Structure theory of manifolds

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Seminar notes
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Contents

1 Structure problems in the theory of manifolds. Survey of results

We introduce here the general method of the structure theory of manifolds. The exposition will be schematic without proofs. We consider the following four categories, which we will try to treat in a unified way.

Diff	=	compact differentiable manifolds and differentiable maps.
PL	=	compact PL manifolds and PL maps.
Top	=	compact topological manifolds and maps.
G	=	Poincaré spaces and homotopy classes of maps.

In all cases one can consider objects with boundaries. But we exclude these in the definitions above.

By a *differentiable manifold* we understand a second countable Hausdorff space M together with a maximal C^∞-atlas on M. For elementary properties of differentiable manifolds we refer to Munkres [1].

We define a piecewise linear space, briefly *PL space*, as a second countable Hausdorff space X together with a maximal PL atlas on X. See Hudson [2] for details. Equivalently a PL structure on X may be defined as a PL equivalence class of triagulations $t : |K| \to X$ where K is a countable, locally finite simplicial complex. If X is an open subset of $\mathbb{R}^n$, then X has a standard PL structure, where the triangulations are linear on each simplex. A PL *manifold* is a PL space locally PL homeomorphic with a fixed Euclidean space.

By a topological manifod we understand a second countable Hausdorff space locally homeomorphic with a fixed Euclidean space. We note that it is still an open problem whether each topological manifold is triangulable, and whether each triangulation of a topological manifold defines a PL manifold. However, it *is* known that not every topological manifold is homeomorphic to a PL manifold. We shall return to this point later.

A finite CW-complex X is called a *Poincaré complex* if there exists a class $[\tilde{X}] \in H_n^{LF}(\tilde{X};\mathbb{Z})$ such that the cap product $\cap[\tilde{X}] : H^q(\tilde{X};\mathbb{Z}) \to H_{n-q}^{LF}(\tilde{X};\mathbb{Z})$ is an isomorphism for all q. H^{LF} denotes homology with locally finite chains and $\tilde{X}$ the universal covering of X. Then n is uniquely determined by X and is called the *formal dimension* of X. By a *Poincaré space* we understand a topological space of the homotopy type of a Poincaré complex.

The isomorphisms of the categories Diff, PL, Top and G are the diffeomorhisms, PL homeomorphisms, homeomorphisms and homotopy equivalences. We denote the sets of isomorphism classes of objects in the categories by $\widehat{\text{Diff}}$, $\widehat{PL}$, $\widehat{\text{Top}}$ and $\widehat{G}$.

Problem 1 *Determine the cardinalities of* $\widehat{\text{Diff}}, \widehat{PL}, \widehat{\text{Top}}$ *and* $\widehat{G}$

All these sets are infinite. Clearly $\widehat{PL}$ is countable since there are only countably many finite simplicial complexes up to simplicial isomorphism. According to J.H.C. Whitehead [3] each finite CW–complex has the homotopy type of a finite simplicial complex. Hence G is countable. There is an elementary argument of Kister [4] to prove that $\widehat{\text{Top}}$ is countable. Finally the countability of $\widehat{PL}$ and structure theory will imply that $\widehat{\text{Diff}}$ is countable.

There are maps

$$\widehat{\text{Diff}} \to \widehat{PL} \to \widehat{\text{Top}} \to \widehat{G} \tag{1}$$

$\widehat{\text{Diff}} \to \widehat{PL}$ is given by the result of J.H.C. Whitehead [5] that each differentiable manifold has PL structure, unique up to PL homeomorphism, compatible with its differentiable structure. $\widehat{PL} \to \widehat{\text{Top}}$ is defined in the obvious way by forgetting the PL structure. Kirby [6] proves that each compact topological manifold has the homotopy type of a finite CW-complex. It follows then from the Poincaré duality theorem that a compact topological manifold is a Poincaré space. Therefore we have a map $\widehat{\text{Top}} \to \widehat{G}$.

Problem 2 *Give counter examples to injectivity and surjectivity of the maps in diagram (1)*

Some beautiful counter examples can be found for all cases. References are Milnor [7], [8], Kervaire [9], Kirby [6] and Gitler, Stasheff [10].

Diagram (1) gives rise to the structure problems of classifying differentiable manifolds within a PL homeomorphism class, etc. The basic method for attacking these structure problems is a procedure which can be divided in the following five steps

a) introduce a fibration concept for each category.

b) define a tangent fibration of an object.

c) construct classifying spaces for the fibrations.

d) transform the structure problems to lifting problems.

e) study the lifting problems by obstruction theory.

We use the following types of fibrations in the different cases

Diff	:	orthogonal bundles.
PL	:	PL bundles.
Top	:	topological bundles.
G	:	spherical fibrations.

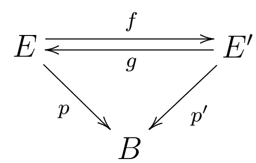
An orthogonal bundle is a fiber bundle in the sense of Steenrod [11], where the base is a topological space, the fiber is $\mathbb{R}^n$ and the structure group is O_n, the orthogonal group over $\mathbb{R}^n$. Bundles are always assumed to be numerable in the sense of Dold [12].

A topological bundle is similarly a fiber bundle, but with the larger structure group Top_n of all homeomorphisms $(\mathbb{R}^n, 0) \hookleftarrow$, in the compact-open topology.

A PL bundle is defined by a projection $p : E \to B$, which is assumed to be a PL map between PL spaces, and local trivializations $\Phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ which are PL homeomorphisms.

A spherical fibration is a Hurewicz fibration with base a topological space and fibers of the homotopy type of a fixed sphere.

The equivalence concept of these fibrations is as follows. For bundles we use bundle equivalence over the identity map at the base, and for spherical fibrations we use fiber homotopy equivalence. Thus in all cases an equivalence between the two fibrations $p : E \to B$ and $p' : E' \to B$ means the existence of a commutative diagram



such that in the different cases f and g are mutually inverse fiberwise orthogonal homeomorphisms, PL homeomorphisms, homeomorphisms or fiber homotopy equivalences. For the cases PL and Top it is also useful to work with microbundles, as introduced by Milnor [13], [14]. By the microbundle representation theorems of Kuiper, Lashof [15] and Holm [16], equivalence classes of microbundles are in bijective correspondence with equivalence classes of bundles.

Let X be a topological space. We denote by $k_{0(n)}(X)$ the set of equivalence classes of orthogonal bundles with fiber $\mathbb{R}^n$ and base X. If $f : X \to Y$ is a map and $\xi \to Y$ an orthogonal bundle, there is a pullback $f^*\xi \to X$, and the equivalence class of $f^*\xi$ depends only on the equivalence class ξ and the homotopy class of f. Hence we get an induced map $f^* : k_{0(n)}(Y) \to k_{0(n)}(X)$ depending only on the homotopy class of f. This gives a contravariant functor

$$k_{0(n)} : \hat{T} \to S$$

where $\hat{T}$ denotes the category of topological spaces and homotopy classes of maps, and S denotes the category of sets. In a similar way we obtain contravariant functors $k_{PL(n)}, k_{\mathrm{Top}(n)}$ and $k_{G(n)}$. We have $k_{PL(n)}$ originally defined on the category of PL spaces and PL homotopy classes of PL maps. However, any map between PL spaces is homotopic to a PL map, and if two PL maps are homotopic, they are PL homotopic. Therefore the PL homotopy classes of PL maps are the same as homotopy classes of maps between PL spaces. By J.H.C. Whitehead [3] each countable CW-complex is homotopy equivalent to a countable, locally finite simplicial complex, hence to a PL space. Therefore we can extend $k_{PL(n)}$ to a contravariant functor over the category $\hat{C}_d$ of spaces having the homotopy type of a countable CW-complex and homotopy classes of maps, where the extension is unique up to natural equivalence of functors.

Whitney sum with a trivial line bundle defines a natural transformation $k_{0(n)} \to k_{0(n+1)}$. We define a functor

$$k_0 = \varinjlim_n k_{0(n)}$$

Thus $k_0(X)$ is the set of stable equivalence classes of orthogonal bundles over X. Similarly we define the functors k_{PL}, k_{Top} and k_G. In the last case the natural transformations $k_{G(n)} \to k_{G(n+1)}$ are defined by means of Whitney join with a trivial zero-sphere fibration.

There are natural transformations of functors

$$k_{0(n)} \to k_{PL(n)} \to k_{\mathrm{Top}(n)} \to k_{G(n)} \tag{2}$$

Here we need a triangulation theorem for orthogonal bundles to define $k_{0(n)} \to k_{PL(n)}$ like that of Lashof, Rothenberg [17], while $k_{PL(n)} \to k_{\mathrm{Top}(n)}$ is defined by forgetting the PL structure, and $k_{\mathrm{Top}(n)} \to k_{G(n)}$ by removing the zero-section of a topological bundle. Passing to the limit with diagram (2) we get the diagram

$$k_0 \to k_{PL} \to k_{\mathrm{Top}} \to k_G \tag{3}$$

Problem 3 *Let M be a differentiable, PL or topological n-manifold or a Poincaré n-space. Define a tangent fibration τ_M in $k_{0(n)}(M), k_{PL(n)}(M), k_{\mathrm{Top}(n)}(M)$ or $k_{G(n)}(M)$.*

If M is a differentiable manifold there is a tangent vector bundle τ_M which has an orthogonal reduction unique up to equivalence, hence we get a well defined element $\tau_M \in k_{0(n)}(M)$. If M is a PL or topological manifold there is a PL or topological tangent microbundle τ_M, and by the micorbundle representation theorems we get a well defined element τ_M of $k_{PL(n)}(M)$ or $k_{\mathrm{Top}(n)}(M)$.

If M is a Poincaré n-space, the stable tangent fibration $\tau_M \in k_G(M)$ is well defined by Spivak [18]. According to Wall (unpublished) it is also possible to define $\tau_M \in k_{G(n)}(M)$, see Dupont [19].

Let $\widehat{C}$ be the category of spaces having the homotopy type of a CW-complex and homotopy classes of maps. By Browns representation theorem the functor $k_{0(n)} : \widehat{G} \to S$ can be represented by a *classifying space* $BO(n)$ and a *universal bundle* γ_n over $BO(n)$. This means there is a bijection

$$k_{0(n)}(X) = [X, BO(n)]$$

where the right side denotes the set of free homotopy classes of maps $f : X \to BO(n)$, and the bijection is given by $[f] \to f^*\gamma_n$. Similarly one obtains classifying spaces $BPL(n), B\mathrm{Top}(n)$ and $BG(n)$ for the functors $k_{PL(n)}, k_{\mathrm{Top}(n)}$ and $k_{G(n)}$, where $k_{PL(n)}$ is extended to $\widehat{C}$ in the process. For this application of Brown's theorem see Siebenmann [20]. The classifying spaces are uniquely defined up to homotopy type by their classifying property. The natural transformation $k_{0(n)} \to k_{0(n+1)}$ defines a map $BO(n) \to BO(n+1)$ uniquely up to homotopy. We define the space $BO = \varinjlim_n BO(n)$ by the telescope construction, and $B0$ is also unique up to homotopy type. The natural map

$$k_0(X) \to [X, BO]$$

is bijective if X has the homotopy type of a finite CW-complex, but not in general. See for instance Siebenmann [20]. Similarly one defines the spaces $BPL, B\mathrm{Top}$ and BG.

Problem 4 *Determine the connectivity of the maps $BO(n) \to BO(n+1)$, $BPL(n) \to BPL(n+1)$, etc.*

This problem is of significance for the stability properties of fibrations and the structure of the stable classifying spaces. One can prove that the maps are n-connected, assuming $n \geq 5$ in the topological case. For the orthogonal case n-connectivity follows from the fibration

$S^n \to BO(n) \to BO(n+1)$, for the PL case it is proved by Haefliger, Wall [21] . Kirby [6] gives the result in the topological case. For the case of spherical fibrations see for instance Milnor [22].

If A is a topological group, there exists a classifying space BA for principal A-bundles. It can be constructed by Brown's theorem or by the infinite join as in Milnor [23]. There is a universal principal A-bundle $A \to EA \to BA$, where EA is a contractible space, Dold [12]. Comparing this bundle with the path fibration over BA we get $A = \Omega BA$ up to homotopy type. If A' is a closed topological subgroup of A with local cross-sections, then there is an A-bundle $A/A' \to BA' \to BA$.

Therefore we introduce the following notation.

If BA is any of the previously considered classifying spaces, we define $A = \Omega BA$. If $BA' \to BA$ is any of the natural maps between classifying spaces, we make it into a fibration and call the fiber A/A'. We get then an infinite fiber sequence

$$\ldots \to \Omega(A/A') \to A' \to A \to A/A' \to BA' \to BA$$

uniquely defined only up to homotopy types of spaces and homotopy of maps.

For the orthogonal group we have $BO_n = BO(n)$ and hence $O_n = O(n)$, because principal O_n-bundles correspond bijectively to orthogonal bundles with fiber $\mathbb{R}^n$. Similarly $B\mathrm{Top}_n = B\mathrm{Top}(n)$ and $\mathrm{Top}_n = \mathrm{Top}(n)$. There is a subgroup PL_n of the PL homeomorphisms in Top_n. But it is an open question whether PL_n can be given the structure of a topological group in such a way that $BPL_n = BPL(n)$ and hence $PL_n = PL(n)$.

We let G_n be the space of homotopy equivalences of S^{n-1} with itself in the compact-open topology. Then $G_n = G(n)$. This follows for example from the existence of a fibration $G_n \to EG_n \to BG_n$ given in Dold, Lashof [24] together with the fact that EG_n is contractible and $BG_n = BG(n)$, which is a consequence of Stasheff [25].

The preliminary distinction between O_n and $O(n)$, etc. has no further significance, once the identifications above have been made. The important point is that we have got additional information about the fibers $O(n), \mathrm{Top}(n)$ and $G(n)$, although not about $PL(n)$.

Diagram (2) induces maps between classifying spaces

$$BO(n) \to BPL(n) \to B\mathrm{Top}(n) \to BG(n) \tag{4}$$

uniquely defined up to homotopy. Passing to the limit with the telescope construction we obtain maps

$$BO \to BPL \to B\mathrm{Top} \to BG \tag{5}$$

The restrictions of these maps to any unstable classifying space $BO(n), BPL(n)$, etc. are unique up to homotopy. This does not determine the maps of diagram (5) up to homotopy, because of the phantom phenomenon. See Siebenmann [20] for a discussion of this point. However the induced maps between homotopy groups are unique. Also the fibers of the maps are well defined up to homotopy type. We have $PL/O = \varinjlim_n PL(n)/O(n)$, etc.

Let M be a PL n-manifold. If M is PL homeomorphic to a differentiable manifold, then there exists a lifting in the diagram

$$\begin{array}{ccc} & & BO(n) \\ & \nearrow & \downarrow \\ M & \xrightarrow[\tau_M]{} & BPL(n) \end{array}$$

This follows immediately from the invariance properties of the tangent fibration. The similar statement holds for the other cases. In other words a lifting of structure gives a lifting of the tangent fibration.

Problem 5 *When is a lifting of the tangent fibration sufficient for the corresponding lifting of structure?*

In general the condition is not sufficient. However in several cases only the assumption that there is a lifting of the stable tangent fibration is enough. Milnor [13] showed that a PL manifold M is PL homeomorphic with a differentiable manifold if there is a lifting in

$$\begin{array}{ccc} & & BO \\ & \nearrow & \downarrow \\ M & \xrightarrow[\tau_M]{} & BPL \end{array}$$

The result follows quickly from the Cairns-Hirsch smoothing theorem, see Hirsch [26].

A topological n-manifold M with $n \geq 5$ is homeomorphic to a PL manifold if there is a lifting in

$$\begin{array}{ccc} & & BPL \\ & \nearrow & \downarrow \\ M & \xrightarrow[\tau_M]{} & B\mathrm{Top} \end{array}$$

See Kirby [6].

If X is a simply connected Poincaré n-space with $n \geq 5$, then X is homotopy equivalent with a PL manifold if there is a lifting in

$$\begin{array}{ccc} & & BPL \\ & \nearrow & \downarrow \\ X & \xrightarrow[\tau_M]{} & BG \end{array}$$

This is proved by PL surgery. The result is due to Browder, Hirsch [27].

In connection with problem 5 there is also the problem of obtaining a lifted structure whose tangent fibration is actually the given lifting in the diagram. Furthermore we have also a uniqueness problem, namely in which sense different liftings of the tangent fibration correspond to different liftings of structure. To attack the lifting problems we need information about the classifying spaces and the maps between them. In particular the following problem is of primary interest.

Problem 6 *Determine the cohomology of the classifying spaces and the homotopy of the fibers.*

We give some of the known results in the stable case only. For rational cohomology we have

$$H^*(BO;\mathbb{Q}) = H^*(BPL;\mathbb{Q}) = H^*(B\mathrm{Top};\mathbb{Q}) = \mathbb{Q}[p_1, p_2, \ldots] \quad \tilde{H}^*(BG;\mathbb{Q}) = 0$$

where p_i is the *universal Pontrjagin class* in degree $4i$. For BO the Pontrjagin classes are integral, i.e. images of integral classes under the coefficient homomorphism induced by $\mathbb{Z} \to \mathbb{Q}$. For BPL and $B\mathrm{Top}$ however, this is not true. For the problem of describing integral classes in $H^*(BPL;\mathbb{Q})$ see Brumfiel [28]. The computation of $H^*(BO;\mathbb{Q})$ is classical. Then the rest follows from the fact that $PL/O, \mathrm{Top}/PL$ and G have only finite homotopy groups.

For $\mathbb{Z}_2$-cohomology we have the classical result

$$H^*(BO;\mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, \ldots]$$

where w_i is the *universal Stiefel-Whitney class* in degree i. From Milgram [29]

$$H^*(BG;\mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, \ldots] \otimes \mathbb{Z}_2\langle e_{01}, e_{11}, \ldots\rangle$$

where in general e_I indexed by $I = (i_0, i_1, \ldots, i_r)$ with $0 \leq i_0 \leq \ldots \leq i_r$ and $r \geq 1$, is as *universal exotic class* in degree $1 + i_0 + 2i_1 + \ldots + 2^r i_r$. The index I satisfies only the condition that if $i_0 = 0$ then $r = 1$ and $i_1 > 0$. The precise structure of $H^*(BPL;\mathbb{Z}_2)$ and $H^*(B\mathrm{Top};\mathbb{Z}_2)$ is not known.

For $\mathbb{Z}_p$-cohomology, where p is an odd prime, the classical result is

$$H^*(BO;\mathbb{Z}_p) = \mathbb{Z}_p[p_1, p_2, \ldots]$$

where p_i is the universal $\mod p$ Pontrjagin class in degree $4i$. About the homotopy groups of the fibers the following is known.

$$\pi_n \mathrm{Top}/PL = \begin{cases} \mathbb{Z}_2 & n = 3 \\ 0 & n \neq 3 \end{cases}$$

Kirby [6].

$$\pi_n G/PL = \begin{cases} \mathbb{Z} & n = 0 \mod 4 \\ \mathbb{Z}_2 & n = 2 \mod 4 \\ 0 & n \text{ odd} \end{cases}$$

Sullivan [30]. The groups

$$\pi_n PL/O = \Gamma_n$$

are finite for all n and in general very complicated. The unknown part of Γ_n is essentially the cokernel of the stable J-homomorphism $J : \pi O_n \to \pi_n$. The groups Γ_n are known for $n \leq 19$.

$\Gamma_0 = 0$	$\Gamma_5 = 0$	$\Gamma_{10} = \mathbb{Z}_6$	$\Gamma_{15} = \mathbb{Z}_{8128} + \mathbb{Z}_2$
$\Gamma_1 = 0$	$\Gamma_6 = 0$	$\Gamma_{11} = \mathbb{Z}_{922}$	$\Gamma_{16} = \mathbb{Z}_2$
$\Gamma_2 = 0$	$\Gamma_7 = \mathbb{Z}_{28}$	$\Gamma_{12} = 0$	$\Gamma_{17} = 4\mathbb{Z}_2$
$\Gamma_3 = 0$	$\Gamma_8 = \mathbb{Z}_2$	$\Gamma_{13} = \mathbb{Z}_3$	$\Gamma_{18} = \mathbb{Z}_8 + \mathbb{Z}_2$
$\Gamma_4 = 0$	$\Gamma_9 = 3\mathbb{Z}_2$	$\Gamma_{14} = \mathbb{Z}_2$	$\Gamma_{19} = \mathbb{Z}_{261632} + \mathbb{Z}_2$

Kervaire, Milnor [31], Brumfiel [32], Cerf [33], Smale [34].

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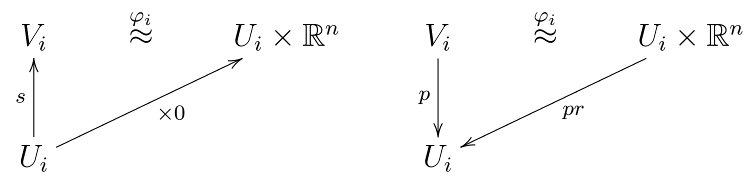
2 Microbundles and bundles

Microbundles were invented by Milnor [1] to provide topological and PL analogues of the tangent and the normal bundle of a manifold in the topological and the PL-case. A *microbundle* is a diagram of maps and spaces

$$X \xrightarrow{s} E \xrightarrow{p} X$$

whose composite is the identity such that the following condition is satisfied:

(Local Triviality) There is a family of homeomorphisms $\varphi_i : V_i \approx U_i \times \mathbb{R}^n$ (called *local trivializations*), where U_i and V_i are open sets of X and E such that $sU_i \subset V_i$ and $pV_i \subset U_i$ making the following diagrams commutative



The family $\{U_i\}$ covers X.

Thus $s : U_i \to V_i$ corresponds under φ_i to the zero-section $x \to (x, 0)$ and $p : V_i \to U_i$ to the projection $(x, v) \to x$. By abuse of language we refer quite generally to $s : X \to E$ as the *zero-section* and $p : E \to X$ as the *projection* of the microbundle. E is called the *total space* and X the *base space*. The integer $n \geq 0$ is the *fibre dimension*. Microbundles will be labelled by small greek letters $\mu, \nu, \ldots$

Remark. The zero-section s imbeds X as a subspace of E. If the fibre dimension n is zero, then s is an open imbedding. If $n > 0$, then sX is never open in E and in general not closed either. However, sX is always closed in some neighborhood in E.

We exhibit some important examples.

Examples. (a) The *standard trivial* $\mathbb{R}^n$*-bundle* given by the diagram

$$\epsilon^n : X \xrightarrow{\times 0} X \times \mathbb{R}^n \xrightarrow{pr} X$$

(b) Any $\mathbb{R}^n$-*bundle*, i.e. any fibre bundle $p : E \to X$ with fibre $\mathbb{R}^n$ and Top_n structure group. Such a bundle has a canonical zero-section $s : X \to E$ and so defines a microbundle

$$X \xrightarrow{s} E \xrightarrow{p} X$$

(c) The *tangent microbundle* of a topological manifold M^m

$$\tau_M : M \xrightarrow{d} M \times M \xrightarrow{pr_1} M$$

Here d is the diagonal map and pr_1 the projection to the first factor. Certainly $pr_1 \circ d$ is the identity map on M. To see that the local triviality condition is satisfied choose an open set $U \subset M$ homeomorphic to $\mathbb{R}^m$ and set $V = U \times U$. Take a homeomorphism $f : U \approx \mathbb{R}^m$ and define $h : V \to U \times \mathbb{R}^m$ by $h(p, v) = (p, f(v) - f(p))$. Then h is a homeomorphism which carries $d : U \to V$ into the zero-section $U \to U \times \mathbb{R}^m$ and $pr_1 : V \to U$ into the projection $U \times \mathbb{R}^m \to U$.

(d) A *microbundle neighborhood* of a submanifold $M \subset N$ is a neighborhood U of M in N which admits a retraction $r : U \to M$ such that the diagram

$$M \subset U \xrightarrow{r} M$$

constitutes a microbundle. Such a microbundle is called a *normal microbundle* of M in N. It is known that normal microbundles do not always exist, even if M is a locally flat submanifold of N, [2]. As an example of a case where it exists, consider the diagonal $\Delta \subset M \times M$, $\Delta = dM$, and its normal microbundle

$$\nu_\Delta : \Delta \subset M \times M \xrightarrow{d \circ pr_1} \Delta$$

After identification of Δ with M by the diagonal map ν_Δ coincides with the tangent microbundle τ_M of M. In short, the tangent microbundle of M equals the normal microbundle of M in $M \times M$, when imbedded as the diagonal. We remark that in general normal microbundles of M in N, even when they exist, are not known to be isomorphic (cf. next).

Two microbundles $\mu : X \xrightarrow{s} E \xrightarrow{p} X$ and $\nu : X \xrightarrow{t} F \xrightarrow{q} X$ over the same base are *isomorphic*, written $\mu \cong \nu$, if there are neighborhoods E' and F' of sX and tX in E and F, respectively, and a homeomorphism $E' \approx F'$ which commutes with zero-sections and projections. A microbundle is *trivial* if it is isomorphic to some standard trivial $\mathbb{R}^n$-bundle.

Most of the important general constructions from bundle theory can be carried directly over to microbundles, e.g. pullbacks, products and Whitney sums. To define the proper notion of a microbundle map note that a microbundle does not essentially depend on its total space E, but only on the germ of E at the zero-section. Given microbundles $\mu : X \xrightarrow{s} E \xrightarrow{p} X$ and $\nu : Y \xrightarrow{t} F \xrightarrow{q} Y$ of the same fibre dimension a mapgerm $\varphi : (E, sX) \Longrightarrow (F, tY)$ is called a *bundle mapgerm* or simply a *mapgerm* if the following is true: There is a neighborhood E' of sX in E and a representative Φ of φ on E' such that Φ maps each fibre in E' injectively into some fibre in F. Clearly a mapgerm $\varphi : \mu \Rightarrow \nu$ covers a map $\varphi|X : X \to Y$ on the base level, and we have $\varphi|X = \Phi|X$ for any representative Φ of φ. If $X = Y$ and φ covers the identity map, then φ is called an *isogerm*. If moreover $\mu = \nu$, φ is an *autogerm*. For each integer $n \geq 0$ there is a category of n-microbundles and isogerms over the space X. It is known that every isogerm has an inverse so that all morphisms are isomorphisms in these categories, [1], lemma 6.4. Clearly two microbundles μ, ν over X are isomorphic if and only if there is an isogerm $\mu \Rightarrow \nu$. Here are some results which help to justify the definitions, cf. [1].

Theorem 1 *Let M be a differentiable manifold with tangent bundle T_M and tangent microbundle τ_M. Then T_M is microbundle isomorphic to τ_M.*

Conversely

Theorem 2 *Let M be a topological manifold and suppose $\tau_M \oplus \epsilon^p \cong \xi \oplus \epsilon^q$ for some vectorbundle ξ on M. Then $M \times \mathbb{R}^r$ can be given a smooth structure for sufficiently large r.*

In short M is stably smoothable if and only if its tangent microbundle is stably isomorphic to a vector bundle.

As remarked above $M \subset N$ need not always have a microbundle neighborhood in N. However M always has a microbundle neighborhood in $N \times \mathbb{R}^r$ for sufficiently large r. And although the resulting normal microbundle ν need not be unique, its stable isomorphism class is uniquely determined, [1], thms 4.3, 5.9 and 5.10.

A microbundle is of *finite type* if it admits finite trivializing covers on the base.

Theorem 3 *Let μ be a microbundle of finite type over a paracompact (Hausdorff) base space X. Then there is a microbundle μ' over X such that $\mu \oplus \mu'$ is trivial.*

Theorem 3 was proved by Milnor for X a finite dimensional simplicial complex (in which case every microbundle over X is of finite type). Proof of the theorem in the above generality is due to Kister, [3].

Remark. Over a compact (Hausdorff) space every microbundle is obviously of finite type. But also over a manifold or a finite dimensional CW-complex every microbundle is of finite type. In fact over a finite dimensional paracompact space or even a retract of such a space every microbundle is of finite type.

Denote by $k_{\text{top}(n)}(X)$ the set of isomorphism classes of n-microbundles over X and by $k_{\text{top}}(X)$ the set of stable isomorphism classes of microbundles over X. Then $k_{\text{top}(n)}(X)$ and $k_{\text{top}}(X)$ are pointed sets, the base points being the class of ϵ^n and the stable class of ϵ, respectively. Furthermore, there are pairings $k_{\text{top}(m)}(X) \times k_{\text{top}(n)}(X) \to k_{\text{top}(m+n)}(X)$ defined by the Whitney sum operation $(\mu, \nu) \to \mu \oplus \nu$, which induce a pairing $k_{\text{top}}(X) \times k_{\text{top}}(X) \to k_{\text{top}}(X)$. With this operation $k_{\text{top}}(X)$ becomes an abelian semigroup with zero-element $[\epsilon]$ (the stable class of ϵ). By theorem 3 and its following remark this semigroup is actually a group if X is compact or finite dimensional paracompact. The pullback construction makes $k_{\text{top}(n)}$ and k_{top} topological functors with values in the categories of pointed sets and semigroups, respectively. (Or for k_{top} suitably restricted even with values in the category of groups.) As in the case of vector bundles these functors are in fact homotopy functors, at least over all paracompact spaces. This follows from the next result.

Theorem 4 *Let μ be a microbundle over a paracompact base space X and $f, g : Y \to X$ homotopic maps. Then $f^*\mu \cong g^*\mu$.*

As usual the proof is in [1].

Although microbundles in many ways resemble bundles we can never directly appeal to bundle theory for a result, a microbundle not actually being a bundle. This is rather awkward in the long run. Even worse, there are important constructions for $\mathbb{R}^n$-bundles that have no immediate counterpart for microbundles, such as the Thom space construction or the associated spherical fibration. The latter construction is for instance the one we need to define a natural transformation $k_{\mathrm{top}(n)} \to k_{G(n)}$.

The main purpose of this section is to establish the following result which removes all these inconveniences.

Theorem 5 *Over a CW-complex every n-microbundle is isomorphic to an $\mathbb{R}^n$-bundle, uniquely determined up to bundle isomorphism.*

Remark. Actually it would be convenient to have theorem 5 even for microbundles over spaces of homotopy type of CW-complexes. Such spaces are not necessarily paracompact. This leads one to consider a class of microbundles called *numerable* microbundles, [4], [5].

It is a fact that theorem 5 is true for numerable microbundles without any conditions on the base X, see again [4]. This general result would more than suffice for our applications, however, its proof is more complicated.

Theorem 5 was proved independently by several people, namely Kister [6] (X finite dimensional), Mazur (unpublished), and later Kuiper/Lashof [7] and Hirsch/Mazur (unpublished) covering also the PL-case. The proof we give below is in the last stage based on an expansion process for open imbeddings of Euclidean spaces, invented by Mazur, and is presumably quite close to Mazur's own proof.

We turn to the proof of theorem 5. This will be carried out by a closer study of the homotopy functor $k_{\mathrm{top}(n)}$. A technical inconvenience (at least) of $k_{\mathrm{top}(n)}$ is that it is defined on a category of spaces and maps rather than on a category of pointed spaces and pointed maps. We first adjust this. Let X be a pointed space with basepoint x_0. A *rooted* microbundle over X is a microbundle

$$X \xrightarrow{s} E \xrightarrow{p} X$$

together with a specific isogerm (*rooting*)

$$x_0 \xrightarrow{s} p^{-1}x_0 \xrightarrow{p} x_0 \Longrightarrow x_0 \xrightarrow{s_0} \mathbb{R}^n \longrightarrow x_0,$$

where n is the fibre dimension. Two rooted microbundles ξ and ξ' over X are *isomorphic* if there exists an isogerm $\xi \Rightarrow \xi'$ commuting with the rootings. Let $\tilde{k}_{\mathrm{top}(n)}(X)$ be the set of isomorphism classes of rooted n-microbundles over X. The pull-back construction establishes $\tilde{k}_{\mathrm{top}(n)}$ as a functor on the category of pointed spaces, and of course $\tilde{k}_{\mathrm{top}(n)}$ is a homotopy functor on the smaller category of pointed paracompact spaces. This is because of the homotopy theorem for rooted microbundles, [1], lemma 7.1:

Lemma 6 *Let μ be a rooted microbundle over a paracompact (pointed) space X. Let $f, g : Y \to X$ be homotopic pointed maps. Then $f^*\mu \cong g^*\mu$ as rooted microbundles.*

There are of course also analogous functors $k_{\mathrm{Top}(n)}$ and $\tilde{k}_{\mathrm{Top}(n)}$ based on bundles rather than on microbundles. These are somewhat more familiar objects. In particular it is well-known that $k_{\mathrm{Top}(n)}$ and $\tilde{k}_{\mathrm{Top}(n)}$ are *representable* over the categories of CW-complexes and pointed CW-complexes, respectively, [8], [9]. For $\tilde{k}_{\mathrm{Top}(n)}$ this means there is pointed CW-complexes $\tilde{B}\mathrm{Top}(n)$ and a natural equivalence of pointed homotopy functors

$$\tilde{k}_{\mathrm{Top}(n)} \cong [\ ,\tilde{B}\mathrm{Top}(n)].$$

It is a less accessible fact that the same is true for the microbundle funtors $\tilde{k}_{\mathrm{top}(n)}$. The proof of this is based on E.H. Brown's representability theorem. Thus we have

Lemma 7 *$\tilde{k}_{\mathrm{Top}(n)}$ and $\tilde{k}_{\mathrm{top}(n)}$ are representable functors on the homotopy category of pointed CW-complexes.*

Since every $\mathbb{R}^n$ bundle is a microbundle, and since isomorphic bundles are isomorphic as microbundles, there is a canonical natural transformation

$$\tau_* : k_{\mathrm{Top}(n)} \to k_{\mathrm{top}(n)}.$$

By the first part of theorem 5 this transformation is surjective and by the second part it is injective. Conversely, if τ_* is surjective, every microbundle is isomorphic to a bundle, and if τ_* is injective, two microisomorphic bundles are isomorphic (as bundles). Hence, to prove theorem 5 it suffices to prove that τ_* is a natural equivalence. Now there is also an analogous natural transformation $\tilde{\tau}_* : \tilde{k}_{\mathrm{Top}(n)} \to \tilde{k}_{\mathrm{top}(n)}$, which is a natural equivalence if and only if the rooted analogue of theorem 5 holds. Moreover, it suffices to look at $\tilde{\tau}_*$ because of

Lemma 8 *If $\tilde{\tau}_*$ is a natural equivalence, so is τ_*.*

Proof. Let X be a CW-complex and let X_+ be X with an isolated base point x_0 adjoined. To each n-microbundle ($\mathbb{R}^n$-bundle) μ over X there is a rooted microbundle ($\mathbb{R}^n$-bundle) μ_+ over X_+, whose fibre over x_0 is $\mathbb{R}^n$ and whose rooting is the identity $1_{\mathbb{R}^n}$. This gives bijections $k_{\mathrm{top}(n)}(X) \cong \tilde{k}_{\mathrm{top}(n)}(X_+)$ and $k_{\mathrm{Top}(n)}(X) \cong \tilde{k}_{\mathrm{Top}(n)}(X_+)$, and a commutative diagram

$$\begin{array}{ccc} k_{\mathrm{Top}(n)}(X) & \cong & \tilde{k}_{\mathrm{Top}(n)}(X_+) \\ \downarrow{\scriptstyle \tau_*} & & \downarrow{\scriptstyle \tilde{\tau}_*} \\ k_{\mathrm{top}(n)}(X) & \cong & \tilde{k}_{\mathrm{top}(n)}(X_+) \end{array}$$

Hence the assertion.

By lemma 7 there exist CW-complexes $\tilde{B}\mathrm{top}(n)$ and $\tilde{B}\mathrm{Top}(n)$ as well as natural equivalences

$$\tilde{k}_{\mathrm{Top}(n)} \ \cong [\ ,\tilde{B}\mathrm{Top}(n)]$$

and

$$\tilde{k}_{\mathrm{top}(n)} \cong [\ ,\tilde{B}\mathrm{top}(n)]$$

Under these identifications the natural transformation $\tilde{\tau}_*$ sends the identity map $1_{\tilde{B}\mathrm{Top}(n)}$ to a map $\tilde{\tau} : \tilde{B}\mathrm{Top}(n) \to \tilde{B}\mathrm{top}(n)$, well determined up to homotopy. Hence we get a commutative diagram of functors and natural equivalences

$$\begin{array}{ccc} \tilde{k}_{\mathrm{Top}(n)} & \cong & [\ ,\tilde{B}\mathrm{Top}(n)]_* \\ {\scriptstyle \tilde{\tau}_*}\downarrow & & \downarrow{\scriptstyle [\tilde{\tau}]\circ} \\ \tilde{k}_{\mathrm{top}(n)} & \cong & [\ ,\tilde{B}\mathrm{top}(n)]_* \end{array}$$

where $[\tilde{\tau}]\circ$ means composition with $[\tilde{\tau}]$. It follows that $\tilde{\tau}_*$ is a natural equivalence if (and only if) $\tilde{\tau} : \tilde{B}\mathrm{Top}(n) \to \tilde{B}\mathrm{top}(n)$ is a homotopy equivalence. By Whitehead's theorem this is true if and only if $\tilde{\tau}$ induces isomorphisms between homotopy groups

$$\tilde{\tau}_* : \pi_i(\tilde{B}\mathrm{Top}(n) \to \pi_i(\tilde{B}\mathrm{top}(n))$$

for $i = 0, 1, \ldots$

Or equivalently, if and only if

$$\tilde{\tau}_* : \tilde{k}_{\mathrm{Top}(n)}(S^i) \to \tilde{k}_{\mathrm{top}(n)}(S^i)$$

are isomorphisms, $i = 0, 1, \ldots$

Conclusion. To show that $\tilde{\tau}_* : \tilde{k}_{\mathrm{Top}(n)} \to \tilde{k}_{\mathrm{top}(n)}$ is a natural equivalence and hence to prove theorem 5 it suffices to show that the homomorphisms

$$\tilde{\tau}_* : \tilde{k}_{\mathrm{Top}(n)}(S^i) \to \tilde{k}_{\mathrm{top}(n)}(S^i), \qquad i = 0, 1, \ldots$$

are isomorphisms. Equivalently, it suffices to show that over S^i every rooted n-microbundle is isomorphic to a rooted $\mathbb{R}^n$-bundle, and every rooted $\mathbb{R}^n$-bundle which is trivial as rooted microbundle is trivial as rooted bundle, $i = 0, 1, \ldots$

For this we shall need the following *germ extension theorem.*

Lemma 9 *Let φ be an autogerm of the trivial bundle ϵ^n over a disk D, and Φ' an automorphism of ϵ^n over ∂D whose germ is $\varphi|\partial D$. Then there is an automorphism Φ of ϵ^n over D whose germ is φ, such that $\Phi|\partial D = \Phi'$.*

Proof. We first treat the case where D is of dimension 0. Then D is a point and ∂D is empty. In this case φ is simply a homeomorphism germ

$$\varphi : (\mathbb{R}^n, 0) \Longrightarrow (\mathbb{R}^n, 0).$$

Clearly φ can be represented by an open imbedding

$$\Phi_0 : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0),$$

because if $\Phi : (U, 0) \to (\mathbb{R}^n, 0)$ is an arbitrary representative of φ and $\theta : (\mathbb{R}^n, 0) \approx (V(\epsilon), 0)$ is a homeomorphism onto a small ϵ-neighborhood $V(\epsilon)$ of the origin which is the identity on $V(\epsilon/2)$, then $\Phi_0 = \Phi \circ \theta$ is defined and is an open imedding whose germ at 0 is φ.

We now alter Φ_0 by stages. Set $\Phi_0 V(1) = W$ and choose $\epsilon > 0$ so that $V(\epsilon) \subset W$. Let $\theta' : (\mathbb{R}^n, 0) \approx (\mathbb{R}^n, 0)$ be a homeomorphism such that θ' maps $V(\epsilon)$ onto $V(2)$ and is the identity on $V(\epsilon/2)$. Then $\Phi_1 = \theta' \circ \Phi_0$ is an open imbedding of $(\mathbb{R}^n, 0)$ whose germ is φ with the property that $\Phi_1 V(1) \supset V(2)$. Now construct inductively imbeddings $\Phi_2, \Phi_3, \ldots :$ $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that

(a) $\qquad \Phi_i V(i) \supset V(i+1), \qquad\qquad i = 2, 3, \ldots$

(b) $\qquad \Phi_i | V(i-1) = \Phi_{i-1} | V(i-1), \qquad i = 2, 3, \ldots$

Then define $\Phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ by $\Phi | V(i) = \Phi_i | V(i)$, $i = 1, 2, \ldots$ Obviously Φ is an imbedding. Moreover Φ is surjective by (a) and germ Φ = germ $\Phi_1 = \varphi$ by (b). This completes the proof in the case where D is 0-dimensional.

Next assume D of dimension > 0. Disregarding the "boundary condition" Φ' the case runs as above, the neighborhoods $V(\epsilon)$ now meaning ϵ disk bundle neighborhoods of $D \times 0$ in $D \times \mathbb{R}^n$. (They form a fundamental system of neighborhoods of $D \times 0$.) It follows that we may assume φ to be the identity autogerm on ϵ^n. Otherwise replace φ with $\varphi \circ \Phi_1^{-1}$ and Φ' with $\Phi' \circ \Phi_1^{-1}$ (over ∂D), where Φ_1 is any automorphism of ϵ^n over D whose germ is φ. Now choose a small open disk $V \subset D$ and set $D' = D - V$. Extend Φ' to an automorphism of ϵ^n over D', also denoted Φ'. (That is possible since ∂D is a retract of D'.) Let $\lambda : D \to [0, 1]$ be a function which is 0 on a closed neighborhood of V and 1 on ∂D. There is then a map $\Pi : D \times \mathbb{R}^n \to D \times \mathbb{R}^n$ given by

$$\Pi(x, y) = (x, \lambda(x) v)$$

which is obviously an automorphism of ϵ^n over $\lambda^{-1}(0, 1]$. Define $\Phi : D \times \mathbb{R}^n \to D \times \mathbb{R}^n$ by

$$\Phi = \begin{cases} \Pi^{-1} \circ \Phi' \circ \Pi & \text{over } \lambda^{-1}(0, 1] \\ \text{Identity} & \text{outside} \end{cases}$$

Then Φ is an automorphism of ϵ^n over D whose germ is the identity, which restricts to Φ' over ∂D.

Remark. a) The limit extension process that gives lemma 9 is due to Mazur. The method itself also yields other important results and has been formalized under the name of "The method of infinite repetition in pure topology", [10], [11]. (However, it is hard to recognize.) b) There is a rooted version of lemma 9 as well, when $(D, \partial D)$ is considered a pointed pair with base point $x_0 \in \partial D \subset D$ ($\dim D > 0$). However, this version follows trivially from lemma 9, since over ∂D the automorphism is already determined.

We can now finish the proof of theorem 5 by showing

Lemma 10 *Let μ be a rooted microbundle over a sphere S. Then there is a rooted $\mathbb{R}^n$-bundle ξ isomorphic to μ. If $\varphi : \xi \Rightarrow \epsilon^n$ is an isogerm to the rooted standard trivial bundle, there is an isomorphism $\Phi : \xi \cong \epsilon^n$ whose germ is φ.*

Proof. Let $\dim S = 0$. Then the claim is partly trivial and follows partly from lemma 9 (the case $\dim D = 0$). Assume next that the claim has been proved for spheres of dimension $\leq n-1$ and let S be an n-sphere, $n \geq 1$. Then S admits a decomposition into halfspheres D^+, D^- with intersection S', a sphere of dimension $n-1$, and the base point x_0 in S'. Since D^+ and D^- are contractible $\operatorname{rel} x_0$, μ is trivial as rooted microbundle over D^+ and over D^-; let φ^-, φ^+

$$\epsilon^n(D^-) \overset{\varphi^-}{\Longrightarrow} \mu \overset{\varphi^+}{\Longleftarrow} \epsilon(D^+)$$

be two "microtrivializations". Over S' this gives an autogerm $\varphi' = (\varphi^+)^{-1} \circ \varphi^-$ of ϵ^n. By our induction assumption there is an automorphism Φ' of ϵ^n over S' whose germ is φ'. From the bundle $\xi = \epsilon^n(D^-) \cup_{\Phi'} \epsilon^n(D^+)$. Then ξ is isomorphic to μ. Finally suppose ξ is any $\mathbb{R}^n$-bundle over S with an isogerm $\varphi : \xi \Rightarrow \epsilon^n$. Over S' there is an isomorphism $\Phi' : \xi \mid S' \cong \epsilon^n(S')$ whose germ is $\varphi \mid S'$. Since $\xi \mid D^+$ is trivial, lemma 9 applies to the pair (φ, Φ') over D^+ and gives a trivalization $\Phi^+ : \xi \mid D^+ \cong \epsilon^n(D^+)$ whose germ is $\varphi \mid D^+$, such that $\Phi^+ \mid S' = \Phi'$. Similarly there is a trivialization $\Phi^- : \xi \mid D^- \cong \epsilon^n(D^-)$. Together with Φ^+ this map gives a trivialization $\Phi : \xi \cong \epsilon^n$ with germ φ. This completes the induction.

Remark. There is also a PL-version of theorem 5 in which only PL-microbundles and PL-bundles enter. The proof given above for the topological case can be modified to give the PL-case. Some extra care is required to remain within the smaller PL-category when performing the necessary constructions, cf. [7]. Also the application of Brown's theorem is less direct.

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3 The fiber sequence of a map, and interpretations

In this chapter a *morphism* between two maps $f : X \to Y$ and $g : U \to V$ will be a pair of maps $\alpha : X \to U$ and $\beta : Y \to V$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[\beta]{} & V \end{array}$$

is homotopy commutative. The morphism will be called a *homotopy equivalence* if α and β are homotopy equivalences.

We start by proving the basic fact that any map $f : X \to Y$ is a fibration up to homotopy equivalence. Then we show that f generates an infinite sequence

$$\cdots \xrightarrow{\Omega j} \Omega E(f) \xrightarrow{\Omega i} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{j} E(f) \xrightarrow{i} X \xrightarrow{f} Y$$

such that each two consecutive maps form the diagram of a fibration, up to homotopy equivalence. Finally we shall consider the homotopy set $[X, G/H]$, where G is a topological group and H a closed subgroup with local cross sections in G, and interpret this set in terms of bundles.

Let $f : X \to Y$ be a map of topological spaces. We make f into a fibration as follows. Define the space

$$T(f) = \{(x, \lambda) \in X \times Y^I : f(x) = \lambda(0)\}$$

with the compact-open topology, and $f' : T(f) \to Y$ by $f'(x, \lambda) = \lambda(1)$.

Proposition 1 *$f' : T(f) \to Y$ is a fibration, and is homotopy equivalent to the map $f : X \to Y$.*

Proof. Let $h_t : A \to Y$ be a homotopy and $g_0 : A \to T(f)$ a lifting of h_0. We can write $g_0(a) = (x_0(a), \lambda_0(a))$ where $f(x_0(a)) = \lambda_0(a)(0)$ and $\lambda_0(a)(1) = h_0(a)$. Let $g_t : A \to T(f)$ be defined by $g_t(a) = (x_0(a), \lambda_t(a))$ where

$$\lambda_t(a)(s) = \begin{cases} \lambda_0(a)\left(\frac{2s}{2-t}\right) & 0 \le s \le 1 - \frac{t}{2} \\ h_{2s+t-2}(a) & 1 - \frac{t}{2} \le s \le 1 \end{cases}$$

Then g_t is a homotopy from g_0, and $f' g_t(a) = h_t(a)$. This proves that $f' : T(f) \to Y$ is a fibration.

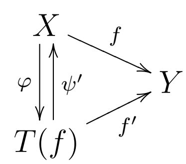
Let $\varphi : X \to T(f)$ and $\psi : T(f) \to X$ be defined by $\varphi(x) = (x, \epsilon_{f(x)})$, where $\epsilon_{f(x)}$ denote the constant path at $f(x)$, and $\psi(x, \lambda) = x$. Then $\psi\varphi = 1$ and $\varphi\psi$ is homotopic to 1 under the deformation $\kappa_t : T(f) \to T(f)$ defined by $\kappa_t(x, \lambda) = (x, \lambda_t)$, where $\lambda_t(s) = \lambda(st)$. This shows that $\varphi : X \to T(f)$ is a homotopy equivalence. There is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \varphi}\downarrow & \nearrow_{f'} & \\ T(f) & & \end{array}$$

which proves the proposition.

Proposition 2 *If $f : X \to Y$ is a fibration, it is fiber homotopy equivalent to $f' : T(f) \to Y$.*

Proof. Consider the homotopy $F_t : T(f) \to Y$ defined by $F_t(x, \lambda) = \lambda(t)$. Then $\psi : T(f) \to X$ given as before by $\psi(x, \lambda) = x$, is a lifting of F_0. Hence there is a homotopy $G_t : T(f) \to X$ such that $G_0 = \psi$ and $fG_t = F_t$. Let $\mu : T(f) \to X^I$ be defined by $\mu(x, \lambda)(t) = G_t(x, \lambda)$. Then $\mu(x, \lambda)(0) = x$ and $f\mu(x, \lambda) = \lambda$. This is the construction of the *path lifting function* for a fibration. We define $\psi' : T(f) \to X$ by $\psi'(x, \lambda) = \mu(x, \lambda)(1)$. There is then a commutative diagram



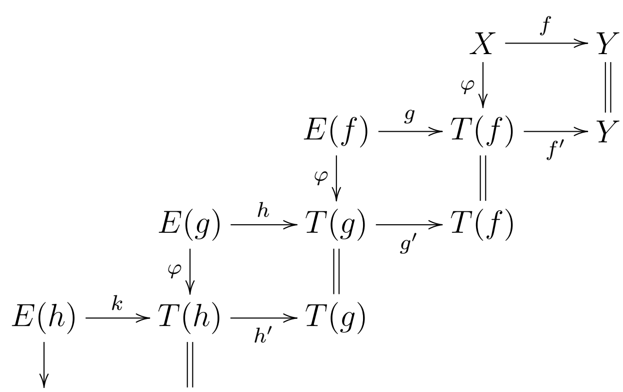
We have to prove the existence of fiber homotopies $\kappa_t : 1 \sim \psi'\varphi$ and $K_t : 1 \sim \varphi\psi'$. Therefore let $\kappa_t(x) = \mu(x, \epsilon_{f(x)})(t)$ and $K_t(x, \lambda) = (\mu(x, \lambda)(t), \lambda_t)$ where $\lambda_t(s) = \lambda(s(1 - t) + t)$. This proves the proposition.

If $f : X \to Y$ is a map of pointed spaces, we take $(*, \epsilon_*)$ as base point in $T(f)$. Then $f' : T(f) \to Y$ is base point preserving, and has the fiber

$$E(f) = \{(x, \lambda) \in X \times Y^I : f(x) = \lambda(0), \lambda(1) = *\}$$

which we call the fiber of f. Also $\varphi : X \to T(f)$ is base point preserving.

If we start with a map $f : X \to Y$ of pointed spaces, we can make it into a fibration, consider the inclusion map $g : E(f) \to T(f)$ of the fiber in the total space, make that into a fibration, and so on. This leads to an infinite commutative diagram as follows, where the rows are fibrations and the vertical maps are homotopy equivalences



(1)

Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration. We have the path lifting function $\mu : T(p) \to E^I$ defined as in the proof of proposition 2. Let $\xi : E^I \to E^I$ be defined by $\xi(\lambda) = \mu(\lambda(0), p\lambda)$. Then

Lemma 3 *$\xi : E^I \to E^I$ is fiber homotopic to the identity in the fibration $p : E^I \to B^I$.*

Proof. Let $A \subseteq I \times I$ be given by $A = \dot{I} \times I \cup I \times 0$. Consider $H : E^I \times I \times I \to B$ defined by $H(\lambda, s, t) = p\lambda(t)$ and let $K : E^I \times A \to E$ be defined by

$$K(\lambda, 0, t) = \mu(\lambda(0), p\lambda)(t)$$
$$K(\lambda, 1, t) = \lambda(t)$$
$$K(\lambda, s, 0) = \lambda(0)$$

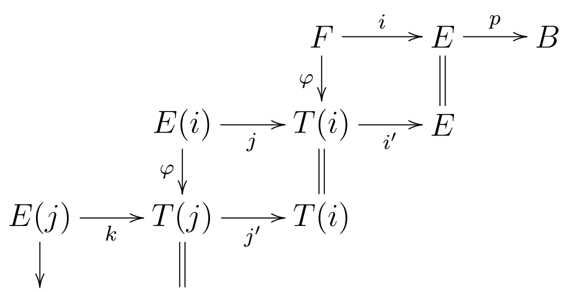
Then $pK = H$ on $E^I \times A$. Hence K can be extended to a lifting of H. Let $K_s : E^I \to E^I$ be defined by $K_s(\lambda)(t) = K(\lambda, s, t)$. Then K_s is a fiber homotopy from ξ to id.

Let $\pi : E(i) \to \Omega B$ be defined by $\pi(x, \lambda) = p\lambda$. Then

Proposition 4 *$\pi : E(i) \to \Omega B$ is a homotopy equivalence.*

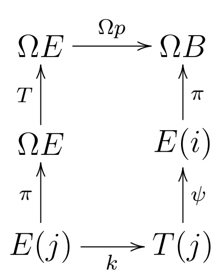
Proof. We define $\iota : \Omega B \to E(i)$ by $\iota(\omega) = (\tilde{\omega}(0), \tilde{\omega})$ where $\tilde{\omega} = \mu(*, \omega^{-1})^{-1}$. Then $\pi\iota = 1$ and $\iota\pi$ is homotopic to 1 by the lemma.

Consider the following diagram where the vertical maps are homotopy equivalences



Then we have

Proposition 5 *The diagram below is homotopy commutative, where T is loop reversal*



Proof. An element $w \in E(j)$ can be written $w = (v, \nu) \in E(i) \times T(i)^I$ where $\nu(0) = v$, $\nu(1) = *$. Then $v = (x, \lambda) \in F \times E^I$ where $\lambda(0) = x$, $\lambda(1) = *$, and $\nu(t) = (x_t, \lambda_t) \in F \times E^I$ where $\lambda_t(0) = x_t$. This gives $x_0 = x$, $\lambda_0 = \lambda$, $x_1 = *$, $\lambda_1 = \epsilon_*$.

Going around the diagram the two ways gives $(\Omega p)T\pi(w)(t) = p\lambda_{1-t}(1)$ and $\pi\psi k(w)(s) = p\lambda(s)$. We define $F : E(j) \times I \times I \to B$ by $F(w,s,t) = p\lambda_t(s)$. Then $F(w,s,0) = p\lambda(s)$ and $F(w,1,1-t) = p\lambda_{1-t}(1)$. Also $F(w,0,t) = F(w,s,1) = *$.

By sweeping the square $I \times I$ with a segment rotating around the corner $(1,0)$ we get a map $H : E(j) \times I \times I \to B$ satisfying $H(w,s,0) = H(w,s,1) = *$, $H(w,0,t) = p\lambda_{1-t}(1)$ and $H(w,1,t) = p\lambda(t)$. This gives a homotopy $H_s : E(j) \to \Omega B$ defined by $H_s(w)(t) = H(w,s,t)$, and we have $H_0 = (\Omega p)T\pi$, $H_1 = \pi\psi k$. This proves the proposition.

Theorem 6 *If $f : X \to Y$ is a map of pointed spaces, there is an infinite sequence of pointed spaces and maps*

$$\dots \longrightarrow \Omega^2 Y \xrightarrow{\Omega j} \Omega E(f) \xrightarrow{\Omega i} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{j} E(f) \xrightarrow{i} X \xrightarrow{f} Y$$

such that any two consecutive maps in the sequence give a diagram which is homotopy equivalent to a fibration, by a homotopy equivalence preserving base points,

Proof. This follows from diagram (1) together with propositions 4 and 5.

Corollary 7 *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. Then there is an infinite sequence of pointed spaces and maps*

$$\dots \longrightarrow \Omega^2 B \xrightarrow{\Omega j} \Omega F \xrightarrow{\Omega i} \Omega E \xrightarrow{\Omega p} \Omega B \xrightarrow{j} F \xrightarrow{i} E \xrightarrow{p} B$$

where any two consecutive maps in the sequence give a diagram which is homotopy equivalent to a fibration by a homotopy equivalence preserving base points.

Proof. This follows from the theorem using proposition 2.

Let $[X,Y]$ be the set of free homotopy classes of maps from X to Y. If Y has a base point, we get a natural base point in $[X,Y]$. Then $[\ ,Y]$ is a contravariant functor from the category of pointed topological spaces to the category of pointed sets.

Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration. Then

$$[\ ,F] \xrightarrow{i_*} [\ ,E] \xrightarrow{p_*} [\ ,B]$$

is an exact sequence of functors. This gives

Corollary 8 *If $f : X \to Y$ is a map of pointed spaces, there is an exact sequence of functors*

$$\cdots \longrightarrow [\ ,\Omega E(f)] \xrightarrow{(\Omega i)_*} [\ ,\Omega X] \xrightarrow{(\Omega f)_*} [\ ,\Omega Y] \xrightarrow{j_*} [\ ,E(f)] \xrightarrow{i_*} [\ ,X] \xrightarrow{f_*} [\ ,Y]$$

Corollary 9 *If $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration, there is an exact sequence of functors*

$$\cdots \longrightarrow [\ ,\Omega F] \xrightarrow{(\Omega i)_*} [\ ,\Omega E] \xrightarrow{(\Omega p)_*} [\ ,\Omega B] \xrightarrow{j_*} [\ ,F] \xrightarrow{i_*} [\ ,E] \xrightarrow{p_*} [\ ,B]$$

Let $p : E \to B$ be a fibration with path lifting function μ. Let $p_B : PB \to B$ be the path fibration over B. We define a map $\beta : PB \to E$ by $\beta(\omega) = \mu(*, \omega)(1)$. Then β is a fiber map.

Proposition 10 *$\beta : PB \to E$ is a fiber homotopy equivalence if E is contractible.*

Proof. Assume E contractible. Then the path fibration $p_E : PE \to E$ has a section $\sigma : E \to PE$. Define $\alpha : E \to PB$ by $\alpha = p\sigma$. We get $\alpha\beta(\omega) = p\sigma p_E \mu(*, \omega)$. From the next lemma $\sigma p_E : PE \to PE$ is fiber homotopic to 1 over E. Therefore $\alpha\beta : PB \to PB$ is fiber homotopic to 1 over B. We have $\beta\alpha(x) = \mu(*, p\sigma(x))(1) = p_E \xi \sigma(x)$, where ξ is defined as in lemma 3. Since ξ is fiber homotopic to 1 over B^I by that lemma, $\beta\alpha : E \to E$ is fiber homotopic to 1 over B. This proves the proposition.

Lemma 11 *Let E be a contractible space and $p_E : PE \to E$ the path fibration over E with a section $\sigma : E \to PE$. Then $\sigma p_E : PE \to PE$ is fiber homotopic to 1 over E.*

Proof. Let $A = \dot{I} \times I \cup I \times \dot{I}$. Then $(I \times I, A)$ is a cofibered pair, hence $(PE \times I \times I, PE \times A)$ is a cofibered pair. We define $H : PE \times A \to E$ by $H(\lambda, s, 0) = \lambda(s)$, $H(\lambda, s, 1) = \sigma(\lambda(1))(s)$, $H(\lambda, 0, t) = *$ and $H(\lambda, 1, t) = \lambda(1)$. Since E is contractible we get an extension $H : PE \times I \times I \to E$. Let $\kappa_t : PE \to PE$ be defined by $\kappa_t(\lambda)(s) = H(\lambda, s, t)$. Then κ_t is a fiber homotopy from 1 to σp_E.

Let G be a topological group. There is then a universal fibration $G \to EG \xrightarrow{p_G} BG$, where EG is a contractible space. Hence it follows from proposition 10 that EG is fiber homotopy equivalent to PBG over BG, and by adjusting $\beta : \Omega BG \to G$ so that it preserves base points we have

Corollary 12 *There is a homotopy equivalence $\beta : \Omega BG \to G$, preserving base points.*

Let H be a closed subgroup of G with local sections. There is then a fibration $G/H \to BH \xrightarrow{\rho} BG$, and we have

Proposition 13 *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega BG & \xrightarrow{\beta} & G \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle q} \\ \Omega BG & \xrightarrow[j]{} & G/H \end{array}$$

where q is the projection and the other maps as before.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} EG & \xrightarrow{p_H} & BH \\ & {\scriptstyle p_G}\searrow \quad \swarrow{\scriptstyle \rho} & \\ & BG & \end{array}$$

G is imbedded as the fiber of p_G, G/H is imbedded as the fiber of ρ, and $p_H = q$ on the fiber. Let μ be a path lifting function for p_G and μ' one for ρ. Then we have $jT(\omega) = \mu'(*, \omega)(1)$ and $q\beta(\omega) = p_H\mu(*, \omega)(1)$. It is clear that the two sections $\mu'(*, \ )$ and $p_H\mu(*, \ )$ in the fibration $PBH \xrightarrow{\rho} PBG$ are fiber homotopic. This implies that jT and $q\beta$ are homotopic.

Proposition 14 *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega BH & \xrightarrow{\beta} & H \\ {\scriptstyle \Omega\rho}\downarrow & & \downarrow{\scriptstyle i} \\ \Omega BG & \xrightarrow[\beta]{} & G \end{array}$$

where i is the inclusion map, and the other maps as before.

Proof. We use the diagram in the previous proof. Let μ'' be a path lifting function for p_H. Then we have $i\beta(\omega) = \mu''(*, \omega)(1)$ and $\beta(\Omega\rho)(\omega) = \mu(*, \rho\omega)(1)$. Because the two maps $\mu''(*, \ )$ and $\mu(*, \rho)$ are fiber homotopic over PBG it follows that $\beta(\Omega\rho)$ and $i\beta$ are homotopic.

Theorem 15 *Let G be a topological group and H a closed subgroup with local sections. Then there is an infinite sequence of pointed spaces and maps, ending with the canonical maps,*

$$\ldots \to \Omega H \xrightarrow{\Omega j} \Omega G \xrightarrow{\Omega q} \Omega(G/H) \xrightarrow{k} H \xrightarrow{j} G \xrightarrow{q} G/H \xrightarrow{i} BH \xrightarrow{\rho} BG$$

such that each two consequtive maps define a diagram which is homotopy equivalent to a fibration, by homotopy equivalences preserving base points.

Corollary 16 *There is an infinite exact sequence of functors*

$$\ldots \to [\ , H] \xrightarrow{j_*} [\ , G] \xrightarrow{q_*} [\ , G/H] \xrightarrow{i_*} [\ , BH] \xrightarrow{\rho_*} [\ , BG]$$

We can interprete the terms of this sequence by means of bundles. $[X, BG]$ is the set of isomorphism classes of principal G-bundles over X, similarly $[X, BH]$. ρ_* is defined by extension of the structure group.

Proposition 17 *If X is a space with a non degenerate base point, then*

$$[SX, BG] = \pi_0(G) \setminus [X, G]/\pi_0(G),$$

where $\pi_0(G)$ operates on $[X, G]$ by left and right multiplication.

Proof. Since BG is path connected, we have canonical isomorphisms

$$\begin{aligned}&[SX, BG] = [SX, BG]_*/\pi_1(BG) = [X, \Omega BG]_*/\pi_1(BG)\\ &= [X, G]_*/\pi_0(G) = \pi_0(G) \setminus [X, G]/\pi_0(G)\end{aligned}$$

Here $\pi_1(BG)$ operates on the right of $[\ ,BG]_*$ as usual. We use the homotopy equivalence $\beta : \Omega BG \to G$ of corollary 12 and the boundary isomorphism $\pi_1(BG) \to \pi_0(G)$ of the homotopy sequence of the fibration $G \to EG \to BG$ for identifications. Then $\pi_0(G)$ operates on the right of $[\ ,G]_*$ by conjugation. We have $[X, G]_* = \pi_0(G) \setminus [X, G]$, where $\pi_0(G)$ operates on the left of $[\ ,G]$ by multiplication. This gives the result.

Hence $[X, G]$ modulo the given action of $\pi_0(G)$ is the set of isomorphism classes of principal G-bundles over SX, similarly for $[X, H]$. j_* is then defined by extension of the structure group.

Next we interpret $[X, G/H]$ by means of bundles. Let $p : E \to X$ be an H-bundle with fiber G, where H acts on G from the left by multiplication. By extending the structure group we can consider E also as principal G-bundle over X. Therefore in particular G acts on E from the right.

By a *G-trivialization* of the given H-bundle E over X we understand a bundle isomorphism $\bar{t} : E \to X \times G$ from the principal G-bundle E over X to the trivial principal G-bundle $X \times G$ over X. Equivalently $\bar{t}$ is a map commuting with the right action of G. There is a bijective correspondence between G-trivializations of E and maps $t : E \to G$ commuting with the right action of G. The correspondence is given by $\bar{t} = (p, t)$.

A *(G, H)-bundle* over X is by definition an H-bundle $p : E \to X$ with fiber G together with a G-trivialization $t : E \to G$. A *morphism* $f : E \to E'$ between (G, H)-bundles over X is defined to be a map between H-bundles such that $t'f$ and t are homotopic through maps commuting with the right action of G. Then each morphism is an isomorphism.

More generally if E and E' are (G, H)-bundles over X and X' a *map* $f : E \to E'$ between (G, H)-bundles is a map of H-bundles such that $t'f$ and t are homotopic as before. Then a morphism in the previous sense is just a map between (G, H)-bundles, covering the identity map.

If E is a (G, H)-bundle over X with G-trivialization $t : E \to G$, and if $f : X^* \to X$ is a map, then the *pullback* $E^* = f^*E$ is a (G, H)-bundle over X^* with G-trivialization $t^* = tf$. A special case is a pullback by an inclusion map which gives a *restriction* of a (G, H)-bundle over X to a (G, H)-bundle over a subspace X^* of X.

Two (G, H)-bundles E_0 and E_1 over X are called *concordant* if there exists a (G, H)-bundle $\tilde{E}$ over $X \times I$ such that $i_0^*\tilde{E} \approx E_0$ and $i_1^*\tilde{E} \approx E_1$, where $i_0, i_1 : X \to X \times I$ are defined by $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$.

Theorem 18 *Two (G, H)-bundles over X are concordant if and only if they are isomorphic.*

Proof. Let $\tilde{E}$ be a (G,H)-bundle over $X \times I$ with a G-trivialization $\tilde{t} : \tilde{E} \to G$. As an H-bundle $\tilde{E}$ is isomorphic with $E \times I$ for some H-bundle E over X. Transfering $\tilde{t}$ by the isomorphism we get a G-trivialization $t : E \times I \to G$. Then $i_0^*\tilde{E}$ is isomorphic to the (G,H)-bundle E with G-trivialization $t_0 = ti_0$, and similarly $i_1^*\tilde{E}$ is isomorphic to the (G,H)-bundle E with G-trivialization $t_1 = ti_1$. But t_0 and t_1 are homotopic through maps commuting with the right action of G. Hence $i_0^*\tilde{E} \approx i_1^*\tilde{E}$. This proves that concordant (G,H)-bundles over X are isomorphic.

The other way is trivial.

Corollary 19 *Let E be a (G,H)-bundle over X, and $f_0, f_1 : X^* \to X$ homotopic. Then $f_0^*E \approx f_1^*E$.*

Proof. If $F : X^* \times I \to X$ is a homotopy from f_0 to f_1, we have $f_0 = Fi_0$ and $f_1 = Fi_1$. Then F^*E is a (G,H)-bundle over $X^* \times I$, and we have $f_0^*E \approx i_0^*F^*E$ and $f_1^*E \approx i_1^*F^*E$. Therefore f_0^*E and f_1^*E are concordant, hence isomorphic.

We let $B(X)$ denote the set of isomorphism classes of (G,H)-bundles over X. Then B is a contravariant functor from the category of topological spaces and homotopy classes of maps.

Let $p : G \to G/H$ be the projection. Since we assume H a closed subgroup of G with local sections, G is a principal H-bundle over G/H. The associated H-bundle with fiber G has a total space $G \times_H G$, where the elements are of the form (g, g') with $g, g' \in G$ and the identification $(gh, h^{-1}g') = (g, g')$ for $h \in H$. The projection $q : G \times_H G \to G/H$ of this bundle is given by $q(g, g') = p(g)$, and the right action of G on the total space $G \times_H G$ is given by $(g, g')g'' = (g, g'g'')$. We have a G-trivialization $t : G \times_H G \to G$ defined by $t(g, g') = gg'$. Therefore $G \times_H G$ is a (G,H)-bundle over G/H. We call it the universal (G,H)-bundle.

We define a natural transformation $T : [\ , G/H] \to B$ of contravariant functors by $Tf = f^*(G \times_H G)$.

Theorem 20 *$T : [\ , G/H] \to B$ is a natural equivalence.*

Proof. Let X be a space. We have a map $T : [X, G/H] \to B(X)$, and define a map $S : B(X) \to [X, G/H]$ as follows. Let E be a (G,H)-bundle over X with G-trivialization $t : E \to G$. We choose a local trivialization $\Phi_U : U \times G \to E$ of the H-bundle E over X, and define $f_U : U \to G/H$ by $f_U(x) = pt\Phi_U(x, 1)$. If $\Phi_V : U \times G \to E$ is another such local trivialization, there is a corresponding coordinate transformation $h_{VU} : U \cap V \to H$ and we have $\Phi_{V,x}^{-1}\Phi_{U,x}(g) = h_{VU}(x)g$ for $x \in U \cap V$ and $g \in G$. Therefore

$$f_U(x) = pt\Phi_U(x, 1) = pt\Phi_{U,x}(1) = pt\Phi_{V,x}h_{VU}(x)$$

$$= pt\Phi_V(x, h_{VU}(x)) = pt(\Phi_V(x, 1)h_{VU}(x)) = p(t\Phi_V(x, 1))h_{VU}(x) = f_V(x)$$

Hence we get a well defined map $f : X \to G/H$ such that $f/U = f_U$ for every trivializing neighborhood U.

Assume E and E' are isomorphic (G, H)-bundles over X with G-trivializations $t : E \to G$ and $t' : E' \to G$. Let $\bar{f} : E \to E'$ be an isomorphism. If $\Phi_U : U \times G \to E$ is a local trivialization of the H-bundle E, then $\Phi'_U = \bar{f}\Phi_U : U \times G \to E'$ is a local trivialization of the H-bundle E'. Therefore we have $f'(x) = pt'\Phi'_U(x, 1) = pt'\bar{f}\Phi_U(x, 1)$ and $f(x) = pt\Phi_U(x, 1)$ for $x \in U$. But $t'\bar{f}$ and t are homotopic through maps commuting with the right action of G. Therefore by using the calculation above, we get that f and f' are homotopic. This proves that $S(E) = f$ gives a well defined map $S : B(X) \to [X, G/H]$. We must prove that S is inverse to T.

Assume E is a given (G, H)-bundle over X as before. Let $S(E) = f$ be the map constructed as above. If $e \in E$ we choose a local trivialization $\Phi_U : U \times G \to E$ around e and define $f_U(x) = (t\Phi_{U,x}(1), \Phi_{U,x}^{-1}(e)) \in G \times_H G$ where $x = p_E(e)$. This gives a map $f_U : p_E^{-1}(U) \to G \times_H G$. Suppose that $\Phi_V : V \times G \to E$ is another local trivialization. Then we have

$$\begin{aligned} f_V(e) &= (t\Phi_{V,x}(1), \Phi_{V,x}^{-1}(e)) = (t(\Phi_{U,x}(1)h_{UV}(x)), h_{VU}(x)\Phi_{U,x}^{-1}(e)) \\ &= ((t\Phi_{U,x}(1))h_{UV}(x), h_{UV}(x)^{-1}\Phi_{U,x}^{-1}(e)) = (t\Phi_{U,x}(1), \Phi_{U,x}^{-1}(e)) = f_U(e) \end{aligned}$$

Hence we get a well defined map $f : E \to G \times_H G$ such that $f/p_E^{-1}(U) = f_U$ for each local trivialization. The diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & G \times_H G \\ {\scriptstyle p_E}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & G/H \end{array}$$

commutes, since we have

$$qf_U(e) = q(t\Phi_{U,x}(1), \Phi_{U,x}^{-1}(e)) = pt\Phi_{U,x}(1) = f(x) = fp_E(e)$$

Also f is a map of H-bundles. The H-bundle structure of $G \times_H G$ over G/H is given as follows. Let $s_i : U_i \to G$ be a local section of $p : G \to G/H$. Then a local trivialization $\Phi_i : U_i \times G \to G \times_H G$ is defined by $\Phi_i(x, g) = (s_i(x), g)$ with inverse $f_i : q^{-1}(U_i) \to U_i \times G$ given by $f_i(g, g') = (p(g), (s_i p(g))^{-1}gg')$. Then

$$\begin{aligned} \Phi_{i,f(x)}^{-1} f\Phi_{U,x}(g) &= pr_G f_i f\Phi_U(x, g) = pr_G f_i(t\Phi_{U,x}(1), g) \\ &= (s_i pt\Phi_{U,x}(1))^{-1}t\Phi_{U,x}(1)g = (s_i f(x))^{-1}t\Phi_{U,x}(1)g \end{aligned}$$

So $\Phi_{i,f(x)}^{-1} f \Phi_{U,x} : G \to G$ is left multiplication by $(s_i f(x))^{-1} t \Phi_{U,x}(1)$ which belongs to H, because applying p we get

$$p((s_i f(x))^{-1} t\Phi_{U,x}(1)) = (s_i f(x))^{-1} pt\Phi_{U,x}(1) = (s_i f(x))^{-1} f(x)$$
$$= (s_i f(x))^{-1} ps_i f(x) = p((s_i f(x))^{-1} s_i f(x)) = p(1)$$

Hence f is a map of H-bundles. Finally

$$tf(e) = tf_U(e) = t\Phi_{U,x}(1)\Phi_{U,x}^{-1}(e) = t(e)$$

so that f is a map of (G,H)-bundles. This proves that we have an isomorphism of (G,H)-bundles $E \approx f^*(G \times_H G)$. So $TS = 1$.

Assume on the other hand $f : X \to G/H$ is given. Let $E^* = f^*(G \times_H G)$ be the (G,H)-bundle over X defined by pullback. If $s_i : U_i \to G$ is a local section defining a local trivialization of $G \times_H G$ as before, let $U_i^* = f^{-1}(U_i)$. Then we have a local trivialization of E^*, namely $\Phi_i^* : U_i^* \times G \to E^*$ defined by $\Phi_i^*(x,g) = (x, \Phi_i(f(x),g))$. Using this we get back a map $f_{U_i^*} : U_i^* \to G/H$ and

$$f_{U_i^*}(x) = pt^*\Phi_i^*(x,1) = ptf\Phi_i^*(x,1) = pt\Phi_i(f(x),1) = f(x)$$

This shows $ST = 1$, and the theorem is proved.

Consequently $[X, G/H]$ can be interpreted geometrically as the isomorphism classes of (G,H)-bundles over X. Then the map $[\ , G/H] \xrightarrow{i_*} [\ , BH]$ in corollary 16 is described by passing from a (G,H)-bundle to its underlying H-bundle. We remark that the projection $p : G \to G/H$ classifies the trivial bundle $G \times G \xrightarrow{pr_1} G$ over G with G-trivialization $t(g,g') = gg'$, because it is evident that this (G,H)-bundle by the previous construction S gives back the map p.

4 Homotopy invariance of fibers and cofibers

If we have given an arbitrary map $f : X \to Y$, we can make f into a fibration or cofibration and construct the fibers and the cofiber pair respectively. We want to prove that these constructions are homotopy invariant.

We consider first the case of fibers and refer back to chapter 3 for notation and some results. We have then a commutative diagram

$$\begin{array}{ccc} X & & \\ {\scriptstyle\varphi}\downarrow & \searrow^{f} & \\ & & Y \\ & \nearrow_{f'} & \\ T(f) & & \end{array}$$

where φ is a homotopy equivalence and f' is a fibration. For each base point $*$ of Y we call $E(f) = f'^{-1}(*)$ a *fiber* of f. The homotopy type of $E(f)$ depends only on the path component of $*$ in Y, not on the actual choice of base point. In particular if Y is path connected, $E(f)$ is uniquely determined up to homotopy type.

If the map $f : X \to Y$ is already a fibration, then φ is a fiber homotopy equivalence, and the fibers of f in the sense defined above coincides up to homotopy type with the ordinary fibers $f^{-1}(*)$ of f.

Theorem 1 *Let*

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & B & \end{array}$$

be a commutative diagram where p and p' are fibrations and f is a homotopy equivalence. Then f is a fiber homotopy equivalence.

Proof. (Dold [1]) Let $g : E' \to E$ be a homotopy inverse to f and $h_t : fg \sim id_{E'}$ a homotopy. We get a homotopy $p'h_t : pg \sim p'$ which may be lifted to a homotopy $k_t : g \sim g'$. Then $pk_t = p'h_t$ and $pg' = p'$. We define a homotopy $l_t : fg' \sim id_{E'}$ by

$$l_t = \begin{cases} fk_{1-2t} & 0 \le t \le \frac{1}{2} \\ h_{2t-1} & \frac{1}{2} \le t \le 1 \end{cases}$$

Then we get a homotopy $p'l_t : p' \sim p'$, and we have

$$p'l_t = \begin{cases} p'h_{1-2t} & 0 \le t \le \frac{1}{2} \\ p'h_{2t-1} & \frac{1}{2} \le t \le 1 \end{cases}$$

which gives the symmetry $p'l_{1-t}$. Now we define $m : E' \times I \times I \to B$ by

$$m_{tu} = \begin{cases} p' & t \le u/2 \text{ or } t \ge 1 - u/2 \\ p'l_{t-u/2} & u/2 \le t \le \frac{1}{2} \\ p'l_{t+u/2} & \frac{1}{2} \le t \le 1 - u/2 \end{cases}$$

and there exists a lifting $n : E' \times I \times I \to E'$ of m such that $n_{t0} = l_t$, $n_{0u} = fg'$ and $n_{1u} = id_{E'}$. Then $n_{t1} : fg' \sim id_{E'}$ is a fiber homotopy. Thus we have proved $fg' \sim_B id_{E'}$, i.e. f has a right fiber homotopy inverse. The same argument applied to g' gives f' such that $g'f' \sim_B id_E$. Then we get

$$g'f \sim_B (g'f)(g'f') = g'(fg')f' \sim_B g'f' \sim_B id_E$$

Hence g' is a fiber homotopy inverse to f, and the theorem is proved.

We introduce the category where the objects are maps between topological spaces, and a morphism from $f : X \to Y$ to $g : U \to V$ is a pair of maps $\alpha : X \to U$ and $\beta : Y \to V$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[\beta]{} & V \end{array}$$

commutes. We denote the morphism by $(\alpha, \beta) : f \to g$. A homotopy between two morphisms $(\alpha_0, \beta_0), (\alpha_1, \beta_1) : f \to g$ is a family of morphisms $(\alpha_t, \beta_t) : f \to g$ such that α_t and β_t are homotopies. (α, β) is called a homotopy equivalence if it has a homotopy inverse in the category.

Theorem 2 *Let*

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[f]{} & B' \end{array}$$

be a commutative diagram, where p and p' are fibrations, $\tilde{f}$ and f homotopy equivalences. Then $(\tilde{f}, f) : p \to p'$ is a homotopy equivalence.

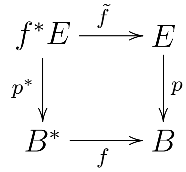
Thus for morphisms between fibrations the concepts of a homotopy equivalence in the strong sense above and in the weak sense of chapter 3 actually coincide.

Proof. Let $\tilde{g} : E' \to E$ and $g : B' \to B$ be homotopy inverses to $\tilde{f}$ and f. Then the diagram

$$\begin{array}{ccc} E & \xleftarrow{\tilde{g}} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xleftarrow[g]{} & B' \end{array}$$

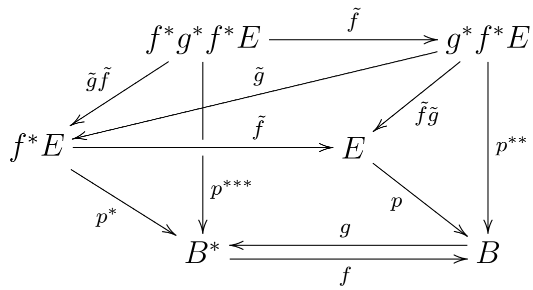
is homotopy commutative. Changing $\tilde{g}$ by a homotopy we may assume the diagram is actually commutative and we have got a morphism $(\tilde{g}, g) : p' \to p$. The homotopy $gf \sim 1$ lifts to a homotopy $\tilde{g}\tilde{f} \sim \tilde{a}$, where $\tilde{a} : E \to E$ is a map over id_B and a homotopy equivalence, hence a fiber homotopy equivalence by theorem 1. Therefore $(\tilde{a}, 1) : p \to p$ is a homotopy equivalence. Since $(\tilde{g}, g)(\tilde{f}, f) = (\tilde{g}\tilde{f}, gf) \sim (\tilde{a}, 1)$, it follows that $(\tilde{g}, g)(\tilde{f}, f)$ is a homotopy equivalence. By symmetry $(\tilde{f}, f)(\tilde{g}, g)$ is a homotopy equivalence. Hence $(\tilde{f}, f)$ is a homotopy equivalence.

Theorem 3 *Let*



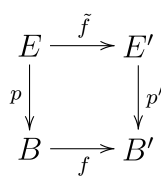
be a pullback diagram where p is a fibration and f is a homotopy equivalence. Then $(\tilde{f}, f) : p^* \to p$ is a homotopy equivalence.

Proof. Let $g : B \to B^*$ be a homotopy inverse to f. We construct the diagram



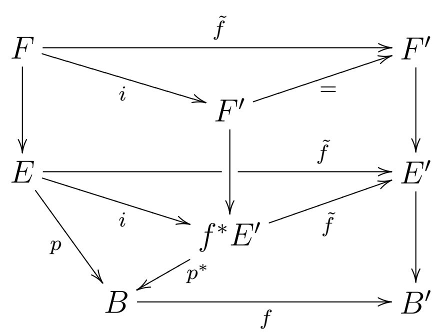
where the maps $\tilde{g}\tilde{f}$ and $\tilde{f}\tilde{g}$ are homotopy equivalences by Spanier [2], p. 102. Hence $\tilde{f} : f^*E \to E$ is a homotopy equivalence and we get the result by theorem 2.

Theorem 4 *Let*



be a commutative diagram where p and p' are fibrations, f and $\tilde{f}$ homotopy equivalences. Then $\tilde{f} : F \to F'$ is a homotopy equivalence for each pair of fibers $F = p^{-1}(x)$ and $F' = p'^{-1}(f(x))$.

Proof. We get a commutative diagram



$\tilde{f} : f^*E' \to E'$ is a homotopy equivalence by theorem 3, hence $i : E \to f^*E'$ is a homotopy equivalence, i.e. by theorem 1 a fiber homotopy equivalence. Therefore $i : F \to F'$, and hence $\tilde{f} : F \to F'$ is a homotopy equivalence.

We can now prove the homotopy invariance of fibers as follows.

Theorem 5 *Let*

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ X_2 & \xrightarrow[f_2]{} & Y_2 \end{array}$$

be a homotopy commutative diagram where α and β are homotopy equivalences. Then the fibers $E(f_1)$ and $E(f_2)$ of base points $y_1 \in Y_1$ and $\beta(y_1) \in Y_2$ are homotopy equivalent.

Proof. Let $h_t : f_2\alpha \sim \beta f_1$ be a homotopy. Define $\alpha' : T(f_1) \to T(f_2)$ by

$$\alpha'(x, \lambda) = (\alpha(x), \mu)$$

$$\mu(t) = \begin{cases} h_{2t}(x) & 0 \le t \le \frac{1}{2} \\ \beta\lambda(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Then the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi_1} & T(f_1) \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\alpha'} \\ X_2 & \xrightarrow[\varphi_2]{} & T(f_2) \end{array}$$

is homotopy commutative. Hence α' is a homotopy equivalence. The diagram

$$\begin{array}{ccc} T(f_1) & \xrightarrow{f_1'} & Y_1 \\ {\scriptstyle\alpha'}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ T(f_2) & \xrightarrow[f_2']{} & Y_2 \end{array}$$

is commutative, and by theorem 4 the restriction $\alpha' : E(f_1) \to E(f_2)$ is a homotopy equivalence.

Next we consider the case of cofibers, where we only have to dualize the previous arguments.

Starting again with a map $f : X \to Y$ we have a commutative diagram

$$\begin{array}{ccc} & & Y \\ & \nearrow^{f} & \big\uparrow{\scriptstyle\pi} \\ X & & \\ & \searrow_{f'} & \\ & & Z_f \end{array}$$

where π is a homotopy equivalence and f' is a cofibration. As usual Z_f denotes the mapping cylinder of f. We call (Z_f, X) the *cofiber pair* of f.

Theorem 6 *Let*

$$\begin{array}{ccccc} & & A & & \\ & {\scriptstyle i}\swarrow & & \searrow{\scriptstyle i'} & \\ X & & \xrightarrow[f]{} & & X' \end{array}$$

be a commutative diagram where i and i' are cofibrations and f is a homotopy equivalence. Then f is a cofiber homotopy equivalence.

Remark. This means of course that there is a map $g : X' \to X$ such that $gi' = i$, and homotopies $h_t : gf \sim id_X$ and $k_t : fg \sim id_{X'}$, such that $h_t i = i$ and $k_t i' = i'$ for all t. The proof of the theorem is dual to that of theorem 1.

Theorem 7 *Let*

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ X & \xrightarrow[\hat{f}]{} & X' \end{array}$$

be a commutative diagram, where i and i' are cofibrations, f and $\hat{f}$ homotopy equivalences. Then $(f, \hat{f}) : i \to i'$ is a homotopy equivalence.

Remark. The proof is dual to that of theorem 2.

The homotopy invariance of the cofiber pair is the following result.

Theorem 8 *Let*

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ X_2 & \xrightarrow[f_2]{} & Y_2 \end{array}$$

be a homotopy commutative diagram where α and β are homotopy equivalences. Then the cofiber pairs (Z_{f_1}, X_1) and (Z_{f_2}, X_2) are homotopy equivalent.

Proof. Let $h_t : f_2\alpha \sim \beta f_1$ be a homotopy. Define $\beta' : (Z_{f_1}, X_1) \to (Z_{f_2}, X_2)$ by

$$\beta'(x,t) = \begin{cases} (\alpha(x), 2t) & 0 \le t \le \frac{1}{2} \\ h_{2t-1}(x) & \frac{1}{2} \le t \le 1 \end{cases}$$

$$\beta'(y) = \beta(y)$$

The diagram

$$\begin{array}{ccc} Z_{f_1} & \xrightarrow{\pi_1} & Y_1 \\ {\scriptstyle \beta'}\downarrow & & {\scriptstyle \beta}\downarrow \\ Z_{f_2} & \xrightarrow[\pi_2]{} & Y_2 \end{array}$$

is homotopy commutative, therefore β' is a homotopy equivalence. The diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1'} & Z_{f_1} \\ {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta'}\downarrow \\ X_2 & \xrightarrow[f_2']{} & Z_{f_2} \end{array}$$

is commutative, hence by theorem 7 the map $\beta' : (Z_{f_1}, X_1) \to (Z_{f_2}, X_2)$ is a homotopy equivalence.

For a broad treatment of homotopy properties of maps see Dieck, e.a. [3].

References

[1] A. Dold, Partitions of unity in the theory of fibrations. *Ann. of Math.* **78** (2) (1963), 223–255.

[2] E. Spanier, Algebraic Topology. McGraw-Hill, 1966.

[3] T. Dieck, K. H. Kamps, D. Puppe, Homotopie theorie. Springer, 1970.

5 Poincaré duality for topological manifolds

The Poincaré duality theorem in its most common form states that if M is a compact, orientable topological n-manifold, then there is an isomorpism

$$H^q(M) \approx H_{n-q}(M)$$

for all q (integer coefficients understood), given by cap multiplication with a fundamental class $[M] \in H_n(M)$.

For orientable topological n-manifolds in general the result is clearly not true, since any contractible manifold of dimension > 0 is a counter example. However, there is a duality isomorphism if we use cohomology with compact support or homology with locally finite chains.

The first generalization is given in Spanier [1]. We consider here the second, i.e. generalization to locally finite chains. The complex of locally finite chains may be obtained as an inverse limit of ordinary chain complexes. Therefore we determine first the homology of an inverse limit of chain complexes. Then we prove the Poincaré duality for locally finite chains by passing to inverse limits. We prove also the relative form.

Homology and inverse limits

In general if $\{A_i, f_{ij}\}$ is a projective system of abelian groups A_i and homomorphisms $f_{ij} : A_j \to A_i$ indexed by a partially ordered set I we may define the *inverse limit* of the system by

$$\varprojlim A_i = \{\{a_i\} \in \prod A_i : f_{ij}(a_j) = a_i \qquad \text{for } i \leq j\}$$

If $I' \subseteq I$ there is a canonical homomorphism

$$\varprojlim_{I'} A_i \to \varprojlim_{I} A_i$$

which is an isomorphism if I' is cofinal in I.

In particular suppose $\{A_n, f_n\}_{\mathbb{N}}$ is a projective system of abelian groups A_n and homomorphisms $f_n : A_{n+1} \to A_n$, indexed by the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. We define a homomorphism

$$\begin{aligned} f : \prod A_n &\to \prod A_n \\ f\{a_n\} &= \{a_n - f_n(a_{n+1})\} \end{aligned}$$

Then we get

$$\begin{aligned} \varprojlim A_n &= \ker f \\ \varprojlim{}' A_n &= \operatorname{coker} f \end{aligned}$$

where the second equation is taken as a definition of $\varprojlim'$ for simplicity. It is easy to verify that also $\varprojlim'$ is unchanged up to isomorphism if we pass to a cofinal indexing set. $\varprojlim$ and $\varprojlim'$ are covariant functors from the category of projective systems of abelian groups to the category of abelian groups. By natural extension they can therefore also be considered as functors from the category of projective systems of chain complexes of abelian groups to the category of chain complexes of abelian groups.

Theorem 1 *Let $\{C^n, f^n\}_{\mathbb{N}}$ be a projective system of chain complexes C^n of abelian groups and surjective chain homomorphisms $f^n : C^{n+1} \to C^n$. Then there is a functorial short exact sequence*

$$0 \to \varprojlim_n{}' H_{p+1}(C^n) \to H_p\Big(\varprojlim_n C^n\Big) \to \varprojlim_n H_p(C^n) \to 0$$

Proof. Let $\prod C^n$ be the product chain complex. Define a chain homomorphism $f : \prod C^n \to \prod C^n$ by $f\{c^n\} = \{c^n - f^n(c^{n+1})\}$ in each degree. Then f is surjective because we can solve the equations $c^n - f^n(c^{n+1}) = d^n$ inductively with respect to c^{n+1}. Therefore we get a short exact sequence of chain complexes

$$0 \to \varprojlim_n C^n \to \prod_n C^n \xrightarrow{f} \prod_n C^n \to 0$$

and hence an exact homology sequence

$$\cdots \to \prod_n H_{p+1}(C^n) \xrightarrow{f_*} \prod_n H_{p+1}(C^n) \xrightarrow{\partial} H_p\Big(\varprojlim_n C^n\Big) \to \prod_n H_p(C^n) \xrightarrow{f_*} \prod_n H_p(C^n) \to \cdots$$

This gives a short exact sequence

$$0 \to \operatorname{coker} f_* \to H_p\Big(\varprojlim_n C^n\Big) \to \ker f_* \to 0$$

which is identical with that of the theorem.

Corollary 2 *Let $\{C_n, f_n\}_{\mathbb{N}}$ be a projective system of cochain complexes C_n of abelian groups and surjective cochain homomorphisms $f_n : C_{n+1} \to C_n$. Then there is a functorial short exact sequence*

$$0 \to \varprojlim_n{}' H^{p-1}(C_n) \to H^p\Big(\varprojlim_n C_n\Big) \to \varprojlim_n H^p(C_n) \to 0$$

Homology with locally finite chains

If X is a topological space, we define a *locally finite singular p-chain* in X as a formal linear combination

$$c = \sum_{\sigma} m(\sigma)\sigma$$

where σ runs through all singular p-simplices of X, $m(\sigma) \in \mathbb{Z}$ and each compact subset of X meets only finitely many σ such that $m(\sigma) \neq 0$.

We let $C_p^{LF}(X)$ denote the abelian group of locally finite singular p-chains in X. Then there is a boundary homomorphism $\partial : C_p^{LF}(X) \to C_{p-1}^{LF}(X)$ defined by

$$\partial c = \sum_{\sigma} m(\sigma)\partial\sigma$$

where the right hand side is obviously a locally finite chain. We get therefore a chain complex $C_*^{LF}(X)$.

When we want to define induced chain homomorphisms we have the problem that the image of a locally finite chain is in general not locally finite. We must therefore restrict attention to a special class of maps. A map $f : X \to Y$ between topological spaces is called *proper* if $f^{-1}(K)$ is compact for each compact subset K of Y. If f is proper we define

$$f_{\#}c = \sum_{\sigma} m(\sigma)f\sigma$$

and get a chain homomorphism $f_{\#} : C_*^{LF}(X) \to C_*^{LF}(Y)$.

If A is a closed subspace of X, the inclusion map $A \to X$ is proper. We get therefore a functor C_*^{LF} from the category of closed topological pairs and proper maps to the category of chain complexes of abelian groups. Composition with homology gives the functor H_*^{LF}, *singular homology with locally finite chains*. It is clear that H_*^{LF} coincides with ordinary singular homology on closed pairs (X, A) where $X - A$ is relatively compact. In general $H_*(X, A) \to H_*^{LF}(X, A)$.

The homology theory H_*^{LF} satisfies the Eilenberg-Steenrod axioms. The homotopy and excision axioms have the following form.

The homotopy axiom. If two proper maps $f, g : (X, A) \to (Y, B)$ between closed topological pairs are properly homotopic, then the induced homomorphisms $f_*, g_* : H_*^{LF}(X, A) \to H_*^{LF}(Y, B)$ are identical.

The excision axiom. If (X, A) is a closed topological pair and U is an open subset of X such that $\operatorname{cl} U \subseteq \operatorname{int} A$, then the excision map $e : (X - U, A - U) \to (X, A)$ induces an isomorphism $e_* : H_*^{LF}(X - U, A - U) \approx H_*^{LF}(X, A)$.

The proofs for ordinary singular homology given in Spanier [1], are valid also for the locally finite case.

We define a cap product $C^q(X) \otimes C_p^{LF}(X) \to C_{p-q}^{LF}(X)$ by the formula

$$u \cap c = \sum_\sigma m(\sigma)\langle u, \sigma_q\rangle_{p-q}\sigma \tag{1}$$

where σ_q denotes the back q-face and ${}_{p-q}\sigma$ the front $(p-q)$-face of σ. By computation

$$\partial(u \cap c) = u \cap \partial c + (-1)^{p-q}\delta u \cap c \tag{2}$$

for $u \in C^q(X)$ and $c \in C_p^{LF}(X)$. Therefore we obtain an induced cap product $H^q(X) \otimes H_p^{LF}(X) \to H_{p-q}^{LF}(X)$.

More generally we can define a relative cap product. Let $\{A_1, A_2\}$ be an excisive couple of closed subsets in X, i.e. $C_*^{LF}(A_1) + C_*^{LF}(A_2) \to C_*^{LF}(A_1 \cup A_2)$ induces an isomorphism in homology. Then the canonical map $C_*^{LF}(X)/(C_*^{LF}(A_1) + C_*^{LF}(A_2)) \to C_*^{LF}(X, A_1 \cup A_2)$ induces an isomorphism in homology. Formula (1) gives a well defined cap product

$$C^q(X, A_1) \otimes (C_p^{LF}(X)/(C_p^{LF}(A_1) + C_p^{LF}(A_2))) \to C_{p-q}^{LF}(X, A_2)$$

satisfying (2). Passing to homology we obtain a cap product

$$H^q(X, A_1) \otimes H_p^{LF}(X, A_1 \cup A_2) \to H_{p-q}^{LF}(X, A_2)$$

The cap product is compatible with that of ordinary singular homology under the natural transformation $H_* \to H_*^{LF}$, since both cap products may be defined by formula (1). If $x \in H^p(X, A_1)$, $y \in H^q(X, A_2)$ and $z \in H_n^{LF}(X, A_1 \cup A_2 \cup A_3)$, then with the appropriate excisiveness conditions satisfied one has in $H_{n-p-q}^{LF}(X, A_3)$ the relation

$$(x \cup y) \cap z = x \cap (y \cap z)$$

Let K be a compact subset of X. We define a chain homomorphism

$$\rho_K : C_*^{LF}(X) \to C_*(X, X-K)$$
$$\rho_K(c) = \sum_\sigma m_K(\sigma)\sigma$$

where

$$m_K(\sigma) = \begin{cases} m(\sigma) & \text{if } \sigma \cap K \neq \emptyset \\ 0 & \text{if } \sigma \cap K = \emptyset \end{cases}$$

We consider the induced chain homomorphism

$$\rho : C_*^{LF}(X) \to \varprojlim_K C_*(X, X-K)$$

where the inverse limit is taken over all compact subsets K of X.

Lemma 3 *ρ is isomorphism.*

Proof. If $\rho(c) = 0$, then $\rho_K(c) = 0$ for each K, hence $c = 0$ because X is covered by compact subsets. Thus ρ is injective. On the other hand suppose $\{c_K\} \in \varprojlim_K C_*(X, X - K)$ is given, where $c_K \in C_*(K)$ is determined up to chains in $X - K$. Let $c_K = \sum_\sigma m(\sigma, K)\sigma$. If $K \subseteq K'$ and $\sigma \cap K \neq \emptyset$, then $m(\sigma, K) = m(\sigma, K')$. Hence if $\sigma \cap K_1$, $\sigma \cap K_2 \neq \emptyset$, then $m(\sigma, K_1) = m(\sigma, K_1 \cup K_2) = m(\sigma, K_2)$. Therefore $m(\sigma) = m(\sigma, K)$ is independent of K as long as $\sigma \cap K \neq \emptyset$. Now we have $c = \sum_\sigma m(\sigma)\sigma \in C_*^{LF}(X)$ and $\rho_K(c) = c_K$ in $C_*(X, X - K)$ for each K. Therefore $\rho(c) = \{c_K\}$ in $\varprojlim_K C_*(X, X - K)$, and ρ is surjective.

Theorem 4 *Let X be a space with a sequence $\{K_i\}$ of compact subsets such that $K_i \subseteq \operatorname{int} K_{i+1}$ and $UK_i = X$. Then there is a short exact sequence*

$$0 \to \varprojlim_i{}' H_{p+1}(X, X - K_i) \to H_p^{LF}(X) \to \varprojlim_i H_p(X, X - K_i) \to 0$$

Proof. The sequence $\{K_i\}$ is cofinal in the family of compact subsets of X. Therefore there is a chain isomorphism

$$\rho : C_*^{LF}(X) \approxeq \varprojlim_i C_*(X, X - K_i)$$

by lemma 3. The natural homomorphisms $C_*(X, X - K_{i+1}) \to C_*(X, X - K_i)$ are surjective. Hence we get the short exact sequence of the theorem from theorem 1.

A short exact sequence for cohomology

If (A, B) is a pair of subsets in a space X, we define a cochain complex

$$\vec{C}^*(A, B) = \varinjlim C^*(U, V)$$

where the direct limit is taken over open pairs (U, V) in X such that $U \supseteq A$ and $V \supseteq B$.

Lemma 5 *Suppose X is a space with a sequence $\{K_i\}$ of compact subsets such that $K_i \subseteq \operatorname{int} K_{i+1}$ and $UK_i = X$. Then the canonical cochain homomorphism*

$$\rho : C^*(X) \to \varprojlim_i \vec{C}^*(K_i)$$

is an isomorphism.

Proof. If $\rho(u) = 0$, then $u/K_i = 0$ for each i, hence $u = 0$, and ρ is injective. Assume on the other hand an element $\{u_i\} \in \varprojlim_i \vec{C}^*(K_i)$ is given, where $u_i \in C^*(U_i)$ for some open neighborhood U_i of K_i. Then $u_{i+1} = u_i$ on a neighborhood of K_i. We define $u \in C^*(X)$ by $u = u_i$ on K_i. It follows that $u = u_i$ on a neighborhood of K_i. Therefore $\rho(u) = \{u_i\}$ in $\varprojlim_i \vec{C}^*(K_i)$ and ρ is surjective.

Theorem 6 *Let X be a space with a sequence $\{K_i\}$ of compact subsets such that $K_i \subseteq \operatorname{int} K_{i+1}$ and $UK_i = X$. Then there is a short exact sequence*

$$0 \to \varprojlim_i{}' \vec{H}^{p-1}(K_i) \to H^p(X) \to \varprojlim_i \vec{H}^p(K_i) \to 0$$

Proof. The canonical homomorphisms $\vec{C}^*(K_{i+1}) \to \vec{C}^*(K_i)$ are surjective. Therefore we get the theorem from lemma 5 together with corollary 2.

Note. Since homology commutes with direct limits we have in general $H(\vec{C}^*(A,B)) = \varinjlim H^*(U,V)$, and this limit is denoted by $\vec{H}^*(A,B)$.

The Poincaré duality theorem

Let M be an oriented topological n-manifold. By definition we assume M has a countable basis. Therefore M is countable at infinity, and theorem 4 gives a short exact sequence

$$0 \to \varprojlim_i{}' H_{p+1}(M, M-K_i) \to H_p^{LF}(M) \to \varprojlim_i H_p(M, M-K_i) \to 0 \qquad (3)$$

By Spanier [1] we have $H_q(M, M-K) = 0$ for $q > n$, so there is an isomorphism

$$\rho : H_n^{LF}(M) \approx \varprojlim_K H_n(M, M-K)$$

Again by Spanier [1] the orientation of M defines a fundamental family $\{z_K\} \in \varprojlim_K H_n(M, M-K)$. We get a correspsonding *fundamental class* $[M] \in H_n^{LF}(M)$ given by $[M] = \rho^{-1}\{z_K\}$. Then we have the theorem announced in the title, as follows

Theorem 7 $\cap[M] : H^q(M) \to H_{n-q}^{LF}(M)$ *is an isomorphism for all q.*

Let U be an open neighborhood of a compact set K in M and $e : (U, U-K) \to (M, M-K)$ the excision map. We define

$$k_* : H^q(U) \to H_{n-q}(M, M-K)$$

by $k_*(v) = e_*(v \cap e_*^{-1} z_K)$. By passing to the direct limit over open neighborhoods U of K we get the homomorphism

$$\vec{k}_* : \vec{H}^q(K) \to H_{n-q}(M, M-K)$$

Lemma 8 *$\vec{k}_*$ is an isomorphism.*

Proof. It is sufficient to prove the lemma for the case where K is contained in a coordinate neighborhood of M, and then use Mayer-Vietoris sequences. We get commutative diagrams of the form

$$\begin{array}{ccccccc}
\cdots \to \vec{H}^{q-1}(K_1 \cap K_2) & \longrightarrow & \vec{H}^q(K_1 \cup K_2) & \longrightarrow & \vec{H}^q(K_1) \oplus \vec{H}^q(K_2) \to \cdots \\
\downarrow{\scriptstyle \vec{k}_*} & & \downarrow{\scriptstyle \vec{k}_*} & & \downarrow{\scriptstyle \vec{k}_*} \\
\cdots \to H_{n-q+1}(M, M - K_1 \cap K_2) & \longrightarrow & H_{n-q}(M, M - K_1 \cup K_2) & \longrightarrow & H_{n-q}(M, M - K_1) \oplus H_{n-q}(M, M - K_2) \to \cdots
\end{array}$$

as in Spanier [1] p. 291, and use the five lemma.

Thus by excision we may assume $M = \mathbb{R}^n$. The family of compact polyhedral neigborhoods is cofinal in the family of neighborhoods of K. Therefore we may assume that K is a compact polyhedron. Then by using Mayer-Vietoris sequences as before we may assume that K is a rectilinear complex. The lemma is now easy to establish.

Proof of theorem. Let $\{K_i\}$ be a sequence of compact subsets of M with $K_i \subsetneq \operatorname{int} K_{i+1}$ and $UK_i = M$. Let $c \in C_n^{LF}(M)$ be a cycle representing the fundamental class $[M]$. Replacing c by a sufficiently fine subdivision we may assume that $\rho_{i-1} c \in C_n(K_i)$ for all i, where $\rho_{i-1} = \rho_{K_{i-1}} : C_n^{LF}(M) \to C_n(M, M - K_{i-1})$ is the chain homomorphism defined earlier.

If U is an open neighborhood of K_i we define the homomorphism

$$l : C^q(U) \to C_{n-q}(M, M - K_{i-1})$$

by $l(u) = u \cap \rho_{i-1} c$. Passing to the direct limit with U we get a homomorphism

$$\vec{l} : \vec{C}^q(K_i) \to C_{n-q}(M, M - K_{i-1})$$

and then by passing to the inverse limit with i we get

$$\cap c : C^q(M) \to C_{n-q}^{LF}(M)$$

There is a commutative diagram

$$\begin{array}{ccc}
 & \overset{\vec{k}_*}{\underset{\approx}{\nearrow}} & H_{n-q}(M, M - K_i) \\
\vec{H}^q(K_i) & & \downarrow \\
 & \underset{\vec{l}_*}{\searrow} & H_{n-q}(M, M - K_{i-1})
\end{array}$$

Therefore the homomorphisms $\vec{l}_*$ induce isomorphisms in $\varprojlim$ and $\varprojlim'$ and we get a commutative diagram

$$\begin{array}{ccccc}
0 \longrightarrow \varprojlim_i{}' \vec{H}^{q-1}(K_1) & \longrightarrow & H^q(M) & \longrightarrow & \varprojlim_i \vec{H}^q(K_1) \longrightarrow 0 \\
{\scriptstyle \vec{l}_*}\downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle \cap[M]} & & {\scriptstyle \approx}\downarrow{\scriptstyle \vec{l}_*} \\
0 \longrightarrow \varprojlim_i{}' H_{n-1-q}(M, M - K_{i-1}) & \longrightarrow & H_{n-q}^{LF}(M) & \longrightarrow & \varprojlim_i H_{n-q}(M, M - K_{i-1}) \longrightarrow 0
\end{array}$$

The theorem then follows from the five lemma.

We establish next the Poincaré duality theorem for manifolds with boundary. Suppose M is a topological n-manifold with boundary ∂M. Then

Lemma 9 *There is an isomorphism*

$$\iota : H_*^{LF}(M-\partial M) \approx H_*^{LF}(M,\partial M)$$

Proof. Let $N=\partial M\times[0,1]$ be a collar on M, where ∂M is identified with $\partial M\times 0$. Let $U=\partial M\times[0,\frac{1}{2}\rangle$. Then we have isomorphisms

$$H_*^{LF}(M-\partial M)\overset{\alpha}{\approx}H_*^{LF}(M-\partial M,N-\partial M)\overset{\beta}{\approx}H_*^{LF}(M-U,N-U)\overset{\gamma}{\approx}H_*^{LF}(M,N)\overset{\delta}{\approx}H_*^{LF}(M,\partial M)$$

α is an isomorphism from the exact homology sequence of the pair $(M-\partial M,N-\partial M)$ because $H_*^{LF}(N-\partial M)=H_*^{LF}(\partial M\times\langle 0,1])=0$, from (3), β and γ are excision isomorphisms and δ is an isomorphism since $\partial M\to N$ is a proper homotopy equivalence. We can now define $\iota=\delta^{-1}\gamma\beta^{-1}\alpha$.

Assume M is orientable, i.e. $M-\partial M$ is orientable. To each fundamental class $[M-\partial M]\in H_n^{LF}(M-\partial M)$ we get a corresponding fundamental class $[M]\in H_n^{LF}(M,\partial M)$ defined by $[M]=\iota[M-\partial M]$. Then we have the relative form of the Poincaré duality theorem as follows

Theorem 10

$$\cap[M]: H^q(M)\to H_{n-q}^{LF}(M,\partial M) \qquad (a)$$

$$\cap[M]: H^q(M,\partial M)\to H_{n-q}^{LF}(M) \qquad (b)$$

are isomorphisms for all q.

To prove that (a) is an isomorphism it is sufficient to cosider the commutative diagram

$$\begin{array}{ccc} H^q(M-\partial M) & \xleftarrow[j^*]{\approx} & H^q(M) \\ {\scriptstyle\cap[M-\partial M]}\Big\downarrow{\scriptstyle\approx} & & \Big\downarrow{\scriptstyle\cap[M]} \\ H_{n-q}^{LF}(M-\partial M) & \xrightarrow[\iota]{\approx} & H_{n-q}^{LF}(M,\partial M) \end{array}$$

where $j: M-\partial M\to M$ is a homotopy equivalence.

To prove that (b) is an isomorphism we need the following

Lemma 11 $\partial: H_n^{LF}(M,\partial M)\to H_{n-1}^{LF}(\partial M)$ *maps $[M]$ to a fundamental class of ∂M.*

Proof. We can define a homomorphism τ by the commutative diagram

$$\begin{array}{ccc} H_*^{LF}(M-\partial M) & \xrightarrow{\tau} & H_*^{LF}(\partial M\times\langle 0,1\rangle) \\ \Big\downarrow & & {\scriptstyle\approx}\Big\downarrow{\scriptstyle\iota} \\ H_*^{LF}(M-\partial M,M-\partial M\times[0,1\rangle) & \xleftarrow[\approx]{} & H_*^{LF}(\partial M\times\langle 0,1],\partial M\times 1) \end{array}$$

By theorem 4 and the characterization of fundamental classes given in Spanier [1], p. 301 we see that $\tau[M - \partial M]$ is a fundamental class of $\partial M \times \langle 0, 1\rangle$.

There is a commutative diagram

$$\begin{array}{ccc} H_n^{LF}(M - \partial M) & \xrightarrow{\tau} & H_n^{LF}(\partial M \times \langle 0,1\rangle) \\ \iota \downarrow \approx & & \approx \downarrow \vee \\ H_n^{LF}(M, \partial M) & & H_n^{LF}(\partial M \times ([0,1\rangle, 0)) \\ \partial \downarrow & & \approx \downarrow \partial \\ H_{n-1}^{LF}(\partial M) & = & H_{n-1}^{LF}(\partial M) \end{array}$$

Hence $\partial[M] = \partial[\partial M \times [0,1\rangle]$. Since it is clear that $\partial[\partial M \times [0,1\rangle]$ is a fundamental class of ∂M, the lemma follows.

By choice of orientation on ∂M we can therefore assume that $\partial[M] = [\partial M]$. Then there is a commutative diagram of exact sequence

$$\begin{array}{ccccccc} \cdots \to H^{q-1}(\partial M) & \longrightarrow & H^q(M, \partial M) & \longrightarrow & H^q(M) \to \cdots \\ \cap[\partial M] \downarrow \approx & & \downarrow \cap[M] & & \approx \downarrow \cap[M] \\ \cdots \to H_{n-q}^{LF}(\partial M) & \longrightarrow & H_{n-q}^{LF}(M) & \longrightarrow & H_{n-q}^{LF}(M, \partial M) \to \cdots \end{array}$$

and the five lemma implies that (b) is an isomorphism.

References

[1] E. Spanier, Algebraic Topology. McGraw-Hill 1968.

6 S-type and S-duality

In this chapter we consider some properties of the stable homotopy category, in particular the existence of duality maps and the construction of duals. We shall view S-duality as an isomorphism of graded functors $(A, B) \to \{X \wedge A, B\}_*$ and $(A, B) \to \{A, Y \wedge B\}_*$. The spaces X and Y are S-duals, and isomorphisms are provided by S-maps $X \wedge Y \to S^0$ or $S^0 \to Y \wedge X$. We prove the classical result that in a sphere a full subcomplex K and its supplement K^- are S-duals. In chapter 7 we prove the Milnor-Spanier/Atiyah duality theorem for spherical fibrations over Poincaré complexes. This contains all earlier versions of the theorem.

The Spanier-Whitehead category SW_* is an additive category whose objects are finite pointed CW-complexes and whose morphisms are limits of homotopy classes. Denote by $\{X, Y\}_* = \mathrm{Hom}_{SW_*}(X, Y)$ the graded abelian group

$$\{X, Y\}_n = \varinjlim [X \wedge S^{n+k}, Y \wedge S^k], \qquad n \in \mathbb{Z}$$

The fact that this limit set is an abelian group is a basic homotopy property of the suspension operation, cf. Spanier [1], p. 44.

A morphism $X \to Y$ in SW_* will be called an *S-map*. Note the following

Lemma 1 $\{X, Y\}_n = 0$ *for* $n < -\dim X$.

Proof. For $k > -n$ we have $\dim(X \wedge S^{n+k}) < k$ and $Y \wedge S^k$ is without cells in dimension $< k$ (except the base point). Hence $[X \wedge S^{n+k}, Y \wedge S^k] = 0$.

In SW_* composition of morphisms gives a bilinear map

$$\{X, Y\}_p \times \{Y, Z\}_q \to \{X, Z\}_{p+q}$$

Similarly, the smash product $\varphi \wedge \psi$ of S-maps gives a bilinear map

$$\{X, Y\}_m \times \{A, B\}_n \to \{X \wedge A, Y \wedge B\}_{m+n}$$

Here $\wedge$ is defined as the limit of the maps

$$\wedge_{pq} : [X \wedge S^{m+p}, Y \wedge S^p] \times [A \wedge S^{n+q}, B \wedge S^q] \to [X \wedge A \wedge S^{m+n+p+q}, Y \wedge B \wedge S^{p+q}]$$

that fit in the commutative diagram

$$\begin{array}{ccc}
X \wedge A \wedge S^{m+n+p+q} & \xrightarrow{(-1)^{np} \wedge_{pq}(\varphi, \psi)} & Y \wedge B \wedge S^{p+q} \\
\| & & \| \\
X \wedge A \wedge S^{m+p} \wedge S^{n+q} & & Y \wedge B \wedge S^p \wedge S^q \\
\uparrow {\scriptstyle \mathrm{id} \wedge \tau \wedge \mathrm{id}} & & \uparrow {\scriptstyle \mathrm{id} \wedge \tau \wedge \mathrm{id}} \\
X \wedge S^{m+p} \wedge A \wedge S^{n+q} & \xrightarrow{\varphi \wedge \psi} & Y \wedge S^p \wedge B \wedge S^q
\end{array}$$

The sign $(-1)^{np}$ is included to make $\wedge_{pq}$ commute with suspensions. τ is the flip-homeomorphism.

From the smash product we derive two *slant products*, denoted / and \. Namely,

$$\{A \wedge X, B\}_m \times \{K, X \wedge L\}_n \xrightarrow{/} \{A \wedge K, B \wedge L\}_{m+n}$$

and

$$\{A, B \wedge X\}_m \times \{X \wedge K, L\}_n \xrightarrow{\backslash} \{A \wedge K, B \wedge L\}_{m+n}$$

defined by

$$\varphi/\psi : A \wedge K \xrightarrow{\mathrm{id}\wedge\psi} A \wedge X \wedge L \xrightarrow{\varphi\wedge\mathrm{id}} B \wedge L$$

$$f\backslash g : A \wedge K \xrightarrow{f\wedge\mathrm{id}} B \wedge X \wedge K \xrightarrow{\mathrm{id}\wedge g} B \wedge L$$

These products are bilinear, and natural in the sense that for given S-maps $A \leftarrow A'$, $B \to B'$, $K \leftarrow K'$, $L \to L'$ the corresponding diagrams

$$\begin{array}{ccc} \{A \wedge X, B\}_* \times \{K, X \wedge L\}_* & \xrightarrow{/} & \{A \wedge K, B \wedge L\}_* \\ \downarrow & & \downarrow \\ \{A' \wedge X, B'\}_* \times \{K', X \wedge L'\}_* & \xrightarrow{/} & \{A' \wedge K', B' \wedge L'\}_* \end{array}$$

$$\begin{array}{ccc} \{A, B \wedge X\}_* \times \{X \wedge K, L\}_* & \xrightarrow{\backslash} & \{A \wedge K, B \wedge L\}_* \\ \downarrow & & \downarrow \\ \{A', B' \wedge X\}_* \times \{X \wedge K', L'\}_* & \xrightarrow{\backslash} & \{A' \wedge K', B' \wedge L'\}_* \end{array}$$

commute. With the identifications $X \wedge S^0 = X = S^0 \wedge X$ we get out special slant products

$$\{X \wedge Y, S^0\}_* \times \{A, Y \wedge B\}_* \xrightarrow{/} \{X \wedge A, B\}_*$$

$$\{S^0, Y \wedge X\}_* \times \{X \wedge A, B\}_* \xrightarrow{\backslash} \{A, Y \wedge B\}_*$$

Definition 2 *X and Y are called S-duals if $\{X \wedge -, -\}_*$ and $\{-, Y \wedge -\}_*$ are isomorphic functors on $SW_* \times SW_*$.*

Here we regard these functors as taking values in the category of graded abelian groups and graded homomorphisms.

It follows from the pairings above that every S-map $u : X \wedge Y \to S^0$ defines a natural transformation

$$\{-, Y \wedge -\}_* \xrightarrow{u/} \{X \wedge -, -\}_*$$

and every S-map $z : S^0 \to Y \wedge X$ a natural transformation

$$\{X \wedge -, -\}_* \xrightarrow{z\backslash} \{-, Y \wedge -\}_*$$

Theorem 3 *The following statements about X and Y are equivalent*

a) X and Y are S-duals.

b) Y and X are S-duals.

c) There exists z such that

$$z\backslash : \{X, S^0\}_* \to \{S^0, Y\}_*$$

is an isomorphism.

d) There exists z such that

$$/z : \{Y, S^0\}_* \to \{S^0, X\}_*$$

is an isomorphism.

e) There exists u such that

$$u/ : \{S^0, Y\}_* \to \{X, S^0\}_*$$

is an isomorphism.

f) There exists u such that

$$\backslash u : \{S^0, X\}_* \to \{Y, S^0\}_*$$

is an isomorphism.

g) There exists u and z such that

$$u/z = \mathrm{id}_X \text{ and } z\backslash u = \mathrm{id}_Y.$$

We will prove the implications b) $\Rightarrow$ f) $\Rightarrow$ g) $\Rightarrow$ b). By symmetry in the proof this will also yield the implications b) $\Rightarrow$ d) $\Rightarrow$ g) $\Rightarrow$ b) and again by symmetry the implications a) $\Rightarrow$ e) $\Rightarrow$ g) $\Rightarrow$ a) and a) $\Rightarrow$ c) $\Rightarrow$ g) $\Rightarrow$ a).

We proceed via some lemmas.

Lemma 4 *Every natural transformation*

$$\mu : \{X \wedge -, -\}_* \to \{-, Y \wedge -\}_*$$

is of the form $\mu(\alpha) = z\backslash\alpha$ for unique $z \in \{S^0, Y \wedge X\}_*$, $z = \mu(\mathrm{id}_X)$.

Proof. Take any S-map $\alpha : X \wedge A \to B$. We get a commutative diagram

$$\begin{array}{ccc}
\{X \wedge A, B\}_* & \xrightarrow{\mu} & \{A, Y \wedge B\}_* \\
\uparrow{\scriptstyle \alpha_*} & & \uparrow{\scriptstyle (\mathrm{id}\wedge\alpha)_*} \\
\{X \wedge A, X \wedge A\}_* & \xrightarrow{\mu} & \{A, Y \wedge X \wedge A\}_* \\
\uparrow{\scriptstyle \wedge\mathrm{id}} & & \uparrow{\scriptstyle \wedge\mathrm{id}} \\
\{X, X\}_* & \xrightarrow{\mu} & \{S^0, Y \wedge X\}_*
\end{array}$$

Hence

$$\begin{aligned}\mu(\alpha) &= \mu(\alpha_*(\mathrm{id}_{X\wedge A})) \\ &= (\mathrm{id}_Y \wedge \alpha)_*\mu(\mathrm{id}_{X\wedge A}) \\ &= (\mathrm{id}_Y \wedge \alpha)_*\mu(\mathrm{id}_X \wedge \mathrm{id}_A) \\ &= (\mathrm{id}_Y \wedge \alpha)_*(\mu(\mathrm{id}_X) \wedge \mathrm{id}_A) \\ &= (\mathrm{id}_Y \wedge \alpha) \circ (\mu(\mathrm{id}_X) \wedge \mathrm{id}_A) \\ &= \mu(\mathrm{id}_X)\backslash\alpha\end{aligned}$$

In other words S-duality arises only from slant products with S-maps. Actually we only need a simple form of lemma 4: If we fix the first variable as S^0, all natural transformations

$$\mu : \{X, -\} \to \{S^0, Y \wedge -\}$$

are of the form $\mu(\alpha) = z\backslash\alpha$.

Lemma 5 *If a natural transformation $\mu : \{X\wedge -, -\}_* \to \{-, Y\wedge -\}_*$ yields an isomorphism at (S^0, S^0), then μ is an isomorphism of functors.*

Proof. Since $S^0 \cong S^n$ in SW_*, we get $\mu(S^m, S^n) : \{X \wedge S^m, S^n\}_* \cong \{S^m, Y \wedge S^n\}_*$. The claim then follows for arbitrary $\mu(A, B)$ by induction on the number of cells in A and B (and the 5 – lemma). As a corollary of lemmas 4 and 5 we get

Corollary 6 *X and Y are S-duals if and only if there is an S-map $z : S^0 \to Y \wedge X$ such that*

$$z\backslash : \{X, S^0\}_* \to \{S^0, Y\}_*$$

is an isomorphism.

We proceed with our proof of theorem 3.

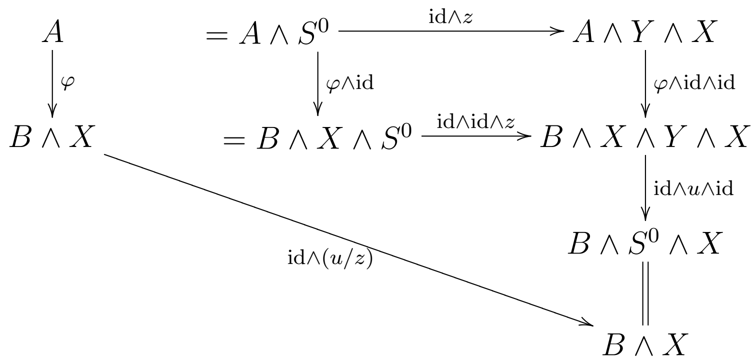
Lemma 7 *The following statements about $z : S^0 \to Y \wedge X$ and $u : X \wedge Y \to S^0$ are equivalent:*

(i) $u/z = \mathrm{id}_X$

(ii) The composite $\{X\wedge A, B\}_* \xrightarrow{z\backslash} \{A, Y \wedge B\}_* \xrightarrow{u/} \{X\wedge A, B\}_*$ *is the identity for all* A, B.

(iii) The composite $\{A, B\wedge X\}_* \xrightarrow{\backslash u} \{A \wedge Y, B\}_* \xrightarrow{/z} \{A, B\wedge X\}_*$ *is the identity for all* A, B.

Proof. (i) $\Rightarrow$ (iii): Given $\varphi : A \to B \wedge X$ there is a commutative diagram



as $u/z = (u \wedge \mathrm{id}_X) \circ (\mathrm{id}_X \wedge z)$. We have

$$\varphi \backslash u = (\mathrm{id}_B \wedge u) \circ (\varphi \wedge \mathrm{id}_Y)$$

and

$$\beta / z = (\beta \wedge \mathrm{id}_X) \circ (\mathrm{id}_A \wedge z)$$

for $\beta : A \wedge Y \to B$. Hence
$(\varphi\backslash u)/z = ((\varphi\backslash u) \wedge \mathrm{id}_X) \circ (\mathrm{id}_A \wedge z) = (\mathrm{id}_B \wedge u \wedge \mathrm{id}_X) \circ (\varphi \wedge \mathrm{id}_Y \wedge \mathrm{id}_X) \circ (\mathrm{id}_A \wedge z)$.
From the diagram we now get

$$(\varphi\backslash u)/z = (\mathrm{id}_B \wedge (u/z)) \circ \varphi$$

Hence (i) $\Rightarrow$ (iii). For $(iii) \Rightarrow (i)$ we need only to choose $A = X, B = S^0$ and $\varphi = id_X$ in the formula above.

The equivalence (i) $\Leftrightarrow$ (ii) is proved analogously, using the formula

$$u/(z\backslash\psi) = \psi \circ ((u/z) \wedge \mathrm{id}_A)$$

for S-maps $X \wedge A \to B$.

Lemma 8 *The following statements about z and u are equivalent:*

(i) $z\backslash u = \mathrm{id}_Y$

(ii) The composite $\{A, Y \wedge B\}_* \xrightarrow{u/} \{X \wedge A, B\}_* \xrightarrow{z\backslash} \{A, Y \wedge B\}_*$ *is the identity for all* A, B.

(iii) The composite $\{A \wedge Y, B\}_* \xrightarrow{/z} \{A, B \wedge X\}_* \xrightarrow{\backslash u} \{A \wedge Y, B\}_*$ *is the identity for all* A, B.

Proof as for lemma 7. We use the identities

$$z\backslash(u/\varphi) = ((z\backslash u) \wedge \mathrm{id}_B) \circ \varphi$$

and

$$(\psi/z)\backslash u = \psi \circ (\mathrm{id}_A \wedge (z\backslash u))$$

valid for arbitrary S-maps $\varphi : A \to Y \wedge B$ and $\psi : A \wedge Y \to B$.

Proof of theorem 3. We organize the proof as explained above.

b) $\Rightarrow$ f): When Y and X are S-duals we have a functorial isomorphism $\{Y \wedge A, B\}_* \cong \{A, X \wedge B\}_*$, in particular

$$\mu : \{Y \wedge A, S^0\}_* \cong \{A, X\}_*$$

and

$$\mu^{-1} : \{A, X\}_* \cong \{Y \wedge A, S^0\}_*$$

with $\mu(\alpha) = z \backslash \alpha$, $z = \mu(\mathrm{id}_Y)$, and $\mu^{-1}(\beta) = w/\beta$, $w = \mu^{-1}(\mathrm{id}_X) : Y \wedge X \to S^0$ (lemma 4 and lemma 7). Set $u = w \circ t$ where $t : X \wedge Y \to Y \wedge X$ is the flip map. Then, if s is the flip $s : Y \wedge A \to A \wedge Y$, we get a commutative diagram

$$\begin{array}{ccc} \{A, X\}_* & = & \{A, X\}_* \\ \downarrow{\scriptstyle \backslash u} & & \downarrow{\scriptstyle w/} \\ \{A \wedge Y, S^0\}_* & \xrightarrow{s^*} & \{Y \wedge A, S^0\}_* \end{array}$$

as $s^*(\varphi \backslash u) = u \circ (\varphi \wedge \mathrm{id}_Y) \circ s = w \circ t \circ (\varphi \wedge \mathrm{id}_Y) \circ s = w \circ (\mathrm{id}_y \wedge \varphi) = w/\varphi$. Because $w/$ and s^* are isomorphisms, $\backslash u$ is also an isomorphism. Choosing $A = S^0$ this gives f).

f) $\Rightarrow$ g): From f) we get that the natural map

$$\backslash u : \{A, B \wedge X\}_* \to \{A \wedge Y, B\}_*$$

is an isomorphism for $A = B = S^0$.

Consequently it is an isomorphism for all A, B by lemma 5. Choose then $z : S^0 \to Y \wedge X$ such that $z \backslash u = \mathrm{id}_Y$ ($A = S^0, B = Y$). From lemma 8 we have that the composite

$$\{A \wedge Y, B\}_* \xrightarrow{/z} \{A, B \wedge X\}_* \xrightarrow{\backslash u} \{A \wedge Y, B\}_*$$

is the identity. Now lemma 7 gives $u/z = \mathrm{id}_X$ which completes g).

g) $\Rightarrow$ b): We are given u and z such that $z \backslash u = \mathrm{id}_Y$ and $u/z = \mathrm{id}_X$. Hence all maps $u/, \backslash z, \backslash u, /z$ occurring in the lemmas 7 and 8 are isomorphisms (as the composites $u/ \circ z\backslash$, $z\backslash \circ u/$, $\backslash u \circ /z$ and $/z \circ \backslash u$ are identity maps). In particular

$$\backslash u : \{A, B \wedge X\}_* \cong \{A \wedge Y, B\}_*$$

Thus we have natural isomorphisms of functors $\{Y \wedge -, -\}_* \cong \{- \wedge Y, -\}_* \cong \{-, - \wedge X\}_* \cong \{-, X \wedge -\}_*$, i.e. Y and X are S-duals, and we have established b). By our earlier remarks (following the statement of theorem 3) this suffices to give theorem 3.

Notice that by theorem 3 the relation "S-duals" is symmetric; this was not immediate from the definition.

Definition 9 *An S-map $z : S^0 \to Y \wedge X$ or $u : X \wedge Y \to S^0$ giving rise to isomorphisms as in theorem 3 will be called an S-duality.*

Corollary 10 *If z (respectively u) is an S-duality, there is an S-duality u (respectively z) such that $z\backslash u = \mathrm{id}_Y$, $u/z = \mathrm{id}_X$.*

Proof. If $u : \{S^0, X\}_* \to \{Y, S^0\}_*$ is an isomorphism, the proof of theorem 3 (f) $\Rightarrow$ g)) shows that there is a z as claimed. The other cases follow similarly.

Let X, Y, K, L be compact CW-complexes and $u : X \wedge Y \to S^0$, $v : K \wedge L \to S^0$ be S-maps. Given S-maps $f : X \to K$ and $g : L \to Y$ we may consider the diagram

$$\begin{array}{ccccc} & & K \wedge L & & \\ & \overset{f\wedge \mathrm{id}}{\nearrow} & & \overset{v}{\searrow} & \\ X \wedge L & & & & S^0 \\ & \underset{\mathrm{id}\wedge g}{\searrow} & & \underset{u}{\nearrow} & \\ & & X \wedge Y & & \end{array}$$

Lemma 11 *If u, v are S-dualities, the relation $v \circ (f \wedge \mathrm{id}_L) = u \circ (\mathrm{id}_X \wedge g)$ determines an isomorphism $\{X, K\}_* \cong \{L, Y\}_*$*

Proof. We get isomorphisms $\{X, K\}_* \xrightarrow{\backslash v} \{X \wedge L, S^0\}_* \xleftarrow{u/} \{L, Y\}_*$. Now observe that $v \circ (f \wedge \mathrm{id}) = f\backslash v$ and $u \circ (\mathrm{id} \wedge g) = u/g$.

Lemma 12 *The following statements about $u : X \wedge Y \to S^0$, $v : K \wedge L \to S^0$, $f : X \to K$ and $g : L \to Y$ are equivalent:*

(i) $u \circ (\mathrm{id}_X \wedge g) = v \circ (f \wedge \mathrm{id}_L)$

(ii) The diagram

$$\begin{array}{ccc} \{A, Y \wedge B\}_* & \xleftarrow{g_*} & \{A, L \wedge B\}_* \\ \Big\downarrow{\scriptstyle u/} & & \Big\downarrow{\scriptstyle v/} \\ \{X \wedge A, B\}_* & \xleftarrow{f^*} & \{K \wedge A, B\}_* \end{array}$$

is commutative for all A, B.

(iii) The diagram

$$\begin{array}{ccc} \{A, B \wedge X\}_* & \xrightarrow{f_*} & \{A, B \wedge K\}_* \\ \Big\downarrow{\scriptstyle \backslash u} & & \Big\downarrow{\scriptstyle \backslash v} \\ \{A \wedge Y, B\}_* & \xrightarrow{g^*} & \{A \wedge L, B\}_* \end{array}$$

is commutative for all A, B.

Proof. Given $\varphi : A \to L \wedge B$ we have

$$u/g_*(\varphi) = (u \circ (\mathrm{id}_X \wedge g) \wedge \mathrm{id}_B) \circ (\mathrm{id}_X \wedge \varphi)$$

and

$$f^*(v/\varphi) = (v \circ (f \wedge \mathrm{id}_L) \wedge \mathrm{id}_B) \circ (\mathrm{id}_X \wedge \varphi)$$

Hence (i) implies (ii). Conversely, taking $B = S^0$, $A = L$, $\varphi = \mathrm{id}_L$ we get from (ii) $v \circ (f \wedge \mathrm{id}_L) = f^*(v/\varphi) = u/g_*(\varphi) = u \circ (\mathrm{id}_X \wedge g)$. (i) $\Leftrightarrow$ (iii) is proved similarly.

If (X, A) is any CW-pair ($A \subset X$), the S-map $\partial = \partial(X, A) \in \{X/A, A\}_{-1}$ is defined by the commutative diagram

$$\begin{array}{ccc} X \cup_A CA & \xrightarrow{p} & X/A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \partial} \\ SA & \xrightarrow{\cong} & A \end{array}$$

where p and q are collapsings. Clearly ∂ is well defined as p is an isomorphism. If $f : (X, A) \to (Y, B)$ is a genuine continuous map, we have a commutative diagram

$$\begin{array}{ccc} X/A & \xrightarrow{\bar{f}} & Y/B \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial} \\ A & \xrightarrow{f} & B \end{array}$$

in SW_*. The sequence

$$A \xrightarrow{i} X \xrightarrow{p} X/A \xrightarrow{\delta} A$$

is called a standard exact triangle. It gives rise to long exact sequences of groups (Puppe sequences, cf chapters 3, 4)

$$\to \{Q, A\}_k \xrightarrow{i_*} \{Q, X\}_k \xrightarrow{p_*} \{Q, X/A\}_k \xrightarrow{\partial_*} \{Q, A\}_{k-1} \to$$

$$\to \{A, Q\}_{k-1} \xrightarrow{\partial^*} \{X/A, Q\}_k \xrightarrow{p^*} \{X, Q\}_k \xrightarrow{i^*} \{A, Q\}_k \to$$

If $g : L \to Y$ is a genuine map, we can form Z_g and C_g, the mapping cylinder and the mapping cone, respectively. We define the S-map $\partial = \partial(g) = \partial(Z_g, L) \in \{C_g, L\}_{-1}$. Then

$$L \xrightarrow{g} Y \xrightarrow{p} C_g \xrightarrow{\partial} L$$

is an *exact triangle*, giving rise to long exact sequences as above.

Consider S-dualities $u : X \wedge Y \to S^0$ and $v : K \wedge L \to S^0$, and let $f : X \to K$, $g : L \to Y$ be maps which are (u, v)-*duals*, i.e.

$$v \circ (f \wedge \mathrm{id}_L) = u \circ (\mathrm{id}_X \wedge g)$$

in $\{X \wedge L, S^0\}_*$ (cf lemma 11). Then there is an S-duality $w : C_f \wedge C_g \to S^0$ such that the S-maps of the triangles

$$X \xrightarrow{f} K \xrightarrow{p} C_f \xrightarrow{\partial} X$$

and

$$Y \xleftarrow{g} L \xleftarrow{\partial} C_g \xleftarrow{p} Y$$

are pairwise (u, v)-duals, (v, w)-duals and (w, u)-duals. We express this briefly as

Theorem 13 *S-duality preserves exact triangles.*

Proof. We may suppose that f and g are inclusions $X \subset K$ and $L \subset Y$, so that $C_f = K/X$ and $C_g = Y/L$. Then $(K \wedge L) \cap (X \wedge Y) = X \wedge L$, and as the diagram

$$\begin{array}{ccccc} & & K \wedge L & & \\ & \overset{\subset}{\nearrow} & & \overset{v}{\searrow} & \\ X \wedge L & & & & S^0 \\ & \underset{\subset}{\searrow} & & \underset{u}{\nearrow} & \\ & & X \wedge Y & & \end{array}$$

commutes, there is an S-map $w' : (K \wedge L) \cup (X \wedge Y) \to S^0$ which restricts to u and v on $K \wedge L$ and $X \wedge Y$, respectively (Mayer-Vietoris gluing). Corresponding to the subcomplexes $K \wedge L$, $X \wedge Y$ and $(K \wedge L) \cup (X \wedge Y)$ in $K \wedge Y$ we then have the following commutative diagram

$$\begin{array}{ccccccc} K \wedge Y & \longrightarrow & K \wedge Y/L & \xrightarrow{\mathrm{id} \wedge \partial_2} & K \wedge L & \xrightarrow{v} & S^0 \\ \| & & \downarrow{\scriptstyle p_1 \wedge \mathrm{id}} & & \downarrow{\scriptstyle \cap} & & \| \\ K \wedge Y & \longrightarrow & K/X \wedge Y/L & \xrightarrow{\partial} & (K \wedge L) \cup (X \wedge Y) & \xrightarrow{w'} & S^0 \\ \| & & \uparrow{\scriptstyle \mathrm{id} \wedge p_2} & & \uparrow{\scriptstyle \cup} & & \| \\ K \wedge Y & \longrightarrow & K/X \wedge Y & \xrightarrow{\partial_1 \wedge \mathrm{id}} & X \wedge Y & \xrightarrow{u} & S^0 \end{array}$$

In fact the right side of the diagram commutes by definition of w', and for the rest it suffices to observe that

$$\partial(K \wedge Y, K \wedge L) = \mathrm{id}_K \wedge \partial(Y, L) = \mathrm{id}_K \wedge \partial_2, \qquad \partial(K \wedge Y, X \wedge Y) = \partial_1 \wedge \mathrm{id}_Y$$

and that $K \wedge Y/(K \wedge L) \cup (X \wedge Y) = K/X \wedge Y/L$. We define

$$w = w' \circ \partial : K/X \wedge Y/L \to S^0$$

and get

$$w \circ (p_1 \wedge \mathrm{id}) = v \circ (\mathrm{id} \wedge \partial_2)$$
$$w \circ (\mathrm{id} \wedge p_2) = u \circ (\partial_1 \wedge \mathrm{id}).$$

Besides

$$v \circ (f \wedge \mathrm{id}) = u \circ (\mathrm{id} \wedge g).$$

From these three equations and lemma 12 we establish the following commutative diagram

$$\begin{array}{ccccccc}
\{S^0, X\}_* & \xrightarrow{f_*} & \{S^0, K\}_* & \xrightarrow{p_{1*}} & \{S^0, K/X\}_* & \xrightarrow{\partial_{1*}} & \{S^0, X\}_* \\
\downarrow{\scriptstyle \backslash u} & & \downarrow{\scriptstyle \backslash v} & & \downarrow{\scriptstyle \backslash w} & & \downarrow{\scriptstyle \backslash u} \\
\{Y, S^0\}_* & \xrightarrow{g^*} & \{L, S^0\}_* & \xrightarrow{\partial_2^*} & \{Y/L, S^0\}_* & \xrightarrow{p_2^*} & \{Y, S^0\}_* \,.
\end{array}$$

As the horizontals are exact triangles and $\backslash u$ and $\backslash v$ are isomorphisms, so is $\backslash w$ (by the 5-lemma). Hence w is an S-duality.

Theorem 14 *Every finite CW-complex has an S-dual.*

Proof. Obviously $S^0 \wedge S^0 \xrightarrow{\cong} S^0$ is an S-duality, so that any two spheres are S-duals (We use that $S^n \cong S^0$). Suppose A has an S-dual B and that $X \supset A$ is obtained from A by attaching a cell. If $\varphi : S^n \to A$ is the attaching map, we have $X = C_\varphi$. Eventually by suspending B we may suppose there is a continuous $\psi : B \to S^m$ which is dual to φ with respect to certain S-dualities $A \wedge B \to S^0$ and $S^n \wedge S^m \to S^0$. By theorem 13 C_ψ is then S-dual to $X = C_\varphi$. It now follows that every connected finite CW-complex has an S-dual. But for any X is $X \cong X \wedge S^1$, which is connected, and an S-dual of $X \wedge S^1$ is an S-dual of X.

We collect some useful observations. The proofs are left to the reader.

Lemma 15 *Suppose X and Y are S-duals and X' and Y'. Then $X \cong X'$ if and only if $Y \cong Y'$.*

(Strictly speaking we need this in the proof of theorem 14.)

Lemma 16 *Suppose X and Y are S-duals and X' and Y'. Then*

(i) $X \wedge X'$ and $Y \wedge Y'$ are S-duals

(ii) $X \vee X'$ and $Y \vee Y'$ are S-duals.

Next an extremely useful result. First recall that by the suspension isomorphisms $H_i(X) \cong H_{i+1}(SX)$ and $H^i(X) \cong H^{i+1}(SX)$ every S-map $f : X \to Y$ induces (graded) isomorphisms $f_* : H_*(X) \to H_*(Y)$ and $f^* : H^*(Y) \to H^*(X)$.

Theorem 17 *Let X and Y be finite CW-complexes and $u : X \wedge Y \to S^0$ an S-map. Then u is an S-duality if and only if*

$$\backslash u : \tilde{H}_*(X) \to \tilde{H}^*(Y)$$

is an isomorphism. Corresponding statements hold for $u/$, $/z$ and $\backslash z$, given $z : S^0 \to Y \wedge X$.

Proof. We first show that $\backslash u : \tilde{H}_*(X) \to \tilde{H}^*(Y)$ is an isomorphism, given an S-duality $u : X \wedge Y \to S^0$. It suffices to demonstrate this for *some* S-dual Y of X and *some* associated S-duality as it then follows generally: If $\backslash u$ is an isomorphism and $v : X \wedge Z \to S^0$ is another S-duality, we have an isomorphism of functors $\epsilon : \{-, Z\}_* \to \{X \wedge -, S^0\}_*$ defined by the commutative diagram

$$\begin{array}{ccc} \{-,Z\}_* & \xrightarrow{v/} & \{X \wedge -, S^0\}_* \\ \downarrow{\scriptstyle \epsilon} & \nearrow{\scriptstyle u/} & \\ \{-,Y\}_* & & \end{array}$$

(as $u/$ is an isomorphism). Plugging in Z, we find $\epsilon = \varphi_*$ for suitable $\varphi : Z \to Y$. Hence $v = v/\mathrm{id}_Z = u/\varphi_*(\mathrm{id}_Z) = u/\varphi = u \circ (\mathrm{id}_X \wedge \varphi)$. Hence the diagram

$$\begin{array}{ccc} \tilde{H}_*(X) & \xrightarrow{\backslash u} & \tilde{H}^*(Y) \\ & {\scriptstyle \backslash v}\searrow & \cong \downarrow {\scriptstyle \varphi^*} \\ & & \tilde{H}^*(Z) \end{array}$$

commutes. Hence $\backslash v$ is in isomorphism (in homology) provided $\backslash u$ is.

This means that for each X it suffices to find *one* S-duality $u : X \wedge Y \to S^0$ (for some Y) such that $\backslash u : \tilde{H}_*(X) \to \tilde{H}^*(Y)$ is an isomorphism. Suppose such a u has been found and consider a $K \supset X$, obtained from X by attaching a cell. By the theorems 13 and 14 there is an S-dual L to K with an S-duality $v : K \wedge L \to S^0$. We may also suppose that the (u, v)-dual to $f : X \subset K$ is an inclusion $g : L \subset Y$. By theorem 13 there is an S-duality $w : K/X \wedge Y/L \to S^0$ such that the S-maps of the exact triangles

$$X \xrightarrow{f} K \xrightarrow{p} K/X \xrightarrow{\partial} X$$
$$Y \xleftarrow{g} L \xleftarrow{\partial'} Y/L \xleftarrow{q} Y$$

are pairwise dual with respect to u, v and w, i.e.

$$\begin{cases} w \circ (p \wedge q) = v \circ (\mathrm{id} \wedge \partial') \\ v \circ (f \wedge \mathrm{id}) = u \circ (\mathrm{id} \wedge g) \\ u \circ (\partial \wedge \mathrm{id}) = w \circ (\mathrm{id} \wedge q)\,. \end{cases}$$

Then the following diagram commutes

$$\begin{array}{ccccccc} \tilde{H}_*(X) & \xrightarrow{f_*} & \tilde{H}_*(K) & \xrightarrow{p_*} & \tilde{H}_*(K/X) & \xrightarrow{\partial_*} & \tilde{H}_*(X) \\ \downarrow{\scriptstyle \backslash u} & & \downarrow{\scriptstyle \backslash v} & & \downarrow{\scriptstyle \backslash w} & & \downarrow{\scriptstyle \backslash u} \\ \tilde{H}^*(Y) & \xrightarrow{g^*} & \tilde{H}^*(L) & \xrightarrow{\partial'^*} & \tilde{H}^*(Y/L) & \xrightarrow{q^*} & \tilde{H}^*(Y) \end{array}$$

But as K/X is a sphere, so is Y/L (up to isomorphism). Hence $\backslash w$ is an isomorphism. By assumption $\backslash u$ is an isomorphism. Therefore the map $\backslash v$ is also an isomorphism.

We are now all set to prove our assertion by induction on the number of cells in X. It remains to prove it for the case $\dim X = 0$, which is straightforward.

The second half of the proof is now easy. Suppose $u : X \wedge Y \to S^0$ is such that $\backslash u : \tilde{H}_*(X) \to \tilde{H}^*(Y)$ is an isomorphism. By theorem 14 Y has an S-dual Z with an S-duality $v : Z \wedge Y \to S^0$. By an argument as above there is an S-map $\psi : X \to Z$ such that the diagram

$$\begin{array}{ccc} \{-, X\}_* & \xrightarrow{\backslash u} & \{- \wedge Y, S^0\}_* \\ {\scriptstyle \psi_*}\downarrow & \nearrow_{\backslash v}^{\cong} & \\ \{-, Z\}_* & & \end{array}$$

commutes. This means that $u = v \circ (\psi \wedge \mathrm{id}_Y)$ and that

$$\begin{array}{ccc} \tilde{H}_*(X) & \xrightarrow[\cong]{\backslash u} & \tilde{H}^*(Y) \\ {\scriptstyle \psi_*}\downarrow & \nearrow_{\backslash v} & \\ \tilde{H}_*(Z) & & \end{array}$$

commutes. But by the first half of the proof $\backslash v$ is an isomorphism and by assumption $\backslash u$ is an isomorphism. Hence ψ_* is an isomorphism. Whitehead's theorem then tells that ψ is an S-isomorphism. But then $\backslash u$ is an isomorphism of functors $\{-, X\}_*$ and $\{- \wedge Y, S^0\}_*$. Hence u is an S-duality.

The next result will not be proved here as it is a consequence of the Milnor-Spanier/Atiyah duality theorem to be proved in the next chapter (or rather of a "relative" version which we leave to the reader).

Theorem 18 *Let M be a compact polyhedral manifold whose tangent microbundle is stably fiber homotopy trivial. Let (K, X) and (Y, L) be pairs of subpolyhedra such that $(Y, L) \subset (M - X, M - K)$ with $H_*(Y, L) \to H_*(M - X, M - K)$ an isomorphism. Then K/X and Y/L are S-dual.*

Corollary 19 (Classical S-duality) *Let A, B be disjoint subpolyhedra of S^n such that $B \subset S^n - A$ induces an isomorphism in homology. Then A and B are S-duals.*

Proof. Apply theorem 18 to show that CA/A and CB/B are S-duals. E.g. choose $M = \mathbb{R}^{n+1}$ (or rather $M = S^{n+1} = \mathbb{R}^{n+1} \cup \{\infty\}$, so as to have M compact) and with $S^n \subset D^{n+1} \subset \mathbb{R}^{n+1}$ the standard inclusions ($S^n = \partial D^{n+1}$) set

$$\begin{aligned} L &= B \\ Y &= CB \\ X &= \partial(2 \cdot D^{n+1}) \\ K &= C(2 \cdot A) \cup \partial(2 \cdot D^{n+1})\,. \end{aligned}$$

Here the cone $C(\ )$ means the *geometric* cone with vertex at the origin in $\mathbb{R}^{n+1}$.

Remark. Theorem 18 actually holds for non-compact manifolds M provided (K, X) and $(Y/L, *)$ are compact pairs.

References

[1] E. Spanier, Algebraic Topology. McGraw-Hill 1968.

7 Poincaré spaces and spherical fibrations

Definitions

At the moment there seems to be no canonically accepted definition of a Poincaré complex X. The definition we shall use in this chapter is to assume X is finite and assume a Poincaré duality property of its universal covering $\tilde{X}$. Since $\tilde{X}$ is in general non finite, we use homology with locally finite chains for $\tilde{X}$. Thus let X be a finite, connected CW-complex with a simply connected covering $p : \tilde{X} \to X$. We say that X is a *Poincaré n-complex* if there exists a class $[\tilde{X}] \in H_n^{LF}(\tilde{X})$, called a *fundamental class* of $\tilde{X}$, such that

$$\cap[\tilde{X}] : H^q(\tilde{X}) \to H_{n-q}^{LF}(\tilde{X})$$

is an isomorphism for all q. It is then clear that n is uniquely determined by X, and the fundamental class $[\tilde{X}]$ is unique up to sign as a generator of $H_n^{LF}(\tilde{X}) = \mathbb{Z}$. We call n the *formal dimension* of X. In general a finite CW-complex is called a Poincaré n-complex if each connected component is a Poincaré n-complex.

With this definition of a Poincaré n-complex, the homotopy invariance of the concept requires a verification. We use the following fact.

Lemma 1 *Let*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array}$$

be a commutative diagram, where the spaces are locally path connected, $\tilde{X}$ and $\tilde{Y}$ are simply connected, p and q are coverings and f is a proper homotopy equivalence. Then $\tilde{f}$ is a proper homotopy equivalence.

Proof. Let $x \in X$ and $y = f(x) \in Y$. From the commutative diagram

$$\begin{array}{ccc} \pi_1(X,x) & \xrightarrow[\approx]{\partial} & p^{-1}(x) \\ {\scriptstyle f_*}\downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle \tilde{f}} \\ \pi_1(Y,y) & \xrightarrow[\partial]{\approx} & q^{-1}(y) \end{array}$$

it follows that $\tilde{f} : p^{-1}(x) \to q^{-1}(y)$ is bijective. In other words $\tilde{f} : \tilde{X} \to \tilde{Y}$ is bijective on fibers.

To prove that $\tilde{f} : \tilde{X} \to \tilde{Y}$ is proper, let $\tilde{L}$ be a compact subset of $\tilde{Y}$ so small that $\tilde{L}$ projects homeomorphically onto a compact subset L of Y, where L has an open, path connected neighborhood V which is evenly covered by q. Let $K = f^{-1}(L)$ and $U = f^{-1}(V)$. Since f is proper, K is compact. We can write $K = \bigcup_{i=1}^{n} K_i$ where K_i is compact and has an open, path connected neighborhood $U_i \subseteq U$ which is evenly covered by p. Then $p^{-1}(K_i) = \bigcup_\alpha \tilde{K}_{i\alpha}$ where each $\tilde{K}_{i\alpha}$ projects homeomorphically onto K_i. Then for a fixed i we have $\tilde{K}_{i\alpha} \cap \tilde{f}^{-1}(\tilde{L}) \neq \emptyset$ for exactly one α. Hence $\tilde{f}^{-1}(\tilde{L}) \subseteq \bigcup_{i\alpha} \tilde{K}_{i\alpha}$ and only finitely many $\tilde{K}_{i\alpha}$ meet $\tilde{f}^{-1}(\tilde{L})$. Therefore $\tilde{f}^{-1}(\tilde{L})$ is compact. It follows that $\tilde{f}$ is proper, because any compact subset of $\tilde{Y}$ is a finite union of compact subsets $\tilde{L}$ of the form considered.

Thus any lifting of a proper homotopy equivalence is a proper map. Similarly any lifting $\tilde{F} : \tilde{X} \times I \to \tilde{Y}$ of a proper homotopy of a proper homotopy equivalence is proper.

Let $g : Y \to X$ be a proper homotopy inverse to f. Since spaces are locally path connected there is a lifting $\tilde{g} : \tilde{Y} \to \tilde{X}$ of g, and $\tilde{g}$ is proper by the first part. Since gf is properly homotopic to id_X, $\tilde{g}\tilde{f}$ is properly homotopic to a covering transformation $\tilde{c}$ of $\tilde{X}$, again by the first part. Therefore $\tilde{c}^{-1}\tilde{g}$ is a left proper homotopy inverse to $\tilde{f}$. By symmetry we get a right inverse. Hence $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a proper homotopy equivalence.

Proposition 2 *If X and Y are homotopy equivalent finite CW-complexes and X is a Poincaré n-complex, then Y is a Poincaré n-complex.*

Proof. Let $f : X \to Y$ be a homotopy equivalence and $\tilde{f} : \tilde{X} \to \tilde{Y}$ a lifting to the simply connected coverings. $\tilde{f}$ is a proper homotopy equivalence by lemma 1. Since $\tilde{X}$ has a fundamental class $[\tilde{X}] \in H_n^{LF}(\tilde{X})$, we get a fundamental class $[\tilde{Y}] = f_*[\tilde{X}] \in H_n^{LF}(\tilde{Y})$, and the proposition follows.

We define a *Poincaré n-space* to be a space which is homotopy equivalent to a Poincaré n-complex.

Theorem 3 *A compact topological n-manifold is a Poincaré n-space.*

Proof. We have to use here the result, see Kirby [1], that if M is a compact topological n-manifold, then there exists a homotopy equivalence $f : M \to X$, where X is a finite CW-complex. Then lemma 1 gives a proper homotopy equivalence $\tilde{f} : \tilde{M} \to \tilde{X}$ between the simply connected coverings. From the Poincaré duality theorem 7 of chapter 5 it follows that $\tilde{X}$ has a fundamental class, so that X is a Poincaré n-complex, hence M a Poincaré n-space.

The Thom isomorphism theorem for spherical fibrations

If $p : E \to B$ is a fibration in general, we define its *Thom pair* to be the cofiber pair (Z_p, E) and its *Thom space* to be the quotient $C_p = Z_p/E$. The constructions are clearly functorial from the category of fibrations.

The Thom pair has the following multiplicative property. Let

$$F_1 \longrightarrow E_1 \xrightarrow{p_1} B_1$$
$$F_2 \longrightarrow E_2 \xrightarrow{p_2} B_2$$

be fibrations, and define their *fiberwise join* $p = p_1 * p_2$ to be the fibration

$$F_1 * F_2 \longrightarrow E \xrightarrow{p} B_1 \times B_2$$

where the total space is the quotient $E = E_1 \times I \times E_2/\sim$ with

$$\begin{aligned} (e_1, 0, e_2) &\sim (e_1', 0, e_2) \qquad \text{if } p_1(e_1) = p_1(e_1') \\ (e_1, 1, e_2) &\sim (e_1, 1, e_2') \qquad \text{if } p_2(e_2) = p_2(e_2') \end{aligned}$$

and $p(e_1, s, e_2) = (p_1(e_1), p_2(e_2))$. It is easy to verify that p is in fact a fibration, and the fiber of p is clearly the join $F_1 * F_2$.

Proposition 4 *There is a homeomorphism* $(Z_p, E) = (Z_{p_1}, E_1) \times (Z_{p_2}, E_2)$.

Proof. There is a homeomorphism $h : Z_p \to Z_{p_1} \times Z_{p_2}$ defined by

$$\begin{aligned} h((e_1, s, e_2), t) &= \begin{cases} ((e_1, 1 - 2s(1-t)), (e_2, t)) & s \leq \frac{1}{2} \\ ((e_1, t), (e_2, t + (2s-1)(1-t))) & s \geq \frac{1}{2} \end{cases} \\ h(b_1, b_2) &= (b_1, b_2) \end{aligned}$$

and h maps E onto $Z_{p_1} \times E_2 \cup E_1 \times Z_{p_2}$.

Corollary 5 *There is a homeomorphism* $C_p = C_{p_1} \wedge C_{p_2}$.

Now we restrict attention to spherical fibrations. Suppose $p : E \to B$ is a $(k-1)$-spherical fibration with fibers $F_b = p^{-1}(b)$. To each path ω from b to b' in B there is a homotopy equivalence $h_{[\omega]} : F_b \to F_{b'}$ and hence also a homotopy equivalence $h_{[\omega]} : (CF_b, F_b) \to (CF_{b'}, F_{b'})$. We get a local system of integers over B defined by

$$O_p = \{H_k(CF_b, F_b)\}_{b \in B}$$

Clearly the fibration is orientable if and only if O_p is constant. In general we call O_p the *orientation system* of p. By a *Thom class* of p we understand a class $U \in H^k(Z_p, E; O_p)$ whose restriction to each fiber is a canonical generator. Then we have the Thom isomorphism theorem in the following form, where $\pi : Z_p \to B$ denotes the projection.

Theorem 6 *Let* $p : E \to B$ *be a* $(k-1)$*-spherical fibration over a locally contractible space* B. *Then* p *has a unique Thom class* $U \in H^k(Z_p, E; O_p)$. *If* S *is any local system of integers over* B *there are isomorphisms*

$$\begin{aligned} \Phi_* &: H_i(Z_p, E; \pi^* S) \approxeq H_{i-k}(B; O_p \otimes S) \\ \Phi^* &: H^i(B; S) \approxeq H^{i+k}(Z_p, E; \pi^*(O_p \otimes S)) \end{aligned}$$

defined by $\Phi_* x = \pi_*(U \cap x)$ *and* $\Phi^* u = \pi^* u \cup U$.

Remark. With the assumption that B is locally contractible we have that p is locally fiber homotopy trivial. The theorem can then be proved by Mayer-Vietoris sequences and a limit argument just as in the case of bundles. Compare Holm [2]. If p is orientable and S is constant, then we can drop the assumption on B, because from the theorem above we get the general case by using a CW-approximation to B.

The following result, in the converse direction of the Thom isomorphism theorem, is useful for recognizing spherical fibrations.

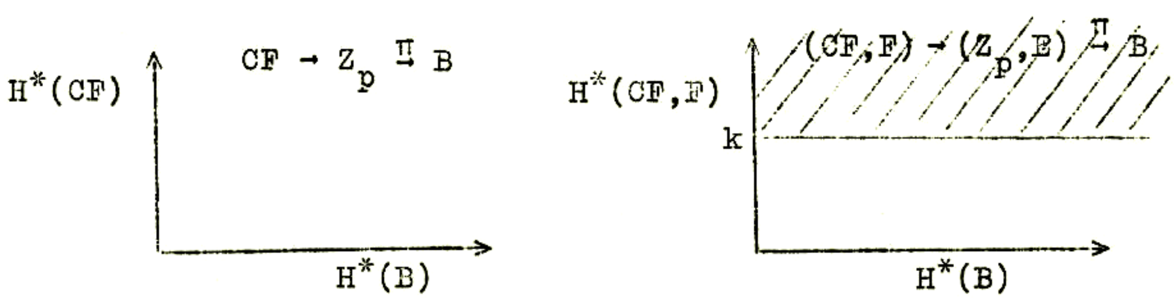
Theorem 7 *Let $F \to E \xrightarrow{p} B$ be a fibration where B and F are simply connected. Assume that there exists a class $U \in H^k(Z_p, E)$ such that*

$$\Phi^* : H^q(B) \to H^{q+k}(Z_p, E)$$

defined by $\Phi^* u = \pi^* u \cup U$ is an isomorphism for all q. Then p is $(k-1)$-spherical.

Proof. We get $H^i(Z_p, E) = 0$ for $i < k$ and $H^k(Z_p, E) = \mathbb{Z}$ with generator U. By universal coefficients $H_i(Z_p, E) = 0$ for $i < k$, and hence by the relative Hurewicz theorem $\pi_{i-1} F = \pi_i(Z_p, E) = 0$ for $i < k$, since we may assume $k \geq 3$. Thus F is $(k-2)$-connected.

Now we consider the cohomology spectral sequences of the fibration $CF \to Z_p \xrightarrow{\pi} B$ and the pair of fibrations $(CF, F) \to (Z_p, E) \xrightarrow{\pi} B$, together with the cup product pairing between them. The E_2-terms are zero except for ${}'E_2^{*0} = H^*(B)$ and ${}''E_2^{*q} = H^*(B) \otimes H^q(CF, F)$ with $q \geq k$ respectively.



We have $H^k(CF, F) = {}''E_2^{0k} = {}''E_\infty^{0k} = H^k(Z_p, E) = \mathbb{Z}$ with generator U.
Therefore $\Phi^* : {}'E_2^{*0} \to {}''E_2^{*k}$ is an isomorphism. By assumption $\Phi^* : {}'E_\infty^{*0} \to {}''E_\infty^{**}$ is an isomorphism. This implies ${}''E_2^{*k} = {}''E_\infty^{*k}$ and ${}''E_\infty^{*q} = 0$ for $q \neq k$. Hence $H^q(CF, F) = {}''E_2^{0q} = 0$ for $q \neq k$. This proves that F is a cohomology $(k-1)$-sphere, hence by universal coefficients and the Whitehead theorem, a homotopy $(k-1)$-sphere.

Normal fibrations of Poincaré polyhedra

Let X be a finite polyhedron with a PL imbedding $X \to \mathbb{R}^{n+k}$. We choose a regular neighborhood N of X and let $r : N \to X$ be a homotopy inverse to the inclusion. Let $p : \partial N \to X$ be the restriction of r. The map p made into a fibration is called a *normal fibration* of X and is denoted by ν. Spivak [3].

Proposition 8 *Up to homotopy type the Thom pair of ν is $(N, \partial N)$, and the fibers of ν are those of the inclusion map $i : \partial N \to N$.*

Proof. This follows from the fact that there is a homotopy commutative diagram

$$\begin{array}{ccc} \partial N & \xrightarrow{i} & N \\ \| & & \downarrow r \\ \partial N & \xrightarrow{q} & X \end{array}$$

By theorem 8 of chapter 4 we have that $(Z_q, \partial N)$ is homotopy equivalent to $(Z_i, \partial N)$ which is homotopy equivalent to $(N, \partial N)$ since the last pair is cofibered. For the fibers we use theorem 5 of chapter 4.

Theorem 9 *Assume that the imbedding $X \hookrightarrow \mathbb{R}^{n+k}$ is of codimension ≥ 3. Then the normal fibration ν is $(k-1)$-spherical if and only if X is a Poincaré n-polyhedron.*

Proof. Up to homotopy type we can write the normal fibration $\nu : F \to \partial N \hookrightarrow N$. For $j = 0, 1$ any map $(D^{j+1}, S^j) \to (N, \partial N)$ can be homotoped off X and hence into ∂N, and we get $\pi_j F = \pi_{j+1}(N, \partial N) = 0$ Hence F is simply connected.

N is an oriented PL $(n+k)$-manifold with boundary. We have a pullback diagram

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\tilde{p}} & N \\ {\scriptstyle \tilde{r}}\downarrow & & \downarrow {\scriptstyle r} \\ \tilde{X} & \xrightarrow{p} & X \end{array}$$

where p is the simply connected covering. Then $\tilde{p}$ is a simply connected covering, and the fibers of the inclusion map $\partial \tilde{N} \hookrightarrow \tilde{N}$ are homeomorphic with those of $\partial N \hookrightarrow N$. Up to homotopy type we have therefore the fibration $\tilde{\nu} : F \to \partial \tilde{N} \hookrightarrow \tilde{N}$.

Assume now that ν is $(k-1)$-spherical. By theorem 6 there is a class $U \in H^k(\tilde{N}, \partial \tilde{N})$ such that $\cup U : H^q(\tilde{N}) \to H^{q+k}(\tilde{N}, \partial \tilde{N})$ is an isomorphism. We get the commutative diagram

$$\begin{array}{ccc} H^q(\tilde{N}) & \xrightarrow{\cup U} & H^{q+k}(\tilde{N}, \partial \tilde{N}) \\ & {\scriptstyle \cap \mu} \searrow \quad \swarrow {\scriptstyle \cap [\tilde{N}]}\ \approx & \\ & H^{LF}_{n-q}(\tilde{N}) & \end{array}$$

where $\mu = U \cap [\tilde{N}] \in H^{LF}_n(\tilde{N})$. Hence $\cap \mu$ is an isomorphism for all q. Since $\tilde{r} : \tilde{N} \to \tilde{X}$ is a proper homotopy equivalence by lemma 1, it follows that there is a fundamental class $[\tilde{X}] = \tilde{r}_*(\mu) \in H^{LF}_n(\tilde{X})$. Hence X is a Poincaré n-polyhedron.

If we assume conversely that X is a Poincaré n-polyhedron, we can reverse the argument to get the class U and then use theorem 7 to conclude that $\tilde{\nu}$ is $(k-1)$-spherical.

For a Poincaré polyhedron there is a duality theorem with local coefficients as follows.

Theorem 10 *Let X be a Poincaré n-polyhedron. Then there exists a local system O_X of integers over X with the following property. There exists a class $[X] \in H_n(X; O_X)$ such that*

$$\cap[X] : H^q(X; S) \to H_{n-q}(X; O_X \otimes S)$$

is an isomorphism for any local system S of integers over X and all q. O_X is uniquely determined up to isomorphism by this property.

Proof. The local system O_X is given by $O_X = O_\nu$ where ν is any $(k-1)$-spherical normal fibration of X. We have the commutative diagram

$$\begin{array}{ccc} H^q(N; q^*S) & \xrightarrow[\approx]{\cup U} & H^{q+k}(N, \partial N; q^*(O_\nu \otimes S)) \\ & {\scriptstyle \cap\mu} \searrow \quad \swarrow {\scriptstyle \cap[N]}\, {\scriptstyle \approx} & \\ & H_{n-q}(N; q^*(O_\nu \otimes S)) & \end{array}$$

where $\mu = U \cap [N] \in H_n(N; q^*O_\nu)$. $\cup U$ is an isomorphism by theorem 6, and $\cap[N]$ from the Poincaré duality theorem with local coefficients for the orientable PL $(n+k)$-manifold N, see Holm [2]. It follows that $\cap\mu$ is an isomorphism, and we define $[X] = r_*\mu \in H_n(X; O_\nu)$. This proves the existence of the local system O_X with the property of the theorem.

To prove uniqueness assume O'_X is another such local system. We may assume X connected without loss of generality. We have

$$H^0(X; O_X \otimes O'_X) = H_n(X; O_X) = H^0(X; O_X \otimes O_X) = H^0(X) = \mathbb{Z}.$$

Thus $O_X \otimes O'_X$ is constant, i.e. $O_X = O'_X$.

We call the local system O_X the *orientation system* of X, and call X *orientable* if O_X is constant.

S-types of Thom spaces and uniqueness of the stable normal fibration

If ξ is a fibration over X, we use also the notation X^ξ for the Thom space of ξ.

If the fibrations ξ and η over X are $(k-1)$-spherical and $(l-1)$-spherical respectively, then $\xi * \eta$ over $X \times X$ is $(k+l-1)$-spherical. The pullback $\xi + \eta = \Delta^*(\xi * \eta)$, where $\Delta : X \to X \times X$ is the diagonal map, is by definition the *Whitney join* of ξ and η.

For spherical fibrations ξ and η over X and Y respectively we have $\xi * \eta = p_1^*\xi + p_2^*\eta$, where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the projections. Since the $(l-1)$-spherical fibration over a point has Thom space S^l we get from corollary 5 that $X^{\xi+1} = S^l \wedge X^\xi$. In other words Whitney join with trivial spherical fibrations corresponds to suspension of the Thom space. Therefore we get

Proposition 11 *If ξ and η are stably fiber homotopy equivalent spherical fibrations over X, then X^ξ and X^η are S-equivalent.*

In other words the S-type of X^ξ depends only on the class of ξ in $k_G(X)$.

We prove next the duality theorem of Atiyah [4], generalized to spherical fibrations, by the method of Holm [2]. Note that if ξ is a spherical fibration over a space X which has the homotopy type of a finite CW-complex, then it follows from Stasheff [5], proposition (O), that the Thom space X^ξ has the homotopy type of a finite CW-complex. Thus the S-duality theory applies to these Thom spaces.

Theorem 12 *Let ξ and η be spherical fibrations over a Poincaré polyhedron X. Assume that $\xi + \eta = \nu$ is a normal fibration of X. Then the Thom spaces X^ξ and X^η are S-dual.*

Proof. Assume ξ is $(k-1)$-spherical, η is $(l-1)$-spherical and that X is a Poincaré n-polyhedron. By proposition 4 we have a commutative diagram

$$\begin{array}{ccc} X^\nu & \xrightarrow{\hat{\Delta}} & X^\xi \wedge X^\eta \\ \uparrow{\scriptstyle c} & & \uparrow{\scriptstyle c} \\ (Z_\nu, E_\nu) & \xrightarrow{\bar{\Delta}} & (Z_\xi, E_\xi) \times (Z_\eta, E_\eta) \\ \downarrow{\scriptstyle \pi_\nu} & & \downarrow{\scriptstyle \pi_\xi \times \pi_\eta} \\ X & \xrightarrow[\Delta]{} & X \times X \end{array}$$

where Δ is the diagonal map, c is collapsing, and $\bar{\Delta}$ and $\hat{\Delta}$ are the induced maps from Δ.

We have the composed map

$$f : S^{n+k+l} \xrightarrow{t} X^\nu \xrightarrow{\hat{\Delta}} X^\xi \wedge X^\eta$$

where t is the collapsing map as before. We let $\iota \in \tilde{H}_{n+k+l}(S^{n+k+l})$ denote the canonical generator and define

$$\begin{aligned} \hat{\alpha} &= f_*(\iota) \in \tilde{H}_{n+k+l}(X^\xi \wedge X^\eta) \\ \alpha &= c_*^{-1}(\hat{\alpha}) \in H_{n+k+l}((Z_\xi, E_\xi) \times (Z_\eta, E_\eta)) \end{aligned}$$

Then we have a commutative diagram

$$\begin{array}{ccc} \tilde{H}^i(X^\xi) & \xrightarrow{\hat{\alpha}/} & \tilde{H}_{n+k+l-i}(X^\eta) \\ {\scriptstyle c^*}\downarrow{\scriptstyle \approx} & & {\scriptstyle c_*}\uparrow{\scriptstyle \approx} \\ H^i(Z_\xi, E_\xi) & \xrightarrow[\alpha/]{} & H_{n+k+l-i}(Z_\eta, E_\eta) \end{array}$$

We want to prove that f is a duality map, i.e. that $\hat{\alpha}/$ or equivalently $\alpha/$ is an isomorphism. Consider the diagram

$$\begin{array}{ccc} H^{q+k}(Z_\xi, E_\xi) & \xrightarrow{\alpha/} & H_{n-q+l}(Z_\eta, E_\eta) \\ \approx \uparrow \Phi_\xi^* & & \approx \downarrow \Phi_{\eta^*} \\ H^q(X; O_\xi) & \underset{\cap [X]}{\xrightarrow{\approx}} & H_{n-q}(X; O_\eta) \end{array}$$

It is easy to verify the commutativity of the diagram. For $u \in H^q(X; O_\xi)$ we get

$$\begin{aligned} u \cap [X] &= p_{2^*}\Delta_*(u \cap [X]) = p_{2*}(u \times 1 \cap \Delta_*[X]) \\ &= p_{2*}(u \times 1 \cap (\pi_\xi \times \pi_\eta)_*(U_\xi \times U_\eta \cap \alpha)) \\ &= p_{2*}(\pi_\xi \times \pi_\eta)_*((\pi_\xi \times \pi_\eta)^*(u \times 1) \cap U_\xi \times U_\eta \cap \alpha) \\ &= \pi_{\eta^*} p_{2^*}(((\pi_\xi^* u \times 1) \cup (U_\xi \times U_\eta)) \cap \alpha) \\ &= \pi_{\eta^*} p_{2^*}((\Phi_\xi^* u \times U_\eta) \cap \alpha) \\ &= \pi_{\eta^*}((\alpha/\Phi_\xi^* u) \cup U_\eta) = \Phi_{\eta^*}(\alpha/\Phi_\xi^* u) \end{aligned}$$

We can now prove the uniqueness of the stable normal fibration by copying the procedure for bundles given in Atiyah [4].

A pointed space $(Y, *)$ is called *reducible* if there is a map $f : (S^n, *) \to (Y, *)$ inducing isomorphisms $f_* : H_p(S^n, *) \approx H_p(Y, *)$ for $p \geq n$, and *coreducible* if there is a map $g : (Y, *) \to (S^n, *)$ iducing isomorphisms $g^* : H^q(S^n, *) \to H^q(Y, *)$ for $q \leq n$. Similarly we define $(Y, *)$ to be *S-reducible* or *S-coreducible* if there are S-maps with the above properties. It is clear that reducibility and coreducibility are properties of the homotopy type of $(Y, *)$, while S-reducibility and S-coreducibility are properties of the S-type.

Proposition 13 *If ν is a normal fibration of a finite polyhedron X, then the Thom space X^ν is reducible.*

Proof. We consider the imbedding $N \hookrightarrow \mathbb{R}^{n+k} \hookrightarrow S^{n+k}$ and the corresponding collapsing map $t : (S^{n+k}, *) \to (N/\partial N, *)$. Then t induces isomorphisms in homology in dimensions $\geq n + k$. The result follows since $(X^\nu, *)$ is homotopy equivalent to $(N/\partial N, *)$ by proposition 8.

Theorem 14 *Let $\xi : E \to X$ be a spherical fibration over a connected, finite CW-complex X. Then X^ξ is S-coreducible if and only if $\xi = 0$ in $k_G(X)$.*

Proof. From the Thom isomorphism theorem we see that if ξ is fiber homotopy trivial, then X^ξ is coreducible. Hence if ξ is stably fiber homotopy trivial, then X^ξ is S-coreducible.

Assume conversely X^ξ is S-coreducible. Replacing ξ with a stably fiber homotopy equivalent fibration we may assume that ξ is an $(n-1)$-spherical fibration with n large, and that X^ξ is coreducible.

There is a commutative diagram of cohomotopy groups

$$\begin{array}{ccc} \pi^{n-1}(E) & \xrightarrow[\approx]{} & \pi^n(Z_\xi, E) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle j^*} \\ \pi^{n-1}(F_x) & \xrightarrow{\approx} & \pi^n(CF_x, F_x) \end{array}$$

where the horizontal maps are isomorphisms from the exact cohomotopy sequences of the pairs, and the vertical maps are induced by inclusions.

Since X^ξ is coreducible we have isomorphisms

$$H^n(S^n, *) \xrightarrow[\approx]{g^*} H^n(X^\xi, *) \xrightarrow[\approx]{c^*} H^n(Z_\xi, E) \xrightarrow[\approx]{j^*} H^n(CF_x, F_x)$$

It follows that $j^* : \pi^n(Z_\xi, E) \to \pi^n(CF_x, F_x)$ is surjective. Hence $i^* : \pi^{n-1}(E) \to \pi^{n-1}(F_x)$ is surjective, i.e. there exists a map $t : E \to S^{n-1}$ whose restriction to F_x is a homotopy equivalence. There is then a map of fibrations with $\bar{t} = (\xi, t)$

$$\begin{array}{ccccc} F_x & \longrightarrow & E & \xrightarrow{\xi} & X \\ {\scriptstyle t}\downarrow & & {\scriptstyle \bar{t}}\downarrow & & \| \\ S^{n-1} & \longrightarrow & X \times S^{n-1} & \longrightarrow & X \end{array}$$

Hence $\bar{t}$ is a weak homotopy equivalence, and therefore a homotopy equivalence, all spaces being of the homotopy type of CW-complexes. By theorem 1 of chapter 3 we get that $\bar{t}$ is a fiber homotopy equivalence, i.e. ξ is fiber homotopy trivial. This proves the theorem.

Theorem 15 *If X is a Poincaré space, there is a unique class in $k_G(X)$ whose Thom space is S-reducible.*

Proof. The elements of $k_G(X)$ have Thom spaces well defined up to S-type by proposition 11. It is sufficient to consider the case where X is a Poincaré polyhedron. Let ν be any normal fibration of X. Then X^ν is S-reducible by proposition 13. To prove uniqueness, suppose ξ is a spherical fibration over X such that X^ξ is S-reducible. From theorem 12 we know that X^ξ and $X^{\nu-\xi}$ are S-dual. Hence $X^{\nu-\xi}$ is S-coreducible, and by theorem 14 we get $\xi = \nu$ in $k_G(X)$, which proves the theorem.

The unique class $k_G(X)$ given by the theorem is called the *stable normal fibration* of X, and is denoted by ν_X. It is clear that ν_X is invariant under homotopy equivalences of Poincaré spaces. We get automatically a *stable tangent fibration* τ_X with the same property, defined by $\tau_X = -\nu_X$ in $k_G(X)$.

References

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