

$K(n)$ -COMPACT SPHERES

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1. INTRODUCTION

For each topological group G there is an associated space BG , called the classifying space of G , such that $G \simeq \Omega BG$, so G is (homotopy equivalent to) a loop space. An explicit construction of BG is given by the bar construction. This will, for non-trivial G , give BG the structure of an infinite dimensional CW-complex. Part of the bar construction can also be carried out for an H -space, depending on how associative the multiplication is. For example, the 2-skeleton of the bar construction can be obtained as the mapping cone of the Hopf construction of the multiplication map $G \times G \rightarrow G$.

A compact Lie group G has the property that $H_*(G; \mathbb{Z})$ is a finite Hopf algebra, and $H^*(BG; \mathbb{Z})$ is a finitely generated algebra. Dwyer and Wilkerson use this to define a generalization of compact Lie groups they call *p-compact groups*, as loop spaces that have finite mod p homology and p -complete classifying space. This definition allows for more exotic examples that have the same homotopy properties as Lie groups, at least to the eyes of mod p homology $H_*(-; \mathbb{F}_p)$. A non-trivial example of this is the Sullivan sphere that exists at odd primes in dimension $2p - 3$.

For each prime p there is a sequence of generalized (co)homology theories, $K(n)$, where $0 \leq n \leq \infty$, called the Morava K -theories. It is common to interpret the theories $K(0)$ and $K(\infty)$ as singular (co)homology with respectively $\mathbb{Q}$ and $\mathbb{F}_p$ coefficients, while $K(1)$ is one of $p-1$ summands of complex mod p topological K -theory. The nature of $K(n)$ for higher n is more subtle, but is related to the formal group laws of height n in characteristic p . A further generalization of p -compact groups is that of $K(n)$ -compact groups. *K(n)-compact groups* are topological groups with finite $K(n)$ -cohomology and such that the classifying space is $K(n)$ -local. By the Atiyah-Hirzebruch spectral sequence all p -compact groups are $K(n)$ -compact, but here there are also exotic examples. In [RW80] the bar spectral sequence is used to compute the $K(n)$ (co)homology of the Eilenberg-Mac Lane spaces $K(\mathbb{Z}/p^j, q)$, and these are examples of $K(n)$ -compact groups when $q < n$.

To do calculations we will use theories that are closely related to $K(n)$, denoted by K_n , E_n and E_n^{Gal} , defined in section 2. The coefficient ring $\pi_0 E_n$ carries the universal deformation of the height n formal group law Γ_n over $\mathbb{F}_{p^n}$ with p -series $[p]_{\Gamma_n}(x) = x^{p^n}$. The Lubin-Tate theory of deformations of formal group laws is explained in section 2. A theorem of Hovey and Strickland [HS99] says that if $K_n^*(X)$ is finite-dimensional over K_n^* and concentrated in even degrees, then $E_n^*(X)$ is finitely generated over E_n^* and concentrated in even degrees. We will use this to algebraically exclude the even dimensional spheres from the $K(n)$ -compact ones.

The classical Adams and Atiyah argument [AA66] uses the Adams operations in K -theory to show that for spheres that are also H -spaces the only possibilities are S^0, S^1, S^3 and S^7 . In section 4 we use the action of the Morava stabilizer group $\mathbb{S}_n$ of stable ring operations on $E_n^*(-)$ (described in section 2), plus the unstable operations constructed in [And95], which extend the Adams operations on $K^*(-)_p^\wedge$ to E_n^* , to give similar bounds on the dimension of a $K(n)$ -local sphere. Here we explain that Adams and Atiyah's argument is equivalent to ours in the case $K(1)$ for $p = 2$. The bounds are more complicated for general n and p , but we are able to show that there are only finitely many dimensions in which a $K(n)$ -local sphere can occur as a $K(n)$ -compact group, for fixed n and p . There is one remaining hypothesis we rely on here, regarding the existence of maximal tori in $K(n)$ -compact groups. The corresponding result for p -compact groups is known ([DW94, 8]), but the non-trivial Sullivan conjecture (now a theorem) is used, and a similar result in the $K(n)$ -local setting is not known. Further, we need the existence of the p^n -skeleton of the bar construction, which is equivalent to the Stasheff's notion of A_{p^n} -spaces ([Sta63]), but we will concentrate on A_∞ -spaces, or topological groups, for simplicity. We need to know the $K(n)$ - and E_n -cohomology of the various skeleta of BG from the bar spectral sequence. We also need some standard formulas for the p -order of an integer.

In section 6 we will try to construct new examples of $K(n)$ -compact groups. To do this we need to calculate the $K(n)$ -cohomology of a loop space, and we do this with a $K(n)$ -based Eilenberg-Moore spectral sequence. The convergence of this spectral sequence is not yet established (see [Sey78], [JO99] for some results concerning this), but we will use it to at least make the results plausible. The first examples of such $K(n)$ -compact groups are the $K(n)$ -compact versions of the Sullivan sphere, where we start with the bar construction on $K(\mathbb{Z}_p, n)$, take homotopy orbits for some suitable action, $K(n)$ -localize, and take the loop space of this. The resulting $K(n)$ -compact groups reside in dimension one less than $2(p^n - 1)/\gcd(p - 1, n)$, which is valid according to the dimension bounds computed earlier. Additional work is required to show that these new groups are equivalent to $K(n)$ -local spheres. An argument that can be used to show that the groups are stably equivalent to spheres is sketched at the end of the paper.

We fix some notation. $\mathbb{Z}_p$ will mean the p -adic integers, and $\mathbb{Z}_{(p)}$ the integers localized at p . $\Lambda_R(S)$, $R[S]$ and $\Gamma_R(S)$ will denote exterior, polynomial and divided power algebras, respectively, with generator set S taken over a ground ring R . Sometimes the subscript R will be left out, when the choice of ring is clear from the context. Write $\Gamma_R(x) = R\{\gamma_n(x) \mid n \geq 0\}$ with $\gamma_i(x)\gamma_j(x) = \binom{i+j}{i}\gamma_{i+j}(x)$.

I would like to express my gratitude to my advisor John Rognes, this thesis would never exist if it were not for his ideas and help, and his patience while helping me learn this material.

2. THE MORAVA K - AND E -THEORIES

Basic constructions. Let MU be the spectrum associated to complex cobordism, with coefficient ring $\pi_*(MU) = \mathbb{Z}[x_1, x_2, \dots]$, x_i having degree $2i$. By localizing at a prime p (see the subsection on localization below), $MU_{(p)}$ splits as an infinite wedge of suspensions of a spectrum known as the Brown-Peterson spectrum, BP ([BP66]), and $\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$, where the v_i 's are taken to be the Araki generators. The degree of v_i is $2(p^i - 1)$, so $\pi_*(BP)$ is much smaller than $\pi_*(MU)$. From this we can construct new spectra by use of the Landweber exact functor theorem ([Lan76]) which states that

$$BP_*(-) \otimes_{\pi_*(BP)} M$$

is a homology theory for nice $\pi_*(BP)$ -modules M . More precisely, multiplication by v_n must be injective in $M \otimes_{\pi_*(BP)} \pi_*(BP)/I_n$ for all n , where $I_n = (p, v_1, \dots, v_{n-1})$. This is the case for the following rings

$$\begin{aligned} \pi_*(\widehat{E(n)}) &= \mathbb{Z}_{(p)}[v_1, \dots, v_n, v_n^{-1}]_{I_n}^\wedge \\ \pi_*(E_n^{Gal}) &= \mathbb{Z}_p[[u_1, \dots, u_{n-1}]] [u, u^{-1}] \\ \pi_*(E_n) &= \mathbb{W}\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]] [u, u^{-1}] \end{aligned}$$

giving the spectra $\widehat{E(n)}$, E_n^{Gal} and E_n , where $\pi_*(E_n^{Gal})$ and $\pi_*(E_n)$ are $\pi_*(BP)$ -modules by the map that sends

$$v_k \mapsto \begin{cases} u_k u^{1-p^k} & k < n \\ u^{1-p^n} & k = n \\ 0 & k > n \end{cases}.$$

Here the u_i have degree 0 and u has degree -2 . Note that $E_n^*(X) = \mathbb{W}\mathbb{F}_{p^n} \otimes_{\mathbb{Z}_p} (E_n^{Gal})^*(X)$, and $(E_n^{Gal})^*(X) = \mathbb{Z}_p \otimes_{\mathbb{W}\mathbb{F}_{p^n}} E_n^*(X)$. A similar relationship exist between $\widehat{E(n)}$ and E_n^{Gal} . This means that the distinction between $\widehat{E(n)}$, E_n^{Gal} and E_n is usually not important. Several of the theorems cited in this paper are originally stated in terms of $\widehat{E(n)}$ or E_n^{Gal} , while we use E_n . The spectrum E_1 is the complex K -theory spectrum completed at p , denoted $KU_p^\wedge$ or $K_p^\wedge$.

The Baas-Sullivan construction developed in [Baa73] can be applied to create versions of complex cobordism where the manifolds are allowed to have singularities, or cones. In this way spectra with coefficients $\pi_*(BP)/I$ can be constructed, where I is any ideal generated by a regular sequence, where a sequence $(a_1, a_2, \dots)$ in a ring R is defined to be a *regular sequence* if multiplication by a_k is injective in $R/(a_1, \dots, a_{k-1})$, for each $k \geq 1$. If we choose this ideal to be $(p, v_1, \dots, v_{n-1}, v_{n+1}, v_{n+2}, \dots)$ we get the connective Morava K -theories $k(n)$ with

$$\pi_*(k(n)) = \mathbb{F}_p[v_n].$$

Let $\Sigma^{2(p^n-1)}k(n) \rightarrow k(n)$ be multiplication by v_n and define

$$K(n) = \operatorname{hocolim}_i \Sigma^{-2i(p^n-1)}k(n)$$

to get the Morava K -theories with $\pi_*(K(n)) = \mathbb{F}_p[v_n, v_n^{-1}]$. Now let

$$K_{n*}(X) = K(n)_*(X) \otimes_{K(n)_*} \mathbb{F}_{p^n}[u, u^{-1}],$$

with $|u| = 2$, where we consider $\mathbb{F}_{p^n}[u, u^{-1}]$ as a $K(n)_*$ -module by the map $v_n \mapsto u^{p^n-1}$. The coefficient rings $\pi_*(K(n))$ etc. are often written as $K(n)_*$ (or $K(n)^* = \pi_{-*}(K(n))$ in cohomology).

The above is the classical construction of these theories. Now we would like to consider products. This used to be rather complicated, but with the new foundations for stable homotopy theory in [EKMM97] this can be done in a more elegant way. We briefly mention how this can be done in the following paragraphs.

Let $\mathcal{M}_S$ be the category of S -modules. $\mathcal{M}_S$ is a closed symmetric monoidal category with a model structure, which means it has a smash product $- \wedge_S - : \mathcal{M}_S \times \mathcal{M}_S \rightarrow \mathcal{M}_S$ and function objects $F_S(-, -) : \mathcal{M}_S^{op} \times \mathcal{M}_S \rightarrow \mathcal{M}_S$ such that $- \wedge_S Y$ and $F_S(Y, -)$ are adjoint functors. The smash product is unital, associative and commutative up to isomorphisms in $\mathcal{M}_S$. The model structure consists of collections of fibrations, cofibrations and weak equivalences satisfying certain axioms. When the weak equivalences in $\mathcal{M}_S$ are inverted, in some sense, we obtain the homotopy category $\bar{h}\mathcal{M}_S$, which is equivalent to the usual stable homotopy category constructed by Adams and Boardman. There is a fully faithful functor $\Sigma^\infty : \mathcal{T} \rightarrow \mathcal{M}_S$ embedding $\mathcal{T}$ in $\mathcal{M}_S$, such that the unit in $\mathcal{M}_S$ is the sphere spectrum $S = \Sigma^\infty S^0$. The corresponding functor $h\mathcal{T} \rightarrow \bar{h}\mathcal{M}_S$ is also written as Σ^∞ ; it is no longer full and faithful. Homotopy and (co)homology theories associated to an S -module E are defined so that $E_*(X) = \pi_*(E \wedge_S X)$ and $E^*(X) = \pi_{-*}(F_S(X, E))$. The smash product and function objects for S -modules will be written as $- \wedge -$ and $F(-, -)$, without the sub- S .

Ring objects can now be defined in $\mathcal{M}_S$ with the usual commutative diagrams, they are called S -algebras. Given a commutative S -algebra R one can also define the notion of R -module spectra. The category $\mathcal{M}_R$ of R -module spectra is again a closed symmetric monoidal category, with smash and function objects $M \wedge_R N$ and $F_R(M, N)$. This is analogous to the situation in algebra where one considers the tensor product over a ring R , or the module of R -linear homomorphisms.

We use the results of [Str99] to construct K - and E -theories in this new setting. All of the above coefficient rings can be written as $(S^{-1}BP_*)/I$, where S is a set of homogeneous elements, and I is an ideal in $S^{-1}BP_*$ generated by a regular sequence of homogeneous elements in non-negative degrees, e.g. $K(n)_* = v_n^{-1}BP_*/(v_i \mid i \neq n)$. Rings of this sort are called *positive localized regular quotients* (PLRQ's) by Strickland. BP_* is a PLRQ of $MU_{(p)}$, so our coefficient rings are also PLRQ's of $MU_{(p)}$. Theorem 2.7 in [Str99] now states that there exist ring spectra $K(n)$, etc. that realize these coefficient rings, at least for $p \neq 2$.

Note that all of the above theories are complex orientable, with orientations coming from MU . Since K_n and E_n are 2-periodic, and since we will be dealing mostly with K_n^0 and E_n^0 , we choose to put the orientation classes in the zeroth cohomology groups $K_n^0(\mathbb{C}P^\infty)$ and $E_n^0(\mathbb{C}P^\infty)$.

We now list certain properties of these theories which we will need later. $K(n)_*$ is a graded field, meaning each graded module over it is free, so there are Künneth isomorphisms

$$K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y).$$

Also, for all spaces X , we have that the map

$$(2.1) \quad K(n)^*(X) \rightarrow \mathrm{Hom}_{K(n)_*}(K(n)_*(X), K(n)_*)$$

is an isomorphism. More generally, for S -algebras E , there is a universal coefficient spectral sequence ([EKMM97, IV])

$$(2.2) \quad \mathrm{Ext}_{E_*}^{**}(E_*(X), E_*) \Longrightarrow E^*(X).$$

We have an Atiyah-Hirzebruch spectral sequence ([Boa99, 12])

$$H_*(X; K(n)_*) \Longrightarrow K(n)_*(X),$$

and a map $f : X \rightarrow Y$ induces a map between two such spectral sequences. If f induces an isomorphism on $H_*(-; \mathbb{F}_p)$, then

$$f_* : H_s(X; K(n)_t) \rightarrow H_s(Y; K(n)_t)$$

is an isomorphism for all s and t , so the induced map on the abutment

$$f_* : K(n)_*(X) \rightarrow K(n)_*(Y)$$

is an isomorphism.

The following theorem will help us determine the Hopf algebra structure of $E_n^*(X)$ from $K_n^*(X)$ and vice versa. Before we state the theorem, let us recall that $E_n^*(X)$ is *pro-free* if and only if it is the I_n -adic completion of a free module, that is

$$E_n^*(X) \cong \varprojlim_k F/I_n^k F$$

for a free E_n^* -module F . For finitely generated modules this is the same as being free, which will always be the case in this paper when we apply the following result.

Theorem 2.3 (Hovey-Strickland [HS99, 2.4–2.5]).

- (1) $E_n^*(X)$ is finitely generated if and only if $K_n^*(X)$ is finitely generated.
- (2) If $K_n^*(X)$ is concentrated in even degrees, then $E_n^*(X)$ is pro-free and concentrated in even degrees.
- (3) If $E_n^*(X)$ is pro-free, then $K_n^*(X) = E_n^*(X)/I_n$.

Note that since $K_n^*(\Sigma X)$ is even when $K_n^*(X)$ is odd, part 2 of the previous theorem also holds when “odd” is substituted for “even”.

Localization. There are two similar constructions of Bousfield localization, one for spaces [Bou75] and one for spectra [Bou79], which are not completely compatible. Here we will mainly use the one for spaces.

Let E be a spectrum. A map $f : A \rightarrow B$ is an E -equivalence if f induces an isomorphism in E -homology, and a space X is called E -local if for any E -equivalence f , the induced map $f^* : [B, X] \rightarrow [A, X]$ is a bijection. Bousfield shows that there is a localization functor $L_E : \mathcal{T} \rightarrow \mathcal{T}$ from the homotopy category of (based) topological spaces to itself, equipped with a universal natural E -equivalence $\eta_X : X \rightarrow L_E X$. $L_E X$ is called the localization of X . The idea is that $L_E X$ is

a “reduced” version of X obtained by “throwing out” the information that E -homology does not see. This is reflected in the fact that if $f : X \rightarrow Y$ induces an isomorphism in E -homology, then $L_E f : L_E X \rightarrow L_E Y$ is a homotopy equivalence. The essential image of L_E is the E -local category, i.e., the full subcategory of $\mathcal{T}$ with objects of the form $L_E X$, up to weak equivalence. Restricted to the E -local category the localization functor is the identity, or more briefly, L_E is idempotent. In particular we have the $H\mathbb{Z}_{(p)}$ -local (called p -local) and $H\mathbb{Z}/p$ -local (called p -complete) categories, and also the $K(n)$ -local category.

Based on Bousfield’s localization of spectra, localization of S -modules are defined in [EKMM97, VIII] in a similar way. An important special case when the localization is understood is when we localize spaces or connective spectra with respect to certain Moore spectra. For instance if $E = M\mathbb{Z}_{(J)}$, where J is a set of primes, then $L_E X = X \wedge M\mathbb{Z}_{(J)}$. When $J = \{p\}$ we have the p -local category. The p -localization of MU above was thus $L_{M\mathbb{Z}_{(p)}} MU$, which is just $MU \wedge M\mathbb{Z}_{(p)}$.

Formal group laws. Recall (e.g., [Rav04, App. 2]) that a (commutative, one-dimensional) formal group law over R is a power series $F(x, y) \in R[[x, y]]$ satisfying

- (1) $F(x, F(y, z)) = F(F(x, y), z)$
- (2) $F(x, y) = F(y, x)$
- (3) $F(x, 0) = F(0, x) = x$.

Here R is a commutative ring with 1. A morphism $f : F \rightarrow G$ of two formal group laws F and G is a power series $f(x) \in R[[x]]$ with $f(0) = 0$ that satisfies $f(F(x, y)) = G(f(x), f(y))$. It is usual to write the expression $F(-, -)$ as $-+_F-$, so the condition becomes $f(x+_F y) = f(x)+_G f(y)$. A morphism f is an isomorphism if and only if $f'(0)$ is invertible in R . The m -series of F is defined inductively to be $[m]_F(x) = [m-1]_F(x) +_F x$ for positive integers m , and $[0]_F(x) = 0$. It is an endomorphism of F . Note that $[m+n]_F(x) = [m]_F(x) +_F [n]_F(x)$, and $[mn]_F(x) = [m]_F([n]_F(x))$. When the ring R is a field of characteristic $p > 0$, as will always be the case for us, a nontrivial endomorphism f of a formal group law must be of the form $g(x^{p^k})$ for an endomorphism g with $g'(0) \neq 0$. If the p -series of a formal group law F over such a field has as first term a unit multiple of x^{p^n} , then F is said to have height n . Over $\mathbb{F}_{p^n}$ there is a unique formal group law that has height n and p -series given by x^{p^n} called the Honda formal group law, that we denote by Γ_n .

Next we describe the Lubin-Tate deformation theory of formal group laws. A reference for this material is Rezk’s lecture notes [Rez98]. Fix a formal group law F over a perfect field k , and a complete local ring B . A deformation of F to B is a pair (G, i) such that G is a formal group law over B and $i : k \rightarrow B/\mathfrak{m}$ is a morphism of fields with $i_* F = \pi_* G$, where $\pi : B \rightarrow B/\mathfrak{m}$ is the projection. Two such deformations (G_1, i_1) and (G_2, i_2) are $\star$ -isomorphic if $i_1 = i_2$ and there is an isomorphism $G_1 \rightarrow G_2$ of formal group laws that is the identity when projected to $B/\mathfrak{m}$. The set of $\star$ -isomorphism classes of deformations is written $\text{Def}_F(B)$. The association $B \mapsto \text{Def}_F(B)$ is a functor; a continuous homomorphism $\phi : B_1 \rightarrow B_2$ induces a morphism $\text{Def}_F(B_1) \rightarrow \text{Def}_F(B_2)$ sending (G, i) to $(\phi_* G, \bar{\phi} i)$.

Theorem 2.4 (Lubin-Tate). *Let F be a formal group law of height n over a perfect field k of characteristic $p > 0$. There exists a complete local ring $LT(F, k)$ such that $LT(F, k)/\mathfrak{m} \cong k$, and a formal group law $\tilde{F}$ over it, with the following property: Given a deformation (G, i) of F to B , there is a unique map $\phi : LT(F, k) \rightarrow B$*

such that $(\phi_* \tilde{F}, i) = (G, i)$ in $\text{Def}_F(B)$.

$\tilde{F}$ is called the universal deformation of F . The ring $LT(F, k)$ can be given explicitly as $LT(F, k) = \mathbb{W}k[[u_1, \dots, u_{n-1}]]$, where $\mathbb{W}k$ is the Witt ring on k , and n is the height of F . Set $LT(F, k)_* = LT(F, k)[u, u^{-1}]$.

Now let $\mathcal{FGL}_n$ be the category with objects the pairs (F, k) , where F is a formal group law of height n over a perfect field k of characteristic p . A morphism from (F_1, k_1) to (F_2, k_2) is a pair (f, j) , where $j : k_1 \rightarrow k_2$ is a homomorphism and $f : j_* F_1 \rightarrow F_2$ is an isomorphism of formal group laws. To each object (F, k) we associate a spectrum $E(F, k)$ by defining

$$E(F, k)_*(X) = MU_*(X) \otimes_{MU_*} LT(F, k)_*,$$

where the MU_* -module structure on $LT(F, k)_*$ comes from the unique map $MU_* \rightarrow LT(F, k)_*$ classifying the universal deformation $\Gamma = \tilde{\Gamma}_n$ over $LT(F, k)_*$. This classifying map is given by Quillen's theorem ([Rez98, 6.2]). There are some details concerning the grading here, since we now work with formal group laws over a graded ring; we will not go into them here. That $E(F, k)$ defines a homology theory follows from the Landweber exact functor theorem, once one has established that $(p, v_1, v_2, \dots)$ is a regular sequence in $LT(F, k)_*$. Given a morphism

$$(f, j) : (F_1, k_1) \rightarrow (F_2, k_2),$$

we want to construct a map $E(F_1, k_1) \rightarrow E(F_2, k_2)$ between the associated spectra. Let $\tilde{F}_1$ over $LT(F_1, k_1)_*$ and $\tilde{F}_2$ over $LT(F_2, k_2)_*$ be the universal deformations. By Theorem 2.4 there exists a unique map

$$\phi : LT(F_1, k_1) \rightarrow LT(F_2, k_2)$$

such that $\phi_* \tilde{F}_1 \cong \tilde{F}_2$ by a $\star$ -isomorphism. We extend ϕ to a map $LT(F_1, k_1)_* \rightarrow LT(F_2, k_2)_*$ by defining $\phi(u) = g'(0)u$. This ϕ now induces a map

$$E(F_1, k_1)_*(X) \rightarrow E(F_2, k_2)_*(X).$$

Again, see [Rez98] for more details.

We now have a functor from $\mathcal{FGL}_n$ to the stable homotopy category of spectra. For the next result, recall that a ring spectrum X is an E_∞ ring spectrum if there are structure maps

$$\xi_j : D_j X \rightarrow X$$

for all $j \geq 0$, making suitable diagrams commute, where $D_j X = E\Sigma_j \wedge_{\Sigma_j} X^j$ is the j 'th extended power of X . The notion of E_∞ ring spectra is essentially the same as commutative S -algebras ([EKMM97, II]).

Theorem 2.5 ([GH04, 7.6]). *The functor $\mathcal{FGL}_n \rightarrow \bar{h}\mathcal{M}_S$ lifts to the category of E_∞ ring spectra.*

The Morava stabilizer group. If we let $F = \Gamma_n$ and $k = \mathbb{F}_{p^n}$, then Theorem 2.5 states that E_n is an E_∞ ring spectrum with an action of the group $\mathbb{S}_n = \text{Aut}(\Gamma_n, \mathbb{F}_{p^n})$, called the Morava stabilizer group. This action is constructed so

that an element $g \in \mathbb{S}_n$ operates on $E_n^*(\mathbb{C}P^\infty)$ by sending the complex orientation x to $\tilde{g}(x)$, where $\tilde{g}$ is a lift of g to $E_n^*[[x]]$.

The p -adic units $\mathbb{Z}_p^\times$ can be embedded in $\mathbb{S}_n$ in the following way. First, we can extend the notion of m -series to all p -adic numbers m . Write $m \in \mathbb{Z}_p$ as $m = \sum_{i \geq 0} a_i p^i$, with $0 \leq a_i \leq p-1$, and define

$$(2.6) \quad [m]_{\Gamma_n}(x) = \sum_{i \geq 0}^{\Gamma_n} [a_i]_{\Gamma_n}([p^i]_{\Gamma_n}(x)),$$

the sum being a formal sum in Γ_n . Since $[p](x) \equiv 0 \pmod{x^{p^n}}$, we have $[p^i](x) = [p]([p^{i-1}](x)) \equiv 0 \pmod{x^{p^{in}}}$. This means that the terms in (2.6) are divisible by higher and higher powers of x , as i increases, so the infinite formal sum is well defined. Note that this definition coincides with the one for integral m -series when m is finite. When $m = \sum a_i p^i$ is a p -adic unit, i.e., $0 < a_0 < p$, then $[m](x) = a_0 x + \dots$ is an automorphism of Γ_n . Thus we get a map $\mathbb{Z}_p^\times \rightarrow \mathbb{S}_n$.

If we now write $n = \sum_{j \geq 0} b_j p^j$, we have $mn = \sum_{i,j \geq 0} a_i b_j p^{i+j}$ and

$$[m]([n](x)) = \sum_{i \geq 0} [a_i p^i] \left(\sum_{j \geq 0} [b_j p^j](x) \right) = \sum_{i,j \geq 0} [a_i b_j p^{i+j}](x) = [mn](x).$$

This shows that the map $\mathbb{Z}_p^\times \rightarrow \mathbb{S}_n$ is a homomorphism. Suppose $[m](x) = \sum_{i \geq 0} [a_i p^i](x) = x$, then since $[a_i p^i](x) \equiv 0 \pmod{x^{p^{in}}}$ it follows that $a_0 = 1$ and $a_i = 0, i \geq 1$. Hence the homomorphism is injective.

3. THE BAR SPECTRAL SEQUENCE

Here we set up a spectral sequence known as the bar spectral sequence, which is needed in the next section to compute the cohomology of BG .

The bar construction. For a topological group G we want to construct a space BG such that $G \simeq \Omega BG$. This is done by the geometric realization of a simplicial space $BG_\bullet$, to be defined next. Let $BG_n = G^n$, the product of G with itself n times, starting with $BG_0 = *$. The face and degeneracy maps are given by

$$d_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & i = 0 \\ (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n) & 0 < i < n \\ (g_1, \dots, g_{n-1}) & i = n \end{cases}$$

$$s_j(g_1, \dots, g_n) = (g_1, \dots, g_j, e, g_{j+1}, \dots, g_n).$$

Now let $BG = |BG_\bullet|$ be the realization

$$\coprod_{n \geq 0} \Delta^n \times G^n / \sim$$

where the relation is generated by $(\delta^i(v), g) \sim (v, d_i(g))$ and $(\sigma^j(v), g) \sim (v, s_j(g))$. δ^i is the inclusion of the i 'th face of the simplex, while σ^j is the degeneracy identifying the j 'th and the $(j+1)$ 'th vertex. This construction is functorial; a topological

homomorphism $G \rightarrow H$ induces a simplicial map $BG_\bullet \rightarrow BH_\bullet$ in the obvious way, and this again induces a map $BG \rightarrow BH$.

Let $EG_\bullet$ be the simplicial space $EG_n = G^{n+1}$, with the same face and degeneracy maps as for $BG_\bullet$, except that there is an extra coordinate g_{n+1} , so that d_n now becomes $d_n(g_1, \dots, g_{n+1}) = (g_1, \dots, g_n g_{n+1})$. Let $EG = |EG_\bullet|$, and let $p : EG \rightarrow BG$ be the map induced by the simplicial projection maps $G^{n+1} \rightarrow G^n$. G acts on the right on $EG_\bullet$ by multiplication in the last coordinate, and this induces a free right G -action on EG . $BG_\bullet$ is the same as the orbit space $EG_\bullet/G$, and also, $BG = EG/G$. The map p is in fact a fibration, at least when $\{e\} \rightarrow G$ is a cofibration, and gives rise to a long exact sequence

$$\cdots \rightarrow \pi_{i+1}(EG) \rightarrow \pi_{i+1}(BG) \rightarrow \pi_i(G) \rightarrow \pi_i(EG) \rightarrow \pi_i(BG) \rightarrow \pi_{i-1}(G) \rightarrow \cdots$$

There is a map $EG \rightarrow PBG$ fitting into the diagram

$$(3.1) \quad \begin{array}{ccccc} G & \longrightarrow & EG & \xrightarrow{p} & BG \\ \downarrow & & \downarrow & & \downarrow = \\ \Omega BG & \longrightarrow & PBG & \longrightarrow & BG, \end{array}$$

obtained by taking a contraction $EG \times I \rightarrow EG$ and composing with p . The bottom row is the path-loop fibration of BG . (3.1) induces maps between the long exact sequences in homotopy of the two fibrations, and by the 5-lemma the map $G \rightarrow \Omega BG$ is a (weak) homotopy equivalence.

There is a similar algebraic construction which gives a complex for computing $\text{Tor}^A(R, R)$, where R is a (graded) commutative ring and A is a (graded) augmented R -algebra. Let $\eta : R \rightarrow A$ be the unit and $\epsilon : A \rightarrow R$ the augmentation of A . We write $\bar{A}$ for $\text{coker } \eta$, and let

$$\beta_n^R(A) = \bar{A} \otimes_R \cdots \otimes_R \bar{A}$$

where $\bar{A}$ occurs n times, and $\beta_0^R(A) = R$. Elements of $\beta_*^R(A)$ are often written $[a_1] \cdots [a_n]$. Let $\bar{a} = (-1)^{1+|a|}a$, and define differentials $\partial_n : \beta_n^R(A) \rightarrow \beta_{n-1}^R(A)$ by

$$\begin{aligned} \partial_n([a_1] \cdots [a_n]) &= \epsilon(a_1)[a_2] \cdots [a_n] + \sum_{i=1}^{n-1} [\bar{a}_1] \cdots [\bar{a}_{i-1}] [\bar{a}_i a_{i+1}] \cdots [a_n] \\ &\quad + [\bar{a}_1] \cdots [\bar{a}_{n-1}] \epsilon(a_n). \end{aligned}$$

Now we will assume that A is a flat R -module. $\beta_*^R(A)$ can then be viewed as a flat resolution of R over A , tensored with R , so $H_*(\beta_*^R(A), \partial_*) = \text{Tor}_{**}^A(R, R)$. $\beta_*^R(A)$ is called the (normalized) bar complex.

We get a coproduct $\Delta : \beta_*^R(A) \rightarrow \beta_*^R(A) \otimes \beta_*^R(A)$, where $\otimes$ now means $\otimes_R$, by defining

$$\Delta([a_1] \cdots [a_n]) = \sum_{i=0}^n [a_1] \cdots [a_i] \otimes [a_{i+1}] \cdots [a_n],$$

where $[]$ is interpreted as 1. It is straight-forward to check that this map is a chain map between $\beta_*^R(A)$ and $\beta_*^R(A) \otimes \beta_*^R(A)$, and hence induces a map

$$\text{Tor}^A(R, R) \rightarrow H_*(\beta_*^R(A) \otimes \beta_*^R(A)) \cong \text{Tor}^A(R, R) \otimes \text{Tor}^A(R, R)$$

in homology, at least if $\text{Tor}^A(R, R)$ is flat over R so the Künneth isomorphism holds. This gives $\text{Tor}^A(R, R)$ a coalgebra structure.

The spectral sequence. Let $BG^{(s)}$ be the image of $\coprod_{0 \leq n \leq s} \Delta^n \times G^n$ in BG , and consider the following unrolled exact couple ([Boa99])

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{i} & E_*(BG^{(s)}) & \xrightarrow{i} & E_*(BG^{(s+1)}) & \xrightarrow{i} & \cdots \\
 & & \downarrow j & & \downarrow j & & \\
 & \swarrow k & E_*(BG^{(s)}, BG^{(s-1)}) & \swarrow k & E_*(BG^{(s+1)}, BG^{(s)}) & &
 \end{array}$$

where E is a homology theory. This gives rise to spectral sequence converging strongly to $\operatorname{colim}_s E_*(BG^{(s)}) = E_*(BG)$, with

$$E_{s,t}^1 = E_{s+t}(BG^{(s)}, BG^{(s-1)}) \cong \tilde{E}_{s+t}(BG^{(s)}/BG^{(s-1)})$$

and $d^1 = j \circ k$. Note that $BG^{(s)}/BG^{(s-1)} \cong S^s \wedge G^{\wedge s}$, since all points in $\Delta^s \times G^s$ either lying on the boundary of Δ^s or containing e are identified to a point, so

$$E_{s,t}^1 \cong \tilde{E}_{s+t}(\Sigma^s G^{\wedge s}) \cong \tilde{E}_t(G^{\wedge s}).$$

To go any further we need the following result.

Proposition 3.2. *Suppose E is a (commutative) ring spectrum and X, Y are spectra. If $E_*(X)$ is a flat E_* -module, then the cross-product map*

$$E_*(X) \otimes_{E_*} E_*(Y) \rightarrow E_*(X \wedge Y)$$

is an isomorphism for all Y . The same is true for cohomology if $E^*(X)$ is finitely generated and free as an E^* -module.

Proof. This can be done by thinking of the expressions $E_*(X) \otimes_{E_*} E_*(Y)$ and $E_*(X \wedge Y)$ as functors in X . The map is an isomorphism when $X = S$; one needs to check that these functors satisfy the axioms for a homology theory. Details can be found in [Swi75, 13.75]. $\square$

If we assume the hypothesis of the previous proposition,

$$E_{s,*}^1 \cong \tilde{E}_*(G) \otimes_{E_*} \cdots \otimes_{E_*} \tilde{E}_*(G),$$

the s -fold tensor product. We want to identify the E^1 -term as the bar complex $\beta(E_*(G)) = \beta_*^{E_*}(E_*(G))$. To this end, we examine the compatibility of the differentials coming from the spectral sequence and the resolution; we need to check the

commutativity of the following diagram

$$\begin{array}{ccc}
 E_*(BG^{(s)}, BG^{(s-1)}) & \xrightarrow{d_s^1} & E_*(BG^{(s-1)}, BG^{(s-2)}) \\
 \cong \downarrow & & \cong \downarrow \\
 \tilde{E}_*(BG^{(s)}/BG^{(s-1)}) & & \tilde{E}_*(BG^{(s-1)}/BG^{(s-2)}) \\
 \cong \downarrow & & \cong \downarrow \\
 \tilde{E}_*(\Sigma^s G^{\wedge s}) & & \tilde{E}_*(\Sigma^{s-1} G^{\wedge s-1}) \\
 \cong \uparrow \Sigma^s & & \cong \uparrow \Sigma^{s-1} \\
 \tilde{E}_*(G^{\wedge s}) & & \tilde{E}_*(G^{\wedge s-1}) \\
 \cong \uparrow \wedge & & \cong \uparrow \wedge \\
 \tilde{E}_*(G)^{\otimes s} & \xrightarrow{\partial_s} & \tilde{E}_*(G)^{\otimes s-1} .
 \end{array}$$

This is standard, and will be omitted.

We can now identify the E^2 -term with the coalgebra $\text{Tor}^{E^*(G)}(E_*, E_*)$. If we assume that each E^r -term is flat, then the coproduct on E^2 induces coproducts on E^r for all r , and also on E^∞ . This means we have a spectral sequence of coalgebras, and we want it to converge to $E_*(BG)$ as a coalgebra. Again we need to assume that $E_*(BG)$ is flat to assure a coproduct.

The above calculations yield part 1 of the following theorem.

Theorem 3.3. *Let E be a commutative ring spectrum and G a topological group.*

- (1) *If $E_*(G)$ is a flat E_* -module, there is a strongly convergent spectral sequence of E_* -modules*

$$\text{Tor}_{**}^{E^*(G)}(E_*, E_*) \implies E_*(BG) .$$

If, in addition, $E_*(BG)$ and each E_{**}^r are flat as E_* -modules, then this is a spectral sequence of E_* -coalgebras.

- (2) *If $E^*(G)$ is a finitely generated and free E^* -module and $E_*(G)$ is a projective E_* -module, then there is a spectral sequence of E^* -algebras*

$$\text{Ext}_{E^*(G)}^{**}(E^*, E^*) \implies E^*(BG) .$$

Proof. It remains to prove part (2). From the long exact sequences of the triples $(BG, BG^{(s)}, BG^{(s-1)})$ we get an exact couple

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{i} & E^*(BG, BG^{(s-1)}) & \xrightarrow{i} & E^*(BG, BG^{(s-2)}) & \xrightarrow{i} & \cdots \\
 & & \downarrow j & & \downarrow j & & \\
 & \swarrow k & E^*(BG^{(s)}, BG^{(s-1)}) & \swarrow k & E^*(BG^{(s-1)}, BG^{(s-2)}) & &
 \end{array}$$

Since $\lim_s E^*(BG, BG^{(s)}) = \lim_s^1 E^*(BG, BG^{(s)}) = 0$ by the Milnor sequence for $E^*(BG, BG^{(s)})$, the spectral sequence arising from this exact couple converges strongly to $\operatorname{colim}_s E^*(BG, BG^{(s)}) = E^*(BG)$. As above

$$E_1^{s,*} \cong \tilde{E}^*(G) \otimes_{E^*} \cdots \otimes_{E^*} \tilde{E}^*(G),$$

the s -fold tensor product.

Given any projective $E_*(G)$ -resolution $P_* \rightarrow E_* \rightarrow 0$ of E_* , note that

$$\operatorname{Hom}_{E_*}(P_* \otimes_{E_*(G)} E_*, E_*) \cong \operatorname{Hom}_{E_*(G)}(P_*, E_*)$$

under the map $f_* \mapsto f_* \circ i_*$, where i_* is the obvious map $P_* \rightarrow P_* \otimes_{E_*(G)} E_*$. Thus since $E_*(G)$ is projective, we can use the bar complex to compute $\operatorname{Ext}_{E_*(G)}(E^*, E^*)$ by applying $\operatorname{Hom}_{E_*}(-, E_*)$. Once we have identified the d_1 differentials as the duals of the differentials in the bar complex, it follows that

$$E_2^{**} \cong \operatorname{Ext}_{E_*(G)}^{**}(E^*, E^*).$$

□

4. RESTRICTIONS ON $K(n)$ -COMPACT SPHERES

The classical argument. Here we recall the argument in [AA66] regarding which spheres are H -spaces. Suppose S^{q-1} is an H -space, $q > 1$. We first exclude the even dimensional spheres. The integral cohomology $H^*(S^{q-1})$ is an exterior algebra $\Lambda(y)$ on a generator in degree $q-1$, and we calculate

$$0 = \Delta(y^2) = (y \otimes 1 + 1 \otimes y)^2 = (1 + (-1)^{|y|})y \otimes y,$$

where $\Delta : \Lambda(y) \rightarrow \Lambda(y) \otimes \Lambda(y)$ is the coproduct. Since $\Lambda(y) \otimes \Lambda(y)$ is free, $|y|$ must be odd, i.e., q is even. Write $q = 2m$.

Recall that the Hopf construction on a map $m : X \times Y \rightarrow Z$ is the map $X * Y \rightarrow \Sigma Z$ defined by $[x, y, t] \mapsto [m(x, y), t]$. We obtain a map $f : S^{4m-1} \rightarrow S^{2m}$ by the Hopf construction on the multiplication map $S^{2m-1} \times S^{2m-1} \rightarrow S^{2m-1}$. The mapping cone C_f is a CW complex obtained by attaching a $4m$ -cell to S^{2m} . We have the short exact sequence

$$0 \rightarrow \tilde{K}^0(C_f/S^{2m}) \rightarrow \tilde{K}^0(C_f) \rightarrow \tilde{K}^0(S^{2m}) \rightarrow 0.$$

Let α be the image in $\tilde{K}^0(C_f)$ of a generator in $\tilde{K}^0(C_f/S^{2m}) = \tilde{K}^0(S^{4m})$, and let β map to a generator in $\tilde{K}^0(S^{2m})$. The cup square β^2 maps to 0 in $\tilde{K}^0(S^{2m})$, so $\beta^2 = H(f)\alpha$ for an integer $H(f)$. $H(f)$ is well-defined up to sign, and depends only on the homotopy class of f ; $H(f)$ is called the Hopf invariant of f . For a map $f : S^{4m-1} \rightarrow S^{2m}$ arising from a H -space operation, the homotopy unitality implies that $H(f) = 1$, see e.g. [Ste62, 1.5].

The Adams operations ψ^k , k a nonzero integer, are ring operations defined on $K^0(-)$ satisfying the following properties

- (1) $\psi^k(L) = L^{\otimes k}$, where L is a line bundle.
- (2) $\psi^p(z) \equiv z^p \pmod{p}$, for a prime p .
- (3) ψ^k and ψ^l commute.
- (4) ψ^k on $\tilde{K}^0(S^{2m})$ is multiplication by k^m .

Now we can use the operations ψ^2 and ψ^3 to prove that $m = 1, 2$ or 4 . Since α is the image of an element in $\tilde{K}^0(S^{4m})$, $\psi^k(\alpha) = k^{2m}\alpha$ by naturality. By similar reasoning, $\psi^k(\beta) = k^m\beta + s_k\alpha$ for some integer s_k , and

$$\begin{aligned}\psi^2\psi^3(\beta) &= \psi^2(3^m\beta + s_3\alpha) = 3^m2^m\beta + (3^ms_2 + 2^{2m}s_3)\alpha \\ \psi^3\psi^2(\beta) &= \psi^3(2^m\beta + s_2\alpha) = 2^m3^m\beta + (2^ms_3 + 3^{2m}s_2)\alpha.\end{aligned}$$

These expressions must be equal, so we obtain the following integer equation

$$(4.1) \quad 3^ms_2 + 2^{2m}s_3 = 2^ms_3 + 3^{2m}s_2,$$

or equivalently,

$$2^m(2^m - 1)s_3 = 3^m(3^m - 1)s_2.$$

We see that 2^m must divide $(3^m - 1)s_2$. But $H(f)\alpha = \beta^2 \equiv_2 \psi^2(\beta) = 2^m\beta + s_2\alpha$, so $s_2 \equiv H(f) \pmod{2}$, i.e., s_2 is odd. Thus 2^m divides $3^m - 1$, and this implies that $m = 1, 2$ or 4 , by a special case of Lemma 4.14.

The above argument can be done with 2-adic K -theory instead of integral K -theory, since only the information of the Hopf invariant modulo powers of 2 is needed. There are 2-adic Adams operations with the same properties as the integral operations, except that property (2) above now only holds for the prime 2. Note that now the Hopf invariant $H(f)$ lies in $\mathbb{Z}_2$, and (4.1) becomes a 2-adic equation; other than that, the argument is exactly the same.

Operations. Next we introduce the operations on $E_n^0(-)$ which will coincide with the p -adic Adams operations on $E_1^0(-)$.

Proposition 4.2. *There exist stable ring operations ψ^k , $k \in \mathbb{Z}_p^\times$, and unstable ring operations ψ^p , acting on $E_n^0(X)$. On $E_n^0(\mathbb{C}P^\infty) \cong E_n^0[[x]]$ these are given by $x \mapsto [k]_\Gamma(x)$ and $x \mapsto [p]_\Gamma(x)$, respectively. Furthermore, ψ^k and ψ^p commute.*

Proof. For a p -adic unit k , the k -series $[k]_\Gamma$ corresponds to an automorphism $E_n \xrightarrow{\kappa} E_n$ of E_∞ ring spectra by Theorem 2.5 and the discussion following it. An element of $E_n^0(X)$ is represented by a map $\Sigma^\infty X_+ \xrightarrow{\alpha} E_n$, and we define ψ^k to be the map that sends this element to

$$\Sigma^\infty X_+ \xrightarrow{\alpha} E_n \xrightarrow{\kappa} E_n.$$

That this is a ring operation is trivial.

In [And95] operations ψ^{p^i} are constructed, and the special case $i = 1$ gives the ψ^p we want. Here is a brief recollection of the construction. Since E_n is an E_∞ ring spectrum, it has power operations

$$P_p : E_n^0(X) \rightarrow E_n^0(D_p X),$$

where D_p is the p -fold extended power. $D_p \Sigma^\infty X_+$ is the same as $\Sigma^\infty(-)_+$ applied to the Borel construction $E\Sigma_p \times_{\Sigma_p} X$, which we also write as $D_p X$. P_p sends $\Sigma^\infty X_+ \xrightarrow{\alpha} E_n$ to the element represented by

$$\Sigma^\infty D_p X_+ = D_p \Sigma^\infty X_+ \xrightarrow{D_p \alpha} D_p E_n \xrightarrow{\xi_p} E_n,$$

where ξ_p is the E_∞ structure map. Let $\Delta : B\Sigma_p \times X \rightarrow D_p X$ be the diagonal map sending (σ, x) to $(\sigma, x, \dots, x)$. There is a generalized character map defined as the composition

$$E_n^0(D_p X) \xrightarrow{\Delta^*} E_n^0(B\Sigma_p \times X) \xleftarrow{\cong} E_n^0(B\Sigma_p) \otimes_{E_n^0} E_n^0(X) \xrightarrow{\chi \otimes 1} L_1 \otimes_{E_n^0} E_n^0(X)$$

where χ is a generalized character as in [HKR00, 6] and L_1 is a ring extension of E_n^0 . The second map is an isomorphism since $E_n^0(B\Sigma_p)$ is finitely generated and free. Now Ando uses the Galois group $\text{Gal}(L_1/E_n^0)$ to show that the composition

$$E_n^0(X) \xrightarrow{P_p} E_n^0(D_p X) \rightarrow L_1 \otimes_{E_n^0} E_n^0(X)$$

actually takes values in $E_n^0(X)$; this composition is the operation ψ^p .

For the commutativity of the operations, observe that since κ is an E_∞ automorphism, κ^* commutes with the E_∞ structure maps and therefore with P_p . Clearly κ^* commutes with Δ^* . $\square$

Corollary 4.3. *On $\tilde{E}_n^0(S^{2m})$ the operations ψ^k and ψ^p are given by multiplication by k^m and p^m .*

Proof. The complex orientation $x \in \tilde{E}_n^0(\mathbb{C}P^\infty)$ restricts to a generator $i^*x \in \tilde{E}_n^0(S^2)$, and the commutative diagram

$$\begin{array}{ccc} \tilde{E}_n^0(S^2) & \xleftarrow{i^*} & \tilde{E}_n^0(\mathbb{C}P^\infty) \\ \downarrow \psi^k & & \downarrow \psi^k \\ \tilde{E}_n^0(S^2) & \xleftarrow{i^*} & \tilde{E}_n^0(\mathbb{C}P^\infty) \end{array}$$

says that ψ^k is multiplication by k on $\tilde{E}_n^0(S^2)$. Consider now the following diagram:

$$\begin{array}{ccccc} \tilde{E}_n^0(S^{2m}) & \xrightarrow{\cong} & \tilde{E}_n^0(S^2 \wedge \dots \wedge S^2) & \xleftarrow{\cong} & \tilde{E}_n^0(S^2) \otimes \dots \otimes \tilde{E}_n^0(S^2) \\ \downarrow \psi^k & & \downarrow \psi^k & & \downarrow \psi^k \otimes \dots \otimes \psi^k \\ \tilde{E}_n^0(S^{2m}) & \xrightarrow{\cong} & \tilde{E}_n^0(S^2 \wedge \dots \wedge S^2) & \xleftarrow{\cong} & \tilde{E}_n^0(S^2) \otimes \dots \otimes \tilde{E}_n^0(S^2) \end{array}$$

The right square commutes since ψ^k is a ring operation, and the cross product map $\wedge$ is an isomorphism by Proposition 3.2. Starting at the top left term $\tilde{E}_n^0(S^{2m})$ and going around the diagram, we see that a generator $y \in \tilde{E}_n^0(S^{2m})$ maps to $k^m y$. The same argument applies to ψ^p . $\square$

$K(n)$ -compact spheres. Now we want to do an argument in the $K(n)$ -local category similar to Adams and Atiyah's, but instead of H -spaces we consider topological groups, or equivalently, loop spaces.

Definition 4.4. A topological group G is *($K(n)$ -locally) stably dualizable* if the $K(n)_*$ -module $K(n)_*(G)$ is finitely generated.

The reason for the above definition comes from the fact that G is stably dualizable if and only if $L_{K(n)}\Sigma^\infty G_+$ is dualizable in the $K(n)$ -local stable category ([HS99, 8.6]). See [Rog05] for a treatment of stably dualizable groups. Note that $K(n)_*(G)$ is finitely generated over $K(n)_*(G)$ if and only if $K_{n*}(G)$ is finitely generated over K_{n*} , by the definition of K_n . These statements imply that $K(n)^*(G)$, $K_n^*(G)$ and $E_n^*(G)$ are finitely generated over their respective coefficient rings, by (2.1), (2.2) and Theorem 2.3.

Proposition 4.5. $E_n^*(G)$ is a Hopf algebra when G is a stably dualizable group and $E_n^*(G)$ is free as an E_n^* -module.

Proof. Since $E_n^*(G)$ is finitely generated and free, Proposition 3.2 tells us that the coproduct can be defined in the usual way to give a Hopf algebra structure. $\square$

Dwyer and Wilkerson define p -compact groups in [DW94] as p -local homotopical variants of compact Lie groups, more precisely they are groups such that $H_*(G; \mathbb{F}_p)$ is finitely generated over $\mathbb{F}_p$ and BG is a p -complete space. In the same manner we define objects which contain the information $K(n)$ sees of a compact Lie group.

Definition 4.6. A topological group G is $K(n)$ -compact if G is stably dualizable and BG is a $K(n)$ -local space.

When X is E -local for any spectrum E , then ΩX is E -local, since

$$\begin{array}{ccc} [\Sigma B, X] & \xrightarrow{(\Sigma f)^*} & [\Sigma A, X] \\ \cong \downarrow & & \cong \downarrow \\ [B, \Omega X] & \xrightarrow{f^*} & [A, \Omega X] \end{array}$$

commutes, and the top map is a bijection when $f : A \rightarrow B$ is an E -equivalence. In our case, this implies that a $K(n)$ -compact group is $K(n)$ -local.

Given a compact Lie group G , a p -compact group can be constructed as $\Omega(BG)_p^\wedge$. This group obviously has p -compact classifying space, and its mod p cohomology can be computed with the Eilenberg-Moore spectral sequence. In this case the spectral sequence takes the form (see the beginning of section 5)

$$\mathrm{Tor}_{**}^{H^*((BG)_p^\wedge; \mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \implies H^*(\Omega(BG)_p^\wedge; \mathbb{F}_p).$$

The canonical map $BG \rightarrow BG_p^\wedge$ induces an isomorphism

$$\mathrm{Tor}_{**}^{H^*((BG)_p^\wedge; \mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathrm{Tor}_{**}^{H^*(BG; \mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p),$$

which in turn induces an isomorphism of the spectral sequences with these two Tor algebras as E_2 -terms, converging to $H^*(\Omega BG; \mathbb{F}_p)$ and $H^*(\Omega(BG)_p^\wedge; \mathbb{F}_p)$, respectively. The strong convergence of these spectral sequences, and the fact that $\Omega BG \simeq G$, now tells us that the induced map $H^*(\Omega(BG)_p^\wedge; \mathbb{F}_p) \rightarrow H^*(G; \mathbb{F}_p)$ is an isomorphism and that the group $\Omega(BG)_p^\wedge$ is indeed p -compact. In the same way p -compact groups G may give examples of $K(n)$ -compact groups, as $\Omega L_{K(n)} BG$.

We now focus our attention on spheres.

Definition 4.7. A $K(n)$ -local sphere is a space X that is homotopy equivalent to $L_{K(n)} S^k$ for some k . k will be called the dimension of X .

The dimension of a $K(n)$ -local sphere is well defined by the following lemma.

Lemma 4.8. $L_{K(n)} S^k \simeq L_{K(n)} S^l$ if and only if $k = l$.

Proof. Assume $L_{K(n)}S^k \simeq L_{K(n)}S^l$. Let ψ^{1+p} be the operation associated to $1+p \in \mathbb{Z}_p^\times$, and consider the following commutative diagram

$$\begin{array}{ccccccc} \tilde{E}_n^0(S^k) & \xleftarrow[\cong]{\eta^*} & \tilde{E}_n^0(L_{K(n)}S^k) & \xrightarrow[\cong]{} & \tilde{E}_n^0(L_{K(n)}S^l) & \xrightarrow[\cong]{\eta^*} & \tilde{E}_n^0(S^l) \\ \downarrow \psi^{1+p} & & \downarrow \psi^{1+p} & & \downarrow \psi^{1+p} & & \downarrow \psi^{1+p} \\ \tilde{E}_n^0(S^k) & \xleftarrow[\cong]{\eta^*} & \tilde{E}_n^0(L_{K(n)}S^k) & \xrightarrow[\cong]{} & \tilde{E}_n^0(L_{K(n)}S^l) & \xrightarrow[\cong]{\eta^*} & \tilde{E}_n^0(S^l). \end{array}$$

Assume k and l have the same parity; if not, the additive structure of $\tilde{E}_n^0(S^k)$ and $\tilde{E}_n^0(S^l)$ differ, and we are done. If k and l are even, the operations on $\tilde{E}_n^0(S^k)$ and $\tilde{E}_n^0(S^l)$ are given by $\psi^{1+p}(y_k) = (1+p)^{k/2}y_k$ and $\psi^{1+p}(y_l) = (1+p)^{l/2}y_l$, where y_k and y_l are the generators of $\tilde{E}_n^0(S^k)$ and $\tilde{E}_n^0(S^l)$. Starting at $\tilde{E}_n^0(S^k)$ and going all the way around the perimeter of the diagram, it is apparent that y_k maps to $(1+p)^{l/2}y_k$ as well, and that $(1+p)^{k/2} = (1+p)^{l/2}$. This implies that $k = l$. If k and l are odd, apply the same argument to the following diagram

$$\begin{array}{ccccccc} \tilde{E}_n^0(\Sigma S^k) & \xleftarrow[\cong]{(\Sigma\eta)^*} & \tilde{E}_n^0(\Sigma L_{K(n)}S^k) & \xrightarrow[\cong]{} & \tilde{E}_n^0(\Sigma L_{K(n)}S^l) & \xrightarrow[\cong]{(\Sigma\eta)^*} & \tilde{E}_n^0(\Sigma S^l) \\ \downarrow \psi^{1+p} & & \downarrow \psi^{1+p} & & \downarrow \psi^{1+p} & & \downarrow \psi^{1+p} \\ \tilde{E}_n^0(\Sigma S^k) & \xleftarrow[\cong]{(\Sigma\eta)^*} & \tilde{E}_n^0(\Sigma L_{K(n)}S^k) & \xrightarrow[\cong]{} & \tilde{E}_n^0(\Sigma L_{K(n)}S^l) & \xrightarrow[\cong]{(\Sigma\eta)^*} & \tilde{E}_n^0(\Sigma S^l) \end{array}$$

to conclude that $k = l$. $\square$

Definition 4.9. A $K(n)$ -compact group that is also a $K(n)$ -local sphere will be called a $K(n)$ -compact sphere.

Proposition 4.10. Let H be a Hopf algebra over E_n^* which is free of rank 2, and assume the product on its augmentation ideal $\ker(\epsilon)$ is trivial. Then H is a primitively generated exterior algebra over E_n^* on a generator of odd degree.

Proof. Let the generators of H be 1 and α with $\epsilon(\alpha) = 0$, where $\epsilon : H \rightarrow E_n^*$ is the augmentation map. Let $\Delta : H \rightarrow H \otimes H$ be the coproduct in H , it has the form $\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + c \cdot \alpha \otimes \alpha$, where $c \in E_n^*$. Then from the fact that Δ is an algebra homomorphism,

$$0 = \Delta(\alpha^2) = \Delta(\alpha)^2 = (\alpha \otimes 1 + 1 \otimes \alpha + c \cdot \alpha \otimes \alpha)^2 = (1 + (-1)^{|\alpha|}) \cdot \alpha \otimes \alpha$$

Equating coefficients of $\alpha \otimes \alpha$ we see that $|\alpha|$ is odd, and since $|c| = -|\alpha|$ from the coproduct formula, c must be 0 since E_n^* is concentrated in even degrees. $\square$

Proposition 4.11. If G is a $K(n)$ -compact sphere of positive dimension, then the dimension of G is odd.

Proof. $K(n)^*(G)$ and $K_n^*(G)$ are free of rank 2, so $\tilde{K}_n^0(G)$ is free of rank 1 and from Theorem 2.3 it follows that the same is true for $\tilde{E}_n^0(G)$. Thus $E_n^*(G)$ is free of rank 2. According to Proposition 4.5 $E_n^*(G)$ is a Hopf algebra. Also, the product on $E_n^*(G)$ is trivial since $E_n^*(G) \cong E_n^*(S^k)$ for $k > 0$, and suspensions in general have trivial cup product structure. Proposition 4.10 applies to give the result. $\square$

Let us use the bar spectral sequence to calculate the E_n -homology and cohomology of BG , in the case where G is a $K(n)$ -compact sphere. We know that $E_{n*}(G) = \Lambda_{E_{n*}}(b)$, where $|b| = 2m - 1$, so the E^2 -term takes the form $\text{Tor}_{**}^{\Lambda(b)}(E_{n*}, E_{n*})$. $\bar{\Lambda}(b) \cong E_{n*}\{b\}$, and the bar complex has $\beta_n(\Lambda(b)) = \bar{\Lambda}(b) \otimes \cdots \otimes \bar{\Lambda}(b)$, a free module on a generator $b \otimes \cdots \otimes b$ in bidegree $(n, n(2m - 1))$. The coproduct on $\beta_*(\Lambda(b))$ coincides with the coproduct on a divided power algebra, so $\beta_*(\Lambda(b)) \cong \Gamma(\sigma b)$ as coalgebras, where σb is a generator in bidegree $(1, 2m - 1)$. Since $b^2 = 0$ in $\Lambda(b)$, the differential in the bar complex is trivial, and

$$\text{Tor}_{**}^{\Lambda(b)}(E_{n*}, E_{n*}) \cong \Gamma(\sigma b).$$

Now we can calculate $\text{Ext}_{\Lambda(b)}^{**}(E_n^*, E_n^*)$ as the dual of $\Gamma(\sigma b)$, denoted by $\Gamma(\sigma b)^*$. Let $\{1, \sigma y, (\sigma y)^2, \dots\}$ be the dual basis of $\{1, \sigma b, \gamma_2(\sigma b), \dots\}$. The product on $\Gamma(\sigma b)^*$ is given by the composition

$$\Gamma(\sigma b)^* \otimes \Gamma(\sigma b)^* \xrightarrow{\cong} (\Gamma(\sigma b) \otimes \Gamma(\sigma b))^* \xrightarrow{\Delta^*} \Gamma(\sigma b)^*.$$

Under the first map, $(\sigma y)^i \otimes (\sigma y)^j$ goes to $(\gamma_i(\sigma b) \otimes \gamma_j(\sigma b))^*$, and precomposing with Δ gives $(\sigma y)^{i+j}$. We conclude that

$$\text{Ext}_{\Lambda(b)}^{**}(E_n^*, E_n^*) \cong E_n^*[\sigma y],$$

where σy has bidegree $(1, 2m - 1)$.

This case satisfies the hypothesis of Theorem 3.3, so we have spectral sequences converging to $E_{n*}(BG)$ and $E_n^*(BG)$. Since the E^2 - and E_2 -terms are free coalgebras / algebras on generators in even degrees, the spectral sequences collapse and we obtain

$$\begin{aligned} E_{n*}(BG) &\cong \Gamma(\sigma b) \\ E_n^*(BG) &\cong E_n^*[[\sigma y]]. \end{aligned}$$

If we truncate the bar filtration at the s 'th level, the spectral sequences calculate the (co)homology of $BG^{(s)}$ as

$$\begin{aligned} E_{n*}(BG^{(s)}) &\cong E_{n*}\{\gamma_i(\sigma b) \mid 0 \leq i \leq s\} \\ E_n^*(BG^{(s)}) &\cong E_n^*[\sigma y]/(\sigma y)^{s+1} \end{aligned}$$

We need to know how the operations ψ^k and ψ^p in Proposition 4.1 operate on $E_n^0(BG)$, so now we put the generator σy in degree 0. It seems like it is necessary to assume existence of a maximal torus, i.e., a map

$$L_{K(n)}\mathbb{C}P^\infty = L_{K(n)}BS^1 \xrightarrow{t} L_{K(n)}BG$$

such that

$$E_n^*(BG) \xrightarrow{t^*} E_n^*(BS^1) \cong E_n^*[[x]]$$

is injective and $t^*(\sigma y) = x^m$. In this case we can consider $E_n^*(BG)$ a subalgebra of $E_n^*(\mathbb{C}P^\infty)$. A quick look at the diagram

$$\begin{array}{ccc} E_n^0(BG) & \xrightarrow{t^*} & E_n^0(\mathbb{C}P^\infty) \\ \downarrow \psi & & \downarrow \psi \\ E_n^0(BG) & \xrightarrow{t^*} & E_n^0(\mathbb{C}P^\infty) \end{array}$$

tells us that

$$(4.12) \quad \begin{aligned} \psi^k(\sigma y) &= (\psi^k(x))^m \\ \psi^p(\sigma y) &= (\psi^p(x))^m. \end{aligned}$$

The existence of such a map $t : BS^1 \rightarrow BG$ holds in the p -complete case (see [DW94, 8]) but the construction involves use of the Sullivan conjecture, which is special to the p -complete setting. A corresponding $K(n)$ -local result is not known.

Let $\text{ord}_p m$ be the p -order of m , i.e. the highest power of p that divides m . More generally let $\text{ord}_{\mathfrak{a}} m$ be the maximal k such that $m \in \mathfrak{a}^k$. For the next theorem we need some calculations of the p -order of factorials.

Lemma 4.13. *Let $m = \sum a_i p^i$, where $0 \leq a_i \leq p-1$ for all i . Then*

$$\text{ord}_p m! = \frac{m - \sum a_i}{p-1}.$$

Proof. First note that

$$\text{ord}_p m! = \sum_{k \geq 1} \left\lfloor \frac{m}{p^k} \right\rfloor,$$

since this counts how many numbers between 1 and m are divisible by p , by p^2 , etc. Now

$$\sum_{k \geq 1} \left\lfloor \frac{m}{p^k} \right\rfloor = \sum_{k \geq 1} \left\lfloor \frac{\sum a_i p^i}{p^k} \right\rfloor = \sum_{k \geq 1} \sum_{i \geq k} a_i p^{i-k} = \sum_{1 \leq k \leq i} a_i p^{i-k},$$

but on the other hand

$$\frac{m - \sum a_i}{p-1} = \frac{\sum_{i \geq 1} a_i (p^i - 1)}{p-1} = \sum_{i \geq 1} a_i \frac{p^i - 1}{p-1} = \sum_{i \geq 1} a_i \sum_{j=0}^{i-1} p^j.$$

Change j to $i-k$ to get

$$\sum_{i \geq 1} \sum_{j=0}^{i-1} a_i p^j = \sum_{i \geq 1} \sum_{k=1}^i a_i p^{i-k} = \sum_{1 \leq k \leq i} a_i p^{i-k}.$$

□

Lemma 4.14. *Let m be a positive integer. Then*

$$\text{ord}_2(3^m - 1) = \begin{cases} 1 & m \text{ odd} \\ \text{ord}_2 m + 2 & m \text{ even} \end{cases},$$

and for $p \neq 2$,

$$\text{ord}_p(g^m - 1) = \begin{cases} 0 & p-1 \nmid m \\ \text{ord}_p m + 1 & p-1 \mid m \end{cases},$$

where g is a topological generator of $\mathbb{Z}_p^\times$.

Proof. We have that $\text{ord}_2(3^m - 1) \geq 1$. When $\text{ord}_2(3^m - 1) \geq 2$, the 2-order is equal to a maximal e such that $2^e \mid 3^m - 1$, i.e., $3^m \equiv 1 \pmod{2^e}$. There is an isomorphism $(\mathbb{Z}/2^e)^\times \cong \mathbb{Z}/2^{e-2} \times \mathbb{Z}/2$ defined by the exponential and Teichmüller maps, such that $3 \mapsto (1, 1)$, so $3^m \equiv 1 \pmod{2^e}$ if and only if $2^{e-2} \mid m$ and $2 \mid m$. Thus the first result follows.

There is an isomorphism $(\mathbb{Z}/p^e)^\times \cong \mathbb{Z}/p^{e-1}(p-1)$ such that $g \mapsto 1$, so $g^m \equiv 1 \pmod{p^e}$ is equivalent to $p^{e-1}(p-1) \mid m$, i.e., $p^{e-1} \mid m$ and $p-1 \mid m$. When $\text{ord}_p(g^m - 1) \geq 1$ is the maximal e such that $p^e \mid g^m - 1$, this is equivalent to the maximal e such that $p^{e-1} \mid m$ and $p-1 \mid m$, and the second result follows. $\square$

Theorem 4.15. *Let G be a $K(n)$ -compact sphere. Then the dimension of G , written as $2m-1$, must satisfy the following:
for $p=2$,*

$$m \leq \begin{cases} 5 \cdot 2^{n-1} - n - 3 & m \text{ odd} \\ (2^n - 1)(\text{ord}_2 m + 3) - n & m \text{ even} \end{cases},$$

and for $p \neq 2$,

$$m \leq (m, p-1) \left(\frac{S-n}{p-1} + S(\text{ord}_p m + 1) \right),$$

where $S = (p^n - 1)/(p - 1)$ and $(m, p-1)$ is the greatest common divisor of m and $p-1$.

Proof. Let x be a complex orientation for E_n , and let ψ^k, ψ^p be the operations in Proposition 4.1. Write

$$\begin{aligned} \psi^k(x) &= [k]_\Gamma(x) = kx + r_2x^2 + r_3x^3 + \cdots \\ \psi^p(x) &= [p]_\Gamma(x) = px + t_2x^2 + t_3x^3 + \cdots \end{aligned}$$

for ψ^k and ψ^p acting on $E_n^0(\mathbb{C}P^\infty) \cong E_n^0[[x]]$. Since Γ is a lift of Γ_n , $t_q \equiv 1 \pmod{\mathfrak{m}}$ and $t_i \in \mathfrak{m}$ when $i \neq q$, where $q = p^n$. We now consider the operations acting on $E_n^0(BG^q)$. For convenience, we write y instead of σy . By (4.12),

$$\begin{aligned} \psi^k(y) &= (\psi^k(x))^m = (kx + r_2x^2 + \cdots)^m \\ &= k^m y + \rho_2 y^2 + \cdots + \rho_q y^q \\ \psi^p(y) &= (\psi^p(x))^m = (px + t_2x^2 + \cdots)^m \\ &= p^m y + \tau_2 y^2 + \cdots + \tau_q y^q, \end{aligned}$$

where $\tau_q \in t_q^m + \mathfrak{m}$, so that $\tau_q \equiv 1 \pmod{\mathfrak{m}}$, and $\tau_i \in \mathfrak{m}$ for $1 < i < q$.

Now we want to use that $\psi^p \psi^k = \psi^k \psi^p$ to find bounds on m .

$$\begin{aligned} \psi^p(\psi^k(y)) &= \psi^p(k^m y + \rho_2 y^2 + \cdots + \rho_q y^q) \\ &= k^m (p^m y + \tau_2 y^2 + \cdots + \tau_q y^q) + \cdots + \rho_q (p^m y + \tau_2 y^2 + \cdots + \tau_q y^q)^q \\ \psi^k(\psi^p(y)) &= \psi^k(p^m y + \tau_2 y^2 + \cdots + \tau_q y^q) \\ &= p^m (k^m y + \rho_2 y^2 + \cdots + \rho_q y^q) + \cdots + \tau_q (k^m y + \rho_2 y^2 + \cdots + \rho_q y^q)^q \end{aligned}$$

Since $\{1, y, y^2, \dots, y^q\}$ is an additive basis for $E_n^0(BG^q)$, these two expressions give the following equations

$$\begin{aligned}
k^m p^m &= p^m k^m \\
k^m \tau_2 + \rho_2 p^{2m} &= p^m \rho_2 + \tau_2 k^{2m} \\
k^m \tau_3 + 2\rho_2 \tau_2 p^m + \rho_3 p^{3m} &= p^m \rho_3 + 2\tau_2 \rho_2 k^m + \tau_3 k^{3m} \\
&\vdots \\
k^m \tau_i + \rho_2 a_2 + \dots + \rho_{i-1} a_{i-1} + \rho_i p^{im} &= p^m \rho_i + \tau_2(\dots) + \dots + \tau_{i-1}(\dots) + \tau_i k^{im} \\
&\vdots \\
k^m \tau_q + \rho_2 a_2 + \dots + \rho_{q-1} a_{q-1} + \rho_q p^{qm} &= p^m \rho_q + \tau_2(\dots) + \dots + \tau_{q-1}(\dots) + \tau_q k^{qm},
\end{aligned}$$

where the a_i 's above are expressions which lie in the ideal $(p^m, \tau_1, \dots, \tau_q)$. We pick the i 'th equation and reorganize the terms to get

$$\begin{aligned}
\tau_i k^m (k^{(i-1)m} - 1) &= \\
\rho_i p^m (p^{(i-1)m} - 1) + \rho_2 a_2 + \dots + \rho_{i-1} a_{i-1} - \tau_2(\dots) - \dots - \tau_{i-1}(\dots).
\end{aligned}$$

We now take the $\mathfrak{m}$ -order of this and end up with

$$\text{ord}_{\mathfrak{m}} \tau_i + \text{ord}_{\mathfrak{m}} k^m + \text{ord}_{\mathfrak{m}} (k^{(i-1)m} - 1) \geq \min\{m, \text{ord}_{\mathfrak{m}} \tau_2, \dots, \text{ord}_{\mathfrak{m}} \tau_{i-1}\}.$$

If we choose $k = g$, where g is prime to p , then $\text{ord}_{\mathfrak{m}} k^m = \text{ord}_{\mathfrak{p}} g^m = 0$ and

$$\text{ord}_{\mathfrak{m}} \tau_i \geq \min\{m, \text{ord}_{\mathfrak{m}} \tau_2, \dots, \text{ord}_{\mathfrak{m}} \tau_{i-1}\} - \text{ord}_{\mathfrak{p}} (g^{(i-1)m} - 1).$$

By induction on i ,

$$\text{ord}_{\mathfrak{m}} \tau_i \geq m - \sum_{j=1}^{i-1} \text{ord}_{\mathfrak{p}} (g^{jm} - 1).$$

We want to look closer at $\sum_{j=1}^{q-1} \text{ord}_{\mathfrak{p}} (g^{jm} - 1)$, and we handle the case $p = 2$ first. Let $g = 3$. It follows from Lemma 4.14 that

$$\sum_{j=1}^{q-1} \text{ord}_2 (3^{jm} - 1) = \sum_{\substack{j=1 \\ 2 \nmid jm}}^{q-1} (\text{ord}_2 jm + 2) + \sum_{\substack{j=1 \\ 2 \nmid jm}}^{q-1} 1.$$

When m is odd, then $2 \mid jm$ if and only if $2 \mid j$, and

$$\begin{aligned}
 \sum_{j=1}^{q-1} \text{ord}_2(3^{jm} - 1) &= \sum_{\substack{j=1 \\ 2 \mid j}}^{q-1} (\text{ord}_2 jm + 2) + \sum_{\substack{j=1 \\ 2 \nmid j}}^{q-1} 1 \\
 &= \sum_{j=1}^{(q-2)/2} (\text{ord}_2 2jm + 2) + \frac{q}{2} \\
 &= \sum_{j=1}^{(q-2)/2} (\text{ord}_2 j + 3) + \frac{q}{2} \\
 &= \text{ord}_2\left(\frac{q-2}{2}\right)! + 3\frac{q-2}{2} + \frac{q}{2} \\
 &= \text{ord}_2(2^{n-1} - 1)! + 3(2^{n-1} - 1) + 2^{n-1} \\
 &= 2^{n-1} - n + 3(2^{n-1} - 1) + 2^{n-1} \\
 &= 5 \cdot 2^{n-1} - n - 3.
 \end{aligned}$$

Here we used the result from Lemma 4.13. For m even we get

$$\begin{aligned}
 \sum_{j=1}^{q-1} \text{ord}_2(3^{jm} - 1) &= \sum_{j=1}^{q-1} (\text{ord}_2 jm + 2) \\
 &= \text{ord}_2(q-1)! + (q-1)(\text{ord}_2 m + 2) \\
 &= 2^n - n - 1 + (2^n - 1)(\text{ord}_2 m + 2) \\
 &= (2^n - 1)(\text{ord}_2 m + 3) - n.
 \end{aligned}$$

Now suppose $p \neq 2$, and choose g to be a topological generator for $\mathbb{Z}_p^\times$, i.e., g generates a dense subgroup of $\mathbb{Z}_p^\times$. From Lemma 4.14, the sum now simplifies to

$$\sum_{j=1}^{q-1} \text{ord}_p(g^{jm} - 1) = \sum_{\substack{j=1 \\ p-1 \mid jm}}^{q-1} \text{ord}_p jm + 1.$$

If we let $a = (p-1)/(m, p-1)$, then $p-1 \mid jm$ if and only if $a \mid j$. Also let

$S = (q - 1)/(p - 1) = \sum_{i=0}^{n-1} p^i$ and $N = (q - 1)/a = (m, p - 1)S$.

$$\begin{aligned}
\sum_{p-1 \mid jm} (\text{ord}_p jm + 1) &= \sum_{a \mid j} (\text{ord}_p jm + 1) \\
&= \sum_{l=1}^N (\text{ord}_p alm + 1) \\
&= \text{ord}_p N! + \sum_{l=1}^N (\text{ord}_p a + \text{ord}_p m) + N \\
&= \text{ord}_p N! + N \cdot \text{ord}_p m + N \\
&= \text{ord}_p \left((m, p - 1) \sum_{i=0}^{n-1} p^i \right)! + N \cdot \text{ord}_p m + N \\
&= \frac{(m, p - 1)S - (m, p - 1)n}{p - 1} + N(\text{ord}_p m + 1) \\
&= (m, p - 1) \left(\frac{S - n}{p - 1} + S(\text{ord}_p m + 1) \right),
\end{aligned}$$

and the stated estimate for m follows. $\square$

Corollary 4.16. *For fixed p and n , there are only finitely many $K(n)$ -compact spheres.*

Proof. Since $(m, p - 1) \leq p - 1$ and $\text{ord}_p m \leq \log_p m$, there are only a finite number of possible m in Theorem 4.15. $\square$

5. THE COBAR SPECTRAL SEQUENCE

In this section we want to construct a spectral sequence that is sort of dual to the bar spectral sequence used above. For singular (co)homology this is a well known spectral sequence with nice convergence properties. More precisely, given a fiber square

$$\begin{array}{ccc}
E & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & B
\end{array}$$

with B e.g. simply-connected, there is a strongly convergent spectral sequence

$$\text{Tor}_*^{H^*(B; R)}(H^*(X; R), H^*(Y; R)) \implies H^*(E; R)$$

called the Eilenberg-Moore spectral sequence. See e.g. [Dwy74]. We want to set up a $K(n)$ version of this to calculate $K(n)^*(\Omega X)$, so we restrict to the fiber square

$$\begin{array}{ccc}
\Omega X & \longrightarrow & PX \\
\downarrow & & \downarrow \\
* & \longrightarrow & X,
\end{array}$$

i.e., the path-loop fibration $\Omega X \rightarrow PX \rightarrow X$.

The cobar construction. First we need a dual version of the bar construction called the cobar construction. Let X be a pointed space, and let $CX^\bullet$ be the cosimplicial space $CX^n = X \times \cdots \times X$ (n times) with coface and codegeneracy maps

$$d^i(x_1, \dots, x_n) = \begin{cases} (*, x_1, \dots, x_n) & i = 0 \\ (x_1, \dots, x_i, x_i, \dots, x_n) & 1 \leq i \leq n \\ (x_1, \dots, x_n, *) & i = n + 1 \end{cases}$$

$$s^j(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x}_{j+1}, \dots, x_{n+1}).$$

For simplicial spaces $X^\bullet$ and $Y^\bullet$, let $\text{Hom}(X^\bullet, Y^\bullet)$ denote the simplicial set with n -simplices the cosimplicial maps $X^\bullet \times \Delta[n] \rightarrow Y^\bullet$, where $\Delta[n]$ is the standard simplicial n -simplex. Let $\Delta^\bullet$ be the cosimplicial space with the standard topological n -simplex Δ^n in degree n , the inclusion δ^i as the cofaces and the collapse map σ^j as the codegeneracies. Now let $CX = \text{Tot}(CX^\bullet) = |\text{Hom}(\Delta^\bullet, CX^\bullet)|$ be the total space of $CX^\bullet$. There is a homeomorphism $CX \cong \Omega X$, see e.g. [BK72, X.3].

The completed tensor product. For the next theorem we need the completed tensor product, and its left derived functors. We describe the basic properties here, details can be found in [JO99, App.]. Say that a graded ring R is a *pro-finite ring* if it can be written as an inverse limit of rings of finite length. A ring of finite length is the same as a ring that is Artinian and Noetherian.

Now fix a graded pro-finite ring R . A *pro-finite R -module* is an R -module that can be written as the inverse limit of R -modules of finite length. Let $\mathcal{M}_R^{pf}$ denote the category of pro-finite R -modules; this is an abelian category with enough projectives. Given two modules $M \cong \lim_{\leftarrow} M_i$ and $N \cong \lim_{\leftarrow} N_j$ in $\mathcal{M}_R^{pf}$, define the completed tensor product of M and N as

$$M \widehat{\otimes}_R N = \lim_{\leftarrow} (M_i \otimes_R N_j).$$

This completed tensor product is an object in $\mathcal{M}_R^{pf}$, and the functor $- \widehat{\otimes}_R N : \mathcal{M}_R^{pf} \rightarrow \mathcal{M}_R^{pf}$ is right exact. Write

$$\widehat{\text{Tor}}_i^R(-, N)$$

for the i 'th left derived functor. As with any left derived functor in an abelian category with enough projectives, we can compute $\widehat{\text{Tor}}_i^R(M, N)$ by finding a projective resolution of M . More precisely, given pro-finite R -modules M and N , a resolution

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

of M , with P_i a projective and pro-finite R -module, then the complex $P_* \widehat{\otimes}_R N$ has as its i 'th homology group $\widehat{\text{Tor}}_i^R(M, N)$.

For the usual Tor functors we had a canonical resolution called the bar resolution, with an associated bar complex $\beta_*^R(A)$, which we used to compute $\text{Tor}^A(R, R)$ when A was a flat augmented R -algebra. We can construct a complex for a projective pro-finite R -algebra $A \cong \lim_{\leftarrow} A_i$ such that each A_i is projective over R . This “completed bar complex” is given by

$$\widehat{\beta}^R(A)_n = A \widehat{\otimes}_R \cdots \widehat{\otimes}_R A,$$

with differentials given by the inverse limit of the differentials of $\beta^R(A_i)$. The exactness of the corresponding completed bar resolution follows by passage to the limit for the A_i , using the exactness of limits for pro-finite R -modules. The homology of this complex is $\widehat{\mathrm{Tor}}^A(R, R)$.

The spectral sequence. For X a CW-complex, let $\{X_i\}$ denote the directed system of finite subcomplexes of X . When E is a spectrum there is a Milnor short exact sequence

$$0 \rightarrow \varprojlim^1 E^*(X_i) \rightarrow E^*(X) \rightarrow \varprojlim E^*(X_i) \rightarrow 0.$$

Suppose that the coefficient ring E^* is a graded field, so that each $E^*(X_i)$ is finitely generated and free, and therefore of finite length. Now the inverse system $\{E^*(X_i)\}$ is easily seen to satisfy the Mittag-Leffler condition, i.e., for each k , the images of the maps $E^*(X_i) \rightarrow E^*(X_k)$, $i \geq k$, satisfies the descending chain condition, and so we have

$$E^*(X) \cong \varprojlim E^*(X_i),$$

which makes $E^*(X)$ a pro-finite E^* -module. We also have a Künneth isomorphism

$$E^*(X) \widehat{\otimes}_{E^*} E^*(X) \cong \varprojlim (E^*(X_i) \otimes_{E^*} E^*(X_j)) \cong \varprojlim E^*(X_i \times X_j) \cong E^*(X \times X).$$

Proposition 5.1. *Let E be an S -algebra such that E^* is a graded field, and let X be an E -local space. There is a spectral sequence of algebras*

$$\widehat{\mathrm{Tor}}_{**}^{E^*(X)}(E^*, E^*) \Longrightarrow \pi_{-*}|F(CX_+^\bullet, E)|.$$

Proof. We construct a simplicial S -algebra $Y_\bullet$, i.e., a simplicial object in the category of S -algebras, as $Y_n = F(CX_+^n, E)$. The face and degeneracy maps are induced from the coface and codegeneracy maps of $CX^\bullet$. As in the case of simplicial spaces there is a geometric realization functor $|-|$, defined as a quotient

$$|Y_\bullet| = \coprod_{n \geq 0} \Sigma^\infty \Delta_+^n \wedge Y_n / \sim.$$

Let $Y = |Y_\bullet|$. Now we can look at the skeletal filtration

$$Y^{(0)} \subset \dots \subset Y^{(s-1)} \subset Y^{(s)} \subset \dots \subset Y,$$

where $Y^{(s)}$ is the quotient of $\coprod_{0 \leq n \leq s} \Sigma^\infty \Delta_+^n \wedge Y_n$. We get the following unrolled exact couple by applying homotopy,

$$\begin{array}{ccccccc} \dots & \xrightarrow{i} & \pi_{-*}(Y^{(s)}) & \xrightarrow{i} & \pi_{-*}(Y^{(s+1)}) & \xrightarrow{i} & \dots \\ & & \downarrow j & & \downarrow j & & \\ & \swarrow k & \pi_{-*}(Y^{(s)}, Y^{(s-1)}) & \swarrow k & \pi_{-*}(Y^{(s+1)}, Y^{(s)}) & & \end{array}$$

resulting in a spectral sequence converging to $\operatorname{colim}_s \pi_{-*}(Y^{(s)}) \cong \pi_{-*}(Y)$. Again $Y^{(s)}/Y^{(s-1)} = S^s \wedge Y_s = \Sigma^s Y_s$, so

$$E_{s,t}^1 \cong \pi_{s-*}(Y^{(s)}/Y^{(s-1)}) \cong \pi_{s-*}(\Sigma^s Y_s) \cong \pi_{-*}(Y_s) = E^*(CX^s).$$

By the paragraph before the statement of the proposition, we have

$$E^*(CX^s) \cong E^*(X) \widehat{\otimes}_{E^*} \cdots \widehat{\otimes}_{E^*} E^*(X),$$

and when the differentials $d^1 : E_{s,*}^1 \rightarrow E_{s-1,*}^1$ are identified as the differentials in the completed bar construction, we can conclude that

$$E_{s,*}^2 \cong \widehat{\operatorname{Tor}}_{s,*}^{E^*(X)}(E^*, E^*).$$

□

There is a map

$$(5.2) \quad |F(CX_+^\bullet, E)| \rightarrow F(\operatorname{Tot}(CX^\bullet)_+, E)$$

defined by sending an element

$$(\xi, f) \in \Delta_+^n \wedge F(CX_+^n, E)$$

to the map

$$\operatorname{Hom}(\Delta^n, X^n)_+ \xrightarrow{g} E$$

given by $g(\sigma) = f(\sigma(\xi))$, where σ is a map $\Delta^n \rightarrow X^n$.

Corollary 5.3. *Assuming the map (5.2) described above is an equivalence, the spectral sequence in Theorem 5.1 converges to $E^*(\Omega X)$.*

As we will see in the next section, the Eilenberg-Mac Lane space $K(\mathbb{Z}/p, n+1)$ has trivial $K(n)$ -cohomology, so it is not $K(n)$ -local. Hence the cobar spectral sequence converging to the $K(n)$ -cohomology of $\Omega K(\mathbb{Z}/p, n+1)$ has trivial input, but $K(n)^*(K(\mathbb{Z}/p, n))$ is non-trivial, so there is no hope of the spectral sequence converging in this case.

Convergence in the p -complete case is discussed in [Dwy74] and [Shi96].

6. CONSTRUCTION OF $K(n)$ -COMPACT SPHERES

The Sullivan sphere. First we recall the construction of a non-trivial example of a p -compact group, known as the Sullivan sphere. Assume now that $p \neq 2$, and let $W = (\mathbb{Z}/p)^\times$ act on $\mathbb{Z}_p$ by multiplication. An element $w \in W$ induces an automorphism $w : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$, which in turn induces a homotopy equivalence $w : K(\mathbb{Z}_p, 2) \rightarrow K(\mathbb{Z}_p, 2)$ such that w_* on $\pi_2 K(\mathbb{Z}_p, 2) \cong \mathbb{Z}_p$ is multiplication by w . To ease the notation we let $X = K(\mathbb{Z}_p, 2)$ from now on. By the Hurewicz isomorphism $\pi_2(X) \rightarrow H_2(X)$, the induced map w_* on $H_2(X)$ is also multiplication by w . Since $H^2(X; \mathbb{F}_p) \cong \operatorname{Hom}(H_2(X), \mathbb{F}_p)$, the same is true for w^* on $H^2(X)$. Now $X = K(\mathbb{Z}, 2)_p^\wedge$, so

$$H^*(X; \mathbb{F}_p) \cong H^*(K(\mathbb{Z}, 2); \mathbb{F}_p) = H^*(\mathbb{C}P^\infty; \mathbb{F}_p) \cong \mathbb{F}_p[x].$$

From this we have that $w^*(x^i) = w^i x^i$, since w^* respects the cup product structure.

Let $G = \Omega(X_{hW})_p^\wedge$, where the subscript hW denotes homotopy orbits, i.e., the space $EW \times_W X = EW \times X/W$. The p -adic completion takes place before the formation of the loop space. Then $BG = (X_{hW})_p^\wedge$, and there is an Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{**}^{H^*(BG; \mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow H^*(G; \mathbb{F}_p).$$

To calculate the E_2 -term, we need to know $H^*(BG; \mathbb{F}_p) \cong H^*(X_{hW}; \mathbb{F}_p)$. There is a fibration

$$X \rightarrow EW \times_W X \rightarrow BW,$$

induced by projection on the first factor, which gives rise to a Serre spectral sequence ([Boa99, 13])

$$H^s(BW; H^t(X; \mathbb{F}_p)) \Longrightarrow H^{s+t}(EW \times_W X; \mathbb{F}_p).$$

Here the E_2 -term is cohomology with twisted coefficients, and can be identified with the s 'th group cohomology of W with coefficients in $H^t(X; \mathbb{F}_p)$, which we denote $H_{gp}^s(W; H^t(X; \mathbb{F}_p))$. Restriction and transfer with respect to $\{1\} \subset W$ gives maps

$$H_{gp}^s(W; H^t(X; \mathbb{F}_p)) \xrightarrow{res} H_{gp}^s(\{1\}; H^t(X; \mathbb{F}_p)) \xrightarrow{trf} H_{gp}^s(W; H^t(X; \mathbb{F}_p))$$

with composite multiplication by $|W| = p - 1$. The middle group is 0 for $s > 0$, and $p - 1$ is a unit in $\mathbb{F}_p$, so also the end groups are 0 for $s > 0$. For $s = 0$ we have

$$H_{gp}^0(W; H^t(X; \mathbb{F}_p)) = H^t(X; \mathbb{F}_p)^W,$$

the group of elements invariant under the action of W . Since the E_2 -term of the Serre spectral sequence is non-trivial only for $s = 0$, we get

$$H^*(X_{hW}; \mathbb{F}_p) \cong H^*(X; \mathbb{F}_p)^W \cong \mathbb{F}_p[x]^W.$$

An element $\sum_i f_i x^i \in \mathbb{F}_p[x]$ is invariant under W if $f_i w^i = f_i$ for all i, w , i.e., if $f_i = 0$ or $p - 1 \mid i$. Hence the invariant polynomials in $\mathbb{F}_p$ are those that are polynomials in $y = x^{p-1}$. Now we have that

$$H^*(BG; \mathbb{F}_p) \cong H^*(X_{hW}; \mathbb{F}_p) \cong \mathbb{F}_p[y],$$

with y in degree $2p - 2$.

To calculate $\mathrm{Tor}_{**}^{\mathbb{F}_p[y]}(\mathbb{F}_p, \mathbb{F}_p)$ we use the following free resolution

$$0 \rightarrow \mathbb{F}_p[y] \xrightarrow{y} \mathbb{F}_p[y] \xrightarrow{\epsilon} \mathbb{F}_p \rightarrow 0,$$

where ϵ is the map that sends y to 0, and gives $\mathbb{F}_p$ its $\mathbb{F}_p[y]$ -module structure. When we tensor this resolution with $\mathbb{F}_p$, as $\mathbb{F}_p[y]$ -modules, we get the complex

$$0 \rightarrow \mathbb{F}_p \xrightarrow{0} \mathbb{F}_p \rightarrow 0.$$

Hence

$$\mathrm{Tor}_{**}^{H^*(BG; \mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \cong \Lambda(\sigma y)$$

with σy in bidegree $(-1, 2p - 2)$. Since there is no room for any differentials in the E_2 -term, the spectral sequence collapses and we arrive at

$$H^*(G; \mathbb{F}_p) \cong \Lambda(\sigma y),$$

σy now in total degree $2p - 3$.

G is called the Sullivan sphere; it is p -compact since BG is p -complete, by construction. Note that G has the same mod p cohomology as a $2p - 3$ sphere, and it is in fact homotopy equivalent to $(S^{2p-3})_p^\wedge$. See the comments after Theorem 6.6. Note also that G really is a new example not arising from a compact Lie group, at least when $p \geq 5$, since its cohomology is different from any of the possible spheres.

Eilenberg-Mac Lane spaces. We cite the results of Ravenel and Wilson concerning the Morava K -theory of Eilenberg-Mac Lane spaces. We make a small modification in that we use $K(\mathbb{Z}_p, q)$ instead of $K(\mathbb{Z}, q)$. This will not change the results since $K(\mathbb{Z}_p, q)$ is the p -completion of $K(\mathbb{Z}, q)$, and a mod p equivalence is a $K(n)$ equivalence. First we need to introduce some notation. Let $\delta : K(\mathbb{Z}/p^j, 1) \rightarrow K(\mathbb{Z}_p, 2)$ be the Bockstein map. $K(n)_*(K(\mathbb{Z}_p, 2)) = K(n)_*(\mathbb{C}P^\infty)$ is free on generators β_i in degree $2i$. δ is a spherical fibration with fibre $K(\mathbb{Z}, 1) = S^1$, and there is a Gysin sequence

$$\begin{aligned} \dots \xrightarrow{\partial} K(n)_m(K(\mathbb{Z}/p^j, 1)) \xrightarrow{\delta_*} K(n)_m(\mathbb{C}P^\infty) \xrightarrow{\Phi} K(n)_{m-2}(\mathbb{C}P^\infty) \\ \xrightarrow{\partial} K(n)_{m-1}(K(\mathbb{Z}/p^j, 1)) \rightarrow \dots \end{aligned}$$

where Φ is given by $\Phi(y) = y \cap [p^j]_{\Gamma_n}(x)$, x being the complex orientation. Since $[p]_{\Gamma_n}(x) = x^{p^n}$ it follows that $[p^j]_{\Gamma_n}(x) = x^{p^{jn}}$. Also, $\beta_{nj+i} \cap x^{p^{nj}} = \beta_i$, so Φ is injective with image generated by $\beta_i, 0 \leq i < nj$. Let $a_i \in K(n)_{2i}(K(\mathbb{Z}/p^j, 1))$ map to β_i under δ_* , and write $a_{(i)} = a_{p^i}$.

Now let

$$\circ : K(\mathbb{Z}/p^j, i) \wedge K(\mathbb{Z}/p^j, k) \rightarrow K(\mathbb{Z}/p^j, i+k)$$

denote the map inducing the cup product map on singular cohomology. This induces a map

$$\circ : K(n)_*(K(\mathbb{Z}/p^j, i)) \otimes_{K(n)_*} K(n)_*(K(\mathbb{Z}/p^j, k)) \rightarrow K(n)_*(K(\mathbb{Z}/p^j, i+k)),$$

the “circle product”. For a sequence $I = (i_1, i_2, \dots, i_k)$, define

$$a_I = a_{(i_1)} \circ a_{(i_2)} \circ \dots \circ a_{(i_k)}.$$

Theorem 6.1 ([RW80, 11.1]). *Let $G = K(\mathbb{Z}/p^j, q)$, $j > 0$, be an Eilenberg-Mac Lane space. When $p \neq 2$ we have the following algebra isomorphisms:*

(1) For $q = 0$,

$$K(n)_*(G) \cong K(n)_*[\mathbb{Z}/p^j],$$

the group ring of $\mathbb{Z}/p^j$ over $K(n)_*$.

(2) For $0 < q < n$,

$$K(n)_*(G) \cong \bigotimes_I K(n)_*[a_I]/(a_I^{p^{k(I)}}),$$

where I ranges over all $(i_1, i_2, \dots, i_q)$ with $n(j-1) < i_1 < i_2 < \dots < i_q < nj$, and $k(I)$ is given by some rather complicated formulas which we do not list here.

(3) For $q = n$,

$$K(n)_*(G) \cong \bigotimes_I K(n)_*[a_I]/(a_I^p + (-1)^q v_n^{c(I)} a_I),$$

where I ranges over $(nk, n(j-1)+1, n(j-1)+2, \dots, nj-1)$ with $0 \leq k \leq j-1$, and $c(I)$ a positive integer.

(4) For $q > n$,

$$K(n)_*(G) \cong K(n)_*.$$

Corollary 6.2. *The Eilenberg-Mac Lane spaces $K(\mathbb{Z}/p^j, q)$, $j > 0$, are $K(n)$ -compact groups when $0 < q < n$.*

Proof. In part (2) of Theorem 6.1, each term $K(n)_*[a_I]/(a_I^{p^{k(I)}})$ is finitely generated as a $K(n)_*$ -module, and since the indexing set I is finite, $K(n)_*(K(\mathbb{Z}/p^j, q))$ is a finitely generated $K(n)_*$ -module. By [Bou82, 7.4], we have

$$\pi_i(L_{K(n)}K(\pi, q)) = \begin{cases} \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/p^\infty, \pi) & i = q \leq n \\ \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/p^\infty, \pi / \text{Tors } \pi) & i = q = n + 1 \\ \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \pi) & i = q + 1 \leq n + 1 \\ 0 & \text{otherwise} \end{cases}$$

for any abelian group π . Recall that $\mathbb{Z}/p^\infty = \mathbb{Z}[1/p]/\mathbb{Z}$. When $\pi = \mathbb{Z}/p^j$ the second group above is trivially zero, and the third group is zero since every element in $\mathbb{Z}[1/p]$ is a multiple of p^j . To calculate the first Ext group above we use the exact sequence

$$0 \rightarrow \varprojlim_i^1 \text{Hom}(\mathbb{Z}/p^i, \pi) \rightarrow \text{Ext}(\mathbb{Z}/p^\infty, \pi) \rightarrow \varprojlim_i \text{Ext}(\mathbb{Z}/p^i, \pi) \rightarrow 0$$

found in [BK72, p. 166]. When $i > j$, it is easy to see that $\text{Hom}(\mathbb{Z}/p^i, \mathbb{Z}/p^j) = 0$ and that $\text{Ext}(\mathbb{Z}/p^i, \mathbb{Z}/p^j) = \mathbb{Z}/p^j$, and so

$$\text{Ext}(\mathbb{Z}/p^\infty, \mathbb{Z}/p^j) = \mathbb{Z}/p^j.$$

Thus

$$\pi_i(L_{K(n)}K(\mathbb{Z}/p^j, q)) = \begin{cases} \mathbb{Z}/p^j & i = q \leq n \\ 0 & \text{otherwise} \end{cases}$$

and $BK(\mathbb{Z}/p^j, q) = K(\mathbb{Z}/p^j, q + 1)$ is $K(n)$ -local when $0 < q < n$. $\square$

These Eilenberg-Mac Lane spaces are examples of groups that are $K(n)$ -compact, but not p -compact, since their mod p homology is known not to be finitely generated, by the calculations by Cartan and Serre. See e.g. [McC01, 6.19].

Next, we state a theorem about the Morava K -theory of integral (or p -adic) Eilenberg-Mac Lane spaces, and we will specialize to the case $K(\mathbb{Z}_p, n + 1)$. Before we state the result, we use the Bockstein map $K(\mathbb{Z}/p^j, n) \rightarrow K(\mathbb{Z}_p, n + 1)$, now denoted δ_j , to define classes in $K(n)_*(K(\mathbb{Z}_p, n + 1))$. For each $k \geq 0$, let

$$b_k = (\delta_{k+1})_*(a_{(kn, kn+1, \dots, kn+n-1)}).$$

Theorem 6.3 ([RW80, 12.1]). *When $p \neq 2$ we have the following algebra isomorphism,*

$$K(n)_*(K(\mathbb{Z}_p, n + 1)) \cong \bigotimes_{k=0}^{\infty} K(n)_*[b_k]/(b_k^p + (-1)^n v_n^{c(k)} b_k).$$

with b_k in degree $2(p^{kn} + p^{kn+1} + \dots + p^{kn+n-1})$, and $c(k)$ a positive integer.

For the construction below, we need the cohomology of the integral Eilenberg-Mac Lane spaces. Let y be dual to b_0 .

Theorem 6.4 ([RW80, 12.4]). *As algebras,*

$$K(n)^*(K(\mathbb{Z}_p, n+1)) \cong K(n)^*[[y]],$$

where y has degree $2(p^n - 1)/(p - 1)$.

New $K(n)$ -compact spheres. Here we do a construction similar to that of the Sullivan sphere, to yield new examples of $K(n)$ -compact groups.

Lemma 6.5. *Let $W = (\mathbb{Z}/p)^\times$ act on $\mathbb{Z}_p$ by multiplication. Then the induced action on $K(n)^*(K(\mathbb{Z}_p, n+1))$ is given by $w \cdot y^i = w^{in} y^i$.*

Proof. We know from the Sullivan sphere construction that an element $w \in W$ induces an automorphism $w_* : H_2(K(\mathbb{Z}_p, 2)) \rightarrow H_2(K(\mathbb{Z}_p, 2))$ which is multiplication by w . By the universal coefficient theorem, the induced automorphism on $H_2(K(\mathbb{Z}_p, 2); \mathbb{F}_p)$ is also multiplication by w . There are maps of ring spectra

$$H\mathbb{F}_p \leftarrow k(n) \rightarrow K(n)$$

which divide out by, and invert, v_n , respectively. They induce the following commutative diagram:

$$\begin{array}{ccccc} H_2(K(\mathbb{Z}_p, 2); \mathbb{F}_p) & \longleftarrow & k(n)_2(K(\mathbb{Z}_p, 2)) & \longrightarrow & K(n)_2(K(\mathbb{Z}_p, 2)) \\ \downarrow w_* & & \downarrow w_* & & \downarrow w_* \\ H_2(K(\mathbb{Z}_p, 2); \mathbb{F}_p) & \longleftarrow & k(n)_2(K(\mathbb{Z}_p, 2)) & \longrightarrow & K(n)_2(K(\mathbb{Z}_p, 2)). \end{array}$$

In all three theories let β_1 denote the homology class that is dual to the orientation class x . β_1 is invariant under the horizontal maps, and the leftmost square tells us that the middle w_* maps β_1 to $w\beta_1$. Now the rightmost square says that on $K(n)_2(K(\mathbb{Z}_p, 2))$, w_* sends β_1 to $w\beta_1$. Since $K(n)_*(\mathbb{C}P^\infty)$ and $K(n)^*(\mathbb{C}P^\infty)$ are dual Hopf algebras, we find that w^* on $K(n)^2(K(\mathbb{Z}_p, 2))$ sends x to wx , and by using the cup product structure, that $w^*(x^i) = w^i x^i$. By dualizing back to homology, $w_*(\beta_i) = w^i \beta_i$.

Now we know the action of W on $K(n)_*(K(\mathbb{Z}_p, 2))$. The Bockstein map used in the definition of the a_i 's gives us the following diagram

$$\begin{array}{ccc} K(n)_*(K(\mathbb{Z}/p^j, 1)) & \xrightarrow{\delta_*} & K(n)_*(K(\mathbb{Z}_p, 2)) \\ \downarrow w_* & & \downarrow w_* \\ K(n)_*(K(\mathbb{Z}/p^j, 1)) & \xrightarrow{\delta_*} & K(n)_*(K(\mathbb{Z}_p, 2)). \end{array}$$

Since a_i maps to β_i under δ_* , and δ_* is injective, we must have $w_*(a_i) = w^i a_i$. It follows that $w_*(a_{(i)}) = w^{p^i} a_{(i)} = w a_{(i)}$, by Fermat's little theorem. The bilinearity of the circle product gives

$$w_*(a_{(i_1, \dots, i_k)}) = w_*(a_{(i_1)}) \circ \dots \circ w_*(a_{(i_k)}) = w^k(a_{(i_1)} \circ \dots \circ a_{(i_k)}) = w^k a_{(i_1, \dots, i_k)}.$$

From the commutative diagram

$$\begin{array}{ccc} K(n)_*(K(\mathbb{Z}/p, n)) & \xrightarrow{(\delta_1)^*} & K(n)_*(K(\mathbb{Z}_p, n+1)) \\ \downarrow w_* & & \downarrow w_* \\ K(n)_*(K(\mathbb{Z}/p, n)) & \xrightarrow{(\delta_1)^*} & K(n)_*(K(\mathbb{Z}_p, n+1)) \end{array}$$

we get that $w_*(b_0) = w^n b_0$. Since y is the dual of b_0 , $w^*(y) = w^n y$, and also $w^*(y^i) = w^{in} y^i$. $\square$

Theorem 6.6. *Assuming the hypothesis in Corollary 5.2, there exist $K(n)$ -compact groups G , for all odd primes p and natural numbers n , such that*

$$K(n)^*(G) \cong \Lambda(\sigma z)$$

where

$$|\sigma z| = 2 \frac{p^n - 1}{(p-1, n)} - 1.$$

Proof. Let W act on $K(\mathbb{Z}_p, n+1)$ as in Lemma 6.5, and let $BG = L_{K(n)}K(\mathbb{Z}_p, n+1)_{hW}$. Then $G = \Omega BG = \Omega L_{K(n)}K(\mathbb{Z}_p, n+1)_{hW}$. To calculate the cohomology of G we use the cobar spectral sequence in Corollary 5.2,

$$\widehat{\text{Tor}}_{**}^{K(n)^*(BG)}(K(n)^*, K(n)^*) \implies K(n)^*(G).$$

First we need to identify $K(n)^*(BG)$. This can be done in the same manner as in the Sullivan sphere case, by using the Serre spectral sequence

$$H^*(BW; K(n)^*(K(\mathbb{Z}_p, n+1))) \implies K(n)^*(BG)$$

coming from the fibration

$$K(\mathbb{Z}_p, n+1) \rightarrow EW \times_W K(\mathbb{Z}_p, n+1) \rightarrow BW.$$

Again, this E_2 -term is the same as the group cohomology of W with coefficients in $K(n)^*(K(\mathbb{Z}_p, n+1))$, which is just the W -invariants of $K(n)^*[[y]]$. A power series $\sum_i c_i y^i$ in $K(n)^*[[y]]$ is W -invariant if and only if

$$c_i(w^{in} - 1) = 0,$$

i.e., $c_i = 0$ or $p-1 \mid in$, for all i . Now $p-1 \mid in$ if and only if $(p-1)/(p-1, n) \mid i$, so the power series is W -invariant if and only if it actually lies in $K(n)^*[[z]]$, where $z = y^{(p-1)/(p-1, n)}$. Thus we have

$$K(n)^*(BG) \cong K(n)^*[[z]],$$

where the degree of z is

$$\frac{p-1}{(p-1, n)}|y| = \frac{p-1}{(p-1, n)} \frac{2(p^n - 1)}{p-1} = 2 \frac{p^n - 1}{(p-1, n)}.$$

We can now calculate the E_2 -term in the cobar spectral sequence. Think of $K(n)^*[[z]]$ as the pro-finite ring

$$\varprojlim_i K(n)^*[z]/(z^i).$$

and take

$$0 \rightarrow K(n)^*[[z]] \xrightarrow{\cdot z} K(n)^*[[z]] \xrightarrow{\epsilon} K(n)^* \rightarrow 0$$

as a projective resolution of $K(n)^*$ in the category of pro-finite $K(n)^*[[z]]$ -modules. The map $\cdot z$ is given as the inverse limit of the maps

$$K(n)^*[z]/(z^i) \xrightarrow{\cdot z} K(n)^*[z]/(z^i)$$

that multiply by z . When we take the completed tensor product of this resolution with $K(n)^*$ over $K(n)^*[[z]]$, we obtain the chain complex

$$0 \rightarrow K(n)^*[[z]] \widehat{\otimes}_{K(n)^*[[z]]} K(n)^* \xrightarrow{\cdot z \otimes 1} K(n)^*[[z]] \widehat{\otimes}_{K(n)^*[[z]]} K(n)^* \rightarrow 0.$$

Now the map

$$\cdot z \widehat{\otimes} 1 = \varprojlim (K(n)^*[z]/(z^i) \otimes K(n)^* \xrightarrow{\cdot z \otimes 1} K(n)^*[z]/(z^i) \otimes K(n)^*)$$

is zero, by the $K(n)^*[[z]]$ -module structure on $K(n)^*$, hence

$$\widehat{\mathrm{Tor}}_{**}^{K(n)^*[[z]]}(K(n)^*, K(n)^*) \cong \Lambda(\sigma z).$$

The bidegree of σz is $(-1, |z|)$. The cobar spectral sequence collapses at the E_2 -term, and we obtain

$$K(n)^*(G) \cong \Lambda(\sigma z),$$

with $|\sigma z| = |z| - 1 = 2(p^n - 1)/(p - 1, n) - 1$. In addition, BG is $K(n)$ -local, so G is $K(n)$ -compact. $\square$

Given a spectrum E , a spectrum in the E -local category is invertible if $L_E(X \wedge Y) = L_E S^0$ for an E -local spectrum Y , and the E -local Picard group Pic_E is defined to be the group of homotopy classes of E -local invertible spectra, with group operation the E -local smash product $L_E(- \wedge -)$.

Let Pic_n denote the $K(n)$ -local Picard group. The suspension spectra of the $K(n)$ -compact groups constructed in Theorem 6.6 yield examples of elements in Pic_n , by [HMS94, 1.3], which says that $L_{K(n)} X \in \mathrm{Pic}_n$ if and only if $\tilde{K}(n)_*(X)$ has rank 1 as a module over $K(n)_*$. In the p -complete situation, it can be shown that $\mathrm{Pic}_{H\mathbb{F}_p} \cong \mathbb{Z}$, where the p -complete n -sphere represents $n \in \mathbb{Z}$. This implies that the Sullivan sphere constructed above is stably homotopy equivalent to $(S^{2p-3})_p^\wedge$. The structure of Pic_n is more complicated, for one, the p -adic integers embed in Pic_n for all p and n ([HMS94, 9.3]). Thus our invertible $K(n)$ -compact groups are not automatically $K(n)$ -local spheres.

If the $K(n)$ -compact groups constructed in Theorem 6.6 are $K(n)$ -local spheres, they have to satisfy the dimension bound in Theorem 4.15, i.e., the equation

$$m \leq (m, p - 1) \left(\frac{S - n}{p - 1} + S(\mathrm{ord}_p m + 1) \right)$$

where $m = (p^n - 1)/(p - 1, n)$, must hold. It will be sufficient to check that

$$(6.7) \quad (p^n - 1)(p - 1) \leq (m, p - 1)(p - 1, n)(Sp - n).$$

Note that $(p - 1)/(p - 1, n)$ divides m and $p - 1$, so

$$p - 1 = \frac{p - 1}{(p - 1, n)}(p - 1, n)$$

divides $(m, p - 1)(p - 1, n)$. Thus $p - 1 \leq (m, p - 1)(p - 1, n)$, and to prove (6.7) it remains to check that

$$p^n - 1 \leq Sp - n = \frac{p^n - 1}{p - 1}p - n = p^n - 1 + \frac{p^n - 1}{p - 1} - n$$

This reduces to $(p - 1)n \leq p^n - 1$, which is easily seen to hold.

In [HMS94, 7] Hopkins *et al* describe an algebraic approximation to Pic_n , more precisely they have a result which says that if two elements in Pic_n have isomorphic E_n -cohomology, as Morava modules, then they are equivalent, provided $n^2 \leq 2p - 2$. A Morava module is an E_n^* -module with an action of $\mathbb{S}_n^0$ commuting with the E_n^* -action, where $\mathbb{S}_n^0$ as the subgroup of $\mathbb{S}_n$ of strict automorphisms. Because of Theorem 2.3, we know that $E_n^*(G)$, G as in Theorem 6.6, is isomorphic to $E_n^*(S^{2m-1})$ as E_n^* -modules, but to obtain knowledge of the $\mathbb{S}_n$ -action we need to carry out the previous construction with E_n instead of $K(n)$, and at each step keep track of the $\mathbb{S}_n$ -action. We would need to generalize the cobar spectral sequence to apply to E_n as well, since E_n^* is not a graded field, and also check that the (currently hypothetical) convergence is as Morava modules, not just E_n^* -modules. Once this is done, we can conclude that G is stably equivalent to a $K(n)$ -local sphere, i.e., $\Sigma^\infty G \simeq \Sigma^\infty L_{K(n)} S^{2m-1}$. Time will not permit us to do this here, but we hope to do more work in this direction in the future.

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