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EXPONENTIALS OF NON-SINGULAR SIMPLICIAL SETS

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(*communicated by J. Daniel Christensen*)

Abstract

A simplicial set is non-singular if the representing map of each non-degenerate simplex is degreewise injective. The simplicial mapping set X^K has n-simplices given by the simplicial maps $\Delta[n] \times K \to X$. We prove that X^K is non-singular whenever X is non-singular. It follows that non-singular simplicial sets form a cartesian closed category with all limits and colimits, but it is not a topos.

1. Introduction

Let *sSet* be the category of simplicial sets, and let *nsSet* denote its full subcategory of *non-singular simplicial sets*, i.e., those X such that for each non-degenerate simplex $x \in X_n$ the representing map $\bar{x} \colon \Delta[n] \to X$ is degreewise injective. The geometric realization $|X|$ of each non-singular simplicial set admits a well-defined PL (piecewise-linear) structure, and the category *nsSet* plays a key role in the passage between simplicial sets and PL manifolds in the proof of the stable parametrized h-cobordism theorem [**WJR13**, Thm. 0.1, §3.4].

The inclusion $U \colon nsSet \to sSet$ admits a left adjoint $D \colon sSet \to nsSet$, called *desingularization*, cf. [**WJR13**, Rmk. 2.2.12] and [**Fje**, Def. 2.2], and the adjunction unit $\eta_X \colon X \to UDX$ is degreewise surjective. The category *nsSet* has all (small) limits and colimits, which are preserved by U and D, respectively. Let (Sd, Ex) denote Kan's adjoint pair [**Kan57**] of endofunctors of *sSet*. The first author [**Fje**, Thm. 1.2] has exhibited a model structure on the category *nsSet*, and has furthermore shown that the adjunction

$$DSd^2 \colon sSet \rightleftarrows nsSet \colon Ex^2U$$

defines a Quillen equivalence from the standard model structure on simplicial sets. The proofs of these two results depend on knowing that the endofunctor of *nsSet* $X \mapsto X \times \Delta[1]$ preserves all colimits, and one purpose of the present paper is to establish this fact.

For any simplicial sets X and K let X^K be the *simplicial mapping set*, with n-simplices the set of maps $\Delta[n] \times K \to X$. Our main result follows.

Theorem 1.1. *Let X and K be any two simplicial sets. If X is non-singular, then so is X^K.*

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It follows that $X \mapsto X^K$ restricts to an endofunctor of *nsSet*. This implies the following generalization of the aforementioned fact.

Proposition 1.2. *Let K be any non-singular simplicial set. Then the endofunctor $X \mapsto X \times K$ of nsSet preserves all colimits.*

The proof of Theorem 1.1 follows easily from the following special case, which also directly implies the case $K = \Delta[1]$ of Proposition 1.2.

Proposition 1.3. *If X is non-singular, then so is $X^{\Delta[1]}$.*

We may restate Theorem 1.1 by saying that the non-singular simplicial sets form an *exponential ideal* in the cartesian closed category [**ML98**, §IV.6] of simplicial sets. The adjunction (D, U) exhibits *nsSet* as a *reflective* full subcategory [**ML98**, §IV.3] of *sSet*, which is *closed under exponentiation* in the sense of [**Day72**]. In this situation, Day's reflection theorem [**Day72**, Thm. 1.2, Cor. 2.1] shows that the reflector $D\colon sSet \to nsSet$ preserves finite products, making *nsSet* a cartesian closed category.

Proposition 1.4. *Desingularization $D\colon sSet \to nsSet$ preserves finite products.*

Remark 1.5. The category *nsSet* is not a *topos* in the sense of [**ML98**, §IV.10], because it does not admit a subobject classifier $t\colon \Delta[0] \to \Omega$. Here Ω_0 would have to consist of precisely two elements, so Ω would be at most 1-dimensional, and could not classify all the subobjects of $\Delta[2]$. This is related to the fact that desingularization does not in general preserve equalizers, as the example of the two maps $\Delta[0] \rightrightarrows \Delta[2]/\delta_1\Delta[1]$ illustrates.

We give the proof of Proposition 1.3 in Section 2, and deduce the remaining results in Section 3.

Remark 1.6. We learned from handling editor Dan Christensen that our Theorem 1.1 has a parallel in earlier work by Michel Zisman [**Zis09**], who defined a class of simplicial sets that he called *regular* (régulier), which properly contains the class of non-singular simplicial sets. Zisman's Theorem 2 states that X^K is regular for each regular simplicial set X, where K is an arbitrary simplicial set, and his key technical Lemma 4 can serve as a replacement for our Lemma 2.4. We explain this relationship in more detail after the proof of the latter lemma.

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2. Rigidity of prisms

Informally, Proposition 1.3 asserts that maps $\Phi\colon \Delta[n] \times \Delta[1] \to X$ from prisms to non-singular simplicial sets are very rigid.

We recall some terminology and notation before turning to the proofs. For each $n \geqslant 0$ let $[n]$ denote the totally ordered set $\{0 < 1 < \cdots < n\}$. Following [**FP90**, §4.1], we shall refer to the functions $\alpha\colon [m] \to [n]$ such that $\alpha(i) \leqslant \alpha(j)$ for all $i \leqslant j$ as *operators*. These are the objects and morphisms of the category Δ. For a simplicial set $X\colon [n] \mapsto X_n$ we write $x\alpha \in X_m$ for the value of the operator $\alpha\colon [m] \to [n]$ on a simplex $x \in X_n$. The *standard n-simplex* $\Delta[n]$ is the simplicial set $[m] \mapsto \Delta([m],[n])$ represented by $[n]$.

An injective operator is said to be a *face operator*, and a surjective operator is said to be a *degeneracy operator*. Special face operators are the *elementary face operators* $\delta_i^n\colon [n-1] \to [n]$ that omit the element i, and the *vertex operators* $\varepsilon_i^n\colon [0] \to [n]$ that hit the element i. Special degeneracy operators are the *elementary degeneracy operators* $\sigma_i^n\colon [n+1] \to [n]$ that send i and its successor $i+1$ to i. Usually, we omit the superscript in the notation.

A face operator or degeneracy operator is *proper* if it is not the identity. A simplex x is a *(proper) face* of a simplex y if $x = y\mu$ for a (proper) face operator μ. Analogously, x is a *(proper) degeneracy* of y if $x = y\rho$ for a (proper) degeneracy operator ρ. A simplex is *degenerate* if it is a proper degeneracy of some simplex. Otherwise, it is said to be *non-degenerate*.

By the Eilenberg–Zilber lemma [**FP90**, Thm. 4.2.3] any simplex x in a simplicial set X can be uniquely expressed as a degeneration $x = x^\sharp x^\flat$ of a non-degenerate simplex. We call the non-degenerate simplex $x^\sharp$ the *non-degenerate part* of x, and will refer to the degeneracy operator $x^\flat$ as the *degenerate part* of x. By the Yoneda lemma, the n-simplices x of a simplicial set X are in natural bijective correspondence with the simplicial maps $\bar{x}\colon \Delta[n] \to X$. The map $\bar{x}$ is the *representing map* of x.

Lemma 2.1. *Let $x \in X_n$ be any simplex. The representing map $\bar{x}\colon \Delta[n] \to X$ is degreewise injective if and only if the $n+1$ vertices $x\varepsilon_0, \dots, x\varepsilon_n \in X_0$ are pairwise distinct.* ☐

Lemma 2.2. *Let x be a simplex in a non-singular simplicial set X, and suppose that $x\varepsilon_k = x\varepsilon_l$ for some $k < l$. Then the degenerate part $x^\flat$ of x factors uniquely through the proper degeneracy operator $\sigma_k \cdots \sigma_{l-1}$.*

Proof. The representing map of the non-degenerate part $x^\sharp$ is degreewise injective, since X is non-singular, so its vertices are pairwise distinct. It follows that $x^\flat(k) = x^\flat(l)$. Since $x^\flat$ is order-preserving, it also follows that $x^\flat(k) = x^\flat(j)$ for all $k \leqslant j \leqslant l$. Let $\rho = \sigma_k \cdots \sigma_{l-1}$. Then $x^\flat(i) = x^\flat(j)$ whenever $\rho(i) = \rho(j)$, and this implies that $x^\flat = (x^\flat\mu)\rho$, where μ is any choice of section to ρ. Thus the asserted factorization exists. Its uniqueness is automatic, since ρ is surjective. ☐

Proof of Proposition 1.3. Suppose that X is non-singular. We must show that each non-degenerate n-simplex Φ in the simplicial mapping set $X^{\Delta[1]}$ has $n+1$ distinct vertices $\Phi\varepsilon_0, \dots, \Phi\varepsilon_n$. Equivalently, we must show that if the k-th and l-th vertices of an n-simplex Φ are equal, for some $0 \leqslant k < l \leqslant n$, then Φ is degenerate. This follows from the two lemmas below. ☐

Lemma 2.3. *Suppose that X is non-singular and Φ is an n-simplex in $X^{\Delta[1]}$ such that $\Phi\varepsilon_k = \Phi\varepsilon_l$, for some $0 \leqslant k < l \leqslant n$. Then*

$$\Phi\varepsilon_k = \Phi\varepsilon_j = \Phi\varepsilon_l$$

for all $k \leqslant j \leqslant l$.

Lemma 2.4. *Suppose that X is non-singular and Φ is an n-simplex in $X^{\Delta[1]}$ such that $\Phi\varepsilon_k = \Phi\varepsilon_{k+1}$, for some $0 \leqslant k < n$. Then there is an $(n-1)$-simplex Ψ in $X^{\Delta[1]}$ for which $\Phi = \Psi\sigma_k$, exhibiting Φ as a degenerate simplex.*

We introduce some more notation before proving these lemmas. By definition, an n-simplex in $X^{\Delta[1]}$ is a simplicial map

$$\Phi\colon \Delta[n] \times \Delta[1] \longrightarrow X .$$

Here, the *prism* $\Delta[n] \times \Delta[1]$ is generated by the non-degenerate $(n+1)$-simplices

$$\gamma_j^{n+1}\colon \Delta[n+1] \longrightarrow \Delta[n] \times \Delta[1] ,$$

for $0 \leqslant j \leqslant n$, given by

$$\gamma_j^{n+1}(i) = \begin{cases} (i,0) & \text{for } 0 \leqslant i \leqslant j, \\ (i-1,1) & \text{for } j+1 \leqslant i \leqslant n+1. \end{cases}$$

Viewing $\Delta[n] \times \Delta[1]$ as the nerve of the partially ordered set $[n] \times [1]$, these generators can be seen as maximal length paths in the diagram below.

$$\begin{tikzcd}
(0,1) \arrow[r] & \dots \arrow[r] & (j,1) \arrow[r] & (j+1,1) \arrow[r] & \dots \arrow[r] & (n,1) \\
(0,0) \arrow[r] \arrow[u] & \dots \arrow[r] & (j,0) \arrow[r] \arrow[u] \arrow[ur] & (j+1,0) \arrow[r] \arrow[u] & \dots \arrow[r] & (n,0). \arrow[u]
\end{tikzcd}$$

In particular, they satisfy the relations

$$\gamma_j^{n+1}\delta_{j+1} = \gamma_{j+1}^{n+1}\delta_{j+1} \tag{2.1}$$

for $0 \leqslant j < n$. Conversely, to specify Φ it suffices to give its values $\Phi\gamma_j^{n+1}$ on these $n+1$ generators, subject to the n relations

$$(\Phi\gamma_j^{n+1})\delta_{j+1} = (\Phi\gamma_{j+1}^{n+1})\delta_{j+1}.$$

Proof of Lemma 2.3. Let X be non-singular and let Φ be an n-simplex in $X^{\Delta[1]}$ with $\Phi\varepsilon_k = \Phi\varepsilon_l$, where $0 \leqslant k < l \leqslant n$. The vertex operators $\varepsilon_i\colon \Delta[0] \to \Delta[1]$ for $i \in \{0,1\}$ induce maps $\varepsilon_i^*\colon X^{\Delta[1]} \to X^{\Delta[0]} \cong X$. Let $x_i = \varepsilon_i^*\Phi$ in X_n be represented by the composite

$$\bar{x}_i\colon \Delta[n] \cong \Delta[n] \times \Delta[0] \xrightarrow{1\times\varepsilon_i} \Delta[n] \times \Delta[1] \xrightarrow{\Phi} X ,$$

restricting Φ to the bottom (for $i = 0$) or the top (for $i = 1$) of the prism. The hypothesis on Φ implies that $x_i\varepsilon_k = x_i\varepsilon_l$ in X_0, so by Lemma 2.2 we can factor the degenerate part $x_i^\flat$ of x_i through $\sigma_k \cdots \sigma_{l-1}$, so that $x_i = y_i\sigma_k \cdots \sigma_{l-1}$ for some $(n+k-l)$-simplices y_i of X.

Consider any j with $k \leqslant j < l$, let $\mu\colon [1] \to [n]$ be the face operator given by $\mu(0) = j$ and $\mu(1) = j+1$, and view the 1-simplex $\Phi\mu$ in $X^{\Delta[1]}$ as the map $\Delta[1] \times \Delta[1] \to X$

indicated by the following square.

$$\begin{tikzcd}
x_1\varepsilon_j \arrow[r, "x_1\mu"] & x_1\varepsilon_{j+1} \\
x_0\varepsilon_j \arrow[u, "\Phi\varepsilon_j"] \arrow[ur] \arrow[ur, phantom, "z_0", shift left=4ex] \arrow[ur, phantom, "z_1"', shift right=4ex] \arrow[r, "x_0\mu"'] & x_0\varepsilon_{j+1}. \arrow[u, "\Phi\varepsilon_{j+1}"']
\end{tikzcd}$$

The factorization of x_i through σ_j shows that $x_i\varepsilon_j = x_i\varepsilon_{j+1}$, for each i. Hence each 2-simplex z_i does not have pairwise distinct vertices, and must therefore be degenerate, since X is non-singular. By Lemma 2.2 we must have $z_0 = w_0\sigma_1$ and $z_1 = w_1\sigma_0$ for some 1-simplices w_i. More precisely, we must have $w_0 = z_0\delta_2 = \Phi\varepsilon_j$ and $w_1 = z_1\delta_0 = \Phi\varepsilon_{j+1}$.

It follows that the diagonal 1-simplex in the figure is simultaneously equal to $z_0\delta_1 = (\Phi\varepsilon_j)\sigma_1\delta_1 = \Phi\varepsilon_j$ and to $z_1\delta_1 = (\Phi\varepsilon_{j+1})\sigma_0\delta_1 = \Phi\varepsilon_{j+1}$. This proves that $\Phi\varepsilon_j = \Phi\varepsilon_{j+1}$ are equal as vertices in $X^{\Delta[1]}$. □

Proof of Lemma 2.4. Let X be non-singular and let Φ be an n-simplex in $X^{\Delta[1]}$ with $\Phi\varepsilon_k = \Phi\varepsilon_{k+1}$, where $0 \leqslant k < n$. We will construct an $(n-1)$-simplex Ψ in $X^{\Delta[1]}$ with $\Phi = \Psi\sigma_k$. Equivalently, we must define $\Psi\colon \Delta[n-1] \times \Delta[1] \to X$ so as to make the right hand triangle commute in the diagram below.

$$\begin{tikzcd}
\Delta[n+1] \arrow[r, "\gamma_j^{n+1}"] & \Delta[n] \times \Delta[1] \arrow[r, "\Phi"] \arrow[d, "\sigma_k \times 1"'] & X \\
 & \Delta[n-1] \times \Delta[1]. \arrow[ur, "\Psi"'] &
\end{tikzcd}$$

The triangle will commute if $\Phi\gamma_j^{n+1} = \Psi(\sigma_k \times 1)\gamma_j^{n+1}$ for each $0 \leqslant j \leqslant n$, since the simplices $\gamma_0^{n+1}, \dots, \gamma_n^{n+1}$ generate the prism $\Delta[n] \times \Delta[1]$. Here

$$(\sigma_k \times 1)\gamma_j^{n+1} = \begin{cases} \gamma_j^n \sigma_{k+1} & \text{for } 0 \leqslant j \leqslant k, \\ \gamma_{j-1}^n \sigma_k & \text{for } k < j \leqslant n. \end{cases} \tag{2.2}$$

Should Ψ exist, it must therefore satisfy

$$\Phi(\gamma_j^{n+1}) = \begin{cases} \Psi(\gamma_j^n)\sigma_{k+1} & \text{for } 0 \leqslant j \leqslant k, \\ \Psi(\gamma_{j-1}^n)\sigma_k & \text{for } k < j \leqslant n. \end{cases}$$

Observing that δ_{k+1} is a section to both σ_k and σ_{k+1}, we are led to define a function

$$\psi\colon \{\gamma_0^n, \dots, \gamma_{n-1}^n\} \longrightarrow X_n$$

by

$$\psi(\gamma_j^n) = \begin{cases} \Phi(\gamma_j^{n+1})\delta_{k+1} & \text{for } 0 \leqslant j \leqslant k, \\ \Phi(\gamma_{j+1}^{n+1})\delta_{k+1} & \text{for } k \leqslant j \leqslant n-1, \end{cases}$$

which specifies where Ψ must send the generators $\gamma_0^n, \dots, \gamma_{n-1}^n$ of $\Delta[n-1] \times \Delta[1]$, should it exist. Note that for $j = k$ the relation

$$\Phi(\gamma_k^{n+1})\delta_{k+1} = \Phi(\gamma_k^{n+1}\delta_{k+1}) = \Phi(\gamma_{k+1}^{n+1}\delta_{k+1}) = \Phi(\gamma_{k+1}^{n+1})\delta_{k+1}$$

holds, by (2.1), so $\psi(\gamma_k^n)$ is unambiguously defined. To verify that $\Psi(\gamma_j^n) = \psi(\gamma_j^n)$ for $0 \leqslant j \leqslant n-1$ defines a map $\Psi\colon \Delta[n-1] \times \Delta[1] \to X$, it is (necessary and) sufficient to confirm the relations

$$\psi(\gamma_j^n)\delta_{j+1} = \psi(\gamma_{j+1}^n)\delta_{j+1} \tag{2.3}$$

for $0 \leqslant j < n-1$. We separate the proof of (2.3) into two cases.

First, for $0 \leqslant j < k$ we use the general rule $\delta_{k+1}\delta_{j+1} = \delta_{j+1}\delta_k$ for $j < k$, together with (2.1), to see that

$$\psi(\gamma_j^n)\delta_{j+1} = \Phi\gamma_j^{n+1}\delta_{k+1}\delta_{j+1} = \Phi\gamma_j^{n+1}\delta_{j+1}\delta_k$$

is equal to

$$\psi(\gamma_{j+1}^n)\delta_{j+1} = \Phi\gamma_{j+1}^{n+1}\delta_{k+1}\delta_{j+1} = \Phi\gamma_{j+1}^{n+1}\delta_{j+1}\delta_k\,.$$

Second, for $k \leqslant j < n-1$ we use the general rule $\delta_{k+1}\delta_{j+1} = \delta_{j+2}\delta_{k+1}$ for $k \leqslant j$, together with (2.1), to see that

$$\psi(\gamma_j^n)\delta_{j+1} = \Phi\gamma_{j+1}^{n+1}\delta_{k+1}\delta_{j+1} = \Phi\gamma_{j+1}^{n+1}\delta_{j+2}\delta_{k+1}$$

is equal to

$$\psi(\gamma_{j+1}^n)\delta_{j+1} = \Phi\gamma_{j+2}^{n+1}\delta_{k+1}\delta_{j+1} = \Phi\gamma_{j+2}^{n+1}\delta_{j+2}\delta_{k+1}\,.$$

This concludes the verification of (2.3), giving us a well-defined map Ψ.

It still remains to argue that $\Phi = \Psi(\sigma_k \times 1)$, and this is where we use the hypotheses on X and Φ. It suffices to check that the equation

$$\Phi\gamma_j^{n+1} = \Psi(\sigma_k \times 1)\gamma_j^{n+1} \tag{2.4}$$

holds for $0 \leqslant j \leqslant n$. Again, we separate the proof into two cases.

First, for $0 \leqslant j \leqslant k$ we must show that the $(n+1)$-simplex $z_j = \Phi(\gamma_j^{n+1})$ in X is equal to

$$\Psi(\sigma_k \times 1)\gamma_j^{n+1} = \Psi\gamma_j^n\sigma_{k+1} = \Phi(\gamma_j^{n+1})\delta_{k+1}\sigma_{k+1} = z_j\delta_{k+1}\sigma_{k+1}\,,$$

where we have used the calculation (2.2). The vertices $z_j\varepsilon_{k+1}$ and $z_j\varepsilon_{k+2}$ in X are equal to $\varepsilon_1^*(\Phi\varepsilon_k)$ and $\varepsilon_1^*(\Phi\varepsilon_{k+1})$, respectively, hence are equal by the assumption that $\Phi\varepsilon_k = \Phi\varepsilon_{k+1}$. It follows by Lemma 2.2 that $z_j = w_j\sigma_{k+1}$ for some n-simplex w_j in X, since X is non-singular. This immediately implies that $z_j\delta_{k+1}\sigma_{k+1} = w_j\sigma_{k+1}\delta_{k+1}\sigma_{k+1} = w_j\sigma_{k+1} = z_j$, since δ_{k+1} is a section to σ_{k+1}.

Second, for $k < j \leqslant n$ we must show that the $(n+1)$-simplex $z_j = \Phi(\gamma_j^{n+1})$ in X is equal to

$$\Psi(\sigma_k \times 1)\gamma_j^{n+1} = \Psi\gamma_{j-1}^n\sigma_k = \Phi(\gamma_j^{n+1})\delta_{k+1}\sigma_k = z_j\delta_{k+1}\sigma_k\,.$$

The vertices $z_j\varepsilon_k$ and $z_j\varepsilon_{k+1}$ in X are equal to $\varepsilon_0^*(\Phi\varepsilon_k)$ and $\varepsilon_0^*(\Phi\varepsilon_{k+1})$, respectively, hence are themselves equal. It follows by Lemma 2.2 that $z_j = w_j\sigma_k$ for some n-simplex w_j in X. This implies that $z_j\delta_{k+1}\sigma_k = w_j\sigma_k\delta_{k+1}\sigma_k = w_j\sigma_k = z_j$, since δ_{k+1} is a section to σ_k. This concludes our verification of (2.4), proving that Φ is a degenerate simplex of $X^{\Delta[1]}$. $\square$

Remark 2.5. As mentioned in Remark 1.6, our Lemma 2.4 is easily deduced from Lemma 4 in [**Zis09**]. A simplicial set X is regular in Zisman's sense if and only if for each non-degenerate simplex $x \in X_n$ and each elementary edge operator $\mu\colon [1] \to [n]$ of the form $\mu(0) = k$ and $\mu(1) = k + 1$, with $0 \leqslant k < n$, the 1-simplex $x\mu \in X_1$ is non-degenerate. (This notion of regularity differs from that defined in [**FP90**, §4.6], which in turn is related to regularity and triangulability for CW complexes.)

Each non-singular simplicial set X is Zisman regular, since the vertices $x\varepsilon_k$ and $x\varepsilon_{k+1}$ of any non-degenerate simplex x in X will be distinct, so that $x\mu$ is non-degenerate, for μ and $0 \leqslant k < n$ as above. If X is non-singular and $\Phi\varepsilon_k = \Phi\varepsilon_{k+1}$ for some n-simplex $\Phi\colon \Delta[n] \times \Delta[1] \to X$ in $X^{\Delta[1]}$, then $x_0\mu$ and $x_1\mu$ in the square diagram in the proof of Lemma 2.3 (with j replaced by k) must both be degenerate as 1-simplices in X. This shows that Φ is *k-almost degenerate* (k-presque dégénéré) in the sense of [**Zis09**, §2.2]. Hence [**Zis09**, Lem. 4] proves that $\Phi = \Psi\sigma_k$ for some $(n-1)$-simplex Ψ in $X^{\Delta[1]}$, which gives the conclusion of our Lemma 2.4.

We choose to retain our proof of this lemma, for the convenience of the reader.

3. Outstanding proofs

Proof of Theorem 1.1. Let X be any non-singular simplicial set. By Proposition 1.3 and induction, $X^{\Delta[1]^n}$ is non-singular, for each $n \geqslant 0$. The inclusion $i\colon \Delta[n] \to \Delta[1]^n$ sending $j \in [n]$ to $(1, \dots, 1, 0, \dots, 0) \in [1]^n$ (with j copies of 1) admits a retraction $r\colon \Delta[1]^n \to \Delta[n]$ sending $(k_1, \dots, k_n)$ to the largest index j such that $k_j = 1$. Hence $r^*\colon X^{\Delta[n]} \to X^{\Delta[1]^n}$ is split injective, and shows that $X^{\Delta[n]}$ is non-singular.

For any simplicial set K, we can find a simplicial set $L = \coprod_\alpha \Delta[n_\alpha]$ and a degreewise surjection $s\colon L \to K$. The induced map

$$s^*\colon X^K \longrightarrow X^L \cong \prod_\alpha X^{\Delta[n_\alpha]}$$

is then degreewise injective, and exhibits X^K as a simplicial subset of a product of non-singular simplicial sets. It follows that X^K is non-singular. □

Proof of Proposition 1.2. When X, K and Y are non-singular, so that $X \times K$ and Y^K are non-singular by Theorem 1.1, the natural bijection

$$sSet(X \times K, Y) \cong sSet(X, Y^K)$$

restricts to a natural bijection $nsSet(X \times K, Y) \cong nsSet(X, Y^K)$. Hence the endofunctor $X \mapsto X \times K$ of $nsSet$ is a left adjoint, and preserves all colimits. □

Proof of Proposition 1.4. Let X and Y be any simplicial sets. Recall that each adjunction unit $\eta_Z\colon Z \to DZ$ is degreewise surjective. Let $a\colon D(X \times Y) \to DX \times DY$ be induced by the two projections from $X \times Y$. The composite

$$X \times Y \overset{\eta_{X\times Y}}{\longrightarrow} D(X \times Y) \overset{a}{\longrightarrow} DX \times DY$$

is then equal to $\eta_X \times \eta_Y$, so a is degreewise surjective. The right adjoint of $\eta_{X\times Y}$ $X \to D(X \times Y)^Y$ factors through $\eta_X\colon X \to DX$, since $D(X \times Y)^Y$ is non-singular by Theorem 1.1. Hence there is a unique factorization

$$X \times Y \xrightarrow{\eta_X \times 1} DX \times Y \xrightarrow{b} D(X \times Y)$$

of $\eta_{X\times Y}$. Similarly, there is a unique factorization

$$DX \times Y \xrightarrow{1\times \eta_Y} DX \times DY \xrightarrow{c} D(X \times Y)$$

of b, again by Theorem 1.1. It follows that $ca\eta_{X\times Y} = c(\eta_X \times \eta_Y) = \eta_{X\times Y}$, so that $ca = 1$, which proves that a is an isomorphism. ☐

References

[Day72] Brian Day, *A reflection theorem for closed categories*, J. Pure Appl. Algebra **2** (1972), no. 1, 1–11, DOI 10.1016/0022-4049(72)90021-7. MR296126

[Fje] Vegard Fjellbo, *Homotopy theory of non-singular simplicial sets.* arXiv:2001.05032.

[FP90] Rudolf Fritsch and Renzo A. Piccinini, *Cellular structures in topology*, Cambridge Stud. Adv. Math., vol. 19, Cambridge University Press, Cambridge, 1990. MR1074175

[Kan57] Daniel M. Kan, *On c. s. s. complexes*, Amer. J. Math. **79** (1957), 449–476, DOI 10.2307/2372558. MR90047

[ML98] Saunders Mac Lane, *Categories for the working mathematician*, 2nd ed., Grad. Texts in Math., vol. 5, Springer-Verlag, New York, 1998. MR1712872

[WJR13] Friedhelm Waldhausen, Bjørn Jahren, and John Rognes, *Spaces of PL manifolds and categories of simple maps*, Ann. of Math. Stud., vol. 186, Princeton University Press, Princeton, NJ, 2013. MR3202834

[Zis09] Michel Zisman, *Ensembles simpliciaux réguliers*, J. Homotopy Relat. Struct. **4** (2009), no. 1, 255–264 (French, with French summary). MR2559642

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