

1.2. Definition. A CW-structure on a space X is a cell decomposition of X with the following properties: Per Holm

- 1. Every p -cell e^p of X admits an homeomorphism $E^p \cong S^{p-1} \times I$ with a continuous extension $E^p \rightarrow X$ mapping S^{p-1} into a finite union of lower dimensional cells. $p = 0, 1, \dots$

ALGEBRAIC TOPOLOGY

Spring 1970

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Additional remarks 1.

- Remarks. (a) C stands for "closure-finite" and W for "weak topology". These terms go back to J.H.C. Whitehead.
- (b) For every p -cell e^p of a CW-complex X there is a map $\tilde{f}: E^p \rightarrow X$ which maps $E^p \cong S^{p-1} \times I$ homeomorphically onto e^p . Since e^p is compact and X is Hausdorff, $\tilde{f}(E^p)$ is closed. Therefore, by the continuity of $\tilde{f}$, $\tilde{f}(S^{p-1}) = \partial e^p$. Similarly $\tilde{f}$ maps

In this section we introduce the general concept of a CW-complex due to J.H.C. Whitehead. These will play a significant role in the sequel.

1.1. Definition. An (open) p-cell is a topological space homeomorphic to the standard open p-cell $E^p \cong S^{p-1} \times I$. A cell decomposition of a space X is a partition of X into cells.

To avoid exceptions we define $E^0 = \{0\}$ and $S^{-1} = \emptyset$. Then definition 1.1 is valid for $p = 0, 1, \dots$.
is called a closed p-cell of X . It is compact since it is the image of any characteristic map for e^p . Therefore if F is a subset of X , then $F \cap e^p$ is closed in X if and only if it

1.2. Definition. A CW-structure on a space X is a cell decomposition of X with the following properties:

C. Every p -cell e^p of X admits an homeomorphism $E^p - S^{p-1} \rightarrow e^p$ with a continuous extension $E^p \rightarrow X$ mapping S^{p-1} into a finite union of lower dimensional cells, $p = 0, 1, \dots$.

W. A subset F of X is closed if and only if $F \cap \bar{e}$ is closed for every cell e in X .

A CW-complex is a Hausdorff space with a CW-structure.

Remarks. (a) C stands for "closure finite" and W for "weak topology". These terms go back to J.H.C. Whitehead.

(b) For every p -cell e^p of a CW-complex X there is a map $\bar{\varphi}: E^p \rightarrow X$ which maps $E^p - S^{p-1}$ homeomorphically onto e^p . Since E^p is compact and X is Hausdorff, $\bar{\varphi}E^p$ is closed. Therefore, by the continuity of $\bar{\varphi}$, $\bar{\varphi}E^p = \bar{e}^p$. Similarly $\bar{\varphi}$ maps $S^{p-1} \subset E^p$ onto the boundary $\dot{e}^p = \bar{e}^p - e^p$. By assumption $\dot{e}^p$ is contained in a finite union of cells of dimensions strictly smaller than p (except when $p = 0$, in which case $\bar{e}^p = e^p$ and $\dot{e}^p = \emptyset$). The map $\varphi: E^p \rightarrow X$ is called a characteristic map for the cell e^p . Except for $p = 0$ it is not determined by e^p . For instance if $h: E^p \rightarrow E^p$ is a homeomorphism, then $hS^{p-1} = S^{p-1}$ and so $\varphi \circ h$ is another characteristic map for e^p .

(c) In a CW-complex X the closed hull of a p -cell e^p is called a closed p -cell of X . It is compact since it is the image of any characteristic map for e^p . Therefore if F is a subset of X , then $F \cap \bar{e}^p$ is closed in X if and only if it

is closed in $\bar{e}^p$. In particular X has the terminal topology with respect to all inclusions of closed cells or, equivalently, with respect to all characteristic maps into X . The open subsets of X are precisely those which meet every closed cell $\bar{e}$ in an open subset of $\bar{e}$.

(d) The closed cells of X are definitely closed subsets of X . The open cells, however, are generally not open subsets. Safe exceptions are the top dimensional cells when such exist: If $\dim X = m$ and e^m is an m -cell of X , then e^m is an open set. To test e^m choose a closed cell $\bar{e}^i$ and form the intersection $e^m \cap \bar{e}^i$. By property C $\dot{e}^i$ is covered by cells disjoint from e^m , hence $e^m \cap \bar{e}^i = e^m \cap e^i$ which is either empty or equal to e^i , an open subset of $\bar{e}^i$.

The p-skeleton of a CW-complex X is the subspace

$$X^p = \bigcup_{i \leq p} e^i = \bigcup_{i \leq p} \bar{e}^i$$

The two unions are equal because the boundary of an i -cell is already covered by lower dimensional cells. Every skeleton of X inherits a cellular decomposition which makes it a CW-complex in its own right. This is immediately clear except perhaps for the W -condition on the topology which we check now: Let F be a subset of X^p such that $F \cap \bar{e}^i$ is closed for every cell e^i in X^p . We show that F is in fact closed in X and therefore in X^p . Pick an arbitrary cell e of X and form $F \cap \bar{e}$. If $e \subset X^p$, then $F \cap \bar{e}$ is closed by assumption. If $e \not\subset X^p$, then $F \cap e = \emptyset$ and so $F \cap \bar{e} = F \cap \dot{e}$. But $\dot{e}$ is covered by lower dimensional cells $e_1, e_2, \dots, e_n$; let $e_1, e_2, \dots, e_k$ be those (if any) which are contained in X^p .

Then

$$\begin{aligned} F \cap \dot{e} &= F \cap (e_1 \cup \dots \cup e_n) \cap \dot{e} \\ &= F \cap (e_1 \cup \dots \cup e_k) \cap \dot{e} \\ &= F \cap (\bar{e}_1 \cup \dots \cup \bar{e}_k) \cap \dot{e} \\ &= ((F \cap \bar{e}_1) \cup \dots \cup (F \cap \bar{e}_k)) \cap \dot{e} \end{aligned}$$

which is closed, since by assumption $F \cap \bar{e}_i$ is closed for $e_i \subset X^p$. Thus we have shown that $F \cap \bar{e}$ is always closed for arbitrary cells e of X . Therefore F is a closed subset of X , since X is a CW-complex, and consequently F is a closed subset of X^p with the subspace topology. This demonstrates the W -property for the subspace topology of X^p and shows that X induces not only a topology and a cell decomposition on any skeleton, but a whole integrated CW-structure.

Incidentally, the argument above shows that if $F \subset X^p$ is a subset which meets every closed cell in X^p in a closed set, then F is closed in X . Applying this to $F = X^p$ we get

1.3 Lemma In a CW-complex every skeleton is a closed subset.]]

The sequence

$$X^0 \subset X^1 \subset X^2 \subset \dots$$

is called the skeleton filtration of X . X is a finite dimensional CW-complex if the skeleton filtration stabilizes, i.e. if it is of the form

$$X^0 \subset X^1 \subset \dots \subset X^m = X$$

The dimension of X is the least integer m such that $X^m = X$; equivalently, it is the maximal dimension of the cells of X .

To avoid exceptions we also introduce the empty complex $\emptyset$, and is the skeleton filtration of a CW-structure on the n -sphere.

we set $X^{-1} = \emptyset$ for any X . The empty complex gets dimension -1 . Note that every non-empty CW-complex must have 0-cells because of condition C.

The skeletons actually determine the topology of a CW-complex. We have the following property trivially satisfied.

1.4 Lemma. A CW-complex X has the terminal topology with respect to all skeletal inclusions $X^p \subset X$, $p = 0, 1, \dots$.]]

If X has only finitely many cells in each dimension, then the skeletons X^p are compact, each skeleton being a finite union of closed cells. A CW-complex is finite if it has a finite number of cells altogether. Clearly a finite CW-complex is compact and finite dimensional. The converse is also true.

1.5 Theorem A CW-complex is compact if and only if it is finite.

Proof. Only the last part needs verification. Suppose X is a compact CW-complex. Choose one point from each cell, and let $F \subset X$ be the set thus obtained. By property C every closed cell of X contains at most a finite number of elements of F . Hence F is closed in X and so is every subset of F , i.e. F is a discrete subset of X . Since X is compact, F is finite.]]

It is time to consider some examples.

1.6. Examples. (a) The sequence

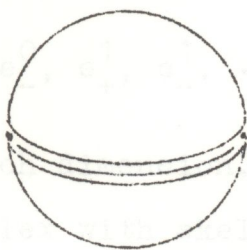
$$S^0 \subset S^1 \subset \dots \subset S^{n-1} \subset S^n$$

is the skeleton filtration of a CW-structure on the n -sphere.

In fact $S^n - S^{n-1}$ is the sum of two n -cells which we denote e_+^n and e_-^n . Similarly $S^{n-1} - S^{n-2}$ is the sum of two $(n-1)$ -cells e_+^{n-1} and e_-^{n-1} . Repeating this splitting process we get a finite collection of cells

$$\{e_+^0, e_-^0, e_+^1, e_-^1, \dots, e_+^n, e_-^n\}$$

which forms a cellular partition of S^n . Moreover, for each cell e^i there is a canonical homeomorphism $E^i \rightarrow \bar{e}^i$ which maps the boundary S^{i-1} identically onto $\partial e^i = S^{i-1}$. Finally the topology of S^n satisfies condition W because if $F \subset S^n$ is a subset such that $F \cap \bar{e}$ is closed for every cell e of S^n , then F is closed since $F = (F \cap \bar{e}_+^0) \cup (F \cap \bar{e}_-^0) \cup \dots \cup (F \cap \bar{e}_+^n) \cup (F \cap \bar{e}_-^n)$. This decomposition is called the regular cell decomposition of S^n .



(b) Let S^∞ be the union of all spheres S^n under the standard inclusions

$$S^0 \subset S^1 \subset \dots$$

The points of S^∞ are sequences $\{x_i\}_{i=1,2,\dots}$ whose coordinates x_i are all 0 except for a finite number and such that

$\sum_{i=1}^{\infty} x_i^2 = 1$. A point $x = \{x_i\}$ in S^{∞} belongs to S^n if and only if $x^{n+1} = x^{n+2} = \dots = 0$.

We give S^{∞} the terminal topology with respect to all inclusions $S^n \subset S^{\infty}$, $n = 0, 1, \dots$, so that a set $F \subset S^{\infty}$ is closed if $F \cap S^n$ is closed for all n . Then S^{∞} becomes a Hausdorff space: If x, y are two points of S^{∞} , then x, y belong to S^n for n sufficiently large. Let f_n be a continuous real valued function on S^n which is 0 on x and 1 on y . By Tietze's theorem there is a continuous extension f_{n+1} to S^{n+1} . Continuing the extension we construct functions $f_{n+2}, f_{n+3}, \dots$, which combine to a function f on S^{∞} such that $f|_{S^{n+1}} = f_{n+1}$. In particular f is continuous (the terminal topology!) and $f(x) = 0$, $f(y) = 1$.

Finally, the cell decompositions given to the spheres S^n in example (a) are compatible for different n and so define a cell decomposition of S^{∞}

$$\{e_+^0, e_-^0, e_+^1, e_-^1, \dots\}$$

with two cells in each dimension. This makes S^{∞} an infinite dimensional CW-complex with skeleton filtration $S^0 \subset S^1 \subset \dots$. The topology satisfies condition W , because if $F \subset S^{\infty}$ meets every closed cell in a closed set, then $F \cap S^n$ is closed in S^n for every n by example (a), hence F is closed in S^{∞} .

(c) In the sphere S^n pick a point e^0 and set $e^n = S^n - e^0$. Then $\{e^0, e^n\}$ is a CW-structure for S^n . Note that with this structure S^n is a CW-complex different from the one exhibited under (a). The skeleton filtration is

$$e^0 = (S^n)^0 = (S^n)^1 = \dots = (S^n)^{n-1} \subset (S^n)^n = S^n.$$

that of example (b). Similarly there is the infinite complex

(d) According to example 10 a real projective n -space P^n is homeomorphic to the adjunction space $E^n \cup_q P^{n-1}$ by a homeomorphism which is the identity on P^{n-1} . The map $q: S^{n-1} \rightarrow P^{n-1}$ is the canonical identification. Hence $e^n = P^n - P^{n-1}$ is an n -cell with a characteristic map

$$\bar{q}: E^n \rightarrow P^n$$

Repeating this procedure resolves P^n into a union of cells $e^0, e^1, \dots, e^n$, which give a CW-structure on P^n . The topology is trivially right because $\bar{e}^n = P^n$ ($\bar{q}$ is surjective). The lower projective spaces form the skeletons. Note that this cell decomposition is just the image of the regular cell decomposition of S^n under the canonical identification $q: S^n \rightarrow P^n$. For each $p \leq n$ the two p -cells e_+^p and e_-^p of S^n are mapped homomorphically onto the single p -cell e^p of P^n .

(e) Complex projective n -space CP^n has a canonical CW-decomposition with precisely one cell in each even dimension up to and including dimension $2n$:

$$CP^n = e^0 \cup e^2 \cup \dots \cup e^{2n}$$

The argument is similar to that given in example (d).

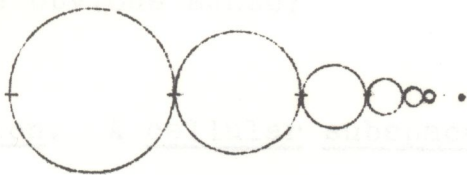
(f) Let P^∞ be the union of all real projective spaces P^n under the standard inclusions

$$P^0 \subset P^1 \subset \dots,$$

and give P^∞ the terminal topology with respect to all inclusions $P^n \subset P^\infty$. With this topology and the cell structure received from all P^n the space P^∞ becomes an infinite CW-complex with one cell in each dimension and with skeletons $P^0 \subset P^1 \subset \dots$. The verification is completely analogous to that of example (b). Similarly there is the infinite complex

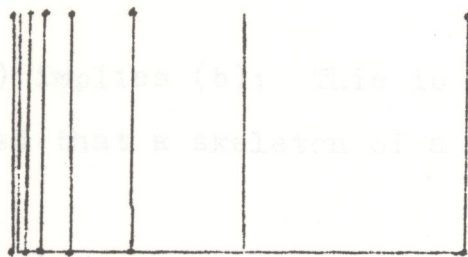
projective space CP^∞ which is a CW-complex with precisely one cell in each even dimension. These spaces play important roles in algebraic topology as classifying spaces for double coverings and complex line bundles respectively, cf. and

(g) Consider the space X below consisting of converging touching circles with the topology from the plane R^2 .



With the limit point adjoined X is compact. Moreover X has an obvious cellular partition into 0-cells and 1-cells indicated on the figure. This is clearly not a CW-structure, however, since it contains infinitely many cells. Because of the bad limit point there is actually no way of making X into a CW-complex, as we shall see. Without the limit point and with the indicated cell structure X would be a 1-dimensional CW-complex.

(h) The Comb Space is another compact subspace of R^2 with a natural cell structure.



For the same reasons as in example (g) above it is not a CW-complex. Nor can it be given a CW-structure since it is not

even locally connected, a topological property inherent in all CW-complexes, cf. theorem

(i) If $\{X_i\}_{i \in J}$ is a family of CW-complexes, then the topological sum $X = \Sigma X_i$ receives in a natural way a CW-structure. A cell of X is a cell of some X_i .

We have already noticed that the skeletons of a CW-complex X are themselves CW-complexes. In fact they are subcomplexes of X in the obvious sense:

1.7 Definition. A cellular subspace of a CW-complex X is a subspace with a cell decomposition by cells from X . A subcomplex of X is a cellular subspace whose cell decomposition is a CW-structure.

To recognize subcomplexes we have the following convenient result.

1.8 Theorem. Let A be a cellular subspace of a CW-complex X . Then the following are equivalent

(a) A is a subcomplex of X

(b) A is closed

(c) If e is a cell and $e \subset A$, then $\bar{e} \subset A$

Proof. (a) implies (b): This is a repetition of the argument which showed that a skeleton of a CW-complex is closed (with $F = A$).

(b) implies (c): Trivial

(c) implies (a): We have to check conditions C and W for A with its given cell decomposition and its subspace

topology. Condition C is immediately verified, while W requires another repetition of the skeleton argument.]]

Theorem 1.8 has the following immediate consequence

1.9 Corollary. In a CW-complex intersection and union of a family of subcomplexes is again a subcomplex.

Proof. This follows trivially from 1.8 (c).]]

If Z is a subset of a CW-complex X , the subcomplex spanned by Z is defined to be the intersection of all subcomplexes containing Z . It is not surprising that compact subsets span finite subcomplexes:

1.10 Lemma. In a CW-complex every compact subset is contained in a finite subcomplex.

Proof. If Z is a compact subset of a CW-complex X , then Z is covered by a finite number of cells $e_1, e_2, \dots, e_m$. (Otherwise we should be able to extract an infinite discrete subset from Z , as in the proof of theorem 1.5.) The boundaries of these cells are covered by a finite number of cells of lower dimensions $e_{m+1}, e_{m+2}, \dots, e_n$ which we add to our collection. Continuing this process we end up after a finite number of steps with a finite collection $\{e_1, e_2, \dots, e_q\}$ such that the boundary of any cell is covered by other cells in the collection. Then $A = \bigcup e_i$ is a finite subcomplex containing Z by theorem 1.8(c)]

A CW-complex is locally finite if every point has a neighborhood which is a finite subcomplex. By Lemma 1.10 a CW-complex is locally finite if and only if it is locally compact.

Examples (d), (e) and (f) above indicate that there exists a more constructive approach to CW-complexes. We now present this.

1.11 Theorem. Let X^p be a p -dimensional CW-complex, and $\varphi_i: S^q \rightarrow X^p$, $i \in J$, a family of maps, where $q \geq p$. Let $E = \sum_i E^{q+1}$ be the corresponding sum of closed standard cells of dimension $q+1$ and denote by S the subset $\sum_i S^q \subset E$. Let $\varphi: S \rightarrow X^p$ be the map defined by the family $\{\varphi_i\}$, and form the adjunction space

$$X^{q+1} = E \cup_{\varphi} X^p.$$

Then X^{q+1} is a $q+1$ -dimensional CW-complex with X^p as p -skeleton and with no cells in dimensions between p and $q+1$.

Proof. Recall that X^{q+1} is by definition the quotient space of $E + X^p$ by the relation which identifies $x \in S \subset E$ with $\varphi(x) \in X^p$. If we denote by $\bar{\psi}: E + X^p \rightarrow X^{q+1}$ the associated identification map, then the restriction $\bar{\varphi} = \bar{\psi}|_E$ is an extension of φ , i.e. we have a commutative diagram of maps and spaces

$$\begin{array}{ccc} E & \xrightarrow{\bar{\varphi}} & X^{q+1} \\ \cup & & \cup \\ S & \xrightarrow{\varphi} & X^p \end{array}$$

Furthermore, we denote the restrictions of $\bar{\varphi}$ to the single cells E^{q+1} of E by $\bar{\varphi}_i$ so that we have the commutative diagrams

$$\begin{array}{ccc} E^{q+1} & \xrightarrow{\bar{\varphi}_i} & X^{q+1} \\ \cup & & \cup \\ S^q & \xrightarrow{\varphi_i} & X^p \end{array}$$

Next observe that S is a closed subset of E since it meets every component E^{q+1} in a closed set S^q . By theorem therefore, $\bar{\varphi}$ restricts to a homeomorphism

$$E - S \approx X^{q+1} - X^p$$

In particular $X^{q+1} - X^p$ is a sum of $q+1$ - cells e_i^{q+1} with characteristic maps $\bar{\varphi}_i$. Together with the cells of X^p they give a cellular decomposition of X^{q+1} . This decomposition satisfies condition C for CW-complexes because the boundary of a $q+1$ - cell is a compact subset of X^p , hence is covered by a finite number of cells from X^p (Lemma 1.10). Furthermore the topology on X^{q+1} satisfies condition W: If $F \subset X^{q+1}$ is a set such that $F \cap \bar{e}$ is closed for every closed cell $\bar{e}$, then F is closed. In fact applying the assumption on F to cells in X^p tells us that $F \cap X^p$ is closed in X^p . Similarly, applying the condition to cells $\bar{e}_i^{q+1}$ tells that $\bar{\varphi}_i^{-1}F$ is closed in E^{q+1} and hence that $\bar{\varphi}^{-1}F$ is closed in E . But then $\bar{\psi}^{-1}F$ is closed in $E + X^p$, because $\bar{\psi}^{-1}F \cap E = \bar{\varphi}^{-1}F$ and $\bar{\psi}^{-1}F \cap X^p = F \cap X^p$. Since $\bar{\psi}$ is an identification, F is closed.

Finally it remains to check that X^{q+1} is a Hausdorff space. First notice that each $q+1$ - cell is an open subset of X^{q+1} . It follows that two different points x, y in X^{q+1} , at least one of which is in $X^{q+1} - X^p$ can be separated by disjoint open sets: Take a small compact neighborhood of the point in $X^{q+1} - X^p$. The interior and the exterior of that neighborhood will provide the open sets. Assume therefore that x, y are in X^p and let U^p, V^p be open disjoint sets in X^p containing x and y respectively. Let $U = \bar{\varphi}^{-1}U^p$ and $V = \bar{\varphi}^{-1}V^p$. Then U and V are disjoint open sets in S . Let $\bar{U}$ and $\bar{V}$ be

disjoint open sets in E which restricts to U and V on S and consider the sets $\bar{\varphi}\tilde{U} \cup U^p$ and $\bar{\varphi}\tilde{V} \cup V^p$ in X^{q+1} . These are certainly disjoint. But they are also open: To check that we pull back by $\bar{\varphi}$ and observe that the result is open in $E + X^p$. For instance $\bar{\varphi}^{-1}(\bar{\varphi}\tilde{U} \cup U^p)$ is open since it meets E in $\tilde{U}$ and X^p in U^p .]]

Theorem 1.11 tells how CW-complexes may be constructed recursively: Start with a non-empty discrete space X^0 . Such a space is certainly a 0-dimensional CW-complex. Then, given any set of maps $\varphi_i: S^q \rightarrow X^0$, $i \in J_q$, we may attach $q+1$ -cells E^{q+1} by means of the maps φ_i to obtain a $q+1$ -dimensional complex X^{q+1} . This complex has X^0 as skeletons up to dimension q , i.e. it has skeleton filtration $X^0 = X^1 = \dots = X^q \subset X^{q+1}$. By repeating the process we can build up a finite dimensional CW-complex of arbitrarily high dimension $q_n + 1$ and with cells in arbitrary dimensions $0 < q_1 + 1 < q_2 + 1 < \dots < q_n + 1$, if the relevant families of maps $\varphi_i: S^{q_j} \rightarrow X^{q_j+1}$ are given. The next result shows that all CW-complexes are essentially of this type.

1.12 Theorem. Let X be a $q+1$ -dimensional CW-complex with $q+1$ -cells $\{e_i^{q+1}\}$. Let $\{\bar{\varphi}_i\}$ be a corresponding family of characteristic maps for the $q+1$ -cells, and form the restrictions $\{\varphi_i\}$, $\varphi_i: S^q \rightarrow X^q$. Then, with notations as above, there is a homeomorphism

$$E \cup_{\varphi} X^q \approx X$$

which is the identity on X^q and maps $q+1$ -cells to $q+1$ -cells.

Proof. The characteristic maps $\bar{\varphi}_i: E^{q+1} \rightarrow X$ combine to a map $\bar{\varphi}: E \rightarrow X$ on the sum $E = \sum_i E^{q+1}$ whose restriction to $S = \sum_i S^q$ is precisely φ . The map $\bar{\varphi}: E \rightarrow X$ together with the inclusion $X^q \subset X$ again combine to a map $E + X^q \rightarrow X$ which we denote $\bar{\psi}$. We claim that $\bar{\psi}$ is an identification: $\bar{\psi}$ is certainly surjective. Suppose $U \subset X$ is a subset such that $\bar{\psi}^{-1}U$ is open. This means that $E \cap \bar{\psi}^{-1}U = \bar{\varphi}^{-1}U$ is open in E and $X^q \cap \bar{\psi}^{-1}U = X^q \cap U$ is open in X^q . From this follows that $\bar{\varphi}_i^{-1}U$ is open in E^{q+1} for all i , hence that $\bar{e}_i^{q+1} \cap U$ is open in $\bar{e}_i^{q+1}$ (since any characteristic map is an identification onto its image), and secondly that $\bar{e} \cap U$ is open in $\bar{e}$ for any cell e of X^q . Hence U is open in X and so $\bar{\psi}$ is an identification. This proves the claim since

$$E \cup_{\varphi} X^q = E + X^q / \bar{\psi} \approx X .]]$$

Thus every finite dimensional CW-complex is isomorphic to a repeated adjunction complex, "isomorphism" in this case meaning "cell preserving homeomorphism". The attaching maps $\varphi_i: S^q \rightarrow X^q$, $q = 0, 1, \dots$, are supplied by characteristic maps (or rather by their restrictions). A general CW-complex is isomorphic to a CW-complex where each skeleton is obtained from the preceeding by attaching (perhaps no) cells.

From theorem 1.12 follows

1.13 Corollary. If X is a CW-complex, then $X^p - X^{p-1}$ is a sum of p -cells.

Remarks. This also follows from remark (d) to definition 1.2: Each p -cell is an open subset of X^p . Therefore, $X^p - X^{p-1}$ being a disjoint union of these open sets is the sum of them.

1.14 Corollary. The 0-skeleton of a CW-complex is a discrete subspace.

We have already characterized the compact and locally compact CW-complexes as precisely those which are finite and locally finite, respectively. It is trivial to give examples of CW-complexes which are not locally finite, however. Thus the question arises what sort of restrictions exist on the topology of a general CW-complex. Our next theorem sums up the most useful general properties of the topology, and the proof illustrates the utility of the combinatorial structure on a CW-complex.

1.15 Theorem. Every CW-complex is a paracompact, perfectly normal k -space.

Remark. Recall that a space X is paracompact if every open cover of X has a locally finite open refinement. It is known that a paracompact Hausdorff space is normal. A space is perfectly normal if it is normal and every closed subset is the zero-set of a continuous real valued function. Finally a space is compactly generated or a k -space if it is Hausdorff and if each subset that meets every compact set in a closed set is itself closed.

Proof: Consider a CW-complex X . We need only show that X is paracompact and that every closed subset $A \subset X$ is the zero-set $f^{-1}\{0\}$ of some continuous function $f: X \rightarrow [0,1]$. To see that X is paracompact observe that X is covered by the family $\{X_i\}$ of all finite subcomplexes, since every closed

cell is contained in a finite subcomplex (lemma 1.10). Moreover, the family $\{X_i\}$ has the property that any union of members of $\{X_i\}$ is closed in X (theorem 1.8 (b) with corollary). Finally each X_i is paracompact, since in fact it is compact. Therefore, by theorem in the appendix X is paracompact. To prove the zero-set property note that this property is certainly shared by all 0-dimensional complexes. Assume that it is satisfied for CW-complexes of dimension $\leq q$, and let X be some $q + 1$ - dimensional complex. Let $\{e_i^{q+1}\}$ be the set of $q + 1$ - cells of X and let $\{\bar{\varphi}_i\}$ be a set of corresponding characteristic maps $\bar{\varphi}_i: E^{q+1} \rightarrow \bar{e}_i^{q+1} \subset X$. Finally let $F \subset X$ be a closed subset. Then, by assumption there is a continuous function $f: X^q \rightarrow [0,1]$ which is 0 precisely on $F \cap X^q$. Consider one of the copies E^{q+1} with its characteristic map $\bar{\varphi}_i: E^{q+1} \rightarrow X$. Form $G_i = \bar{\varphi}_i^{-1}F$ and $g_i = f \circ \varphi_i: S^q \rightarrow [0,1]$, with $\varphi_i = \bar{\varphi}_i|_{S^q}$. Then g_i is 0 precisely on $\varphi_i^{-1}F = G_i \cap S^q$. Assume that g_i can be extended to a function $\bar{g}_i: E^{q+1} \rightarrow [0,1]$ which is 0 precisely on G_i . Then g_i factorizes uniquely through $\bar{\varphi}_i$, i.e. $g_i = \bar{f}_i \circ \bar{\varphi}_i$, and $\bar{f}_i: \bar{e}_i^{q+1} \rightarrow [0,1]$ is a continuous extension of $f|_{\dot{e}_i^{q+1}}$ which is zero precisely on $F \cap \bar{e}_i^{q+1}$. Given such a function $\bar{f}_i$ for each i , since they coincide with f on X^q (wherever defined), they combine with f to a function $\bar{f}: X \rightarrow [0,1]$ which is 0 precisely on F . It is clear that $\bar{f}$ is continuous because $\bar{f}|_{X^q} = f$ is continuous and $\bar{f}|_{\bar{e}_i^{q+1}} = \bar{f}_i$ is continuous for every $q + 1$ - cell e_i^{q+1} .

Finally suppose X is an arbitrary CW-complex and $F \subset X$ a closed subset. Let $f_0: X^0 \rightarrow [0,1]$ be the map which is 0 on $F \cap X^0$ and 1 outside. Then f is continuous since X^0 is discrete. By the argument above there exists a sequence of

continuous extensions $f_1, f_2, \dots$ with $f_q: X^q \rightarrow [0,1]$ and such that f_q is 0 precisely on $F \cap X^q$. Let $f: X \rightarrow [0,1]$ be the map defined by $f|X^q = f_q$. Then f is continuous (lemma 1.4) and is 0 precisely on F . This completes the proof except we have to verify the following special case.

1.16 Lemma. Let G be a closed subset of E^{q+1} and $g: S^q \rightarrow [0,1]$ a function which is 0 precisely on $G \cap S^q$. Then there is an extension $\bar{g}: E^{q+1} \rightarrow [0,1]$ of g which is 0 precisely on G .

Proof. Let $g_1: S^q \cup G \rightarrow [0,\infty)$ be the function which equals g on S^q and is 0 on G . Then g_1 is continuous; let $\bar{g}_1$ be a continuous extension of g_1 to E^{q+1} . Next, let $\bar{h}: E^{q+1} \rightarrow [0,\infty)$ be a map which is 0 precisely on $S^q \cup G$ (e.g. $\bar{h}(x) = \text{dist}(x, S^q \cup G)$). Then $\bar{g}_1 + \bar{h}: E^{q+1} \rightarrow [0,\infty)$ is an extension of g_1 which is 0 precisely on G . Hence $\bar{g} = \max(\bar{g}_1 + \bar{h}, 1)$ will be a required extension of g .]]