$ is the natural covering map. \square

Remark 8.3. If we pass from the ring of two-adic integers to the field of two-adic numbers, there is still a Galois reduction map $K(\hat{\mathbb{Q}}_2) \rightarrow \text{Im} J_{\mathbb{C}}$ with homotopy fiber $K^{\text{red}}(\hat{\mathbb{Q}}_2)$, and a fiber sequence $B \text{Im} J_{\mathbb{C}} \rightarrow K^{\text{red}}(\hat{\mathbb{Q}}_2) \rightarrow U$. (We use Remark 2.7 to identify the infinite loop cofiber as U .)

$$\begin{array}{ccccc}
 & & \mathbb{Z} \times BU & & \\
 & \swarrow 1-\psi^{-1} & \downarrow \partial_1 & & \\
 BU & \xrightarrow{B\partial} & B \text{Im} J_{\mathbb{C}} & & \\
 & & \downarrow f_{\mathbb{C}} & & \\
 \Omega \text{Im} J_{\mathbb{C}} & \xrightarrow{\partial_2} & K^{\text{red}}(\hat{\mathbb{Q}}_2) & \longrightarrow & K(\hat{\mathbb{Q}}_2) \xrightarrow{\text{red}} \text{Im} J_{\mathbb{C}} \\
 \downarrow \Omega i & & \downarrow & & \\
 \Omega(\mathbb{Z} \times BU) & \xrightarrow{1-\Omega\psi^{-1}} & U & &
 \end{array}$$

Up to two-adic units, the connecting map $\partial_1 : \Omega U \simeq \mathbb{Z} \times BU \rightarrow B \text{Im} J_{\mathbb{C}}$ is the composite $B\partial \circ (1 - \psi^{-1})$, while the connecting map $\partial_2 : \Omega \text{Im} J_{\mathbb{C}} \rightarrow K^{\text{red}}(\hat{\mathbb{Q}}_2)$ uniquely lifts the composite $(1 - \Omega\psi^{-1}) \circ \Omega i$ over $K^{\text{red}}(\hat{\mathbb{Q}}_2) \rightarrow U$.

9. Comparison with $K(\mathbb{Z})$

We can now consider $K(\hat{\mathbb{Z}}_2)$ as an invariant of $K(\mathbb{Z})$, i.e., as a target for detecting classes in $K(\mathbb{Z})$. We round off with a brief discussion of this map under the assumption that the Lichtenbaum–Quillen conjecture holds for $K(\mathbb{Z})$ at two, that is, if Φ is an equivalence, or at least admits a spectrum-level section.

Let $\alpha_{8k-1} \in \pi_{8k-1} Q(S^0)$ be the generator of the image of J in this degree, and let $\mu_{8k+1} \in \pi_{8k+1} Q(S^0)$ be the order two class detected by KO -theory, also in the image of J .

Lemma 9.1. *The composite homomorphism*

$$JK_*(\mathbb{Z}) \xrightarrow{\phi} K_*(\mathbb{Z}) \rightarrow K_*(\hat{\mathbb{Z}}_2)$$

has kernel $\mathbb{Z}/2$ in degrees $8k+3$ for $k \geq 0$, and is injective otherwise. When nontrivial, the kernel is generated by $\mu_{8k+1}\eta^2$, which is of order two in $\pi_{8k+3}(\text{Im} J_{\mathbb{R}}) \subset JK_{8k+3}(\mathbb{Z})$.

Proof. The homotopy of $JK_*(\mathbb{Z})$ is given by the fiber sequence

$$\text{Im } J_{\mathbb{R}} \xrightarrow{j} JK(\mathbb{Z}) \rightarrow BBSO$$

from (2.3), where the boundary map hits the classes $\alpha_{8k-1}\eta$ and $\alpha_{8k-1}\eta^2$ in degrees $8k$ and $8k + 1$ for $k \geq 1$. So for $* \geq 2$

$$(9.2) \quad JK_*(\mathbb{Z}) = \begin{cases} 0 & \text{for } * \equiv 0, 4, 6 \pmod{8}, \\ \mathbb{Z} \oplus \mathbb{Z}/2 & \text{for } * \equiv 1 \pmod{8}, \\ \mathbb{Z}/2 & \text{for } * \equiv 2 \pmod{8}, \\ \mathbb{Z}/16 & \text{for } * \equiv 3 \pmod{8}, \\ \mathbb{Z} & \text{for } * \equiv 5 \pmod{8}, \\ \mathbb{Z}/2^{v_2(k)+4} & \text{for } * = 8k - 1. \end{cases}$$

The homotopy of $K_*(\hat{\mathbb{Z}}_2)$ is given by Theorem 5.4. So for $* \geq 1$

$$(9.3) \quad K_*(\hat{\mathbb{Z}}_2) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & \text{for } * \equiv 1 \pmod{4}, \\ \mathbb{Z}/2 & \text{for } * \equiv 2 \pmod{4}, \\ \mathbb{Z} \oplus \mathbb{Z}/2^{v_2(k)+3} & \text{for } * = 4k - 1, \\ \mathbb{Z}/2^{v_2(k)+3} & \text{for } * = 4k. \end{cases}$$

By Theorem 7.7 only torsion classes can be in the kernel of the given homomorphism. In degrees $* = 8k + 1$ (with $k \geq 1$) the $\mathbb{Z}/2$ in $JK_*(\mathbb{Z})$ comes from $\text{Im } J_{\mathbb{R}}$ and is detected by KO -theory, hence also by the complex e -invariant. (See Proposition 7.19 of [1].)

In degrees $* = 8k + 2$ the $\mathbb{Z}/2$ in $JK_*(\mathbb{Z})$ is generated by $\mu_{8k+1}\eta$, which maps to $\mu_{8k+1}t_1 \neq 0$ in $\pi_{8k+2}(B \text{Im } J_{\mathbb{C}}) \subseteq K_{8k+2}(\hat{\mathbb{Z}}_2)$. (Here t_1 is the generator of $\pi_1(B \text{Im } J_{\mathbb{C}})$.) This is because η^2 maps to $\eta \cdot \{3\}$ detected as ηt_1 , and in general multiplication by v_1^{4k} takes η to μ_{8k+1} .

In degrees $* = 8k + 3$ the homomorphism is $\mathbb{Z}/16 \rightarrow \mathbb{Z} \oplus \mathbb{Z}/8$ with $\mathbb{Z}/8 \subset \mathbb{Z}/16$ represented by the real image of J . So twice the generator in $\mathbb{Z}/16$ maps to twice the generator in $\mathbb{Z}/8$, and so the kernel is the $\mathbb{Z}/2$ generated by $\mu_{8k+1}\eta^2$.

In degrees $* \equiv 4, 6, 8 \pmod{8}$ the source is trivial and in degrees $* \equiv 5 \pmod{8}$ there is no torsion in the source, so in all these degrees the homomorphism is injective. Finally, in degrees $* = 8k - 1$ with $k \geq 1$ the real and complex e -invariants agree, so the torsion group $\mathbb{Z}/2^{v_2(k)+4}$ is mapped isomorphically onto the torsion in $K_{8k-1}(\hat{\mathbb{Z}}_2)$. \square

Now we discuss a possible normalization of the identification of the infinite loop space cofiber of $f_{\mathbb{C}}$ with BBU , and a description of the map $K(\mathbb{Z}) \rightarrow K(\hat{\mathbb{Z}}_2)$. Consider

the following diagram of vertical fiber sequences:

$$\begin{array}{ccccc}
 BBO & \longleftarrow & K^{\text{red}}(\mathbb{Z}) & \longrightarrow & K^{\text{red}}(\hat{\mathbb{Z}}_2) \\
 \downarrow & & \downarrow & & \downarrow \\
 JK(\mathbb{Z}) & \xleftarrow{\Phi} & K(\mathbb{Z}) & \longrightarrow & K(\hat{\mathbb{Z}}_2) \\
 \downarrow & & \downarrow \pi & & \downarrow \text{red} \\
 \text{Im } J_{\mathbb{C}} & = & \text{Im } J_{\mathbb{C}} & = & \text{Im } J_{\mathbb{C}}.
 \end{array}
 \tag{9.4}$$

The fiber sequence on the left is from (2.2). The middle sequence is induced from the map $\pi : K(\mathbb{Z}) \rightarrow K(\mathbb{F}_3) \simeq \text{Im } J_{\mathbb{C}}$, and defines $K^{\text{red}}(\mathbb{Z})$. The fiber sequence on the right is from Definition 3.4. We do not know if the bottom right hand square homotopy commutes.

Hypothesis 9.5. *Assume that the reduction map $K(\hat{\mathbb{Z}}_2) \rightarrow \text{Im } J_{\mathbb{C}}$ can be chosen to extend the natural map $K(\mathbb{Z}) \rightarrow K(\mathbb{F}_3) \simeq \text{Im } J_{\mathbb{C}}$. Also assume that Φ admits an infinite loop section over the maps to $\text{Im } J_{\mathbb{C}}$.*

(In fact Φ is a homotopy equivalence, by [34, 35], so the latter assumption holds.)
Then there is a diagram

$$\begin{array}{ccccc}
 & & & & B \text{Im } J_{\mathbb{C}} \\
 & & & & \downarrow f_{\mathbb{C}} \\
 (9.6) & BBO & \longrightarrow & K^{\text{red}}(\mathbb{Z}) & \longrightarrow & K^{\text{red}}(\hat{\mathbb{Z}}_2) \\
 & & \searrow f & & \downarrow & \\
 & & & & & BBU.
 \end{array}$$

Proposition 9.7. *Under the hypothesis above, the infinite loop space cofiber of $f_{\mathbb{C}}$ may be identified with BBU in such a way that the composite map f above is homotopic to Bc , where $c : BO \rightarrow BU$ is the complexification map.*

There are no essential infinite loop maps $BBO \rightarrow B \text{Im } J_{\mathbb{C}}$, so this determines the composite $BBO \rightarrow K^{\text{red}}(\hat{\mathbb{Z}}_2)$ up to homotopy.

Proof. By Theorem 7.7 and (9.2) the map $f : BBO \rightarrow BBU$ takes a generator in degree $4k + 1$ to a generator in the same degree when k is even, and to twice a generator when k is odd, just like the delooped complexification map Bc .

Since BBO is one-connected and BBU agrees with its K -localization above degree one, there is a bijection between the infinite loop maps $BBO \rightarrow BBU$ and the spectrum maps $\Sigma KO \rightarrow \Sigma KU$. Likewise there is a bijection of between the infinite loop maps $BBO \rightarrow B \operatorname{Im} J_{\mathbb{C}}$ and the spectrum maps $\Sigma KO \rightarrow \Sigma JU$, since $B \operatorname{Im} J_{\mathbb{C}}$ agrees with its K -localization above degree one. Hence we may work with the corresponding K -local spectra (and avoid encumbering the notation to account for low-dimensional corrections).

We are then considering maps from ΣKO to (the suspension of) the fiber sequence (6.4). There are no essential spectrum maps from ΣKO to $\Sigma^2 KO$ or $\Sigma^3 KO$ by Proposition 2.2 of [25], so $\Sigma c: \Sigma KO \rightarrow \Sigma KU$ induces a bijection of maps from ΣKO . The given map $f: BBO \rightarrow BBU$ agrees with Bc on homotopy, up to two-adic units, so the corresponding spectrum map $\Sigma KO \rightarrow \Sigma KU$ factors up to homotopy as

$$\Sigma KO \xrightarrow{\simeq} \Sigma KO \xrightarrow{\Sigma c} \Sigma KU$$

where the left map is an equivalence because it induces an isomorphism on homotopy groups. Such an equivalence is a power series in the real Adams operations ψ^k (with k odd), which commutes with complexification. Thus f is also homotopic to (the one-connected cover of) a composite

$$\Sigma KO \xrightarrow{\Sigma c} \Sigma KU \xrightarrow{\simeq} \Sigma KU.$$

By adjusting the identification of the infinite loop space cofiber of $f_{\mathbb{C}}$ with BBU by the right hand equivalence, the first conclusion follows.

For the second claim, consider maps from ΣKO into the (co)fiber sequence

$$KU \rightarrow \Sigma JU \rightarrow \Sigma KU \xrightarrow{\Sigma \psi^3 - 1} \Sigma KU$$

defining ΣJU . There are no essential spectrum maps $\Sigma KO \rightarrow KU$ by Corollary 2.3 of [25]. Since composition with $\Sigma \psi^3 - 1$ acts injectively on the self maps of ΣKO , and any map $\Sigma KO \rightarrow \Sigma KU$ lifts uniquely over Σc , it follows that any composite $\Sigma KO \rightarrow \Sigma JU \rightarrow \Sigma KU$ is null homotopic, proving the second conclusion. Hence f is uniquely determined by its composite to BBU , which we may take to be Bc . \square

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