LOCAL CODERIVATIVES AND APPROXIMATION OF HODGE LAPLACE PROBLEMS

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Abstract. The standard mixed finite element approximations of Hodge Laplace problems associated with the de Rham complex are based on proper discrete subcomplexes. As a consequence, the exterior derivatives, which are local operators, are computed exactly. However, the approximations of the associated coderivatives are nonlocal. In fact, this nonlocal property is an inherent consequence of the mixed formulation of these methods, and can be argued to be an undesired effect of these schemes. As a consequence, it has been argued, at least in special settings, that more local methods may have improved properties. In the present paper, we construct such methods by relying on a careful balance between the choice of finite element spaces, degrees of freedom, and numerical integration rules. Furthermore, we establish key convergence estimates based on a standard approach of variational crimes.

1. Introduction

The purpose of this paper is to discuss finite element methods for the Hodge Laplace problems of the de Rham complex where both the approximation of the exterior derivative and the associated coderivative are local operators. This is in contrast to the more standard mixed methods for these problems, as described in [7, 8], where the coderivative is approximated by a nonlocal operator d_h^*. To discuss this phenomenon in a more familiar setting, consider the mixed method for the Dirichlet problem associated to a second order elliptic equation of the form

$$-\operatorname{div}(\boldsymbol{K}\operatorname{grad}u) = f \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0, \tag{1.1}$$

where the unknown function u is a scalar field defined on a bounded domain Ω in $\mathbb{R}^n$, and $\partial\Omega$ is its boundary. The coefficient $\boldsymbol{K}$ is matrix valued, spatially varying, and uniformly positive definite. When $\boldsymbol{K}$ is the identity, this problem corresponds to the Hodge Laplace problem studied below in the case when the unknown is an n–form. The standard mixed finite element method for this problem, cf. [12], takes the form:

Find $(\sigma_h, u_h) \in \Sigma_h \times V_h$ such that

$$\begin{aligned} \left\langle \boldsymbol{K}^{-1}\sigma_h, \tau \right\rangle - \langle u_h, \operatorname{div}\tau\rangle &= 0, \qquad && \tau \in \Sigma_h, \\ \langle \operatorname{div}\sigma_h, v\rangle &= \langle f, v\rangle\,, && v \in V_h, \end{aligned} \tag{1.2}$$

where Σ_h and V_h are finite element spaces which are subspaces of $H(\operatorname{div}, \Omega)$ and $L^2(\Omega)$, respectively, and where σ_h is an approximation of $-\boldsymbol{K}\operatorname{grad}u$. Here the

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notation $\langle \cdot, \cdot \rangle$ is used to denote the L^2 inner product for both scalar fields and vector fields defined on Ω.

For the typical examples we have in mind the finite element space V_h will consist of discontinuous piecewise polynomials with respect to a nonoverlapping partition $\mathcal{T}_h$ of the domain Ω. In this case the finite element method (1.2) is referred to as a locally conservative or volume preserving method, since

$$\int_{\Omega_0} f\, dx = \int_{\partial \Omega_0} \sigma_h \cdot \nu \, ds \tag{1.3}$$

for any subdomain Ω_0 of Ω which is a union of elements of $\mathcal{T}_h$. Here ν is the outward unit normal to the boundary of Ω_0. In particular, (1.3) holds if Ω_0 consists of a single element of $\mathcal{T}_h$, and reflects a locally conservation property of the continuous problem. In contrast to this, standard finite element methods for problems of the form (1.1), based on the Dirichlet principle and subspaces of the Sobolev space $H^1(\Omega)$, will not admit a corresponding local conservative property. In fact, for certain problems it can be argued that locally conservative methods are more compatible with the continuous problem than the more standard H^1–method with H^1 conforming elements. For example, for porous medium flow in a strongly heterogeneous and anisotropic setting it has often been argued that locally conservative numerical methods give a better local representation of the physics of the problem, and therefore a qualitatively better approximation, cf. [1, 2]. As a consequence, there has been a substantial interest in developing conservative schemes. In addition to the mixed method (1.2) this includes various schemes referred to as finite volume schemes [22, 24], in particular the multi–point flux approximation schemes [3], and mimetic finite differences [16, 17].

The mixed method (1.2) is a volume preserving discretization in the sense of (1.3), and it is based on a sound variational principle, the principle of complementary energy. On the other hand, the mixed method (1.2) fails to have another local property of the continuous problem since the operator, $u_h \mapsto \sigma_h$, defined by the first equation of (1.2), and which approximates the operator $-\boldsymbol{K}\,\mathrm{grad}$, is nonlocal. This is basically due to the continuity requirements of the finite element spaces Σ_h. Since Σ_h is required to be a subset of $H(\mathrm{div}, \Omega)$, the inverse of the so–called "mass matrix", derived from the L^2 inner product $\langle \sigma_h, \tau \rangle$ of the first equation of (1.2), will be nonlocal. In other words, a local perturbation of u_h will in general lead to a global perturbation of σ_h, and this purely numerical effect is sometimes considered to be undesirable. In fact, in many physical applications, the map $u_h \mapsto \sigma_h$ approximates a constitutive law which is represented as a local operator. For example, in porous medium flow, this corresponds to Darcy's law. Therefore, a central issue in the construction of many of the alternative finite volume schemes is to obtain volume preserving methods which are also based on local approximations of the fluxes $\sigma_h \cdot \nu$, cf. (1.3). We should also mention that there is a relation between the desired local properties described above and so-called mass lumping. This is a procedure which is often performed in the setting of time dependent problems, to remove the effect of mass matrices, to obtain explicit or simplified time stepping schemes. For examples of such studies we refer to [20] and references given there. However, we will not study time dependent problems in this paper, even if our results can potentially be used in this context.

An early attempt to overcome the locality problem of the mixed method (1.2) in the two dimensional case, and using the lowest order Raviart-Thomas space, was done in [9]. In this case the unknown u_h is a piecewise constant, while the fluxes $\sigma_h \cdot \nu$ are constant on each edge of the triangulation. The discussion in [9] was restricted to the case $\boldsymbol{K}$ equal to the identity. The first equation of (1.2) is approximated by a numerical integration rule based on the fluxes at the edges. This approach leads to a so-called two-point flux method, i.e., the flux across an edge is proportional to the difference of u_h at the two neighbouring triangles. However, this method has serious defects. In particular, in the general setting, where $\boldsymbol{K}$ is matrix valued and spatially varying, the two-point flux method will not always be consistent, cf. [3, 4].

The multi-point flux approximation schemes were derived to overcome this problem, and with Darcy flow and reservoir simulation as the main area of application. We refer to the survey paper [3] by Aavatsmark for more details. The multi-point flux schemes are usually described in the setting of finite difference methods. However, for the analysis of these finite volume schemes it seems that the most useful approach is to be able to relate the schemes properly to a perturbed mixed finite element method, cf. [10, 23, 30, 31, 34]. An alternative approach to overcome the defects of the two-point flux method was proposed by Brezzi et al. [13]. They proposed to use the lowest order Brezzi-Douglas-Marini space instead of the Raviart-Thomas space, and to perturb that mixed method by introducing a quadrature rule based on vertex values instead of edge values. They also showed satisfactory results in the three dimensional case. A similar method was proposed by Wheeler and Yotov [34], where also quadrilateral grids are studied, and further extensions to hexahedral grids are studied in [29, 33].

The results of the present paper can be seen as further generalizations of the results of [13, 34]. In fact, the mixed method (1.2) corresponds to a special case of the finite element methods studied in [7, 8] for the more general Hodge Laplace problems. Furthermore, the lack of locality described above is a common feature of almost all of these finite element methods. Therefore, the purpose of the present paper is to construct corresponding perturbations of the mixed methods for the Hodge Laplace problems which will overcome the problem of lack of locality in this more general setting. As a consequence, the potential applications of the results of this paper are not restricted to Darcy flow and similar problems, but may for example also be used to localize various methods for Maxwell's equations. We refer to [7, 8] for more details on the various realizations of the Hodge Laplace problems.

The present paper is organized as follows. In the next section we will present a brief review of exterior calculus, the de Rham complex and its discretizations. In Section 3 we will discuss an abstract error analysis, in the setting of Hilbert complexes, which we will find useful in more concrete applications below. Such applications, in the setting of finite element discretizations with respect to simplicial meshes, will be discussed in Section 4, while corresponding results for cubical meshes are discussed in Section 5.

2. Preliminaries

Throughout this paper we will adopt the language of finite element exterior calculus as in [7, 8]. We assume that $\Omega \subset \mathbb{R}^n$ is bounded polyhedral domain, and we will study finite element approximations of differential forms defined on Ω. More precisely, we consider maps defined on Ω with values in the space $\mathrm{Alt}^k(\mathbb{R}^n)$, the space of alternating k–linear maps on $\mathbb{R}^n$. For $0 \leq k \leq n$ this is a real vector space with dimension

$$\dim \mathrm{Alt}^k(\mathbb{R}^n) = \binom{n}{k}.$$

When $k = 0$, $\mathrm{Alt}^0(\mathbb{R}^n) = \mathbb{R}$. For $1 \le k \le n$ let $\Sigma(k)$ be the set of increasing injective maps from $\{1, ..., k\}$ to $\{1, ..., n\}$. Then we can define an inner product on $\mathrm{Alt}^k(\mathbb{R}^n)$ by the formula

$$\langle a, b \rangle_{\mathrm{Alt}} = \sum_{\sigma \in \Sigma(k)} a(e_{\sigma_1}, \ldots, e_{\sigma_k}) b(e_{\sigma_1}, \ldots, e_{\sigma_k}), \quad a, b \in \mathrm{Alt}^k(\mathbb{R}^n),$$

where σ_i denotes $\sigma(i)$ for $1 \le i \le k$ and $\{e_1, \ldots, e_n\}$ is any orthonormal basis of $\mathbb{R}^n$.

Differential forms are maps defined on a spatial domain Ω with values in $\mathrm{Alt}^k(\mathbb{R}^n)$. If u is a differential k-form and $t_1, \ldots, t_k$ are vectors in $\mathbb{R}^n$, then $u_x(t_1, \ldots, t_k)$ denotes the value of u applied to the vectors $t_1, \ldots, t_k$ at the point $x \in \Omega$. The differential form u is an element of the space $L^2\Lambda^k(\Omega)$ if and only if the map

$$x \mapsto u_x(t_1, \ldots, t_k)$$

is in $L^2(\Omega)$ for all tuples $t_1, \ldots, t_k$. In fact, $L^2\Lambda^k(\Omega)$ is a Hilbert space with inner product given by

$$\langle u, v \rangle = \int_\Omega \langle u_x, v_x \rangle_{\mathrm{Alt}} \, dx.$$

The exterior derivative of a k-form u is a $(k+1)$-form du given by

$$du_x(t_1, \ldots t_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} \partial_{t_j} u_x(t_1, \ldots, \hat{t}_j, \ldots, t_{k+1}),$$

where $\hat{t}_j$ implies that t_j is not included, and ∂_{t_j} denote the directional derivative. The Hilbert space $H\Lambda^k(\Omega)$ is the corresponding space of k-forms u on Ω, which is in $L^2\Lambda^k(\Omega)$, and where its exterior derivative, $du = d^k u$, is also in $L^2\Lambda^{k+1}(\Omega)$. The L^2 version of the de Rham complex then takes the form

$$H\Lambda^0(\Omega) \xrightarrow{d^0} H\Lambda^1(\Omega) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H\Lambda^n(\Omega).$$

In the setting of k–forms, the Hodge Laplace problem takes the form

$$Lu = (d^* d + d d^*) u = f, \tag{2.1}$$

where $d = d^k$ is the exterior derivative mapping k–forms to $(k+1)$–forms, and the coderivative $d^* = d_k^*$ can be seen as the formal adjoint of d^{k-1}. Hence, the Hodge Laplace operator L above is more precisely expressed as $L = d_{k+1}^* d^k + d^{k-1} d_k^*$. A typical model problem studied in [7, 8] is of the form (2.1) and with appropriate boundary conditions. The mixed finite element methods are derived from a weak formulation, where $\sigma = d^* u$ is introduced as an auxiliary variable. It is of the form:

Find $(\sigma, u) \in H\Lambda^{k-1}(\Omega) \times H\Lambda^k(\Omega)$ such that

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, d^{k-1} \tau \rangle &= 0, & \tau &\in H\Lambda^{k-1}(\Omega), \\ \langle d^{k-1} \sigma, v \rangle + \langle d^k u, d^k v \rangle &= \langle f, v \rangle, & v &\in H\Lambda^k(\Omega). \end{aligned} \tag{2.2}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner products of all the spaces of the form $L^2\Lambda^j(\Omega)$ which appears in the formulation, i.e., $j = k-1, k, k+1$. We refer to Sections 2 and 7 of [7] for more details. We note that only the exterior derivate d is used explicitly in the weak formulation above, while the relation $\sigma = d_k^* u$ is formulated weakly in the first equation. The formulation also contains the two boundary conditions that the traces of $\star u$ and $\star du$ vanish on the boundary, where $\star$ is the Hodge star operator mapping k–forms to $(n-k)$–forms. The problem (2.2) with $k = n-1$ corresponds to a weak formulation of the elliptic equation (1.1) in the case when the coefficient $\boldsymbol{K}$ is the identity matrix. In fact, variable coefficients can also easily be included in the weak formulations (2.2) by changing the L^2 inner products, see [7, Section 7.3]. However, throughout the rest of the discussion below we will restrict the discussion to the constant coefficient case. But we emphasize that the extension of the discussion to problems with variable coefficients of the form studied in [7], and which are piecewise constants with respect to the mesh we consider, is indeed straightforward.

In general, the solution of the system (2.2) may not be unique. Depending on the topology of the domain Ω there may exist nontrivial harmonic forms, i.e., nontrivial elements of the space

$$\mathfrak{H}^k(\Omega) = \{v \in H\Lambda^k(\Omega) \ : \ dv = 0 \text{ and } \langle v, d\tau\rangle = 0 \text{ for all } \tau \in H\Lambda^{k-1}(\Omega)\}.$$

Hence, to obtain a system with a unique solution, an extra condition requiring orthogonality with respect to the harmonic forms, is usually included.

The basic construction in finite element exterior calculus is of a corresponding subcomplex

$$V_h^0 \xrightarrow{d} V_h^1 \xrightarrow{d} \cdots \xrightarrow{d} V_h^n,$$

where the spaces V_h^k are finite dimensional subspaces of $H\Lambda^k(\Omega)$. In particular, the discrete spaces should have the property that $d(V_h^{k-1}) \subset V_h^k$. The finite element methods studied in [7, 8] are based on the weak formulation (2.2). These methods are obtained by simply replacing the Sobolev spaces $H\Lambda^{k-1}(\Omega)$ and $H\Lambda^k(\Omega)$ by the finite element spaces V_h^{k-1} and V_h^k. More precisely, we are searching for a triple $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ such that

$$\begin{aligned} \langle \sigma_h, \tau\rangle - \langle d\tau, u_h\rangle &= 0, && \tau \in V_h^{k-1}, \\ \langle d\sigma_h, v\rangle + \langle du_h, dv\rangle + \langle p_h, v\rangle &= \langle f, v\rangle, && v \in V_h^k, \\ \langle u_h, q\rangle &= 0, && q \in \mathfrak{H}_h^k, \end{aligned} \tag{2.3}$$

where the space $\mathfrak{H}_h^k$, approximating the harmonic forms, is given by

$$\mathfrak{H}_h^k = \{v \in V_h^k \ : \ dv = 0 \text{ and } \langle v, d\tau\rangle = 0 \text{ for all } \tau \in V_h^{k-1}\}.$$

In particular, the exterior derivative appearing in the method is the exact operator d, restricted to the spaces V_h^{k-1} and V_h^k, while no d^* operator appears. Instead, an approximation of d^* is implicitly defined by the system (2.3). More precisely, the operator $d_h^* : V_h^k \to V_h^{k-1}$ is defined by the first equation of the system (2.3), i.e.,

$$\langle d_h^* u, \tau\rangle = \langle u, d\tau\rangle, \quad u \in V_h^k, \ \tau \in V_h^{k-1}. \tag{2.4}$$

In fact, just as we have explained for the special discrete problem (1.2) above, the continuity requirements of the spaces V_h^{k-1} will in general have the effect the operator d_h^* is nonlocal, in contrast to the continuous case where d^* is a local operator. Motivated by this our purpose in this paper is to construct perturbations of the standard mixed methods which are converging, but also have the property that the corresponding operator d_h^* is local. We will achieve this by replacing the L^2 inner product $\langle d_h^* u, \tau\rangle$ in (2.4) by a proper approximation, and by choosing the spaces V_h^{k-1} and V_h^k carefully. In all the examples presented below the finite element spaces V_h^{k-1} and V_h^k will be of "low-order". We also recall that in the continuous setting the coderivative d^* can, up to a sign, be represented on the form $\star d \star$, where $\star$ denotes Hodge operators. Therefore, the theory below can also be related to the discussion by Hiptmair in [26] on discrete Hodge operators and local approximation of constitutive laws.

In the theoretical analysis of the stability of numerical methods constructed from the discrete complex, bounded projections $\pi_h^k : H\Lambda^k(\Omega) \to V_h^k$ are utilized, such that the diagram

$$\begin{array}{ccccccc} H\Lambda^0(\Omega) & \xrightarrow{d} & H\Lambda^1(\Omega) & \xrightarrow{d} & \cdots & \xrightarrow{d} & H\Lambda^n(\Omega) \\ \downarrow \pi_h^0 & & \downarrow \pi_h^1 & & & & \downarrow \pi_h^n \\ V_h^0 & \xrightarrow{d} & V_h^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & V_h^n \end{array} \tag{2.5}$$

commutes. Such commuting projections are referred to as cochain projections. The importance of bounded cochain projections is related to the stability of the discretizations of the Hodge Laplace problems. It follows from the results of [8, Section 3.3] that the existence of bounded cochain projections is equivalent to stability of the associated finite element method. As a consequence, the most common stability criteria are obtained by showing the existence of such projections.

If $\{\mathcal{T}_h\}$ is a family of simplicial meshes, as described for example in [7, Section 5], then the spaces V_h^k are taken from two main families. Either V_h^k is of the form $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$, consisting of all elements of $H\Lambda^k(\Omega)$ which restrict to polynomial k-forms of degree at most r on each simplex T in the partition $\mathcal{T}_h$, or $V_h^k = \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$, which is a space which sits between $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h)$. In addition, both spaces have the property that the elements have continuous traces on each simplex in $\Delta_{n-1}(\mathcal{T}_h)$, and as a consequence they are subspaces of $H\Lambda^k(\Omega)$. Here we adopt the notation that $\Delta_k(\mathcal{T}_h)$ denotes the set of all the k–dimensional subsimplexes of the triangulation $\mathcal{T}_h$. The spaces $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ are generalizations of the Raviart-Thomas and Brezzi-Douglas-Marini spaces, used to discretize $H(\mathrm{div})$ and $H(\mathrm{rot})$ in two space dimensions, and the Nédélec edge and face spaces of the first and second kind, used to discretize $H(\mathrm{curl})$ and $H(\mathrm{div})$ in three space dimensions.

The simplest stable discretization of the Hodge Laplace problem is obtained by choosing both spaces V_h^{k-1} and V_h^k to be the classical Whitney forms, i.e., we take $V_h^{k-1} = \mathcal{P}_1^-\Lambda^{k-1}(\mathcal{T}_h)$ and $V_h^k = \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$. For the space $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$ the degrees of freedom are simply the integrals of the traces over each element of $\Delta_k(\mathcal{T}_h)$. The corresponding degrees of freedom for the corresponding linear space, $\mathcal{P}_1\Lambda^k(\mathcal{T}_h)$, is the corresponding integrals over each element of $\Delta_k(\mathcal{T}_h)$ against all scalar linear test functions, cf. [7, Theorem 4.10]. As we will see below, this extra local freedom, represented by linear test functions on the k-dimensional subsimplexes, will be crucial for our construction of local methods below.

We recall that another family of numerical methods that have been proposed for Hodge Laplace problems are based on "discrete exterior calculus" as presented in [21, 27]. By construction these methods utilize local approximation of both the exterior derivative d and the coderivative d^*. However, for these methods a satisfactory convergence theory seems still to be lacking. In contrast to this, for the methods constructed here we derive convergence results based on a standard approach of finite elements and variational crimes. The study of such variational crimes in the general setting of Hilbert complexes will be given in the next section.

3. Abstract error analysis and variational crimes

We will find it useful to base our analysis below on some abstract error estimates in the general setting of Hilbert complexes and variational crimes. In this respect our discussion in this section resembles parts of the theory presented in [28]. However, compared to the results of [28], covering discrete Hilbert complex with nonconforming finite elements, we only consider conforming approximations, but we provide more explicit conditions for consistency and convergence. Our notation and set–up are basically taken from [8, Chapter 3].

A closed Hilbert complex (W, d) consists of a sequence of Hilbert spaces W^k with index k and a sequence of closed, densely-defined linear operators $d^k : W^k \to W^{k+1}$ such that $d^{k+1} \circ d^k = 0$. The sequence of operators d is called a differential. A Hilbert subcomplex of (W, d) is a Hilbert complex $(\bar{W}, \bar{d})$ such that $\bar{W}^k$ is a subspace of W^k and $\bar{d}^k = d^k|_{\bar{W}^k}$ for each k. The domain complex of a closed Hilbert complex (W, d) is the Hilbert subcomplex (V, d) such that $V^k \subset W^k$ is the domain of d^k for all k. We use $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ to denote the inner product and the corresponding norm on W^k, respectively, but we omit index k since it is usually clear from the context. Similarly, we use $\langle \cdot, \cdot \rangle_V$ to denote the inner product

$$\langle \omega, \omega' \rangle_V := \langle \omega, \omega' \rangle + \left\langle d^k\omega, d^k\omega' \right\rangle, \qquad \omega, \omega' \in V^k \tag{3.1}$$

and $\|\cdot\|_V$ is the associated norm.

The dual complex (W, d^*), associated to (W, d), is the Hilbert complex with same W^k as Hilbert spaces and $d^*_{k+1} : W^{k+1} \to W^k$, the adjoint of d^k, as differential. The d^* is also called the coderivative of d. We say that d^k is closed and densely-defined if the range of d^k is closed in W^{k+1} and the domain of d^k is dense in W^k.

Let us define subspaces of V^k as

$$\begin{aligned}
\mathfrak{Z}^k &= \{\omega \in V^k \ : \ d\omega = 0\},\\
\mathfrak{B}^k &= d(V^{k-1}) = \{\omega \in V^k \ : \ \omega = d\eta \text{ for some } \eta \in V^{k-1}\},\\
\mathfrak{H}^k &= \{\omega \in V^k \ : \ d\omega = 0 \text{ and } \langle \omega, d\eta \rangle = 0 \text{ for all } \eta \in V^{k-1}\}.
\end{aligned}$$

We can consider a Hodge decomposition of W^k,

$$W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_W},$$

where $\oplus$ stands for orthogonal decomposition of subspaces with the W-inner product and $\mathfrak{Z}^{k\perp_W}$ is the orthogonal complement of $\mathfrak{Z}^k$ in W^k with the W-inner product. Similarly, there is a Hodge decomposition

$$V^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_V}$$

with $\mathfrak{Z}^{k\perp_V}$, the orthogonal complement of $\mathfrak{Z}^k$ in V^k with the V-inner product in (3.1). Since (W, d) is a closed Hilbert complex, $\mathfrak{B}^k$, $\mathfrak{H}^k$, $\mathfrak{Z}^k$, $\mathfrak{Z}^{k\perp_V}$ are closed subspaces of V^k. For a closed subspace W' of W^k, $P_{W'}$ denotes the W-orthogonal projections of W^k into W'.

In an abstract Hilbert complex, for a given $f \in W^k$ with $f \perp \mathfrak{H}^k$, a variational mixed form of the Hodge Laplace problem (2.1) is to find $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$

such that

$$
\begin{aligned}
\langle \sigma, \tau \rangle - \langle d\tau, u \rangle &= 0, && \tau \in V^{k-1}, \\
\langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle, && v \in V^k, \\
\langle u, q \rangle &= 0, && q \in \mathfrak{H}^k.
\end{aligned} \tag{3.2}
$$

To discretize (3.2), we assume that (V_h, d) is a family of finite dimensional subcomplexes parametrized by a discretization parameter $h \in (0, 1]$. So $V_h^k \subset V^k$ and $d(V_h^{k-1}) \subset V_h^k$. Furthermore, we assume that the discretization is stable in the sense that there exists uniformly bounded cochain projections, cf. [8, Section 3.3]. If we define function spaces

$$
\mathfrak{Z}_h^k = \{\omega \in V_h^k \ : \ d\omega = 0\}, \qquad \mathfrak{B}_h^k = dV_h^{k-1},
$$
$$
\mathfrak{H}_h^k = \{\omega \in V_h^k \ : \ d\omega = 0, \ \langle \omega, d\tau \rangle = 0 \quad \tau \in V_h^{k-1}\},
$$

then there is a discrete Hodge decomposition of V_h^k for each k

$$
V_h^k = \mathfrak{Z}_h^k \oplus \mathfrak{Z}_h^{k\perp} = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus \mathfrak{Z}_h^{k\perp} \tag{3.3}
$$

with $\mathfrak{Z}_h^{k\perp}$, the orthogonal complement of $\mathfrak{Z}_h^k$ in V_h^k [8]. Furthermore, a discrete Poincaré inequality holds, i.e., there exists $c_P > 0$, independent of h, such that

$$
\|v\| \leq c_P \|dv\|, \qquad v \in \mathfrak{Z}_h^{k\perp}. \tag{3.4}
$$

The discrete problem corresponding to (3.2) is to find $(\tilde{\sigma}_h, \tilde{u}_h, \tilde{p}_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ such that

$$
\begin{aligned}
\langle \tilde{\sigma}_h, \tau \rangle - \langle d\tau, \tilde{u}_h \rangle &= 0, && \tau \in V_h^{k-1}, \\
\langle d\tilde{\sigma}_h, v \rangle + \langle d\tilde{u}_h, dv \rangle + \langle \tilde{p}_h, v \rangle &= \langle f, v \rangle, && v \in V_h^k, \\
\langle \tilde{u}_h, q \rangle &= 0, && q \in \mathfrak{H}_h^k.
\end{aligned} \tag{3.5}
$$

For simplicity we will use $\mathcal{X}^k$ and $\mathcal{X}_h^k$ to denote $V^{k-1} \times V^k \times \mathfrak{H}^k$ and $V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$, respectively, and define $\|(\tau, v, q)\|_{\mathcal{X}}$ by

$$
\|(\tau, v, q)\|_{\mathcal{X}} := \|\tau\|_V + \|v\|_V + \|q\|, \qquad (\tau, v, q) \in \mathcal{X}^k. \tag{3.6}
$$

For future reference we rewrite the system (3.2) as

$$
\mathcal{B}(\sigma, u, p; \tau, v, q) = \langle f, v \rangle, \qquad (\tau, v, q) \in \mathcal{X}^k, \tag{3.7}
$$

with the bilinear form

$$
\mathcal{B}(\tau, v, q; \tau', v', q') = \langle \tau, \tau' \rangle - \langle d\tau', v \rangle + \langle d\tau, v' \rangle + \langle dv, dv' \rangle + \langle q, v' \rangle - \langle q', v \rangle. \tag{3.8}
$$

Stability and error estimates for the discrete approximations (3.5) are given in [8, Theorem 3.9]. An estimate of the form

$$
\|(\sigma, u, p) - (\tilde{\sigma}_h, \tilde{u}_h, \tilde{p}_h)\|_{\mathcal{X}} \quad \lesssim \inf_{(\tau, v, q) \in \mathcal{X}_h^k} \|(\sigma, u, p) - (\tau, v, q)\|_{\mathcal{X}} + \mathcal{E}_h(u) \tag{3.9}
$$

holds with an extra error term $\mathcal{E}_h(u)$ explained below. Here, and below, the notation $X \lesssim Y$ is used to state that $X \leq CY$, with a constant $C > 0$ independent of the discretization parameter h. Furthermore, the extra error term $\mathcal{E}_h(u)$ appears as a consequence of the fact that the space of discrete harmonic forms, $\mathfrak{H}_h^k$ is not a subspace of $\mathfrak{H}^k$. In fact, if there are no nontrivial harmonic forms, i.e., if $\mathfrak{H}^k$ only consists of the zero element, then $\mathcal{E}_h(u) = 0$. Otherwise, $\mathcal{E}_h(u)$ will usually be of higher order than the other terms on the right-hand side of (3.9). We refer to [7, Section 7.6] and [8, Section 3.4] for more details.

As is already mentioned in the previous section, our purpose is developing a convergent numerical method for (3.7) such that the d_h^* operator defined by $d_h^* u_h := \sigma_h$, is a local operator. To achieve a numerical method which results in a local d_h^* operator, we consider discrete mixed Hodge Laplace problems with a bilinear form $\mathcal{B}_h$, a variant of $\mathcal{B}$. In $\mathcal{B}_h$ the first equation in (3.5) is modified to make the numerical solutions σ_h and u_h have a local relation. To define $\mathcal{B}_h$, we suppose that there is a bilinear form $\langle \cdot, \cdot \rangle_h$ satisfying the following assumption:

- **(A)** $\langle \cdot, \cdot \rangle_h$ is a symmetric bounded coercive bilinear form on $V_h^{k-1} \times V_h^{k-1}$ such that the norm $\|\tau\|_h := \langle \tau, \tau \rangle_h^{1/2}$ is equivalent to $\|\tau\|$ for $\tau \in V_h^{k-1}$ with constants independent of h.

In this section we remain $\langle \cdot, \cdot \rangle_h$ undefined, and proceed discussions on the numerical method in abstract setting. Explicit examples of $\langle \cdot, \cdot \rangle_h$ on simplicial and cubical meshes will be given in the next two sections. We now define $\mathcal{B}_h(\tau, v, q; \tau', v', q') : \mathcal{X}_h^k \times \mathcal{X}_h^k \to \mathbb{R}$ by

$$\begin{aligned} \mathcal{B}_h(\tau, v, q; \tau', v', q') \\ = \langle \tau, \tau' \rangle_h - \langle d\tau', v \rangle + \langle d\tau, v' \rangle + \langle dv, dv' \rangle + \langle q, v' \rangle - \langle q', v_h \rangle , \end{aligned} \tag{3.10}$$

and consider the problem to find $(\sigma_h, u_h, p_h) \in \mathcal{X}_h^k$ such that

$$\mathcal{B}_h(\sigma_h, u_h, p_h; \tau, v, q) = \langle f, v \rangle , \qquad (\tau, v, q) \in \mathcal{X}_h^k. \tag{3.11}$$

Let us define the norm $\|(\tau, v, q)\|_{\mathcal{X}_h}$ for $(\tau, v, q) \in \mathcal{X}_h^k$ by

$$\|(\tau, v, q)\|_{\mathcal{X}_h} := \left(\|\tau\|_h^2 + \|d\tau\|^2 + \|v\|_V^2 + \|q\|^2 \right)^{\frac{1}{2}} .$$

From the assumption **(A)** it is easy to see that $\|\cdot\|_{\mathcal{X}_h}$ is equivalent to $\|\cdot\|_{\mathcal{X}}$ in $\mathcal{X}_h^k$. Due to **(A)** and the discrete Poincaré inequality (3.4), there exists a positive constant, again denoted by c_P, that

$$\|\rho\|_h \le c_P \|d\rho\|, \quad \rho \in \mathfrak{Z}_h^{k-1\perp}. \tag{3.12}$$

Theorem 3.1. *Suppose that $\mathcal{B}_h$ is defined as in* (3.10) *with $\langle \cdot, \cdot \rangle_h$ satisfying* **(A)**. *Then $\mathcal{B}_h$ satisfies*

$$\inf_{0 \neq (\tau, v, q) \in \mathcal{X}_h^k} \sup_{0 \neq (\tau', v', q') \in \mathcal{X}_h^k} \frac{\mathcal{B}_h(\tau, v, q; \tau', v', q')}{\|(\tau, v, q)\|_{\mathcal{X}_h} \|(\tau', v', q')\|_{\mathcal{X}_h}} \gtrsim 1. \tag{3.13}$$

The proof of this theorem is completely analogous to the proof of Theorem 3.2 in [8] with the discrete Poincaré inequality (3.12), so we do not prove it here.

By Theorem 3.1, (3.11) has a unique solution $(\sigma_h, u_h, p_h) \in \mathcal{X}_h^k$. To show the convergence of $\|(\sigma - \sigma_h, u - u_h, p - p_h)\|_{\mathcal{X}}$, we use a standard method with the triangle inequality and an interpolation of (σ, u, p) in $\mathcal{X}_h$. Since we already know the good approximation result (3.9), we only need to estimate $\|(\sigma_h - \tilde{\sigma}_h, u_h - \tilde{u}_h, p_h - \tilde{p}_h)\|_{\mathcal{X}}$.

We first observe that

$$\begin{aligned} \mathcal{B}_h(\tilde{\sigma}_h, \tilde{u}_h, \tilde{p}_h; \tau, v, q) &= \mathcal{B}(\tilde{\sigma}_h, \tilde{u}_h, \tilde{p}_h; \tau, v, q) + \langle \tilde{\sigma}_h, \tau \rangle - \langle \tilde{\sigma}_h, \tau \rangle_h \\ &= \langle f, v \rangle + \langle \tilde{\sigma}_h, \tau \rangle - \langle \tilde{\sigma}_h, \tau \rangle_h , \end{aligned} \tag{3.14}$$

by the definitions of $\mathcal{B}_h$, $\mathcal{B}$, and (3.5). To estimate $\|(\sigma_h - \tilde{\sigma}_h, u_h - \tilde{u}_h, p_h - \tilde{p}_h)\|_{\mathcal{X}}$, we use the equivalence of $\|\cdot\|_{\mathcal{X}_h}$ and $\|\cdot\|_{\mathcal{X}}$ on $\mathcal{X}_h^k$ and the Strang lemma in the variational crimes [11, Chap. 10]. More specifically, the equivalence of $\|\cdot\|_{\mathcal{X}_h}$ and $\|\cdot\|_{\mathcal{X}}$, (3.13), and (3.11) lead us to

$$\begin{aligned}
&\|(\sigma_h - \tilde{\sigma}_h, u_h - \tilde{u}_h, p_h - \tilde{p}_h)\|_{\mathcal{X}} \\
&\lesssim \sup_{(\tau,v,q)\in\mathcal{X}_h^k} \frac{\mathcal{B}_h(\sigma_h - \tilde{\sigma}_h, u_h - \tilde{u}_h, p_h - \tilde{p}_h; \tau, v, q)}{\|(\tau,v,q)\|_{\mathcal{X}_h}} \\
&= \sup_{(\tau,v,q)\in\mathcal{X}_h^k} \frac{\langle f, v\rangle - \mathcal{B}_h(\tilde{\sigma}_h, \tilde{u}_h, \tilde{p}_h; \tau, v, q)}{\|(\tau,v,q)\|_{\mathcal{X}_h}} \\
&= \sup_{\tau\in V_h^{k-1}} \frac{\langle\tilde{\sigma}_h, \tau\rangle_h - \langle\tilde{\sigma}_h, \tau\rangle}{\|\tau\|_V}.
\end{aligned} \tag{3.15}$$

This shows that convergence of $\|(\sigma_h - \tilde{\sigma}_h, u_h - \tilde{u}_h, p_h - \tilde{p}_h)\|_{\mathcal{X}}$ is related to the consistency error from the discrete bilinear form $\langle\cdot,\cdot\rangle_h$. To have a consistency error estimate, we need another assumption for $\langle\cdot,\cdot\rangle_h$:

(B) There exist discrete subspaces $W_h^{k-1} \subset W^{k-1}$ and $\tilde{V}_h^{k-1} \subset V_h^{k-1}$ that

$$\langle\tau, \tau_0\rangle = \langle\tau, \tau_0\rangle_h, \qquad \tau\in\tilde{V}_h^{k-1}, \tau_0 \in W_h^{k-1}, \tag{3.16}$$

and a linear map $\Pi_h : V_h^{k-1} \to \tilde{V}_h^{k-1}$ such that $d\Pi_h\tau = d\tau$, $\|\Pi_h\tau\| \lesssim \|\tau\|$, and

$$\langle\Pi_h\tau, \tau_0\rangle_h = \langle\tau, \tau_0\rangle_h, \quad \tau_0 \in W_h^{k-1}. \tag{3.17}$$

Note that if (3.16) holds with $\tilde{V}_h^{k-1} = V_h^{k-1}$, then all other conditions are satisfied with Π_h as the identity map.

The error bound of $\|(\sigma - \sigma_h, u - u_h, p - p_h)\|_{\mathcal{X}}$ follows from the estimate of $\|(\sigma_h - \tilde{\sigma}_h, u_h - \tilde{u}_h, p_h - \tilde{p}_h)\|_{\mathcal{X}}$ obtained in the theorem below.

Theorem 3.2. *Suppose that $\mathcal{B}_h$ is given as in* (3.10) *with $\langle\cdot,\cdot\rangle_h$ satisfying* **(A)** *and* **(B)**. *Then, for (σ_h, u_h, p_h), the solution of* (3.11),

$$\|(\sigma_h - \tilde{\sigma}_h, u_h - \tilde{u}_h, p_h - \tilde{p}_h)\|_{\mathcal{X}} \lesssim \|\sigma - P_{W_h}\sigma\| + \|\sigma - \tilde{\sigma}_h\| \tag{3.18}$$

holds in which P_{W_h} is the W-orthogonal projection onto W_h^{k-1}.

Proof. By (3.15), it suffices to show

$$|\langle\tilde{\sigma}_h, \tau\rangle - \langle\tilde{\sigma}_h, \tau\rangle_h| \lesssim (\|\sigma - P_{W_h}\sigma\| + \|\sigma - \tilde{\sigma}_h\|)\|\tau\|_V. \tag{3.19}$$

We first observe that $\langle\tilde{\sigma}_h, \tau'\rangle = \langle\tilde{\sigma}_h, \Pi_h\tau'\rangle$ for $\tau' \in V_h^{k-1}$ holds by taking $\tau = \tau' - \Pi_h\tau'$ in the first equation of (3.5). Using this equality, we have

$$\begin{aligned}
\langle\tilde{\sigma}_h, \tau\rangle - \langle\tilde{\sigma}_h, \tau\rangle_h &= \langle\tilde{\sigma}_h, \Pi_h\tau\rangle - \langle\tilde{\sigma}_h, \tau\rangle_h \\
&= \langle\tilde{\sigma}_h, \Pi_h\tau\rangle - \langle\tilde{\sigma}_h, \Pi_h\tau\rangle_h + \langle\tilde{\sigma}_h, \Pi_h\tau - \tau\rangle_h \\
&= \langle\tilde{\sigma}_h - P_{W_h}\sigma, \Pi_h\tau\rangle - \langle\tilde{\sigma}_h - P_{W_h}\sigma, \Pi_h\tau\rangle_h \\
&\quad + \langle\tilde{\sigma}_h - P_{W_h}\sigma, \Pi_h\tau - \tau\rangle_h,
\end{aligned} \tag{3.20}$$

where, to get the last equality, we used (3.16) for the first two terms and (3.17) for the last term, respectively. By the Cauchy–Schwarz inequality and the boundedness of $\langle\cdot,\cdot\rangle_h$ in **(A)**, we have

$$|\langle\tilde{\sigma}_h - P_{W_h}\sigma, \Pi_h\tau\rangle - \langle\tilde{\sigma}_h - P_{W_h}\sigma, \Pi_h\tau\rangle_h| \lesssim (\|\tilde{\sigma}_h - \sigma\| + \|\sigma - P_{W_h}\sigma\|)\|\tau\|. \tag{3.21}$$

A similar argument with $\|\Pi\tau - \tau\| \lesssim \|\tau\|$ from the boundedness of Π_h, gives

$$\begin{aligned} |\langle \tilde{\sigma}_h - P_{W_h}\sigma, \Pi_h\tau - \tau\rangle_h| &\lesssim \|\tilde{\sigma}_h - P_{W_h}\sigma\|\|\Pi_h\tau - \tau\| \\ &\lesssim (\|\tilde{\sigma}_h - \sigma\| + \|\sigma - P_{W_h}\sigma\|)\|\tau\|. \end{aligned} \tag{3.22}$$

Then, (3.19) follows from (3.20), the triangle inequality, and the estimates (3.21) and (3.22). This completes the proof. □

In summary, we have presented perturbation results for mixed approximations of abstract Hodge Laplace problems with sufficient conditions for well-posedness and error estimates. If the method is based on a standard mixed method of the form (3.5), which is stable, then the extra error introduced by the modification of the bilinear form $\mathcal{B}$ into $\mathcal{B}_h$, cf. (3.8) and (3.10), is controlled by Theorem 3.2 above. Hence, the extra conditions to check are conditions **(A)** and **(B)**. This result will be utilized in the next two sections to establish error estimates for proper perturbations constructed such that the associated discrete coderivatives are local.

4. The simplicial case

In this section we apply the abstract framework in the previous section to mixed Hodge Laplace problems of the de Rham complex on simplicial meshes. We let Ω be a bounded polyhedral domain in $\mathbb{R}^n$. Recall that the de Rham complex on Ω is the Hilbert complex (W, d) with $W^k = L^2\Lambda^k(\Omega)$, $0 \le k \le n$, and with corresponding domain complex (V, d), where $V^k = H\Lambda^k(\Omega)$. Here, $d = d^k : H\Lambda^k(\Omega) \to L^2\Lambda^{k+1}(\Omega)$ is the exterior derivative. Let $\{\mathcal{T}_h\}$ be a family of shape-regular simplicial meshes of Ω, indexed by the parameter $h = \max\{\operatorname{diam} T \ : \ T \in \mathcal{T}_h\}$. Associated to the mesh $\mathcal{T}_h$ there are basically two families of finite element spaces of differential forms, $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ for $0 \le k \le n$, where r is the local polynomial degree.

In our discussion below we will study concrete realizations of discretizations of the form (3.11), where the discrete spaces V_h^{k-1} and V_h^k are chosen as $\mathcal{P}_1\Lambda^{k-1}(\mathcal{T}_h)$ and $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$, respectively. In other words, we are combining the lowest order finite element spaces of the two basic families. The exterior derivative d maps $\mathcal{P}_1\Lambda^{k-1}(\mathcal{T}_h)$ into $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$. In fact, this pair leads to a stable discretization of the corresponding Hodge Laplace problem in the form of (3.5), where the inner product $\langle\cdot,\cdot\rangle$ corresponds to L^2 inner products. Furthermore, the right-hand side of (3.9) is of order $O(h)$ under the assumption that the solution is sufficiently regular, cf. [7, Section 7.6] or [8, Chapter 5]. The corresponding discrete coderivative d_h^*, defined by (2.4), is a map from $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$ to $\mathcal{P}_1\Lambda^{k-1}(\mathcal{T}_h)$. However, as discussed in Sections 1 and 2 above, this operator will be nonlocal as a consequence of the continuity properties of the space $\mathcal{P}_1\Lambda^{k-1}(\mathcal{T}_h)$. Therefore, to achieve a local d_h^* operator we will follow the approach of Section 3 above, and modify the inner product on $\mathcal{P}_1\Lambda^{k-1}(\mathcal{T}_h)$, cf. (3.10). More precisely, we will replace the L^2 inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{P}_1\Lambda^{k-1}(\mathcal{T}_h)$ by a modified inner product $\langle\cdot,\cdot\rangle_h$. The main purpose of this modification is to obtain a local coderivative d_h^*, mapping $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$ to $\mathcal{P}_1\Lambda^{k-1}(\mathcal{T}_h)$, defined by

$$\langle d_h^* u, \tau\rangle_h = \langle u, d\tau\rangle, \quad u \in \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h),\ \tau \in \mathcal{P}_1\Lambda^{k-1}(\mathcal{T}_h). \tag{4.1}$$

To apply the convergence theory of Section 3 we need to verify that the stability condition **(A)** and the consistency condition **(B)** hold. In the present case we will verify condition **(B)** with the space $\tilde{V}_h^{k-1}$ taken to be V_h^{k-1}. Hence, to verify this condition we only need to show that condition (3.16) holds for a proper space

$W_h^{k-1} \subset L^2\Lambda^{k-1}(\Omega)$. Furthermore, to preserve the linear convergence of the method the space W_h^{k-1} should have the property that the L^2 error of the orthogonal projection onto W_h^{k-1} is of order $O(h)$, cf. Theorem 3.2. In fact, throughout the discussion of this section we will take W_h^{k-1} to be the space of piecewise constant forms, i.e.,

$$W_h^{k-1} := \{\tau \in L^2\Lambda^{k-1}(\Omega) \,|\, \tau|_T \in \mathcal{P}_0\Lambda^{k-1}(T) \text{ for all } T \in \mathcal{T}_h\}, \tag{4.2}$$

and as a consequence the desired accuracy of the projection is achieved.

Instead of discussing how to construct the modified inner product $\langle d_h^* u, \tau\rangle_h$ on one specific space, $\mathcal{P}_1\Lambda^{k-1}(\mathcal{T}_h)$, we will consider the construction of such modified inner products on all the spaces of the form $\mathcal{P}_1\Lambda^k(\mathcal{T}_h)$, where $0 \le k \le n$. We recall that the space $\mathcal{P}_r\Lambda^k(\mathbb{R}^n)$, i.e., the space of polynomial k-forms of degree r, consists of all polynomials of degree r with values in $\mathrm{Alt}^k(\mathbb{R}^n)$, and its dimension is given by

$$\dim \mathcal{P}_r\Lambda^k(\mathbb{R}^n) = \binom{n+r}{r}\binom{n}{k}.$$

Furthermore, an element u of $\mathcal{P}_1\Lambda^k(\mathbb{R}^n)$ is of the form

$$u(x) = a_0 + \sum_{j=1}^n a_j x_j, \qquad a_j \in \mathrm{Alt}^k(\mathbb{R}^n).$$

Hence, to determine u on a simplex $T \in \mathcal{T}_h$ we need $(n+1)\binom{n}{k}$ degrees of freedom. The standard degrees of freedom for the space $\mathcal{P}_1\Lambda^k(\mathcal{T}_h)$ is given by

$$\int_f \mathrm{tr}_f\, u \wedge v, \qquad v \in \mathcal{P}_1\Lambda^0(f), f \in \Delta_k(\mathcal{T}_h), \tag{4.3}$$

cf. [7, Theorem 4.10]. In fact, for any $f \in \Delta_k(\mathcal{T}_h)$ an element in $\mathcal{P}_1\Lambda^k(f)$ can be uniquely identified with an element in $\mathcal{P}_1\Lambda^0(f)$ through the Hodge star operator on f. Therefore, the degrees of freedom given by (4.3), on a fixed $f \in \Delta_k(\mathcal{T}_h)$, determines the $\mathrm{tr}_f\, u$ uniquely. This means that

$$\dim \mathcal{P}_1\Lambda^k(\mathcal{T}_h) = (k+1)|\Delta_k(\mathcal{T}_h)|,$$

where $|\Delta_k(\mathcal{T}_h)|$ is the cardinality of the set $\Delta_k(\mathcal{T}_h)$. Furthermore, the degrees of freedom given by (4.3) can be replaced by any other set of degrees of freedom which determines $\mathrm{tr}_f\, u$ uniquely on each $f \in \Delta_k(\mathcal{T}_h)$.

If $u \in \mathcal{P}_1\Lambda^k(\mathcal{T}_h)$ and $f \in \Delta_k(\mathcal{T}_h)$ then $\mathrm{tr}_f\, u$ is uniquely determined by $\mathrm{tr}_f\, u$ evaluated at each vertex of f. In particular, if f has vertices $x_0, x_1, \dots, x_k$, i.e., $f = [x_0, x_1, \dots x_k]$, then $\mathrm{tr}_f\, u$ at vertex x_i is determined by the functional $\phi_{f,x_i}(u)$ given by

$$\phi_{f,x_i}(u) = u_{x_i}(x_0 - x_i, \dots, x_{i-1} - x_i, x_{i+1} - x_i, \dots, x_k - x_i).$$

In other words, at the point x_i we apply the k-form u to the k vectors $x_j - x_i$, $j \ne i$, which all are tangential to f. By letting the index i run from 0 to k we obtain $k+1$ functionals which determines $\mathrm{tr}_f\, u$ uniquely. Furthermore, an element $u \in \mathcal{P}_1\Lambda^k(\mathcal{T}_h)$ is uniquely determined by the degrees of freedom $\{\phi_{f,x}(u)\}$, where (f, x) runs over all pairs such that $f \in \Delta_k(\mathcal{T}_h)$ and $x \in \Delta_0(f)$. Of course, this is again $(k+1)|\Delta_k(\mathcal{T}_h)|$ linearly independent degrees of freedom.

If $u \in \mathcal{P}_1\Lambda^k(\mathcal{T}_h)$ and x_i is a vertex, i.e., $x_i \in \Delta_0(\mathcal{T}_h)$, then u is not continuous at x_i. In general, u will have a separate value for each $T \in \mathcal{T}_h$ which touches x_i. However, the value of u at x_i, taken in the simplex T, is uniquely determined by the $\binom{n}{k}$ degrees of freedom given by $\phi_{f,x_i}(u)$ for all $f \in \Delta_k(T)$ such that $x_i \in \Delta_0(f)$. As a consequence, it follows that all the values of u at x_i are determined by $\phi_{f,x_i}(u)$ for $f \in \Delta_k(\mathcal{T}_h)$ with $x_i \in \Delta_0(f)$. In particular, if $\phi_{f,x}(u) = 0$ for a fixed $x \in \Delta_0(\mathcal{T}_h)$ and all $f \in \Delta_k(\mathcal{T}_h)$ such that $x \in \Delta_0(f)$, then $u = 0$ at x.

Given a set of degrees of freedom we can also define the corresponding dual basis $\{\psi_{f,x}\}$ for the space $\mathcal{P}_1\Lambda^k(\mathcal{T}_h)$ defined by

$$\phi_{g,y}(\psi_{f,x}) = \delta_{(f,x),(g,y)}, \qquad f, g \in \Delta_k(\mathcal{T}_h),\, x \in \Delta_0(f),\, y \in \Delta_0(g), \tag{4.4}$$

with the obvious interpretation that $\delta_{(f,x),(g,y)} = 1$ if $(f,x) = (g,y)$ and zero otherwise. It is clear from the above observation that $\psi_{f,x} = 0$ at all $y \in \Delta_0(\mathcal{T}_h)$ such that $y \neq x$. In fact, the piecewise linear form $\psi_{f,x}$ has a simple explicit representation in terms of barycentric coordinates. To see this, for $x_j \in \Delta_0(\mathcal{T}_h)$ we let λ_j be the piecewise linear function determined by $\lambda_j(x_j) = 1$, while λ_j vanish on all other vertices. In other words, λ_j corresponds to the barycentric coordinate associate the vertex x_j for all $T \in \mathcal{T}_h$ such that $x_j \in \Delta_0(T)$, and it is extended by zero elsewhere. Note that the corresponding 1-form, $d\lambda_j$, is piecewise constant and vanish outside the macroelement Ω_{x_j}. Here we use the notation that for any $f \in \Delta(\mathcal{T}_h)$, the associated macroelement Ω_f is given by

$$\Omega_f = \bigcup \{\, T \,|\, T \in \mathcal{T}_h,\ f \in \Delta(T)\,\}.$$

In particular, we note that if $[x_j, x_i] \in \Delta_1(\mathcal{T}_h)$ then $d\lambda_j(x_j - x_i) = 1$. On the other hand, $d\lambda_j(x_l - x_i) = 0$ if $[x_i, x_l] \in \Delta_1(\mathcal{T}_h)$ with both endpoints different from x_j.

Assume now that $f = [x_0, x_1, \dots x_k] \in \Delta_k(\mathcal{T}_h)$. The corresponding functions ψ_{f,x_i}, $i = 0, 1, \dots, k$ are given by

$$\psi_{f,x_i} = \lambda_i d\lambda_0 \wedge \cdots \wedge d\lambda_{i-1} \wedge d\lambda_{i+1} \wedge \cdots \wedge d\lambda_k,$$

where $\wedge$ denotes the wedge product. The functions ψ_{f,x_i} given above are obviously in $\mathcal{P}_1\Lambda^k(\mathcal{T}_h)$ and it is straightforward to check that they satisfy the conditions (4.4). Observe also that the basis functions ψ_{f,x_i} have local support. In fact, $\operatorname{supp}(\psi_{f,x_i}) \subset \Omega_{x_i}$. The basis $\{\psi_{f,x}\}$ for the space $\mathcal{P}_1\Lambda^k(\mathcal{T}_h)$, just introduced, is related to point values of traces via the dual relation (4.4). Furthermore, the modified inner product $\langle \cdot, \cdot \rangle_h$ will also be defined by point values. In fact, the modified inner product will be constructed such that the matrix $\langle \psi_{f,x_i}, \psi_{g,x_j} \rangle_h$ is block diagonal, and this is the key property we will use below to show that the constructed coderivative d_h^* is local.

To define the modified inner product $\langle \cdot, \cdot \rangle_h$ on $\mathcal{P}_1\Lambda^k(\mathcal{T}_h)$ we first recall that if $T = [x_0, x_1, \dots, x_n] \in \mathcal{T}_h$, then the identity

$$\int_T u(x)\, dx = \sum_{i=0}^{n} \frac{|T|}{n+1} u(x_i)$$

holds for all linear and scalar valued functions u. Here $|T|$ denotes the volume of T. Therefore, the bilinear form $\langle \cdot, \cdot \rangle_{h,T}$, given by

$$\langle u, v \rangle_{h,T} = \sum_{i=0}^{n} \frac{|T|}{n+1} \langle u_{x_i}, v_{x_i} \rangle_{\mathrm{Alt}},$$

defines an inner product on $\mathcal{P}_1\Lambda^k(T)$ which is exactly equal to the inner product on $L^2\Lambda^k(T)$ for $u \in \mathcal{P}_1\Lambda^k(T)$ and $v \in \mathcal{P}_0\Lambda^k(T)$. As a consequence, if we define $\langle \cdot, \cdot \rangle_h$ by

$$\langle u, v \rangle_h = \sum_{T \in \mathcal{T}_h} \langle u, v \rangle_{h,T}, \qquad u, v \in \mathcal{P}_1\Lambda^k(\mathcal{T}_h), \tag{4.5}$$

then this is an inner product on $\mathcal{P}_1\Lambda^k(\mathcal{T}_h)$ which is identical to the standard L^2 inner product if $u \in \mathcal{P}_1\Lambda^k(\mathcal{T}_h)$ and $v \in W_h^k$, cf. (4.2). Furthermore, it follows from standard scaling arguments and shape regularity that the inner product $\langle \cdot, \cdot \rangle_h$ is equivalent to the standard L^2 inner product on $\mathcal{P}_1\Lambda^k(\mathcal{T}_h)$, with constants independent of h.

We can summarize the discussion so far as follows.

Theorem 4.1. *For $1 \le k \le n$ let $V_h^{k-1} = \mathcal{P}_1\Lambda^{k-1}(\mathcal{T}_h)$ and $V_h^k = \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$. Furthermore, let the bilinear form $\mathcal{B}_h$ be defined as in* (3.10) *with $\langle \cdot, \cdot \rangle$ being the appropriate L^2 inner products and the modified inner product $\langle \cdot, \cdot \rangle_h$ on V_h^{k-1} defined as in* (4.5)*. Then the stability condition* **(A)** *and the consistency condition* **(B)** *holds, where W_h^{k-1} is given as in* (4.2)*.*

As we have already observed above the solution (σ_h, u_h, p_h) of the problem (3.11), with the set up given in the theorem above, will in general converge to the corresponding exact solution of the Hodge Laplace problem. This is just a consequence of the estimate (3.9) combined with Theorem 3.2. Furthermore, under the appropriate regularity assumptions on the exact solution the convergence will be linear with respect to the mesh size h, i.e., the error will be $O(h)$.

Next, we will consider the operator d_h^* defined by (4.1), and show that this operator is indeed a local operator. This will basically follow from the fact that the matrix $\langle \psi_{f,x_i}, \psi_{g,x_j} \rangle_h$ is block diagonal.

Theorem 4.2. *For $1 \le k \le n$ let $d_h^* : \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h) \to \mathcal{P}_1\Lambda^{k-1}(\mathcal{T}_h)$ be the operator defined by* (4.1)*. This operator is local. More precisely, for any vertex $x_i \in \Delta_0(\mathcal{T}_h)$ the values $(d_h^* u)_{x_i}$ only depends on u restricted to the macroelement Ω_{x_i}.*

Proof. For any $u \in \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$ we can express $d_h^* u$ in terms of the basis functions ψ_{f,x_i} in $\mathcal{P}_1\Lambda^{k-1}(\mathcal{T}_h)$, i.e.,

$$d_h^* u = \sum_{(f,x_i)} c_{f,x_i} \psi_{f,x_i}, \qquad c_{f,x_i} \in \mathbb{R},$$

where the sum runs over all pairs (f, x_i) such that $f \in \Delta_{k-1}(\mathcal{T}_h)$ and $x_i \in \Delta_0(f)$. Furthermore, the matrix $\langle \psi_{f,x_i}, \psi_{g,x_j} \rangle_h$ is block diagonal, where the blocks are of the form $\langle \psi_{f,x_i}, \psi_{g,x_i} \rangle_h$, i.e., they correspond to the vertices x_i in $\Delta_0(\mathcal{T}_h)$. Therefore, if we fix a vertex x_i, then all the coefficients of the form c_{f,x_i} is determined by the subsystem of (4.1) of the form

$$\sum_f c_{f,x_i} \langle \psi_{f,x_i}, \psi_{g,x_i} \rangle_h = \langle u, d\psi_{g,x_i} \rangle, \tag{4.6}$$

where f and g runs over all elements of $\Delta_{k-1}(\mathcal{T}_h)$ which contains the vertex x_i. However, this represents a square positive definite system which determines the coefficients c_{f,x_i} uniquely, and hence all the values $(d_h^* u)_{x_i}$. Finally, since the support of the basis functions ψ_{g,x_i} is contained in Ω_{x_i} it follows that the right hand side of (4.6) only depends on u restricted to Ω_{x_i}. □

It follows from the proof above that the coefficients c_{f,x_i} can be computed from the local systems (4.6). When x_i runs over all the vertices of the mesh, these matrices represent the diagonal blocks of the full matrix $\left\langle \psi_{f,x_i}, \psi_{g,x_j} \right\rangle_h$. In fact, the elements of the block associated the vertex x_i can be explicitly expressed in terms of the volumes of the n simplexes T touching x_i, the volumes of f and g, and the principal angles between f and g.

To see this, and to have the simplest notation, we perform this discussion of the matrix $\left\langle \psi_{f,x_i}, \psi_{g,x_j} \right\rangle_h$ in the setting of k-forms instead of $(k-1)$-forms. We fix a vertex in $\Delta_0(\mathcal{T}_h)$, and call it x_0. To compute the elements of the diagonal block of the matrix $\left\langle \psi_{f,x_i}, \psi_{g,x_j} \right\rangle_h$, associated the vertex x_0, we let $f = [x_0, x_1, \cdots, x_k] \in \Delta_k(\mathcal{T}_h)$. If we assume that the vertices are ordered, such that the vectors $x_1 - x_0, \cdots, x_k - x_0$ are positively oriented, then

$$d\lambda_1 \wedge \cdots \wedge d\lambda_k = \frac{1}{k!\,|f|} \operatorname{vol}_f, \tag{4.7}$$

when the forms are restricted to vectors which are tangential to f, cf. [7, Section 4.1]. Here vol_f denotes the standard volume form on f. If f and g are two k–dimensional simplexes containing x_0 we then obtain that

$$\begin{aligned} \langle \psi_{f,x_0}, \psi_{g,x_0} \rangle_h &= \sum_T \frac{|T|}{n+1} \langle (\psi_{f,x_0})_{x_0}, (\psi_{g,x_0})_{x_0} \rangle_{\mathrm{Alt}} \\ &= \frac{1}{(n+1)(k!)^2} \sum_T \frac{|T|}{|f|\,|g|} \langle \operatorname{vol}_f, \operatorname{vol}_g \rangle_{\mathrm{Alt}}\,, \end{aligned}$$

where the sum is over all $T \in \mathcal{T}_h$ such that both f and g are in $\Delta_k(T)$. Furthermore, we assume that vol_f has been properly extended to a k-form on $\mathbb{R}^n$ such that (4.7) holds for all vectors and for $x \in T$. However, the inner product $\langle \operatorname{vol}_f, \operatorname{vol}_g \rangle_{\mathrm{Alt}}$ is related to the principal angles of the two k–dimensional subspaces of $\mathbb{R}^n$ containing f and g, cf. for example [32, Theorem 5]. Therefore, we have indeed obtained the desired representation of the elements of the matrix $\left\langle \psi_{f,x_j}, \psi_{g,x_j} \right\rangle_h$.

5. The cubical case

In this section we present concrete realizations of the abstract framework studied in Section 3 above for approximations of the mixed Hodge Laplace problems on cubical meshes. Here a cubical mesh $\mathcal{T}_h$ on the domain Ω is a mesh where each element is a Cartesian product of intervals.

Mixed finite element methods with local coderivatives on cubical meshes have been studied by Wheeler and collaborators for the Darcy flow problems in two and three dimensions (i.e., $k = n$ and $n = 2, 3$), see [29, 33]. In the two dimensional case the arguments are rather similar to the simplicial case. By choosing $V_h^{k-1} = V_h^1$ as the lowest order Brezzi–Douglas–Marini space (BDM_1), cf. [15], and piecewise constant functions for $V_h^k = V_h^2$, combined with an integration rule based on vertex values, a local coderivative d_h^* is obtained. However, the natural analog of this approach for $n = 3$, where the lowest order Brezzi–Douglas–Duran–Fortin space (BDDF_1, [14]) is chosen for $V_h^{k-1} = V_h^2$ and $V_h^k = V_h^3$ consist of piecewise constants, will not lead to a corresponding local method. To overcome this problem Wheeler et al. replaced the standard BDDF_1 space by an enriched space. The discussion in this section can be seen as a generalization of the discussion given in [29, 33] to general k-forms in any dimension n. The most natural analogs of the BDM_1 and

BDDF_1 spaces for the case of differential forms and higher space dimensions are the $\mathcal{S}_1\Lambda^k(\mathcal{T}_h)$ spaces introduced by Arnold and Awanou in [5], cf. the discussion given in the introduction of that paper. We will give a brief review of these spaces below. However, to obtain the finite element spaces we need to obtain local approximations of the coderivatives, we will enrich the finite element spaces $\mathcal{S}_1\Lambda^k(\mathcal{T}_h)$ to obtain a larger spaces, which we will denote $\mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$.

For our discussion below we introduce some additional notation. Recall the definition of the set $\Sigma(k)$ given in Section 2 above, i.e., $\sigma \in \Sigma(k)$ is an increasing sequence with values σ_i, $1 \le i \le k$, such that

$$1 \le \sigma_1 < \sigma_2 < \ldots < \sigma_k \le n.$$

The set $\Sigma(k)$ has $\binom{n}{k}$ elements. We will use $[\![\sigma]\!]$ to denote the range of σ, i.e.,

$$[\![\sigma]\!] = \{\sigma_1, \sigma_2, \ldots, \sigma_k\} \subset \{1, 2, \ldots, n\},$$

and σ^* is the complementary sequence in $\Sigma(n-k)$ such that

$$[\![\sigma]\!] \cup [\![\sigma^*]\!] = \{1, 2, \ldots, n\}.$$

For each $\sigma \in \Sigma(k)$ we define $dx_\sigma = dx_{\sigma_1} \wedge \cdots \wedge dx_{\sigma_k}$ and the set $\{dx_\sigma \,:\, \sigma \in \Sigma(k)\}$ is a basis of $\mathrm{Alt}^k(\mathbb{R}^n)$. A differential k-form u then admits the representation

$$u = \sum_{\sigma \in \Sigma(k)} u_\sigma dx_\sigma,$$

where the coefficients u_σ are scalar functions on Ω. Furthermore, the exterior derivative du can be expressed as

$$du = \sum_{\sigma \in \Sigma(k)} \sum_{i=1}^{n} \partial_i u_\sigma dx_i \wedge dx_\sigma,$$

if $\partial_i u_\sigma$ is well-defined as a function on Ω. The Koszul operator $\kappa : \mathrm{Alt}^k(\mathbb{R}^n) \to \mathrm{Alt}^{k-1}(\mathbb{R}^n)$ is defined by contraction with the vector x, i.e., $(\kappa u)_x = u_x \lrcorner x$. As a consequence of the alternating property of u it therefore follows that $\kappa \circ \kappa = 0$. It also follows that

$$\kappa(dx_\sigma) = \kappa(dx_{\sigma_1} \wedge \cdots \wedge dx_{\sigma_k}) = \sum_{i=1}^{k} (-1)^{i+1} x_{\sigma_i} dx_{\sigma_1} \wedge \cdots \wedge \widehat{dx_{\sigma_i}} \wedge \cdots \wedge dx_{\sigma_k},$$

where $\widehat{dx_{\sigma_i}}$ means that the term dx_{σ_i} is omitted. This definition is extended to the space of differential k-form on Ω by linearity, i.e.,

$$\kappa u = \kappa \sum_{\sigma \in \Sigma(k)} u_\sigma dx_\sigma = \sum_{\sigma \in \Sigma(k)} u_\sigma \kappa(dx_\sigma).$$

If f is an $(n-1)$-dimensional hyperplane of $\mathbb{R}^n$, obtained by fixing one coordinate, for example

$$f = \{\, x \in \mathbb{R}^n \,:\, x_1 = c \,\},$$

then we can define the Koszul operator κ_f for forms defined on f by $(\kappa_f v)_x = v_x \lrcorner (x - x^f)$, where $x^f = (c, 0, \ldots, 0)$. We note that the vector $x - x^f$ is in the tangent space of f for $x \in f$. Since $\mathrm{tr}_f(u \lrcorner (x - x^f)) = \mathrm{tr}_f\, u \lrcorner (x - x^f)$ for $x \in f$ and $(\kappa u)_x = u_x \lrcorner (x - x^f) + u_x \lrcorner x^f$, we can conclude that

$$\mathrm{tr}_f\, \kappa u = \kappa_f \,\mathrm{tr}_f\, u + \mathrm{tr}_f(u \lrcorner x^f). \tag{5.1}$$

For a multi-index α of n nonnegative integers, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Hence, if u is in $\mathcal{P}_r\Lambda^k$, the space of polynomial k-forms of order r, then u can be expressed as

$$u = \sum_{\sigma \in \Sigma(k)} u_\sigma dx_\sigma, \qquad u_\sigma = \sum_{|\alpha| \le r} c_\alpha x^\alpha \in \mathcal{P}_r,$$

where $|\alpha| = \sum_i \alpha_i$. In other words, the coefficients u_σ are ordinary real valued polynomials of degree less than or equal to r. The corresponding tensor product space, $\mathcal{Q}_r\Lambda^k$, consists of k-forms u where the coefficients u_σ is a tensor product of polynomials of degree less than or equal to r, i.e.,

$$u_\sigma = \sum_{\alpha_i \le r, 1 \le i \le n} c_\alpha x^\alpha \in \mathcal{Q}_r.$$

Denoting $\mathcal{H}_r\Lambda^k$ the space of differential k-forms with homogeneous polynomial coefficients of degree r, we also have the identity

$$(\kappa d + d\kappa)u = (r + k)u, \qquad u \in \mathcal{H}_r\Lambda^k, \tag{5.2}$$

cf. [7, Section 3].

5.1. **The families $\mathcal{Q}_r^-\Lambda^k$ and $\mathcal{S}_r\Lambda^k$.** Our discussion below relates two of the previously constructed families of finite element spaces with respect to cubical meshes, the $\mathcal{Q}_r^-\Lambda^k$-family of [6] and the $\mathcal{S}_r\Lambda^k$-family of [5]. Here the parameter $r \ge 1$ is related to the local polynomial degree, and for each k, $0 \le k \le n$, the spaces $\mathcal{Q}_r^-\Lambda^k(\mathcal{T}_h)$ and $\mathcal{S}_r\Lambda^k(\mathcal{T}_h)$ are subspaces of $H\Lambda^k(\Omega)$. Furthermore, each family is nested, i.e., $\mathcal{Q}_{r-1}^-\Lambda^k(\mathcal{T}_h) \subset \mathcal{Q}_r^-\Lambda^k(\mathcal{T}_h)$ and $\mathcal{S}_{r-1}\Lambda^k(\mathcal{T}_h) \subset \mathcal{S}_r\Lambda^k(\mathcal{T}_h)$. There are also other families of cubical finite element differential forms proposed in the literature, cf. for example [18, 19, 25], but these spaces will not be used here.

The families $\mathcal{Q}_r^-\Lambda^k$ and $\mathcal{S}_r\Lambda^k$ lead to subcomplexes of the de Rham complex of the form

$$\begin{aligned} &\mathbb{R} \longrightarrow \mathcal{Q}_r^-\Lambda^0(\mathcal{T}_h) \xrightarrow{d^0} \mathcal{Q}_r^-\Lambda^1(\mathcal{T}_h) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \mathcal{Q}_r^-\Lambda^n(\mathcal{T}_h) \longrightarrow 0, \\ &\mathbb{R} \longrightarrow \mathcal{S}_r\Lambda^0(\mathcal{T}_h) \xrightarrow{d^0} \mathcal{S}_{r-1}\Lambda^1(\mathcal{T}_h) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \mathcal{S}_{r-n}\Lambda^n(\mathcal{T}_h) \longrightarrow 0. \end{aligned} \tag{5.3}$$

For a given k, $0 \le k \le n$, and a given cubical mesh $\mathcal{T}_h$ the space $\mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h)$ is the simplest space in the two families above. In fact, we will see below that for any $r \ge 1$ we also have

$$\mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h) \subset \mathcal{Q}_r^-\Lambda^k(\mathcal{T}_h), \mathcal{S}_r\Lambda^k(\mathcal{T}_h).$$

Furthermore, in complete analogy with the Whitney forms, $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$ in the case of simplicial meshes, the spaces $\mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h)$ has a single degree of freedom associated each subsimplex of dimension k. More precisely, the degrees of freedom for an element $u \in \mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h)$ are given by

$$\int_f \operatorname{tr}_f u, \quad f \in \Delta_k(\mathcal{T}_h). \tag{5.4}$$

If $T = I_1 \times I_2 \times \ldots \times I_n \in \mathcal{T}_h$ and $u \in \mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h)$, then $u|_T$ is of the form

$$u|_T = \sum_{\sigma \in \Sigma(k)} \Big(\sum_{\alpha_j \le 1 - \delta_{j,\sigma}} c_\alpha x^\alpha \Big) dx_\sigma, \tag{5.5}$$

where $\delta_{j,\sigma} = 1$ if $j \in [\![\sigma]\!]$ and zero otherwise. The local spaces $\mathcal{Q}_1^-\Lambda^k(T)$ has dimension $2^{n-k}\binom{n}{k}$, and together with degrees of freedom (5.4) this defines the space

$\mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h)$. In particular, the space $\mathcal{Q}_1^-\Lambda^0(\mathcal{T}_h) = \mathcal{Q}_1\Lambda^0(\mathcal{T}_h)$, while $\mathcal{Q}_1^-\Lambda^n(\mathcal{T}_h) = \mathcal{P}_0\Lambda^n(\mathcal{T}_h)$, i.e., the space of piecewise constant n-forms. Furthermore, for each k with $0 < k < n$, the space $\mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h)$ is strictly contained in $\mathcal{Q}_1\Lambda^k(\mathcal{T}_h)$.

In [5] the definition of the spaces $\mathcal{S}_r\Lambda^k(\mathcal{T}_h)$ was based on the concept of linear degree. However, a simple and more explicit charaterization of these spaces can be given when $r = 1$. By utilizing the definition given in [5] in this special case we can derive that a function u in the space $\mathcal{S}_1\Lambda^k(\mathcal{T}_h)$ is locally of the form

$$u|_T = u^- + d\kappa \sum_{\sigma\in\Sigma(k)} \sum_{i\in[\![\sigma]\!]} \Big(\sum_{\alpha_j \le 1-\delta_{j,\sigma}} c_\alpha x^\alpha \Big) x_i \, dx_\sigma, \quad T \in \mathcal{T}_h, \tag{5.6}$$

where $u^- \in \mathcal{Q}_1^-\Lambda^k(T)$. The local space $\mathcal{S}_1\Lambda^k(T)$ has dimension $2^{n-k}\binom{n}{k}(k+1)$, which should be compared with the fact that $\dim \mathcal{Q}_1\Lambda^k(T) = 2^n\binom{n}{k}$. Furthermore, the degrees of freedom of the space $\mathcal{S}_1\Lambda^k(\mathcal{T}_h)$ is given by

$$\int_f \operatorname{tr}_f u \wedge v, \qquad v \in \mathcal{P}_1\Lambda^0(f), \quad f \in \Delta_k(\mathcal{T}_h). \tag{5.7}$$

In the special cases $k = 0$ and $k = n$ we have

$$\mathcal{S}_1\Lambda^0(\mathcal{T}_h) = \mathcal{Q}_1\Lambda^0(\mathcal{T}_h) \quad \text{and } \mathcal{S}_1\Lambda^n(\mathcal{T}_h) = \mathcal{P}_1\Lambda^n(\mathcal{T}_h).$$

Furthermore, when $n = 2$ or 3 the degrees of freedom of the space $\mathcal{S}_1\Lambda^{n-1}(\mathcal{T}_h)$ corresponds to the degrees of freedom of the BDM_1 and the BDDF_1 spaces.

It follows from (5.6) that $d\mathcal{S}_1\Lambda^k(\mathcal{T}_h) = d\mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h)$. Since it is well known that the pair $(\mathcal{Q}_1^-\Lambda^{k-1}(\mathcal{T}_h), \mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h))$ is a stable pair for the standard mixed formulation (2.3), cf. [6], it is an easy consequence of this property that the pair $(\mathcal{S}_1\Lambda^{k-1}(\mathcal{T}_h), \mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h))$ also leads to a stable method. However, as we have already indicated above, the spaces $\mathcal{S}_1\Lambda^{k-1}(\mathcal{T}_h)$ has to be enriched in order to be useful in the present setting, i.e., to give rise to a method with a local coderivative d_h^*. This larger space is introduced below and denoted $\mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$.

5.2. **The spaces $\mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$.** For $\mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$ we first define the space of shape functions $\mathcal{S}_1^+\Lambda^k$. We prove that this space is invariant under dilation and translation, then the space of local shape functions on T is well-defined as the restriction of $\mathcal{S}_1^+\Lambda^k$ on T.

To define $\mathcal{S}_1^+\Lambda^k$, let $\mathcal{B}\Lambda^k$ be the span of forms $\{p_{\sigma^*} p_\sigma dx_\sigma \,|\, \sigma \in \Sigma(k)\}$, where $p_\sigma \in \mathcal{Q}_1(\mathbb{R}^k)$ and $p_{\sigma^*} \in \mathcal{Q}_1(\mathbb{R}^{n-k})$ are polynomials in the variables $\{x_j\}_{j\in[\![\sigma]\!]}$ and $\{x_j\}_{j\in[\![\sigma^*]\!]}$, respectively, and where $p_\sigma(0) = 0$. From this definition of $\mathcal{B}\Lambda^k$ it is obvious that

$$\mathcal{Q}_1\Lambda^k = \mathcal{Q}_1^-\Lambda^k \oplus \mathcal{B}\Lambda^k. \tag{5.8}$$

Furthermore, it follows directly from the definition of $\mathcal{B}\Lambda^k$ that $d\mathcal{B}\Lambda^k \subset \mathcal{B}\Lambda^{k+1}$. In fact, if $u \in \mathcal{H}_r\Lambda^k \cap \mathcal{B}\Lambda^k$ then, by (5.2), $du = (r+k)d\kappa du \in d\kappa\mathcal{B}\Lambda^{k+1}$. Therefore, we can conclude that

$$d\mathcal{B}\Lambda^k \subset \mathcal{B}\Lambda^{k+1} \cap d\kappa\mathcal{B}\Lambda^{k+1}. \tag{5.9}$$

We define $\mathcal{S}_1^+\Lambda^k$ by

$$\mathcal{S}_1^+\Lambda^k = \mathcal{Q}_1^-\Lambda^k + d\kappa\mathcal{B}\Lambda^k, \tag{5.10}$$

and we now prove that this is invariant under dilation and translation.

Lemma 5.1. *If $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a composition of dilation and translation, then $\phi^*\mathcal{S}_1^+\Lambda^k \subset \mathcal{S}_1^+\Lambda^k$, where ϕ^* is the corresponding pullback.*

Proof. Let $\phi(x) = Dx + b$ for a given invertible $n \times n$ diagonal matrix D and a vector $b \in \mathbb{R}^n$. To show $\phi^*\mathcal{S}_1^+\Lambda^k \subset \mathcal{S}_1^+\Lambda^k$, assume that $u \in \mathcal{S}_1^+\Lambda^k$ is written as $u = u^- + d\kappa u^+$ with $u^- \in \mathcal{Q}_1^-\Lambda^k$ and $u^+ \in \mathcal{B}\Lambda^k$. Then we have

$$\phi^*u = \phi^*u^- + \phi^*d\kappa u^+ = \phi^*u^- + d\phi^*\kappa u^+ = \phi^*u^- + d\kappa\phi^*u^+ + d((\phi^*u^+)\lrcorner b)$$

where we used $\phi^*\kappa u^+ = \kappa\phi^*u^+ + (\phi^*u^+)\lrcorner b$ in the last equality (cf. [7, Section 3.2]). We easily check that $\phi^*u^- \in \mathcal{Q}_1^-\Lambda^k$ and from (5.8) we have

$$d\kappa\phi^*u^+ \in d\kappa\mathcal{Q}_1\Lambda^k = d\kappa(\mathcal{Q}_1^-\Lambda^k \oplus \mathcal{B}\Lambda^k) \subset \mathcal{Q}_1^-\Lambda^k + d\kappa\mathcal{B}\Lambda^k = \mathcal{S}_1^+\Lambda^k.$$

It remains to show that

$$d((\phi^*u^+)\lrcorner b) \in \mathcal{S}_1^+\Lambda^k. \tag{5.11}$$

To see this, note that $(\phi^*u^+)\lrcorner b \in \mathcal{Q}_1\Lambda^{k-1}$, so $(\phi^*u^+)\lrcorner b \in \mathcal{Q}_1^-\Lambda^{k-1} \oplus \mathcal{B}\Lambda^{k-1}$ by (5.8). By (5.9) we therefore have

$$d((\phi^*u^+)\lrcorner b) \in \mathcal{Q}_1^-\Lambda^k + d\kappa\mathcal{B}\Lambda^k = \mathcal{S}_1^+\Lambda^k,$$

so (5.11) is established. □

We define the space of shape functions of $\mathcal{S}_1^+\Lambda^k(T)$ on an element $T \in \mathcal{T}_h$ as the restriction of the functions in the class $\mathcal{S}_1^+\Lambda^k$ such that

$$\mathcal{S}_1^+\Lambda^k(T) = \mathcal{Q}_1^-\Lambda^k(T) + d\kappa\mathcal{B}\Lambda^k(T). \tag{5.12}$$

By comparing the definition above with the characterizations of the spaces $\mathcal{Q}_1^-\Lambda^k(T)$ and $\mathcal{S}_1\Lambda^k(T)$ we can conclude that $\mathcal{S}_1^+\Lambda^k(T)$ contains these spaces. It also follows directly from the definition that $\dim\mathcal{B}\Lambda^k(T) = \binom{n}{k}2^{n-k}(2^k-1)$, and therefore we must have

$$\dim\mathcal{S}_1^+\Lambda^k(T) \le \dim\mathcal{Q}_1^-\Lambda^k(T) + \dim\mathcal{B}\Lambda^k(T) = 2^n\binom{n}{k}. \tag{5.13}$$

In fact, we will show below that this inequality is an equality, and that the degrees of freedom for this space is

$$\int_f \operatorname{tr}_f u \wedge v, \qquad v \in \mathcal{Q}_1\Lambda^0(f), \quad f \in \Delta_k(T). \tag{5.14}$$

Furthermore, it will follow from the discussion below that for any $u \in \mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$ the degrees of freedom associated an interface $f \in \Delta_{n-1}(\mathcal{T}_h)$ determines $\operatorname{tr}_f u$ uniquely. As a consequence, $\mathcal{S}_1^+\Lambda^k(\mathcal{T}_h) \subset H\Lambda^k(\Omega)$, and

$$\mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h) \subset \mathcal{S}_1\Lambda^k(\mathcal{T}_h) \subset \mathcal{S}_1^+\Lambda^k(\mathcal{T}_h).$$

From the definition above we can easily derive that $\mathcal{S}_1^+\Lambda^0(\mathcal{T}_h) = \mathcal{Q}_1\Lambda^0(\mathcal{T}_h) = \mathcal{Q}_1^-\Lambda^0(\mathcal{T}_h)$, and for $k = n$ it is a consequence of (5.2) that $d\kappa\mathcal{B}\Lambda^n(T) = \mathcal{B}\Lambda^n(T)$. Therefore, $\mathcal{S}_1^+\Lambda^n(\mathcal{T}_h) = \mathcal{Q}_1\Lambda^n(\mathcal{T}_h)$. Moreover, from (5.12) we have that $d\mathcal{S}_1^+\Lambda^k(\mathcal{T}_h) = d\mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h)$. As above, we can therefore conclude that the pair $(\mathcal{S}_1^+\Lambda^{k-1}(\mathcal{T}_h), \mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h))$ is a stable pair for the mixed formulation (2.3). Moreover, in the present case we will be able to construct a suitable integration rule such that conditions **(A)** and **(B)** of Section 3 are fulfilled, and which leads to a local coderivative d_h^*. However, first we need to analyze the spaces $\mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$ just introduced.

If m is a k-form given by $m = p dx_\sigma$, where $\sigma \in \Sigma(k)$ and the coefficient polynomial $p(x)$ is a monomial, then we will refer to m as a form monomial.

Lemma 5.2. *The following hold:*

(a) *For a form monomial $m \neq 0$ in $\mathcal{B}\Lambda^k(T)$, κm generates at least one form monomial such that its coefficient contains a quadratic factor.*
(b) *For $u \in \mathcal{B}\Lambda^k(T)$, the coefficient of each form monomial of $d\kappa u$ has at most one quadratic factor.*
(c) *For $u \in \mathcal{B}\Lambda^k(T)$ and $f \in \Delta_k(T)$, $\operatorname{tr}_f(d\kappa u) \in \mathcal{Q}_1\Lambda^k(f)$.*
(d) *The operator $d\kappa$ is injective on $\mathcal{B}\Lambda^k(T)$.*

Proof. Let us define $\mathcal{B}$ as

$$\mathcal{B} = \{p_{\sigma^*} p_\sigma dx_\sigma \in \mathcal{B}\Lambda^k(T) \mid \sigma \in \Sigma(k)\}, \tag{5.15}$$

where $p_\sigma(x) = x^\alpha$ and $p_{\sigma^*}(x) = x^\beta$ are monomials in $\mathcal{Q}_1$ of the variables $\{x_j\}_{j \in [\![\sigma]\!]}$ and $\{x_j\}_{j \in [\![\sigma^*]\!]}$, respectively, and where $|\alpha| \geq 1$. The set $\mathcal{B}$ is a basis for $\mathcal{B}\Lambda^k(T)$.

For $p_{\sigma^*} p_\sigma dx_\sigma \in \mathcal{B}$, $\kappa(p_{\sigma^*} p_\sigma dx_\sigma)$ is a linear combination of

$$m_i = (-1)^{i+1} p_{\sigma^*} p_\sigma x_{\sigma_i} dx_{\sigma_1} \wedge \cdots \wedge \widehat{dx_{\sigma_i}} \wedge \cdots \wedge dx_{\sigma_k}, \qquad 1 \leq i \leq k. \tag{5.16}$$

Since p_σ has a factor x_{σ_j} for some $\sigma_j \in [\![\sigma]\!]$, the coefficient of m_j has $x_{\sigma_j}^2$ as factor, so claim (a) is proved. Furthermore, a direct computation gives

$$\begin{aligned} dm_i &= \partial_{\sigma_i}(p_{\sigma^*} p_\sigma x_{\sigma_i}) dx_\sigma \\ &\quad + (-1)^{i+1} \sum_{j \in [\![\sigma^*]\!]} (\partial_j p_{\sigma^*}) p_\sigma x_{\sigma_i} dx_j \wedge dx_{\sigma_1} \wedge \cdots \wedge \widehat{dx_{\sigma_i}} \wedge \cdots \wedge dx_{\sigma_k}. \end{aligned}$$

and each of these coefficients has at most one quadratic factor. Therefore, claim (b) is proved.

To prove (c), it is enough to show that $\operatorname{tr}_f dm_i \in \mathcal{Q}_1\Lambda^k(f)$ for any $f \in \Delta_k(T)$ and $1 \leq i \leq k$. Recall that $f \in \Delta_k(T)$ is determined by fixing values of $n-k$ variables. Let $I(f) \subset \{1, \ldots, n\}$ be the set of indices such that f is determined by fixing x_l for all $l \in I(f)$. By letting vol_f be the volume form on f, we have, up to a sign, that

$$\operatorname{tr}_f(dm_i) = \begin{cases} (\partial_{\sigma_i}(p_{\sigma^*} p_\sigma x_{\sigma_i}))|_f \operatorname{vol}_f, & \text{if } I(f) = [\![\sigma^*]\!], \\ (x_{\sigma_i}(\partial_j p_{\sigma^*}) p_\sigma)|_f \operatorname{vol}_f, & \text{if } I(f) = \{\sigma_i\} \cup [\![\sigma^*]\!] \setminus \{j\}, \\ 0, & \text{otherwise.} \end{cases}$$

Since all variables in $p_{\sigma^*} p_\sigma$ have degree at most 1, the same is the case for $\partial_{\sigma_i}(p_{\sigma^*} p_\sigma x_{\sigma_i})$, while all variables in $x_{\sigma_i}(\partial_j p_{\sigma^*}) p_\sigma$ have degree at most 1, with a possible exception for x_{σ_i} which is constant on f. So claim (c) follows.

To prove the injectivity of $d\kappa$ on $\mathcal{B}\Lambda^k(T)$, we first prove that κ is injective. To see this consider two distinct elements $m = p_{\sigma^*} p_\sigma dx_\sigma$ and $\tilde{m} = p_{\tilde{\sigma}^*} p_{\tilde{\sigma}} dx_{\tilde{\sigma}}$ of $\mathcal{B}$. We claim that all the monomials generated by κm and $\kappa \tilde{m}$ which have a quadratic factor, are also distinct. To see this we assume the contrary, i.e., that there are $i \in [\![\sigma]\!]$ and $\tilde{i} \in [\![\tilde{\sigma}]\!]$ such that $[\![\sigma]\!] \setminus \{i\} = [\![\tilde{\sigma}]\!] \setminus \{\tilde{i}\}$ and

$$x_i p_{\sigma^*} p_\sigma = \pm x_{\tilde{i}} p_{\tilde{\sigma}^*} p_{\tilde{\sigma}}, \tag{5.17}$$

where the left-hand side is quadratic in x_i and the right-hand side is quadratic in $x_{\tilde{i}}$. Since $p_{\sigma^*} p_\sigma, p_{\tilde{\sigma}^*} p_{\tilde{\sigma}} \in \mathcal{Q}_1$, this can be true only when $i = \tilde{i}$, i.e., $\sigma = \tilde{\sigma}$. However, (5.17) implies that $m = \tilde{m}$, which is a contradiction. This implies that the elements of $\kappa(\mathcal{B})$ are linearly independent, and therefore κ is injective on $\mathcal{B}\Lambda^k(T)$. Finally, since d is injective on the image of κ by (5.2), $d\kappa$ is injective on $\mathcal{B}\Lambda^k(T)$. □

The following key result is a consequence of the lemma just established.

Theorem 5.3. *For $T \in \mathcal{T}_h$ and $0 \le k \le n$ we have*

$$\dim \mathcal{S}_1^+\Lambda^k(T) = 2^n \binom{n}{k}.$$

Proof. By Lemma 5.2 (d), the spaces $d\kappa\mathcal{B}\Lambda^k(T)$ and $\mathcal{B}\Lambda^k(T)$ have the same dimension, therefore the conclusion will follow if we show that the sum (5.12) is a direct sum, cf. (5.13). To show that the sum (5.12) is direct, it is enough to show

$$D \cap d\kappa\mathcal{B}\Lambda^k(T) = \{0\}, \qquad D := \ker d \cap \mathcal{Q}_1^-\Lambda^k(T). \tag{5.18}$$

Note that $D\cap\mathcal{B}\Lambda^k(T) = \{0\}$ due to (5.8). Furthermore, by (5.2), $D = (d\kappa+\kappa d)D = d\kappa D$. Therefore, (5.18) will follow if we can show that $d\kappa$ is injective on $D\oplus\mathcal{B}\Lambda^k(T)$. However, by (5.2) this will follow if we can show that κ is injective on $D \oplus \mathcal{B}\Lambda^k(T)$. To see why this is the case we observe that $\kappa(\mathcal{Q}_1^-\Lambda^k(T)) \subset \mathcal{Q}_1^-\Lambda^{k-1}(T)$. Combined with Lemma 5.2 (a) this implies that $\kappa D \cap \kappa\mathcal{B}\Lambda^k(T) = \{0\}$. As a consequence, κ is injective on $D \oplus \mathcal{B}\Lambda^k(T)$ if it is injective on each of the spaces D and $\mathcal{B}\Lambda^k(T)$ separately. The latter statement follows from Lemma 5.2 (d), while the injectivity on D follows from the fact that $\ker d \cap \ker \kappa = \{0\}$ by (5.2). □

We are now ready to prove unisolvency of $\mathcal{S}_1^+\Lambda^k(T)$ with the degrees of freedom (5.14).

Theorem 5.4. *An element $u \in \mathcal{S}_1^+\Lambda^k(T)$ is uniquely determined by the degrees of freedom* (5.14).

Proof. Since $\dim \mathcal{Q}_1(f) = 2^k$ for $f \in \Delta_k(T)$, the number of degrees of freedom in (5.14) is

$$|\Delta_k(T)|\times 2^k = \binom{n}{k} 2^{n-k} \times 2^k = 2^n \binom{n}{k},$$

which is same as $\dim \mathcal{S}_1^+\Lambda^k(T)$. Therefore, it is enough to show that if $u \in \mathcal{S}_1^+\Lambda^k(T)$, and all the degrees of freedom (5.14) vanish, then $u = 0$.

By dilation and translation, we may assume that T is the unit hypercube $[0,1]^n \subset \mathbb{R}^n$, cf. Lemma 5.1. Suppose that $u = \sum_{\sigma\in\Sigma(k)} u_\sigma dx_\sigma \in \mathcal{S}_1^+\Lambda^k(T)$ and all the degrees of freedom (5.14) of u are zero. From Lemma 5.2 (c) we can conclude that $\operatorname{tr}_f u = 0$ for all $f \in \Delta_k(T)$. In particular, the coefficient u_σ vanish for all faces f where x_i is fixed in $\{0,1\}$ for $i \in [\![\sigma^*]\!]$, and as a consequence u_σ has $\prod_{i\in[\![\sigma^*]\!]} x_i(1-x_i)$ as a factor. Therefore, if $k < n-1$, it follows from Lemma 5.2 (b) that $u = 0$. Furthermore, if $k = n$, then $\mathcal{S}_1^+\Lambda^n(T) = \mathcal{Q}_1\Lambda^n(T)$, so $u = 0$ is a direct consequence of the degrees of freedom in this case.

It remains to cover the case $k = n-1$. In this case u can be written as

$$u = \sum_{i=1}^n u_i dx_{\sigma(i)}, \quad \text{where } dx_{\sigma(i)} := dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

From the discussion above we already know that the coefficients u_i have $x_i(1-x_i)$ as factors. In other words,

$$u_i = c_i x_i(1-x_i), \quad i = 1, 2, \ldots, n, \tag{5.19}$$

where the coefficients c_i may depend on x, but they are independent of x_i. Furthermore, since $\operatorname{tr} u = 0$ on the boundary of T and du is a constant n-form, we can conclude from Stokes theorem that $du \equiv 0$. Furthermore, u is of the form

$$u = \sum_{i=1}^{n} (a_i + b_i x_i) dx_{\sigma(i)} + d\kappa u^+,$$

with constant coefficients a_i, b_i, and $u^+ \in \mathcal{B}\Lambda^{n-1}(T)$. From the definition of the space $\mathcal{B}\Lambda^{n-1}(T)$ we have that

$$u^+ = \sum_{i=1}^{n} u_i^+ dx_{\sigma(i)}, \qquad \text{with } u_i^+ = p_i + x_i q_i,$$

where the polynomials p_i and q_i are in $\mathcal{Q}_1$, independent of the variable x_i, and satisfies $p_i(0) = q_i(0) = 0$.

Note that

$$\kappa u_i^+ dx_{\sigma(i)} = \sum_{j=1}^{i-1} (-1)^{j+1} x_j u_i^+ dx_{\sigma(j,i)} - \sum_{j=i+1}^{n} (-1)^{j+1} x_j u_i^+ dx_{\sigma(i,j)},$$

where $dx_{\sigma(i,j)}$, for $i < j$, is the $n-2$ form obtained from $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ by omitting dx_i and dx_j. If we let $v = d\kappa u^+$ then a further calculation using the definition of the exterior derivative gives

$$v = \sum_{i=1}^{n} v_i dx_{\sigma(i)}, \quad \text{where } v_i = \sum_{j=1}^{n} \partial_j (x_j u_i^+ + (-1)^{i+j+1} x_i u_j^+).$$

From this it follows that the coefficients u_i of u can be represented as $u_i = u_i^1 + u_i^2$ where

$$u_i^1 = a_i + \sum_{j=1}^{n} \partial_j (x_j p_i + (-1)^{i+j+1} x_i p_j) \quad \text{and } u_i^2 = x_i [b_i + \sum_{j=1}^{n} \partial_j (x_j q_i + (-1)^{i+j+1} x_j q_j)].$$

We observe that all terms in this expression for u_i, except for $a_i + \sum_{j \neq i} \partial_j (x_j p_i)$, has x_i as a factor. In fact, this term is independent of the variable x_i, and therefore we must have

$$a_i + \sum_{j \neq i} \partial_j (x_j p_i) = a_i + (n-1) p_i + \sum_{j=1}^{n} x_j \partial_j p_i \equiv 0.$$

However, by using (5.2) in the special case of zero forms, we easily see that the only possible solution is $p_i = -a_i/(n-1)$. In particular, since $p_i(0) = 0$, we can conclude that both p_i and a_i are zero. Therefore, $u_i = u_i^2 = x_i \tilde{u}_i$, where

$$\tilde{u}_i = b_i + \sum_{j=1}^{n} \partial_j (x_j q_i + (-1)^{i+j+1} x_j q_j), \quad i = 1, 2, \ldots, n. \tag{5.20}$$

As a consequence, we obtain

$$\partial_i \tilde{u}_i = \sum_{j=1}^{n} \partial_i \partial_j (x_j q_i + (-1)^{i+j+1} x_j q_j) = \sum_{j=1}^{n} (-1)^{i+j+1} \partial_i q_j.$$

However, by (5.19) we also have $\tilde{u}_i = c_i(1 - x_i)$ and $\partial_i \tilde{u}_i = -c_i$, and therefore we obtain

$$c_i = (-1)^i \partial_i Q, \quad \text{where } Q = \sum_{j=1}^{n} (-1)^j q_j.$$

The equation we obtain from the two representations of $\tilde{u}_i$ can be written

$$(-1)^i [b_i + n q_i + \sum_{j=1}^{n} x_j \partial_j q_i] - Q = (1 - x_i) \partial_i Q.$$

Note that $du = \sum_i (-1)^{i+1} b_i = 0$ and therefore, by summing the equation above over i, we obtain

$$\sum_{i=1}^{n} (x_i - \frac{1}{2}) \partial_i Q = 0.$$

However $Q \in \mathcal{Q}_1$, and by expanding Q in monomials with respect to the variables $x_i - \frac{1}{2}$ we can conclude that Q is a constant. Furthermore, it vanishes at the origin, so $Q \equiv 0$. From (5.20) we then obtain that

$$\frac{u_i}{x_i} = \tilde{u}_i = b_i + \sum_{j=1}^{n} \partial_j (x_j q_i),$$

which is independent of the variable x_i. By (5.19) this implies that each u_i is zero. This completes the proof. □

Next we consider the traces of elements in $\mathcal{S}_1^+ \Lambda^k(T)$ on $f \in \Delta_{n-1}(T)$. Since f is defined by fixing one coordinate, the other $n-1$ variables define a coordinate system on f. In particular, we can define the corresponding Koszul operator κ_f for differential forms on f, cf. (5.1), and as a consequence the space $\mathcal{S}_1^+ \Lambda^k(f)$ is defined by the embedding of f into $\mathbb{R}^{n-1}$.

Theorem 5.5. *If $f \in \Delta_{n-1}(T)$ and $k \le n-1$, then*

$$\operatorname{tr}_f \mathcal{S}_1^+ \Lambda^k(T) \subset \mathcal{S}_1^+ \Lambda^k(f). \tag{5.21}$$

Proof. Since the trace operator maps $\mathcal{Q}_1^- \Lambda^k(T)$ into $\mathcal{Q}_1^- \Lambda^k(f)$, we only have to show that $\operatorname{tr}_f (d\kappa \mathcal{B} \Lambda^k(T)) \subset \mathcal{S}_1^+ \Lambda^k(f)$.

Without loss of generality, we may assume that $f = \{x \in \mathbb{R}^n : x_1 = c\}$ for a constant c. Note that the definition of $\mathcal{B}\Lambda^k(T)$ then implies that

$$\operatorname{tr}_f \mathcal{B}\Lambda^k(T) \subset \mathcal{B}\Lambda^k(f). \tag{5.22}$$

Furthermore, any nonzero form monomial $u = u_\sigma dx_\sigma \in \mathcal{B}\Lambda^k(T)$ satisfies one of the following conditions:

i) $1 \notin [\![\sigma]\!]$,
ii) $1 \in [\![\sigma]\!]$ and there exists $i \in [\![\sigma]\!]$, $i \ne 1$, such that u_σ has x_i as a factor,
iii) $1 \in [\![\sigma]\!]$ and there exists no $i \in [\![\sigma]\!]$, $i \ne 1$, such that u_σ has x_i as a factor.

We will prove that $\operatorname{tr}_f d\kappa u \in \mathcal{S}_1^+ \Lambda^k(f)$ in each of these cases. In case i) it follows from (5.1) that $\operatorname{tr}_f \kappa u = \kappa_f \operatorname{tr}_f u$, since $u \lrcorner x^f = 0$. Therefore, $\operatorname{tr}_f d\kappa u = d\kappa_f \operatorname{tr}_f u$, and this is in $d\kappa_f \mathcal{B}\Lambda^k(f) \subset \mathcal{S}_1^+ \Lambda^k(f)$ by (5.22). In case ii) and iii) we write u as $u_\sigma dx_1 \wedge dx_\eta$, where $[\![\eta]\!] \subset \{2, \dots, n\}$. A direct computation shows that $\operatorname{tr}_f \kappa u = c u_\sigma dx_\eta$. In case ii) there is $i \in [\![\eta]\!]$ such that u_σ has x_i as a factor. Therefore,

$\operatorname{tr}_f \kappa u \in \mathcal{B}\Lambda^{k-1}(f)$, and (5.9) implies that $\operatorname{tr}_f d\kappa u = d \operatorname{tr}_f \kappa u$ is in $d\kappa_f \mathcal{B}\Lambda^k(f) \subset \mathcal{S}_1^+\Lambda^k(f)$. Finally, in case iii) $\operatorname{tr}_f \kappa u \in \mathcal{Q}_1^-\Lambda^{k-1}(f)$, and therefore

$$\operatorname{tr}_f d\kappa u = d \operatorname{tr}_f \kappa u \in \mathcal{Q}_1^-\Lambda^k(f) \subset \mathcal{S}_1^+\Lambda^k(f).$$

This completes the proof. □

The inclusion (5.21) is indeed an equality. In fact, this follows since an element of $\mathcal{S}_1^+\Lambda^k(f)$ is uniquely determined by degrees of freedom associated the elements of $\Delta_k(f)$. Furthermore, the trace result can be used repeatedly to conclude that

$$\operatorname{tr}_f \mathcal{S}_1^+\Lambda^k(T) = \mathcal{S}_1^+\Lambda^k(f), \quad f \in \Delta(T),\ n \geq \dim f \geq k.$$

In particular, if $\dim f = k$ we have

$$\operatorname{tr}_f \mathcal{S}_1^+\Lambda^k(T) = \mathcal{Q}_1\Lambda^k(T), \quad f \in \Delta_k(T). \tag{5.23}$$

An important consequence of the combination of the Theorems 5.4 and 5.5 is also that the space $\mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$ is a subspace of $H\Lambda^k(\Omega)$, since the traces are continuous over elements of $\Delta_{n-1}(\mathcal{T}_h)$. Furthermore, as we have already indicated above, it is a consequence of the fact that $d\mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h) = d\mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$ and the stability of the method derived from the $\mathcal{Q}_1^-\Lambda^k$ spaces, that the pair $(\mathcal{S}_1^+\Lambda^{k-1}(\mathcal{T}_h), \mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h))$ is a stable pair for the mixed formulation (2.3). Therefore, according to the abstract theory in Section 3, to obtain a convergent method with a local coderivative d_h^*, we need to define a proper integration rule such that conditions **(A)** and **(B)** holds.

5.3. **The local method.** It is a consequence of the standard error estimate (3.9) that the choices $V_h^{k-1} = \mathcal{S}_1^+\Lambda^{k-1}(\mathcal{T}_h)$ and $V_h^k = \mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h)$ for the standard mixed method (2.3) will, under the assumption of a sufficiently regular solution, lead to an estimate for the error in the energy norm of order $O(h)$. Therefore, the goal is to perturb the method such that we preserve this convergence order, and also local coderivatives d_h^*. As in the simplicial case the discussion is based on the abstract theory of Section 3. Furthermore, $W^k = L^2\Lambda^k(\Omega)$, $V^k = H\Lambda^k(\Omega)$, and $\langle \cdot, \cdot \rangle$ is used to denote appropriate L^2 inner products.

In the present case condition **(B)** will appear slightly more complicated than in the simplicial case, since the space $\tilde{V}_h^{k-1}$ is strictly contained in V_h^{k-1}. In fact, we will take $\tilde{V}_h^{k-1} = \mathcal{Q}_1^-\Lambda^{k-1}(\mathcal{T}_h)$ and as in the simplicial case the space W_h^{k-1} is given by (4.2), i.e., it consists of piecewise constant $(k-1)$-forms. As a consquence, it follows from Theorem 3.2 that if we are able to define a modified inner product on $\mathcal{S}_1^+\Lambda^{k-1}(\mathcal{T}_h)$ such that conditions **(A)** and **(B)** hold with these choices, then the linear convergence is obtained.

As in the simplical case our choice of modified inner product can be motivated from an alternative set of degrees of freedom for the spaces $\mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$. The degrees of freedom for this space given by (5.14) shows that the global dimension of this space is given by

$$\dim \mathcal{S}_1^+\Lambda^k(\mathcal{T}_h) = 2^k |\Delta_k(\mathcal{T}_h)|.$$

In particular, $\operatorname{tr}_f u$ for $f \in \Delta_k(\mathcal{T}_h)$ and $u \in \mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$ is uniquely determined by the 2^k degrees of freedom associated f. However, elements of $\mathcal{Q}_1\Lambda^k(f)$ can be identified by an element in $\mathcal{Q}_1\Lambda^0(f)$, and therefore $\operatorname{tr}_f u$ is also determined by the values of $\operatorname{tr}_f u$ at the 2^k vertices of f. More precisely, for each $f \in \Delta_k(\mathcal{T}_h)$ and each $x_0 \in \Delta_0(f)$ we define the functional $\phi_{f,x}$ by

$$\phi_{f,x_0}(u) = u_x(x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0),$$

where $\{x_j\}_{j=1}^k$ are the k vertices of f such that $[x_0, x_j] \in \Delta_1(f)$. The functionals $\phi_{f,x}$ for $x \in \Delta_0(f)$ will determine $\operatorname{tr}_f u$ uniquely, and the set $\{\phi_{f,x} \,|\, f \in \Delta_k(\mathcal{T}_h),\, x \in \Delta_0(f)\,\}$ will be a set of global degrees of freedom of $\mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$. Furthermore, if $T \in \mathcal{T}_h$, then restriction of $u \in \mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$ to T at the vertex $x_0 \in \Delta_0(T)$ is determined by the $\binom{n}{k}$ possible choices of $\phi_{f,x_0}(u)$ for $f \in \Delta_k(T)$ such that $x_0 \in f$.

We will let $\{\psi_{f,x}\}$ be the corresponding dual basis for the space $\mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$, defined by

$$\phi_{g,y}(\psi_{f,x}) = \delta_{(f,x),(g,y)}, \qquad f, g \in \Delta_k(\mathcal{T}_h),\, x \in \Delta_0(f),\, y \in \Delta_0(g).$$

The modified inner product $\langle \cdot, \cdot \rangle_h$ on $\mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$ is now defined by

$$\langle u, v \rangle_h = \sum_{T \in \mathcal{T}_h} \langle u, v \rangle_{h,T}, \quad \text{where } \langle u, v \rangle_{h,T} = 2^{-n}|T| \sum_{x \in \Delta_0(T)} \langle u_x, v_x \rangle_{\mathrm{Alt}}.$$

It follows from the discussion of degrees of freedom above that the quadratic form $\langle \cdot, \cdot \rangle_h$ is an inner product $\mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$, and a standard scaling argument shows that it is equivalent to the standard L^2 inner product. So condition **(A)** holds.

Next, we will verify condition **(B)**, but with $k-1$ replaced by k to simplify the notation. We observe that the inner product $\langle \cdot, \cdot \rangle_h$ satisfies

$$\langle u, v \rangle_h = \langle u, v \rangle, \quad u \in \mathcal{Q}_1\Lambda^k(\mathcal{T}_h),\, v \in W_h^k.$$

As a consequence, (3.16) holds for $\tilde{V}_h^k = \mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h) \subset \mathcal{Q}_1\Lambda^k(\mathcal{T}_h)$. To complete the verification of condition **(B)** we have to define a projection $\Pi_h : \mathcal{S}_1^+\Lambda^k(\mathcal{T}_h) \to \mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h)$ which satisfies $d\Pi_h u = du$. We define this projection by the degrees of freedom (5.4), i.e.,

$$\int_f \operatorname{tr}_f \Pi_h u = \int_f \operatorname{tr}_f u, \quad f \in \Delta_k(\mathcal{T}_h).$$

Note that it follows from the definition of the spaces $\mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$ and $\mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h)$ that both $d\Pi_h u$ and du are piecewise constant forms, and by Stokes' theorem they are equal. Furthermore, since the degrees of freedom of $\mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h)$ is a subset of the degrees of freedom of $\mathcal{S}_1^+\Lambda^k(\mathcal{T}_h)$, the uniform L^2 boundedness of Π_h is a consequence of equivalence of the L^2 norm and a discrete norm defined by the degrees of freedom on each of these spaces. Finally, it remains to verify (3.17), i.e., we need verify that

$$\langle \Pi_h u, v \rangle_h = \langle u, v \rangle_h, \quad u \in \mathcal{S}_1^+\Lambda^k(\mathcal{T}_h), v \in W_h^k. \tag{5.24}$$

To see this we observe that

$$\begin{aligned}\langle u, v \rangle_{h,T} &= 2^{-n}|T| \sum_{x \in \Delta_0(T)} \langle u_x, v_x \rangle_{\mathrm{Alt}} \\ &= 2^{-n}|T| \sum_{f \in \Delta_k(T)} \sum_{x \in \Delta_0(f)} \langle (\operatorname{tr}_f u)_x, (\operatorname{tr}_f v)_x \rangle_{\mathrm{Alt}(f)},\end{aligned}$$

where the subscript $\mathrm{Alt}(f)$ indicates the inner product of alternating k-forms on f. Furthermore, since $\mathrm{tr}_f\, u \in \mathcal{Q}_1\Lambda^k(f)$ and $\mathrm{tr}_f\, v \in \mathcal{P}_0\Lambda^k(f)$, we have

$$\begin{aligned} 2^{-k} \sum_{x\in\Delta_0(f)} \langle (\mathrm{tr}_f\, u)_x, (\mathrm{tr}_f\, v)_x \rangle_{\mathrm{Alt}(f)} &= |f|^{-1} \int_f \langle \mathrm{tr}_f\, u, \mathrm{tr}_f\, v \rangle_{\mathrm{Alt}(f)} \ \mathrm{vol}_f \\ &= |f|^{-1} \int_f \langle \mathrm{tr}_f\, \Pi_h u, \mathrm{tr}_f\, v \rangle_{\mathrm{Alt}(f)} \ \mathrm{vol}_f \\ &= 2^{-k} \sum_{x\in\Delta_0(f)} \langle (\mathrm{tr}_f\, \Pi_h u)_x, (\mathrm{tr}_f\, v)_x \rangle_{\mathrm{Alt}(f)}, \end{aligned}$$

and hence the desired identity (5.24) holds. We have therefore verified condition **(B)** .

Finally, we need to convince ourselves that the corresponding operator d_h^*, defined by

$$\langle d_h^* u, \tau \rangle_h = \langle u, d\tau \rangle, \quad u \in \mathcal{Q}_1^- \Lambda^k(\mathcal{T}_h), \tau \in \mathcal{S}_1^+ \Lambda^k(\mathcal{T}_h),$$

is local. However, since the mass matrix $\langle \psi_{f,x}, \psi_{g,y} \rangle_h$ is block diagonal, where the blocks correspond to the vertices of $\mathcal{T}_h$, we can argue exactly as we did in the proof of Theorem 4.2 above to establish this property.

6. Conclusion

We have carried out the construction of finite element methods for the Hodge Laplace problems that admit local approximations of the coderivatives. Constructions are performed both with respect to simplicial and cubical meshes. These methods will therefore correspond to methods where the approximation of local constitutive laws are local, in contrast to the properties of more standard mixed finite element methods. The methods are of low order, and can also be seen as finite difference methods. However, an advantage of our approach is that there is a natural path to convergence estimates, based on standard finite element theory and variational crimes.

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